On one irreducible component in the variety of Leibniz algebras

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Аннотация

Maqolada bitta xosil qiluvchiga ega bo'lgan Leibniz algebralari to'plami ochiq to'plam bo'lishi ko'rsatilgan va uning yopilmasi keltirilmaydigan komponenta bo'lishi isbotlangan.

В данной работе показано, что множество однопорожденных алгебр Лейбница является открытым множеством и ее замыкание является неприводимой компонентой многообразия алгебр Лейбница.

The theory of deformations originated with the problem of classifying all possible pairwise non-isomorphic complex structures on a given differentiable real manifold. Formal deformations of arbitrary rings and associative algebras, and the related cohomology questions, were first investigated by Gerstenhaber [4]. Firstly the notion of deformation was applied to Lie algebras by Nijenhuis and Richardson [7]. Because various fields in mathematics and physics exist in which deformations are used. They studied one-parameter deformations and established connection between Lie algebra cohomology and infinitesimal deformations.

Recall, that Leibniz algebras are generalization of Lie algebras [5], [6] and it is natural to apply the theory of deformations for the study of Leibniz algebras.

Let V be the underlying vector space of the Leibniz algebra L of dimension n and let $GL(V)$ be the group of the invertible linear mappings f such that $f \in GL_n(F)$. The action of the group $GL(V)$ on the variety of Leibniz algebras induces an action on the Leibniz algebras variety: two laws μ_1 and μ_2 are isomorphic, if there exists a linear mapping f, such that

$$\mu_2(x, y) = f^{-1}(\mu_1(f(x), f(y))) \text{ for all } x \in V_\alpha, y \in V_\beta.$$

The orbit under this action denoted by $Orb(\mu)$ and consists of all algebras isomorphic to the algebra μ. Therefore the description of n-dimensional algebras with dimensions n (further denoted by $Leib_n$) can be reduced to a geometric problem of classification of orbits under the action of the group $GL(V)$.

From algebraic geometry it is known that an algebraic variety is an union of irreducible components. The algebras with open orbits in the variety of Leibniz algebras are *called rigid*. The closures of these open orbits give irreducible components of the variety. Therefore studying of the rigid algebras is crucial problem from the geometrical point of view. The problem of finding such algebras is crucial for the description of the variety $Leib_n$.

In this paper we calculate the second group of cohomology of the null-filiform Leibniz algebra and show that the set of single-generated Leibniz algebras forms an irreducible component of the variety of Leibniz algebras.

Moreover, it established that any single-generated algebra is linear integrable deformation of null-filiform algebra.

Throughout the paper we consider finite-dimensional vector spaces and algebras over the field of zero characteristic. Moreover, in the multiplication table of a Leibniz algebra the omitted products assumed to be zero.

Definition 1. A Leibniz algebra over F is a vector space L equipped with a bilinear map, called bracket, $[-,-] : L \times L \to L$ satisfying the Leibniz identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y],$$

for all $x, y, z \in L$.

We call a vector space M a module over L if there are two bilinear maps:

$$[-,-] : L \times M \to M \qquad \text{and} \qquad [-,-] : M \times L \to M$$

satisfying the following three axioms

$$\begin{aligned} [m,[x,y]] &= [[m,x],y] - [[m,y],x], \\ [x,[m,y]] &= [[x,m],y] - [[x,y],m], \\ [x,[y,m]] &= [[x,y],m] - [[x,m],y], \end{aligned}$$

for any $m \in M$, $x, y \in L$.

Given a Leibniz algebra L, let $C^n(L,M)$ be the space of all F-linear homogeneous mapping $L^{\otimes n} \to M$, $n \geq 0$ and $C^0(L,M) = M$.

