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Phase Transition and Critical Values of a Nearest-Neighbor System with Uncountable Local State Space on Cayley Trees

Benedikt Jahnel · Christof Külske · Golibjon I. Botirov

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Abstract We consider a ferromagnetic nearest-neighbor model on a Cayley tree of degree $k \geqslant 2$ with uncountable local state space $[0, 1]$ where the energy function depends on a parameter $\theta \in [0, 1)$. We show that for $0 \leqslant \theta \leqslant \frac{5}{3k}$ the model has a unique translation-invariant Gibbs measure. If $\frac{5}{3k} < \theta < 1$ there is a phase transition, in particular there are three translation-invariant Gibbs measures.

Keywords Cayley tree · Hammerstein's integral operator · Bifurcation analysis · Gibbs measures · Phase transition

Mathematics Subject Classification (2010) 82B20 · 82B26 · 82B27

1 Introduction

The notion of a phase transition describes the coexistence of more then one equilibrium phase of a given system. Usually models depend on one or more parameters and there exist some critical values of the parameters where the number of equilibria jump from one to more then one. This jump can be detected by looking at appropriate

B. Jahnel (✉) · C. Külske
Fakultät für Mathematik, Ruhr-Universität Bochum, D44801 Bochum, Germany
e-mail: Benedikt.Jahnel@ruhr-uni-bochum.de
URL: http://www.ruhr-uni-bochum.de/ffm/Lehrstuehle/Kuelske/jahnel.html

C. Külske
e-mail: Christof.Kuelske@ruhr-uni-bochum.de
URL: http://www.ruhr-uni-bochum.de/ffm/Lehrsttuehle/Kuelske/kuelske.html/~kuelske/

G. I. Botirov
Faculty of Physics and Mathematics, Bukhara State University, Bukhara, Uzbekistan
e-mail: botirovg@yandex.ru

observables as functions of the parameters and there points of discontinuity. From a physical perspective this amounts to a sudden change of at least one property of the modelled material. In mathematical terms existence of a phase transition for a system is given by non-uniqueness of Gibbs measures see for example [5]. In case of Cayley trees one usually employs the theory of Markov random fields and their consistency equations for boundary laws [5].

In the present note we continue the investigation from [2] and consider a model with nearest-neighbor interactions and local state space given by the uncountable set $[0, 1]$ on a Cayley tree Γ^k of general order $k \geqslant 2$. The translation-invariant Gibbs measures are studied via a non-linear functional equation and we prove non-uniqueness of translation-invariant Gibbs measures in the right parameter regime for all $k \geqslant 2$ and not only for $k \in \{2, 3\}$ as in [2].

It is known that the spin systems on trees have produced the first and most tractable examples of certain qualitative phenomena. The function ξ_{xy} is interpreted as *energy* function, which is as usual nonconstant, symmetric and continuous. In [8] the authors studied several models on general infinite trees, including the classical Heisenberg and Potts models. The model which we consider in this paper is similar to the known models (for example rotor [8], spherical [12] and many other models) with nearest-neighbor interactions which have uncountable set of spin values.

To be more specific, the Hamiltonian of the model depends on a single parameter $\theta \in [0, 1)$ and we prove that if $0 \leqslant \theta \leqslant \frac{5}{3k}$ there is a unique translational-invariant Gibbs measure and if $\frac{5}{3k} < \theta < 1$ there are three translational-invariant Gibbs measures. The design of the Hamiltonian is the result of a sequence of papers [3, 4, 11] providing examples of tree indexed models with uncountable local state space showing critical behavior of parameters.

2 The Setup

Cayley Tree and Configurations The *Cayley tree* (Bethe lattice) Γ^k of order $k \geqslant 1$ is an infinite tree, i.e a graph without cycles, such that exactly $k+1$ edges originate from each vertex (see for example [1]). Let $\Gamma^k = (V, L)$ where V is the set of *vertices* and L the set of *edges*. Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them and we denote $l = \langle x, y\rangle$. On this tree, there is a natural distance to be denoted $d(x, y)$, being the number of nearest neighbor pairs of the minimal path between the vertices x and y (by *path* one means a collection of nearest neighbor pairs, two consecutive pairs sharing at least a given vertex). For a fixed $x^0 \in V$, the *root*, let

$$W_n = \{x \in V : d\left(x, x^0\right) = n\}, \qquad V_n = \{x \in V : d\left(x, x^0\right) \leqslant n\};$$

be respectively the sphere and the ball of radius n with center at x^0, and for $x \in W_n$ let

