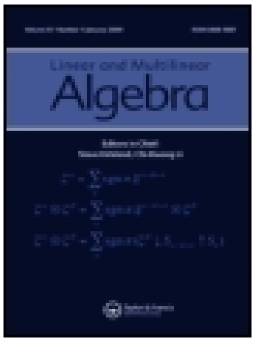

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Local derivations on Solvable Lie algebras

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ABSTRACT

We show that in the class of solvable Lie algebras there exist algebras which admit local derivations which are not ordinary derivation and also algebras for which every local derivation is a derivation. We found necessary and sufficient conditions under which any local derivation of solvable Lie algebras with abelian nilradical and one-dimensional complementary space is a derivation. Moreover, we prove that every local derivation on a finite-dimensional solvable Lie algebra with model nilradical and maximal dimension of complementary space is a derivation.

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1. Introduction

The notions of local derivations were first introduced in 1990 by Kadison [1] and Larson and Sourour [2]. The main problems concerning this notion are to find conditions under which local derivations become derivations and to present examples of algebras with local derivations that are not derivations. Kadison proved that each continuous local derivation of a von Neumann algebra M into a dual Banach M-bimodule is a derivation. In 2001 Johnson culminated the studies on local derivations, showing that every local derivation from a C^*-algebra A into a Banach A-bimodule is a derivation [3].

The study of local derivations on algebras of measurable operators was initiated in papers [4–6] and others. In particular, the paper [4] is devoted to the study of local derivations on the algebra $S(M,\tau)$ of τ-measurable operators affiliated with a von Neumann algebra M and a faithful normal semi-finite trace τ. It is proved that every local derivation on $S(M,\tau)$ which is continuous in the measure topology automatically becomes a derivation. The paper [4] also deals with the problem of existence of local derivations which are not derivations on algebras of measurable operators. Namely, necessary and sufficient conditions were obtained for the algebras of measurable and τ-measurable operators affiliated with a commutative von Neumann algebra to admit local derivations that are not derivations.

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Later, several papers have been devoted to similar notions and corresponding problems for derivations and automorphisms of Lie algebras [7,8]. In [7] Ayupov and Kudaybergenov have proved that every local derivation on semi-simple Lie algebras is a derivation and gave examples of finite-dimensional nilpotent Lie algebras with local derivations which are not derivations. The paper [9] is devoted to the study of so-called 2-local derivations on finite-dimensional Lie algebras and it is proved that every 2-local derivation on a semi-simple Lie algebra is a derivation and that each finite-dimensional nilpotent Lie algebra with dimension larger than two admits 2-local derivation which is not a derivation.

It is well known that any finite-dimensional Lie algebra over a field of characteristic zero is decomposed into the semidirect sum of semi-simple subalgebra and solvable radical. The semi-simple part is a direct sum of simple Lie algebras which are completely classified, and solvable Lie algebras can be classified by means of its nilradical. There are several papers which deal with the problem of classification of all solvable Lie algebras with a given nilradical, for example Abelian, Heisenberg, filiform, quasi-filiform nilradicals, etc. [10–12]. It should be noted that any derivation on a semi-simple Lie algebra is inner and any nilpotent Lie algebra has a derivation which is not inner. In the class of solvable Lie algebras there exist algebras which any derivation is inner and also algebras which admits not inner derivation.

In this paper, we investigate local derivations of solvable Lie algebras. We show that in the class of solvable Lie algebras there exists a solvable algebra admitting local derivations which are not ordinary derivation and also there exist algebras for which every local derivation is a derivation.

More precisely, local derivations of solvable Lie algebras with abelian nilradical and one-dimensional complementary space are investigated. The necessary and sufficient conditions under which every local derivation of such Lie algebras becomes a derivation are found. We also consider solvable Lie algebra with model nilradical and maximal dimension of complementary space, i.e. in sense of Mubarakzjanov [13] correspond to maximal number of nil-independent derivations and prove that any local derivation of such type of algebras is a derivation.

2. Preliminaries

In this section, we present some known facts about Lie algebras and their derivations.