Let $d^n : C^n(L,M) \to C^{n+1}(L,M)$ be an F-homomorphism defined by

$$(d^n f)(x_1,\dots,x_{n+1}) := [x_1, f(x_2,\dots,x_{n+1})] + \sum_{i=2}^{n+1}(-1)^i[f(x_1,\dots,\widehat{x}_i,\dots,x_{n+1}),x_i]$$

$$+ \sum_{1\leq i<j\leq n+1}(-1)^{j+1} f(x_1,\dots,x_{i-1},[x_i,x_j],x_{i+1},\dots,\widehat{x}_j,\dots,x_{n+1}),$$

where $f \in C^n(L,M)$ and $x_i \in L$. Since the derivative operator $d = \sum\limits_{i\geq 0} d^i$ satisfies the property $d \circ d = 0$, the n-th cohomology group is well defined and

$$HL^n(L,M) = ZL^n(L,M)/BL^n(L,M),$$

where the elements $ZL^n(L,M)$ and $BL^n(L,M))$ are called *n-cocycles* and *n-coboundaries*, respectively.

Usually a 2-cocycle is called infinitesimal deformation.

A deformation of a Leibniz algebra L is a one-parameter family L_t of Leibniz algebras with the bracket

$$\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \cdots,$$

where φ_i are 2-cochains, i.e., elements of $Hom(L \otimes L, L) = C^2(L,L)$.

Note that a linear integrable deformation φ satisfies the condition

$$\varphi(x, \varphi(y,z)) - \varphi(\varphi(x,y),z) + \varphi(\varphi(x,z),y) = 0. \tag{1}$$

For a Leibniz algebra L consider the following central lower series:

$$L^1 = L, \quad L^{k+1} = [L^k, L^1], \quad k \geq 1.$$

Definition 2. A Leibniz algebra L is said to be nilpotent, if there exists $p \in \mathbb{N}$ such that $L^p = 0$.

Now we give the notion of null-filiform Leibniz algebra.

Definition 3. An n-dimensional Leibniz algebra is said to be null-filiform if $\dim L^i = n+1-i, \ 1 \leq i \leq n+1$.

Theorem 1 [1]. An arbitrary n-dimensional null-filiform Leibniz algebra is isomorphic to the algebra:

$$NF_n: \quad [e_i, e_1] = e_{i+1}, \ 1 \leq i \leq n-1,$$

where $\{e_1, e_2, \ldots, e_n\}$ is a basis of the algebra NF_n.

Note that any derivation of null-filiform Leibniz algebra NF^n has the following matrix form [3]:

$$\begin{pmatrix} a_1 & a_2 & a_3 & \ldots & a_n \\ 0 & 2a_1 & a_2 & \ldots & a_{n-1} \\ 0 & 0 & 3a_1 & \ldots & a_{n-2} \\ \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & 0 & \ldots & na_1 \end{pmatrix}.$$

From this we conclude that $DimBL^2(NF^n, NF^n) = n^2 - n$.

In general, 2-cocycle is a bilinear map from $NF^n \otimes NF^n$ to NF^n such that $d^2\varphi = 0$, i.e.

$$d^2\varphi(x,y,z) = [x,\varphi(y,z)] - [\varphi(x,y),z] + [\varphi(x,z),y] + \varphi(x,[y,z]) - \varphi([x,y],z) + \varphi([x,z],y).$$

Proposition 1. The following cochains:

$$\varphi_{j,k}(x_j, x_1) = x_k, \ 1 \leq j \leq n, \ 2 \leq k \leq n,$$

$$\psi_j \ (1 \leq j \leq n-1) = \begin{cases} \psi_j(x_j, x_1) = x_1, \\ \psi_j(x_i, x_{j+1}) = -x_{i+1}, \quad 1 \leq i \leq n-1, \end{cases}$$

form the basis of $ZL^2(NF^n, NF^n)$.

Proof. Using the Leibniz 2-cocycle property $(d^2\varphi)(x_i, x_1, x_1) = 0$, we have

$$\varphi(x_i, x_2) = -[x_i, \varphi(x_1, x_1)], \quad 1 \leq i \leq n-1, \quad \varphi(x_n, x_2) = 0 \qquad (2)$$

The conditions $(d^2\varphi)(x_i, x_1, x_j) = 0$, $(d^2\varphi)(x_i, x_j, x_1) = 0$ for $1 \leq i \leq n$, $2 \leq j \leq n$ imply

$$[x_i, \varphi(x_1, x_j)] + [\varphi(x_i, x_j), x_1] - \varphi([x_i, x_1], x_j) = 0,$$

$$[x_i, \varphi(x_j, x_1)] - [\varphi(x_i, x_j), x_1] + \varphi(x_i, [x_j, x_1]) + \varphi([x_i, x_1], x_j) = 0.$$