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}$$

be the set of direct successors of x. There exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geqslant 1$ and the group G_k of the free products of $k+1$ cyclic groups of second order with generators $a_1, a_2, \ldots, a_{k+1}$. A *configuration* σ on V is then defined as a function $x \in V \to [0, 1]$; the set of all configurations is $[0, 1]^V$. For $A \subset V$ a configuration σ_A on A is an arbitrary function $\sigma_A : A \to [0, 1]$. Denote $\Omega_A = [0, 1]^A$ the set of all configurations on A.

Hamiltonians and Gibbs Measures We consider a model where the spin takes values in the set $[0, 1]$, and is assigned to the vertices of the Cayley tree. The (formal) *Hamiltonian* of the model is given by

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \xi_{\sigma(x)\sigma(y)}, \tag{2.1}$$

where $J \in \mathbb{R} \setminus \{0\}$ and $\xi : (u, v) \in [0; 1]^2 \to \xi_{uv} \in \mathbb{R}$ is a given bounded, measurable function.[1] Let λ be the Lebesgue measure on $[0, 1]$. On the set of all configurations on A the a priori measure λ_A is introduced as the $|A|$ fold product of the measure λ. Here and further on $|A|$ denotes the cardinality of A. We consider a standard sigma-algebra $\mathbb{B}$ of subsets of $\Omega = [0, 1]^V$ generated by the measurable cylinder subsets. A probability measure μ on $(\Omega, \mathbb{B})$ is called a *Gibbs measure* (with Hamiltonian H) if it satisfies the Dobrushin-Lanford-Ruelle (DLR) equation, namely for any $n = 1, 2, \ldots,$ and $\sigma_n \in \Omega_{V_n}$:

$$\mu\left(\{\sigma \in \Omega : \sigma|_{V_n} = \sigma_n\}\right) = \int_{\Omega} \mu(d\omega)\nu^{V_n}_{\omega|W_{n+1}}(\sigma_n),$$

where $\nu^{V_n}_{\omega|W_{n+1}}$ is the conditional Gibbs density

$$\nu^{V_n}_{\omega|W_{n+1}}(\sigma_n) = \frac{1}{Z_n(\omega|_{W_{n+1}})} \exp\left(-\beta H(\sigma_n, \omega|_{W_{n+1}})\right),$$

and $\beta = \frac{1}{T},\ T > 0$ is the temperature.

Denote by $L_n = \{\langle x, y\rangle \in L : x, y \in V_n\}$ the set of edges in a ball with radius n. Let Ω_{V_n} be the set of configurations in V_n and Ω_{W_n} the set of configurations in W_n. Furthermore, $\sigma|_{V_n}$ and $\omega|_{W_n}$ denote the restrictions of configurations $\sigma, \omega \in \Omega$ to V_n and W_{n+1}, respectively. Next, $\sigma_n : x \in V_n \mapsto \sigma_n(x)$ is a configuration in V_n and $H\left(\sigma_n, \omega|_{W_{n+1}}\right)$ is defined as the sum $H(\sigma_n) + U\left(\sigma_n, \omega|_{W_{n+1}}\right)$ where

$$H(\sigma_n) = -J \sum_{\langle x,y\rangle \in L_n} \xi_{\sigma_n(x)\sigma_n(y)},$$

$$U\left(\sigma_n, \omega|_{W_{n+1}}\right) = -J \sum_{\substack{\langle x,y\rangle:\\ x\in V_n,\ y\in W_{n+1}}} \xi_{\sigma_n(x)\omega(y)}.$$

[1] We note that the reason to study such models is their simplicity and these models may have non-uniqueness of Gibbs measures, i.e. provide examples of models with uncountable spin values with phase transitions [10].