Let L be a Lie algebra. For a Lie algebra L consider the following central lower and derived sequences:

$$L^1 = L, \quad L^{k+1} = [L^k, L^1], \quad k \geq 1,$$
$$L^{[1]} = 1, \quad L^{[s+1]} = [L^{[s]}, L^{[s]}], \quad s \geq 1.$$

Definition 2.1: A Lie algebra L is called nilpotent (respectively, solvable), if there exists $p \in \mathbb{N}$ ($q \in \mathbb{N}$) such that $L^p = 0$ (respectively, $L^{[q]} = 0$).

Any Lie algebra L contains a unique maximal solvable (resp. nilpotent) ideal, called the radical (resp. nilradical) of the algebra. A non-trivial Lie algebra is called semi-simple if its radical is zero.

A derivation on a Lie algebra L is a linear map $d : L \to L$ which satisfies the Leibniz rule:

$$\mathrm{d}([x,y]) = [\mathrm{d}(x),y] + [x,\mathrm{d}(y)] \quad \text{forany } x,y \in L.$$

The set of all derivations of a Lie algebra L is a Lie algebra with respect to the usual matrix commutator and it is denoted by $Der(L)$.

For any element $x \in L$ the operator of right multiplication $ad_x : L \to L$, defined as $ad_x(z) = [z,x]$ is a derivation, and derivations of this form are called inner derivation. The set of all inner derivations of L, denoted $ad(L)$, is an ideal in $Der(L)$.

Definition 2.2: A linear operator Δ is called a local derivation if for any $x \in L$, there exists a derivation $d_x : L \to L$ (depending on x) such that $\Delta(x) = d_x(x)$. The set of all local derivations on L we denote by $LocDer(L)$.

We have the following theorem for the local derivation on semi-simple Lie algebras.

Theorem 2.3 ([7]): *Let L be a finite-dimensional semi-simple Lie algebra. Then any local derivation Δ on L is a derivation.*

Let N be a finite-dimensional nilpotent Lie algebra. For the matrix of linear operator ad_x denote by $C(x)$ the descending sequence of its Jordan blocks' dimensions. Consider the lexicographical order on the set $C(N) = \{C(x) \mid x \in N\}$.

Definition 2.4: A sequence

$$\left(\max_{x \in N \setminus N^2} C(x)\right)$$

is said to be the characteristic sequence of the nilpotent Lie algebra N.

Definition 2.5: A nilpotent Lie algebra with characteristic sequence $(n_1, n_2, \ldots, n_k, 1)$ is said to be model if there exists a basis $\{e_1, e_2, \ldots, e_n\}$ such that

$$\begin{aligned} &[e_i, e_1] = e_{i+1}, \quad 2 \le i \le n_1, \\ &[e_{n_1+\cdots+n_{j-1}+i}, e_1] = e_{n_1+\cdots+n_{j-1}+i+1}, \quad 2 \le j \le k,\ 2 \le i \le n_j, \end{aligned} \tag{1}$$

where omitted products are equal to zero.

3. Local derivation of solvable Lie algebras with abelian nilradical

In this section, we investigate solvable Lie algebras with abelian nilradical. First we consider the following example.

Example 3.1: Consider the following three-dimensional solvable Lie algebras with two-dimensional abelian nilradical [14].

$$\begin{aligned} L_1 : &\quad [e_2, e_1] = e_2, \qquad [e_3, e_1] = e_3; \\ L_2 : &\quad [e_2, e_1] = e_2 + e_3, \quad [e_3, e_1] = e_3. \end{aligned}$$

Any local derivation of L_1 is a derivation, but L_2 admits a local derivation which is not a derivation.

Indeed, by the direct calculation we obtain that the matrix form of the derivations of algebras L_1 and L_2, respectively have the following forms:

$$Der(L_1) = \begin{pmatrix} 0 & \xi_{1,2} & \xi_{1,3} \\ 0 & \xi_{2,2} & \xi_{2,3} \\ 0 & \xi_{3,2} & \xi_{3,3} \end{pmatrix}, \quad Der(L_2) = \begin{pmatrix} 0 & \xi_{1,2} & \xi_{1,3} \\ 0 & \xi_{2,2} & \xi_{2,3} \\ 0 & 0 & \xi_{2,2} \end{pmatrix}.$$

It is not difficult to show that any local derivation on L_1 is a derivation and linear operator Δ defined as $\Delta(e_1) = 0,\ \Delta(e_2) = 0,\ \Delta(e_3) = e_3$ on L_2 is a local derivation which is not a derivation.