Summarizing above equalities, we derive

$$\begin{cases} \varphi(x_i, x_{j+1}) = -[x_i, \varphi(x_1, x_j) + \varphi(x_j, x_1)], & 1 \leq i \leq n-1, \\ & 2 \leq j \leq n-1, \\ \varphi(x_n, x_{j+1}) = 0, & 2 \leq j \leq n-1, \\ [x_i, \varphi(x_1, x_n) + \varphi(x_n, x_1)] = 0, & 1 \leq i \leq n. \end{cases} \qquad (3)$$

Set $\varphi(x_j, x_1) = \sum\limits_{k=1}^{n} a_{j,k} x_k$ for $1 \leq i \leq n$.

Using inductively method from equalities (2) and (3) we get $a_{n,1} = 0$ and

$$\varphi(x_i, x_{j+1}) = -a_{j,1} x_{i+1}, \quad 1 \leq i \leq n-1, \ 1 \leq j \leq n-1.$$

Therefore, we obtain that any infinitesimal deformation of NF^n has the following form:

$$\begin{cases} \varphi(x_j, x_1) = a_{j,1}x_1 + a_{j,2}x_2 + \cdots + a_{j,n}x_n, & 1 \le j \le n-1 \\ \varphi(x_n, x_1) = a_{n,2}x_2 + \cdots + a_{n,n}x_n, & \\ \varphi(x_i, x_{j+1}) = -a_{j,1}x_{i+1}, & 1 \le i \le n-1, \ 1 \le j \le n-1. \end{cases}$$

Therefore, $\varphi_{j,k}$ and ψ_j form a basis of $ZL^2(NF^n, NF^n)$. $\square$

Corollary 1. $dim(ZL^2(NF^n, NF^n)) = n^2 - 1$.

Below we describe a basis of subspace $BL^2(NF^n, NF^n)$ in terms of $\varphi_{j,k}$ and ψ_j.

Proposition 3. The cocycles

$$\xi_{j,k}: \quad \begin{cases} \xi_{j,1} = \psi_{j-1} - \varphi_{j,2}, & 2 \le j \le n, \\ \xi_{j,k} = \varphi_{j-1,k}, & 2 \le j \le k \le n, \\ \xi_{j,k} = \varphi_{j-1,k} - \varphi_{j,k+1}, & 2 \le k < j \le n \end{cases}$$

form a basis of $BL^2(NF^n, NF^n)$.

Proof. Consider endomorphisms $f_{j,k}$ defined as follows:

$$f_{j,k}(x_j) = x_k, \ 2 \le j \le n, \ 1 \le k \le n.$$

It is easy to see that $f_{j,k}$ are complement of derivations to $C^1(NF^n, NF^n)$. Therefore, the elements of the space $BL^2(NF^n, NF^n)$ are $d^1 f_{j,k}$ such that $d^1 f_{j,k} = f_{j,k}([x, y]) - [f_{j,k}(x), y] - [x, f_{j,k}(y)]$.

Then we obtain

$$d^1 f_{j,1} \ (2 \le j \le n) = \begin{cases} d^1 f_{j,1}(x_{j-1}, x_1) = x_1, \\ d^1 f_{k,1}(x_j, x_1) = -x_2, \\ d^1 f_{j,1}(x_i, x_j) = -x_{i+1}, \quad 2 \le i \le n-1, \end{cases}$$

$$d^1 f_{j,k} \begin{pmatrix} 2 \le j \le n, \\ 2 \le k \le n-1 \end{pmatrix} = \begin{cases} d^1 f_{j,k}(x_{j-1}, x_1) = x_k, \\ d^1 f_{j,k}(x_j, x_1) = -x_{k+1}, \end{cases}$$

$$d^1 f_{k,n} \ (2 \le k \le n) = \{ d^1 f_{k,n}(x_{k-1}, x_1) = x_n.$$

It should be noted that

$$\begin{cases} d^1 f_{j,1} = \psi_{j-1} - \varphi_{j,2} & 2 \le j \le n, \\ d^1 f_{j,k} = \varphi_{j-1,k} - \varphi_{j,k+1}, & 2 \le j \le n, \ 2 \le k \le n-1, \\ d^1 f_{j,n} = \varphi_{j-1,n}, & 2 \le j \le n. \end{cases}$$