Finally, $Z_n\left(\omega\mid_{W_{n+1}}\right)$ stands for the partition function in V_n, with the boundary condition $\omega\mid_{W_{n+1}}$:

$$Z_n\left(\omega\mid_{W_{n+1}}\right)=\int_{\Omega_{V_n}}\exp\left(-\beta H\left(\widetilde{\sigma}_n,\omega\mid_{W_{n+1}}\right)\right)\lambda_{V_n}(d\widetilde{\sigma}_n).$$

An Integral Equation Write $x<y$ if the path from x^0 to y goes through x. Call vertex y a *direct successor* of x if $y>x$ and x,y are nearest neighbors. Denote by $S(x)$ the set of direct successors of x. Observe that any vertex $x\neq x^0$ has k direct successors and x^0 has $k+1$. Let $h:\ x\in V\mapsto h_x=(h_{t,x},t\in[0,1])\in\mathbb{R}^{[0,1]}$ be a mapping of $x\in V\setminus\{x^0\}$. Given $n=1,2,\ldots$, consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n)=Z_n^{-1}\exp\left(-\beta H(\sigma_n)+\sum_{x\in W_n}h_{\sigma(x),x}\right). \tag{2.2}$$

Here, as before, $\sigma_n:x\in V_n\mapsto\sigma(x)$ and Z_n is the corresponding partition function:

$$Z_n=\int_{\Omega_{V_n}}\exp\left(-\beta H(\widetilde{\sigma}_n)+\sum_{x\in W_n}h_{\widetilde{\sigma}(x),x}\right)\lambda_{V_n}(d\widetilde{\sigma}_n). \tag{2.3}$$

The probability distributions $\mu^{(n)}$ are called *compatible* if for any $n\geqslant 1$ and $\sigma_{n-1}\in\Omega_{V_{n-1}}$:

$$\int_{\Omega_{W_n}}\mu^{(n)}(\sigma_{n-1}\vee\omega_n)\lambda_{W_n}(d(\omega_n))=\mu^{(n-1)}(\sigma_{n-1}). \tag{2.4}$$

Here $\sigma_{n-1}\vee\omega_n\in\Omega_{V_n}$ is the concatenation of σ_{n-1} and ω_n. In this case, because of the Kolmogorov extension theorem, there exists a unique measure μ on Ω_V such that, for any n and $\sigma_n\in\Omega_{V_n}$, $\mu\left(\left\{\sigma\Big|_{V_n}=\sigma_n\right\}\right)=\mu^{(n)}(\sigma_n)$. The following proposition is proved in [11].

Proposition 2.1 *The probability distributions $\mu^{(n)}(\sigma_n)$, $n=1,2,\ldots$, in (2.2) are compatible iff for any $x\in V\setminus\{x^0\}$ the following equation holds:*

$$f(t,x)=\prod_{y\in S(x)}\frac{\int_0^1\exp(J\beta\xi_{tu})f(u,y)du}{\int_0^1\exp(J\beta\xi_{0u})f(u,y)du}. \tag{2.5}$$

Here, and below $f(t,x)=\exp(h_{t,x}-h_{0,x})$, $t\in[0,1]$ and $du=\lambda(du)$ is the Lebesgue measure.

From Proposition 2.1 it follows that for any $h=\{h_x\in\mathbb{R}^{[0,1]},x\in V\}$ satisfying (2.5) there exists a unique Gibbs measure μ and vice versa. However, the analysis of solutions to (2.5) is not easy. This difficulty depends on the given function ξ.

In [2] a function ξ_{tu} is constructed under which the (2.5) has at least two solutions (on the Cayley tree of order $k = 2$ and 3) in the class of translation-invariant functions $f(t, x)$, i.e $f(t, x) = f(t)$, for any $x \in V$. In this paper we generate this result for any $k \geqslant 2$. For translation-invariant functions (2.5) can be written as

$$f(t) = \left(\frac{\int_0^1 K(t, u) f(u) du}{\int_0^1 K(0, u) f(u) du} \right)^k, \tag{2.6}$$

where $K(t, u) = \exp(J\beta\xi_{tu})$, $f(t) > 0$, $t, u \in [0, 1]$. We put

$$C^+[0, 1] = \{f \in C[0, 1] : f(x) \geqslant 0\}.$$

We are interested in positive continuous solutions to (2.6).

A Representation of Solutions For every $k \in \mathbb{N}$ we consider an integral operator H_k acting in the cone $C^+[0, 1]$ as

$$(H_k f)(t) = \int_0^1 K(t, u) f^k(u) du, \ \ k \in \mathbb{N}.$$

The operator H_k is called *Hammerstein's integral operator* of order k. This operator is well known to generate ill-posed problems [6, 7]. Clearly, if $k \geqslant 2$ then H_k is a nonlinear operator.