Let L be a solvable Lie algebra with abelian nilradical N and let $dimN = n$, $dimL = n + 1$. Take a basis $\{x, e_1, e_2, \dots, e_n\}$ of L such that $\{e_1, e_2, \dots, e_n\}$ a basis of N. It is known that operator of right multiplication ad_x is a non-nilpotent operator on N [13]. Moreover, such solvable algebras characterized by the operator ad_x, i.e. two solvable algebras with abelian nilradical N and one-dimensional complementary space are isomorphic if and only if the corresponding operators of right multiplication have the same Jordan forms.

Theorem 3.2: *Let L be a solvable Lie algebra with the abelian nilradical N and the dimension of the complementary space to the nilradical is equal to one. Any local derivation on L is a derivation if and only if ad_x is a diagonalizable operator.*

Proof: Let ad_x be a diagonalizable operator. Then there exists a basis $\{e_1, e_2, \dots, e_n\}$ of N such that the Jordan form of the operator ad_x on this basis has the form:

$$\begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

Consequently,

$$[e_i, x] = \lambda_i e_i, \quad 1 \le i \le n.$$

Let $d \in Der(L)$, then

$$d(x) = \beta_1 e_1 + \beta_2 e_2 + \dots + \beta_n e_n,$$
$$d(e_i) = \alpha_{i,1} e_1 + \alpha_{i,2} e_2 + \dots + \alpha_{i,n} e_n, \quad 1 \le i \le n.$$

From the property of derivation we have

$$\begin{aligned} d([e_i, x]) &= [d(e_i), x] + [e_i, d(x)] \\ &= [\alpha_{i,1} e_1 + \alpha_{i,2} e_2 + \dots + \alpha_{i,n} e_n, x] + [e_i, \beta_1 e_1 + \beta_2 e_2 + \dots + \beta_n e_n] \\ &= \alpha_{i,1}\lambda_1 e_1 + \alpha_{i,2}\lambda_2 e_2 + \dots + \alpha_{i,n}\lambda_n e_n. \end{aligned}$$

On the other hand,

$$d([e_i, x]) = \lambda_i d(e_i) = \lambda_i(\alpha_{i,1} e_1 + \alpha_{i,2} e_2 + \dots + \alpha_{i,n} e_n).$$

Comparing the coefficients at the basis elements we obtain

$$\alpha_{i,j}(\lambda_i - \lambda_j) = 0, \quad 1 \le j \le n. \tag{2}$$

Case 1. Let $\lambda_i \neq \lambda_j$, for any $i, j (i \neq j)$, then we get $\alpha_{i,j} = 0$ for $i \neq j$. Thus, we have that the matrix of the derivations of L has the form:

$$Der(L) = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \dots & \beta_n \\ 0 & \alpha_{1,1} & 0 & \dots & 0 \\ 0 & 0 & \alpha_{2,2} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \alpha_{n,n} \end{pmatrix}.$$

Case 2. Let $\lambda_i = \lambda_j$ for some i and j. Without loss of generality, we can assume that

$$\lambda_1 = \dots = \lambda_s,\ \lambda_{s+1} = \dots = \lambda_{s+p}, \quad \dots \quad \lambda_{n-q} = \dots = \lambda_n.$$

From the equality (2) we obtain that the matrix form of $Der(L)$ is the following:

$$\begin{pmatrix} 0 & \beta_1 & \cdots & \beta_s & \beta_{s+1} & \dots & \beta_{s+p} & \dots & \beta_{n-q} & \dots & \beta_n \\ 0 & \alpha_{1,1} & \dots & \alpha_{1,s} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \alpha_{s,1} & \dots & \alpha_{s,s} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \alpha_{s+1,s+1} & \dots & \alpha_{s+1,s+p} & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \alpha_{s+p,s+1} & \dots & \alpha_{s+p,s+p} & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & \alpha_{n-q,n-q} & \dots & \alpha_{n-q,n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & \alpha_{n,n-q} & \dots & \alpha_{n,n} \end{pmatrix}.$$