From the condition $d^1 f_{k,s} + d^1 f_{k+1,s+1} + \cdots + d^1 f_{n+k-s,n} = \varphi_{k-1,s}$ for $2 \leq k \leq s \leq n$, we conclude that the maps $\xi_{k,s},\ 2 \leq k \leq n,\ 1 \leq s \leq n$ form the basis of $BL^2(NF^n, NF^n)$. $\square$

Corollary 2. The adjoint classes $\overline{\varphi_{n,k}}$ $(2 \leq k \leq n)$ form a basis of $HL^2(NF^n, NF^n)$. Consequently, $dim HL^2(NF^n, NF^n) = n - 1$.

In the following proposition we describe infinitesimal deformations of NF^n satisfying the equality (1).

Proposition 4. A 2-cocycle of NF^n satisfy the equality (1) if and only if it has the form:

$$\sum_{j,k} a_{j,k}\varphi_{j,k}.$$

Proof. It is easy to check that 2-cocycles of the form $\sum\limits_{j,k} a_{j,k}\varphi_{j,k}$ are satisfy the equality (1).

Let $\varphi \in ZL^2(NF^n, NF^n)$, then $\varphi = \sum\limits_{j,k} a_{j,k}\varphi_{k,s} + \sum\limits_{j=1}^{n-1} b_j\psi_k$.

From condition

$$\varphi(x_1, \varphi(x_1, x_1)) - \varphi(\varphi(x_1, x_1), x_1) + \varphi(\varphi(x_1, x_1), x_1) = 0,$$

we get $b_1 = 0$.

The following chain of equalities

$$\varphi(x_i, \varphi(x_j, x_{j+1})) - \varphi(\varphi(x_i, x_j), x_{j+1}) + \varphi(\varphi(x_i, x_{j+1}), x_j) =$$
$$\varphi(x_i, \psi_j(x_j, x_{j+1})) - \varphi(\psi_{j-1}(x_i, x_j), x_{j+1}) + \varphi(\psi_j(x_i, x_{j+1}), x_j) =$$
$$-\psi_j(x_i, b_j x_{j+1}) + \psi_j(b_{j-1}x_{i+1}, x_{j+1}) - \psi_{j-1}(b_j x_{j+1}, x_j) =$$
$$b_j^2 x_{i+1} - b_j b_{j-1} x_{i+2} + b_j b_{j-1} x_{i+2} = b_j^2 x_{i+1}$$

imply $b_j = 0,\ 2 \leq j \leq n-1$.

Consider linear integrable deformations $\mu_t = NF^n + t\sum\limits_{j,k} a_{j,k}\varphi_{j,k}$ of NF^n.

Since every non-trivial equivalence class of deformations defines uniquely an element of $HL^2(L, L)$, due to Corollary 2 it is sufficient to consider $\mu_t(a_2, a_3, \ldots, a_n) = NF^n + t\sum\limits_{k=2}^{n} a_k\varphi_{n,k}$, where $(a_2, a_3, \ldots, a_n) \neq (0, 0, \ldots, 0)$.

Thus, the table of multiplication of $\mu_t(a_2, a_3, \ldots, a_n)$ has the form

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n-1, \\ [x_n, x_1] = t\sum\limits_{k=2}^{n} a_k x_k. & \end{cases}$$

Putting $a'_k = ta_k$, we can assume $t = 1$.

Proposition 5. An arbitrary single-generated Leibniz algebra admit a basis $\{x_1, x_2, \ldots, x_n\}$ such that the table of multiplication has the form of $\mu_1(a_2, a_3, \ldots, a_n)$.

Proof Let L be a single generated Leibniz algebras and let x be a generator of L. We put

$$x_1 = x, \quad x_2 = [x, x], \quad x_3 = [[x, x], x], \quad \ldots, \quad x_n = [[x, x], ..., x].$$

Since x is a generator, $\{x_1, x_2, \ldots, x_n\}$ form a basis of L. Evidently $\{x_2, \ldots, x_n\}$ belong to right annihilator of L. Hence, we have $[x_i, x_j] = 0,\ 2 \leq j \leq n-1$. Let $[x_n, x_1] = \sum\limits_{k=1}^{n} a_k x_k$.