It is known that the set of translation-invariant Gibbs measures of the model (2.1) is described by the fixed points of Hammerstein's operator (see [11]).

Let $k \geqslant 2$ in the model (2.1) and

$$\xi_{t,u} = \xi_{t,u}(\theta, \beta) = \frac{1}{\beta} \ln \left(1 + \theta \sqrt[3]{4 \left(t - \frac{1}{2} \right) \left(u - \frac{1}{2} \right)} \right), \ t, u \in [0, 1] \tag{2.7}$$

where $0 \leqslant \theta < 1$. Then for the kernel $K(t, u)$ of the Hammerstein operator H_k we have

$$K(t, u) = 1 + \theta \sqrt[3]{4 \left(t - \frac{1}{2} \right) \left(u - \frac{1}{2} \right)}.$$

Note that this model was first considered in [2] and for $k = 2, 3$ it was shown non-uniqueness of translation-invariant Gibbs measures. In this paper we generalize the results of [2] to arbitrary $k \geqslant 2$. In [2] we defined the operator $V_k : (x, y) \in \mathbb{R}^2 \to (x', y') \in \mathbb{R}^2$ by

$$V_k : \begin{cases} x' = 3\Bigg(\dfrac{\left(x+y\theta\sqrt[3]{2}\right)^{k+1}-\left(x-y\theta\sqrt[3]{2}\right)^{k+1}}{2\sqrt[3]{2}(k+1)y\theta} - \dfrac{\left(x+y\theta\sqrt[3]{2}\right)^{k+2}+\left(x-y\theta\sqrt[3]{2}\right)^{k+2}}{\sqrt[3]{4}(k+1)(k+2)y^2\theta^2} + \\ \qquad + \dfrac{\left(x+y\theta\sqrt[3]{2}\right)^{k+3}-\left(x-y\theta\sqrt[3]{2}\right)^{k+3}}{2(k+1)(k+2)(k+3)y^3\theta^3}\Bigg) \\ y' = 3\Bigg(\dfrac{\left(x+y\theta\sqrt[3]{2}\right)^{k+1}+\left(x-y\theta\sqrt[3]{2}\right)^{k+1}}{2\sqrt[3]{4}(k+1)y\theta} - \dfrac{3\left(\left(x+y\theta\sqrt[3]{2}\right)^{k+2}-\left(x-y\theta\sqrt[3]{2}\right)^{k+2}\right)}{4(k+1)(k+2)y^2\theta^2} + \\ \qquad + \dfrac{3\left(\left(x+y\theta\sqrt[3]{2}\right)^{k+3}+\left(x-y\theta\sqrt[3]{2}\right)^{k+3}\right)}{2\sqrt[3]{2}(k+1)(k+2)(k+3)y^3\theta^3} - \dfrac{3\left(\left(x+y\theta\sqrt[3]{2}\right)^{k+4}-\left(x-y\theta\sqrt[3]{2}\right)^{k+4}\right)}{2\sqrt[3]{4}(k+1)(k+2)(k+3)(k+4)y^4\theta^4}\Bigg) \end{cases} \tag{2.8}$$

and the following proposition was proved.

Proposition 2.2 *A function $\varphi \in C[0, 1]$ is a solution of Hammerstein's equation*

$$(H_k f)(t) = f(t) \tag{2.9}$$

iff $\varphi(t)$ has the following form

$$\varphi(t) = x' + y'\theta\sqrt[3]{4\left(t - \frac{1}{2}\right)},$$

where $(x', y') \in \mathbb{R}^2$ is a fixed point of the operator V_k (2.8).

For $k = 2, 3$ and in the right parameter regime non-uniqueness of translation-invariant Gibbs measures was proved.

3 Bifurcation Analysis of the System

The function V_k can be written in the following way. First for even $k \geqslant 2$:

$$V_k(x, y) = \begin{cases} x' = 3 \sum\limits_{l=0,2,\ldots,k} \binom{k}{l} x^l \left(\sqrt[3]{2}\theta y\right)^{k-l} A_k(l) \\ y' = 3 \sum\limits_{l=1,3,\ldots,k-1} \binom{k}{l} x^l \left(\sqrt[3]{2}\theta y\right)^{k-l} B_k(l) \end{cases} \tag{3.1}$$