Let Δ be a local derivation on L and let

$$\Delta(x) = \xi x + \xi_1 e_1 + \xi_2 e_2 + \dots + \xi_n e_n,$$
$$\Delta(e_i) = \zeta_i x + \zeta_{i,1} e_1 + \zeta_{i,2} e_2 + \dots + \zeta_{i,n} e_n,\ 1 \le i \le n.$$

Considering the equalities $\Delta(x) = d_x(x)$ and $\Delta(e_i) = d_{e_i}(e_i)$ for $1 \le i \le n$, we conclude that Δ is a derivation. Therefore, any local derivation on L is a derivation.

Now let Jordan form of the operator ad_x be

$$ad_x = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_s \end{pmatrix}$$

and suppose that there exists a Jordan block with order $k (k \ge 2)$. Without loss of generality one can assume that J_1 has order $k \ge 2$. Then the table of multiplication of L has the form:

$$e_i x = \lambda_1 e_i + e_{i+1},\ 1 \le i \le k-1,$$

$$e_k x = \lambda_1 e_k,$$
$$e_i x = \lambda_i e_i + \mu_i e_{i+1}, \; k+1 \le i \le n,$$

where $\mu_i = 0; 1$.

By the direct verification of the property of derivation we obtain that the general form of the matrix of $Der(L)$ is

$$\begin{pmatrix} 0 & \beta_1 & \beta_2 & \dots & \beta_{k-1} & \beta_k & \beta_{k+1} & \dots & \beta_n \\ 0 & \alpha_{1,1} & \alpha_{1,2} & \dots & \alpha_{1,k-1} & \alpha_{1,k} & 0 & \dots & 0 \\ 0 & 0 & \alpha_{1,1} & \dots & \alpha_{1,k-2} & \alpha_{1,k-1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & \alpha_{1,1} & \alpha_{1,2} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & \alpha_{1,1} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & H_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & H_s \end{pmatrix},$$

where H_i are the block matrices with the same dimension of Jordan blocks J_i.

Consider the linear operator $\Delta : L \to L$ defined by

$$\Delta(x) = 0, \quad \Delta(e_i) = e_i, \quad 1 \le i \le k-1,$$
$$\Delta(e_k) = 2e_k, \quad \Delta(e_i) = 0, \quad k+1 \le i \le n.$$

It is obvious that Δ is not a derivation. We show that Δ is a local derivation.

Indeed, for any element $y = \gamma x + \eta_1 e_1 + \eta_2 e_2 + \dots + \eta_n e_n \in L$ we consider

$$\Delta(y) = \Delta(\gamma x + \eta_1 e_1 + \eta_2 e_2 + \dots + \eta_n e_n) = \eta_1 e_1 + \eta_2 e_2 + \dots + \eta_{k-1} e_{k-1} + 2\eta_k e_k.$$

Consider the derivation d_y such that

$$d_y(x) = 0, \quad d_y(e_i) = 0, \quad k+1 \le i \le n,$$
$$d_y(e_i) = \alpha_{1,1} e_i + \alpha_{1,2} e_{i+1} + \dots + \alpha_{1,k-i+1} e_k, \quad 1 \le i \le k.$$

Then

$$d_y(y) = d_y(\gamma x + \eta_1 e_1 + \eta_2 e_2 + \dots + \eta_n e_n)$$
$$= \eta_1 \alpha_{1,1} e_1 + (\eta_2 \alpha_{1,1} + \eta_1 \alpha_{1,2}) e_2 + \dots + (\eta_k \alpha_{1,1} + \eta_{k-1} \alpha_{1,2} + \dots + \eta_1 \alpha_{1,k}) e_k.$$