From the Leibniz identity $[x_1, [x_n, x_1]] = [[x_1, x_n], x_1] - [[x_1, x_n], x_1] = 0$, we conclude $a_1 = 0$. Therefore, we obtain the existence a basis $\{x_1, x_2, \ldots, x_n\}$ in any single-generated Leibniz algebra such that the table of multiplication in this basis has the form:

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n-1, \\ [x_n, x_1] = \sum\limits_{k=2}^{n} a_k x_k. & \end{cases}$$

Let a_j is the first non vanishing parameter in algebra $\mu(a_2, a_3, \ldots, a_n)$, then by scaling $x'_i = \frac{1}{\sqrt[n-j+1]{a_j}^{\,i}} x_i,\ 1 \leq i \leq n$, we can assume $a_j = 1$, i.e the first non-vanishing parameter can be taken equal to 1.

Note that the set of single-generated Leibniz algebras is open. Indeed, if q-generated $(q > 1)$ Leibniz algebra with a basis $\{e_1, e_2, \ldots, e_n\}$, then for any $e_i \in L$ the elements $e_i, e_i^2, \ldots, e_i^n$ are linear depended. That is determinants of the matrices $A_i,\ 1 \leq i \leq n$ which consists of the rows $e_i, e_i^2, \ldots, e_i^n$ are zero, hence we get n-times of polynomials with structure constants of the algebra. Therefore, q-generated $(q > 1)$ Leibniz algebras form a closed set. Taking into account that the set of all single-generated Leibniz algebras is complemented set to the closed set, we conclude that the set of single-generated Leibniz algebras is open.

It is easy to see that an algebra $\mu_1(a_2, a_3, \ldots, a_n)$ is a linear deformation of an algebra $\mu_1(a'_2, a'_3, \ldots, a'_n)$.

Since $dim(Der(\mu_1(a_2, a_3, \ldots, a_n))) = n - 1$, $(a_2, a_3, \ldots, a_n) \neq (0, 0, \ldots, 0)$, then by arguments used in [2] for non-isomorphic algebras

$\overline{\mu_1(a_2, a_3, \dots, a_n)}$ and $\mu_1(a'_2, a'_3, \dots, a'_n)$ we derive $\mu_1(a_2, a_3, \dots, a_n) \notin \overline{Orb(\mu_1(a'_2, a'_3, \dots, a'_n))}$.

Summarizing results on single-generated Leibniz algebras, we obtain the main result of the paper.

Theorem 2. $\overline{\bigcup\limits_{a_2,\dots,a_n} Orb(\mu_1(a_2, a_3, \dots, a_n))}$ forms an irreducible component of the variety of n-dimensional Leibniz algebras.

References

1. Ayupov Sh.A., Omirov B.A. *On some classes of nilpotent Leibniz algebras,* Sib. Math. J., vol. 42(1), 2001, p. 18–29.

2. Burde D. *Degeneration of* 7*-dimensional nilpotent Lie algebras,* Comm. Algebra, vol. 33(4), 2005, p. 1259–1277.

3. Casas J. M., Ladra M., Omirov B.A., Karimjanov I.K. *Classification of solvable Leibniz algebras with null-filiform nilradical*, Linear and Multilinear Algebra 61(6) (2013), p. 758–774.

4. Gerstenhaber M. *On the deformation of rings and algebras,* I, III,Ann. Math. 79 (1964), 59-103; 88(1968), 1–34

5. Loday J.-L., *Une version non commutative des algèbres de Lie: les algèbres de Leibniz,* L'Enseignement Mathématique, vol. 39, 1993, p. 269–292.

6. Loday J.-L., *Pirashvili T. Universal enveloping algebras of Leibniz algebras and (co)homology.* Math. Ann., 296, 1993, p.139–158.

7. Nijenhuis A., Richardson R. W. *Cohomology and deformations in graded Lie algebras,* Bull. Amer. Math. Soc. 72(1966), p. 1–29.