And for odd $k \geqslant 3$:

$$V_k(x, y) = \begin{cases} x' = 3 \sum\limits_{l=1,3,\ldots,k} \binom{k}{l} x^l \left(\sqrt[3]{2}\theta y\right)^{k-l} A_k(l) \\ y' = 3 \sum\limits_{l=0,2,\ldots,k-1} \binom{k}{l} x^l \left(\sqrt[3]{2}\theta y\right)^{k-l} B_k(l) \end{cases} \tag{3.2}$$

Here we wrote

$$\begin{aligned} A_k(l) &:= \tfrac{1}{(k+1-l)} - \tfrac{2}{(k+2-l)(k+1-l)} + \tfrac{2}{(k+3-l)(k+2-l)(k+1-l)} \\ B_k(l) &:= \tfrac{\sqrt[3]{4}}{2(k+1-l)} - \tfrac{3\sqrt[3]{4}}{2(k+2-l)(k+1-l)} + \tfrac{3\sqrt[3]{4}}{(k+3-l)(k+2-l)(k+1-l)} \\ &\quad - \tfrac{3\sqrt[3]{4}}{(k+4-l)(k+3-l)(k+2-l)(k+1-l)} \end{aligned}$$

We prove the following theorem.

Theorem 3.1 *a) If* $0 \leqslant \theta \leqslant \frac{5}{3k}$, *then the Hammerstein operator* H_k *has a unique (nontrivial) positive fixed point in* $C[0,1]$*;*

b) If $\frac{5}{3k} < \theta < 1$, *then there are exactly three positive fixed points in C[0,1] of Hammerstein's operator.*

Proof **Case** $y = 0$. In this case from (3.1) and (3.2) for $V_k(x, y) = (x, y)$ we get

$$x = x^k$$

this equation has two solutions $x = 0, \quad x = 1$ if k is even and three solutions $x = 0, \quad x = \pm 1$ if k is odd. Thus $V_k(x, y)$ has two (three) fixed points $(0,0),\ (1,0) \quad ((0,0),\ (1,0),\ (-1,0))$. Hence by Proposition 2.1 we have one positive solution $\varphi(t) = 1$ of Hammerstein's equation for any $k \geqslant 1$.

Case $y \neq 0$. Let us start with k even say $k = 2m$. For $z = \frac{x}{y}$ this fixed point equation reads

$$z = \frac{\sum\limits_{l=0,2,\dots,2m} \binom{k}{l} z^l \left(\sqrt[3]{2}\theta\right)^{k-l} A_k(l)}{\sum\limits_{l=1,3,\dots,2m-1} \binom{k}{l} z^l \left(\sqrt[3]{2}\theta\right)^{k-l} B_k(l)} =: f(z) \tag{3.3}$$

In order to determine the number of fixed points we need to determine the number of solutions of the equation

$$\sum_{l=0,2,\dots,k} \binom{k}{l} z^l \left(\sqrt[3]{2}\theta\right)^{k-l} A_k(l) = \sum_{l=2,4,\dots,k} \binom{k}{l-1} z^l \left(\sqrt[3]{2}\theta\right)^{k-l+1} B_k(l-1) \tag{3.4}$$

which is equivalent to finding number of positive roots of the polynomial

$$r_{k,0}(\theta) + \sum_{1,2,\dots,m} r_{k,2l}(\theta) t^l = 0 \tag{3.5}$$

where $t = z^2$ and $r_{k,0}(\theta) := \left(\sqrt[3]{2}\theta\right)^k A_k(0) \geqslant 0$,

$$r_{k,l}(\theta) := \left(\sqrt[3]{2}\theta\right)^{k-l} \left[\binom{k}{l} A_k(l) - \binom{k}{l-1} \left(\sqrt[3]{2}\theta\right) B_k(l-1)\right]. \tag{3.6}$$

Notice that the sign of the coefficient $r_{k,l}(\theta)$ is determined by the sign of

$$m_{k,l}(\theta) := \tfrac{1}{l}((k-l)(k+3-l)+2) - \theta\left(k-l+\tfrac{6}{k+4-l}-\tfrac{6}{(k+5-l)(k+4-l)}\right)\Big].$$