Let us show that for an appropriate choice of the parameters $\alpha_{i,j}$ that $\Delta(y) = d_y(y)$. This is satisfied if

$$\eta_1 = \eta_1 \alpha_{1,1},$$
$$\eta_2 = \eta_2 \alpha_{1,1} + \eta_1 \alpha_{1,2},$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$\eta_{k-1} = \eta_{k-1} \alpha_{1,1} + \eta_{k-2} \alpha_{1,2} + \dots + \eta_1 \alpha_{1,k-1},$$
$$2\eta_k = \eta_k \alpha_{1,1} + \eta_{k-1} \alpha_{1,2} + \dots + \eta_1 \alpha_{1,k}.$$

Note that this system of equations has a solution with respect to $\alpha_{i,j}$ for any parameters η_i. Indeed:

- if $\eta_1 \neq 0$, then

$$\alpha_{1,1} = 1,\ \alpha_{1,2} = \cdots = \alpha_{1,k-1} = 0,\ \alpha_{1,k} = \frac{\eta_k}{\eta_1},$$

- if $\eta_1 = \cdots = \eta_{s-1} = 0$ è $\eta_s \neq 0,\ 2 \leq s \leq k-1$, then

$$\alpha_{1,1} = 1,\ \alpha_{1,2} = \cdots = \alpha_{1,k-s} = 0,\ \alpha_{1,k-s+1} = \frac{\eta_k}{\eta_s},$$

- if $\eta_1 = \cdots = \eta_{k-1} = 0$ è $\eta_k \neq 0$ then we have $\alpha_{1,1} = 2$.

Hence, Δ is a local derivation. ■

Now we consider solvable Lie algebras with abelian nilradical and maximal complementary vector space. It is known that the maximal dimension of complementary space for solvable Lie algebras with n-dimensional abelian nilradical is equal to n. Moreover, up to isomorphism there exists only one such solvable Lie algebra with the following non-zero multiplications:

$$L_n : [e_i, x_i] = e_i, \quad 1 \leq i \leq n.$$

Theorem 3.3: *Any local derivation on L_n is a derivation.*

Proof: First, we describe the derivations of the algebra L_n. Let $d \in Der(L_n)$ then we have

$$d(e_i) = \sum_{j=1}^{n} \alpha_{i,j} e_j, \quad d(x_i) = \sum_{j=1}^{n} \beta_{i,j} e_j, \quad 1 \leq i \leq n.$$

Using the property of derivation $d([e_i, x_j]) = [d(e_i), x_j] + [e_i, d(x_j)]$ we obtain that $\alpha_{i,j} = 0$ for $i \neq j$. From the equality $d([x_i, x_j]) = [d(x_i), x_j] + [x_i, d(x_j)]$ we get $\beta_{i,j} = 0$ for $i \neq j$. Therefore, we have that any derivation of L_n has the following form:

$$d(e_i) = \alpha_i e_i, \quad d(x_i) = \beta_i e_i, \quad 1 \leq i \leq n.$$

Let Δ be a local derivation on L_n, then $\Delta(e_i) = d_{e_i}(e_i) = \gamma_i e_i$ and $\Delta(x_i) = d_{x_i}(x_i) = \delta_i e_i$. Thus, Δ is a derivation. ■

4. Local derivation of solvable Lie algebras with model nilradical

Let L be a solvable Lie algebra and its nilradical is a model algebra N. Let the characteristic sequence of N be equal to $(n_1, n_2, \ldots, n_k, 1)$. Then the multiplication of N has the following form:

$$[e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n_1,$$
$$[e_{n_1+\cdots+n_{j-1}+i}, e_1] = e_{n_1+\cdots+n_{j-1}+i+1}, \quad 2 \leq j \leq k,\ 2 \leq i \leq n_j.$$

It is known that the maximal dimension of the complementary space of solvable Lie algebra whose nilradical is a model nilpotent Lie algebra with characteristic sequence $(n_1, n_2, \ldots, n_k, 1)$ is equal to $k+1$. Let $L = Q + N$ be a solvable Lie algebra with $dimQ =$

$k+1$. Then L has a basis $\{x_1, x_2, \dots, x_{k+1}, e_1, e_2, \dots, e_n\}$ such that the table of multiplication is the following:

$$L_{k+1}(N): \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n_1, \\ [e_{n_1+\dots+n_{j-1}+i}, e_1] = e_{n_1+\dots+n_{j-1}+i+1}, & 2 \le j \le k, \quad 2 \le i \le n_j, \\ [e_i, x_1] = ie_i, & 1 \le i \le n, \\ [e_i, x_2] = e_i, & 2 \le i \le n_1+1, \\ [e_{n_1+\dots+n_{j-2}+i}, x_j] = e_{n_1+\dots+n_{j-2}+i}, & 3 \le j \le k+1, \quad 2 \le i \le n_{j-1}+1. \end{cases}$$

In [15] it is proved that any derivation of $L_{k+1}(N)$ is inner for any characteristic sequence $(n_1, n_2, \dots, n_k, 1)$.

In this section, we investigate local derivations on $L_{k+1}(N)$. For any local derivation Δ on $L_{k+1}(N)$ we have

$$\Delta(x_j) = d_{x_j}(x_j) = [x_j, a_j], \quad 1 \le j \le k+1,$$
$$\Delta(e_i) = d_{e_i}(e_i) = [e_i, b_i], \quad 1 \le i \le n.$$

Put

$$a_j = \sum_{p=1}^{k+1} \alpha_{j,p} x_p + \sum_{p=1}^{n} \beta_{j,p} e_p, \quad b_i = \sum_{p=1}^{k+1} \gamma_{i,p} x_p + \sum_{p=1}^{n} \delta_{i,p} e_p.$$

Then we obtain

$$\Delta(x_1) = -\sum_{p=1}^{n} p\beta_{1,p} e_p, \quad \Delta(x_j) = -\sum_{p=n_1+\dots+n_{j-2}+2}^{n_1+\dots+n_{j-1}+1} \beta_{j,p} e_p, \quad 2 \le j \le k+1,$$

$$\Delta(e_1) = \gamma_{1,1} e_1 - \sum_{j=2}^{k+1} \sum_{p=n_1+\dots+n_{j-2}+2}^{n_1+\dots+n_{j-1}} \delta_{1,i} e_{p+1},$$

$$\Delta(e_i) = (i\gamma_{i,1} + \gamma_{i,j})e_i + \delta_{i,1} e_{i+1}, \quad n_1 + \dots + n_{j-2} + 2 \le i \le n_1 + \dots + n_{j-1},$$
$$2 \le j \le k-1,$$
$$\Delta(e_i) = (i\gamma_{i,1} + \gamma_{i,j})e_i, \quad i = n_1 + \dots + n_{j-1} + 1,\ 2 \le j \le k-1.$$

Proposition 4.1: *There exists $y \in L_{k+1}(N)$, such that $\Delta(x_j) = [x_j, y]$ for $1 \le j \le k+1$.*

Proof: For any j $(2 \le j \le k+1)$ and for the fixed s $(n_1 + \dots + n_{j-2} + 2 \le s \le n_1 + \dots + n_{j-1} + 1)$ we consider

$$\Delta(sx_j - x_1) = s\Delta(x_j) - \Delta(x_1) = \sum_{p=1}^{n} p\beta_{1,p} e_p - s \sum_{p=n_1+\dots+n_{j-2}+2}^{n_1+\dots+n_{j-1}+1} \beta_{j,p} e_p.$$

On the other hand

$$\Delta(sx_j - x_1) = [sx_j - x_1, y_{sx_j - x_1}] = \left[sx_j - x_1, \sum_{p=1}^{k+1} A_{sx_j-x_1,p} x_p + \sum_{p=1}^{n} B_{sx_j-x_1,p} e_p \right]$$

$$= \sum_{p=1}^{n} p B_{sx_j-x_1,p} e_p - s \sum_{p=n_1+\cdots+n_{j-2}+2}^{n_1+\cdots+n_{j-1}+1} B_{sx_j-x_1,p} e_p.$$

Comparing coefficients at the basis element of e_s we have $s(\beta_{1,s} - \beta_{j,s}) = 0$ which implies $\beta_{j,s} = \beta_{1,s}$.

Since $j(2 \leq j \leq k+1)$ and $s(n_1 + \cdots + n_{j-2} + 2 \leq s \leq n_1 + \cdots + n_{j-1} + 1)$ we have that $\beta_{j,p} = \beta_{1,p}$ for any j and p.