In other words, for fixed even k and $l \in \{2,4,\dots,k\}$, $m_{k,l}(\theta_0)=0$ marks the value of θ where the sign changes. Notice the derivative of m is negative, hence m is positive for $\theta < \theta_0$ and negative for $\theta > \theta_0$. Also notice, the critical value $\theta_0(l)$ is decreasing in l, indeed, the function

$$f_k(j) := \frac{(k-2j)(k+3-2j)+2}{2j\left(\left(k-2j+\frac{6}{k+4-2j}-\frac{6}{(k+5-2j)(k+4-2j)}\right)\right.},$$

where we substituted $2j := l$ is strictly decreasing which can be verified in the following way:

$$f_k'(j) = -\tfrac{j^2(4k+12)-j(24k+20+4k^2)+(23k+9k^2+k^3+15)}{(3+k-2j)^2j^2}$$

Since the denominator is positive we verify that the function $g_k(j) := j^2(4k+12) - j(24k+20+4k^2)+(23k+9k^2+k^3+15) \geqslant 0$. But this is true since it attains its global minimum at $j_0 = \frac{k^2+6k+5}{2k+6} \geqslant \frac{k}{2}$ with $g_k(j_0) = \frac{4(5+6k+k^2)}{3+k} \geqslant 0$. So we have a decreasing sequence of critical $(\theta_0(l))_{l\in\{2,4,\dots,k\}}$ with the lowest $\theta_0(k) = \theta_{cr} = \frac{5}{3k}$. Below θ_{cr} all coefficients $r_{k,l}(\theta)$ are positive and hence, there is no nontrivial root for the polynomial. Above the critical value θ_{cr} the bifurcation picture changes and the polynomial has exactly one real root t_0 by Descartes' rule of signs. Hence by $z^2 = t_0$ we have $z = \pm\sqrt{t_0}$.

For even k we have $z_0 = \sqrt{t_0}$, $z_1 = -\sqrt{t_0}$, so $z_0 = f(z_0) =: \frac{F_1(z_0)}{F_2(z_0)}$ one can recover solutions for the two-dimensional fixed point equation:

$$\begin{cases} (x_0, y_0) = \left(\frac{F_1(z_0)}{F_2(z_0)} G_0^{\frac{1}{1-k}},\ G_0\right) \\ (x_1, y_1) = \left(\frac{F_1(z_1)}{F_2(z_1)} G_1^{\frac{1}{1-k}},\ G_1\right) = (x_0, -y_0). \end{cases} \tag{3.7}$$

where $G_0 = 3\sum\limits_{l=0,2,\dots,k}\binom{k}{l}\left(\frac{F_1(z_0)}{F_2(z_0)}\right)^{l-1}\left(\sqrt[3]{2}\theta\right)^{k-l}A_k(l)$ and
$G_1 = 3\sum\limits_{l=1,\dots,k}\binom{k}{l}\left(\frac{F_1(z_1)}{F_2(z_1)}\right)^{l-1}\left(\sqrt[3]{2}\theta\right)^{k-l}A_k(l).$

Consequently by Proposition 2.1 the operator H_k has a unique positive fixed point $\varphi_1(t) \equiv 1$ if $0 \leqslant \theta \leqslant \frac{5}{3k}$. In the case $\frac{5}{3k} < \theta < 1$ the functions

$$\varphi_1(t) \equiv 1,\ \varphi_2(t) = x_0 + y_0\theta\sqrt[3]{4\left(t-\frac{1}{2}\right)},\ \ \varphi_3(t) = x_0 - y_0\theta\sqrt[3]{4\left(t-\frac{1}{2}\right)} \tag{3.8}$$

φ_1, φ_2 and φ_3 all are fixed points (2.9).

Since $(0, 0)$ is always a solution of the fixed point equation and V_k is odd, above the critical value we have exactly three solutions.

For odd k, (3.5) reads $\sum_{l=1,3,\dots,k} r_{k,l}(\theta)z^l = 0$. Since we are looking for positive roots, we can divide by z and look at $r_{k,1}(\theta) + \sum_{l=2,\dots,k-1} r_{k,l+1}(\theta)z^l = 0$. By the same calculations as in the even case we find that the bifurcation picture changes at θ_{cr}. □

From Proposition 2.1 and Theorem 3.1 it follows that

Theorem 3.2 *a) If* $0 \leqslant \theta \leqslant \frac{5}{3k}$ *for the model* (2.1) *on the Cayley tree* Γ^k*, then there exists a unique translation-invariant Gibbs measure;*

b) If $\frac{5}{3k} < \theta < 1$*, then there exist three translation-invariant Gibbs measures.*

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