If we take an element $y = \sum_{p=1}^{n} \beta_{1,p} e_p$, then we have

$$\Delta(x_j) = [x_j, y], \quad 1 \leq j \leq k+1.$$

■

From the previous proposition we conclude that without loss of generality it can be used β_p instead of $\beta_{1,p}$. Thus, we have

$$\Delta(x_1) = -\sum_{p=1}^{n} p\beta_p e_p, \quad \Delta(x_j) = -\sum_{p=n_1+\cdots+n_{j-2}+2}^{n_1+\cdots+n_{j-1}+1} \beta_p e_p, \quad 2 \leq j \leq k+1. \tag{3}$$

Now we consider the value of local derivations on the generators of N (which algebraically generate the basis), i.e. $e_1, e_2, e_{n_1+2}, \ldots, e_{n_1+\cdots+n_{k-1}+2}$.

Proposition 4.2: *There exists $z \in L_{k+1}(N)$, such that*

$$\Delta(x_j) = [x_j, z], \quad \Delta(e_1) = [e_1, z], \quad \Delta(e_{n_1+\cdots+n_{j-2}+2}) = [e_{n_1+\cdots+n_{j-2}+2}, z],$$
$$2 \leq j \leq k+1.$$

Proof: By Proposition 4.1 we have that there exists $y \in L_{k+1}(N)$ such that $\Delta(x_j) = [x_j, y]$.

Consider $\Delta(x_1 - 3x_2 - e_1)$ Using the equality (3) we have

$$\Delta(x_1 - 3x_2 - e_1) = -\sum_{p=1}^{n} p\beta_p e_p + 3\sum_{p=2}^{n_1+1} \beta_p e_p - \gamma_{1,1} e_1 + \sum_{j=2}^{k+1} \sum_{p=n_1+\cdots+n_{j-2}+2}^{n_1+\cdots+n_{j-1}} \delta_{1,i} e_{p+1}.$$

On the other hand

$$\Delta(x_1 - 3x_2 - e_1) = [x_1 - 3x_2 - e_1, y_{x_1-3x_2-e_1}]$$
$$= \left[x_1 - 3x_2 - e_1, \sum_{p=1}^{k+1} A_{x_1-3x_2-e_1,p} x_p + \sum_{p=1}^{n} B_{x_1-3x_2-e_1,p} e_p \right]$$

$$= -\sum_{p=1}^{n} pB_{x_1-3x_2-e_1,p}e_p + 3\sum_{p=2}^{n_1+1} B_{x_1-3x_2-e_1,p}e_p - A_{x_1-3x_2-e_1,1}x_1$$
$$+\sum_{j=2}^{k+1}\sum_{p=n_1+\dots+n_{j-2}+2}^{n_1+\dots+n_{j-1}} B_{x_1-3x_2-e_1,p}e_{p+1}.$$

Comparing the coefficients at the basis elements e_2 and e_3, we get

$$B_{x_1-3x_2-e_1,2} = \beta_2, \quad B_{x_1-3x_2-e_1,2} = \delta_{1,2},$$

which implies $\delta_{1,2} = \beta_2$.

Considering $\Delta(x_1-(i+1)x_2-e_1)$ inductively we obtain $\delta_{1,i} = \beta_i$ for $2 \le i \le n_1$. Indeed,

$$\Delta(x_1-(i+1)x_2-e_1)$$
$$= -\sum_{p=1}^{n} p\beta_p e_p + (i+1)\sum_{p=2}^{n_1+1} \beta_p e_p - \gamma_{1,1}e_1 + \sum_{j=2}^{k+1}\sum_{p=n_1+\dots+n_{j-2}+2}^{n_1+\dots+n_{j-1}} \delta_{1,p}e_{p+1}.$$

On the other hand

$$\Delta(x_1-(i+1)x_2-e_1)$$
$$= [x_1-(i+1)x_2-e_1, y_{x_1-(i+1)x_2-e_1}]$$
$$\times \left[x_1-(i+1)x_2-e_1, \sum_{p=1}^{k+1} A_{x_1-(i+1)x_2-e_1,p}x_p + \sum_{p=1}^{n} B_{x_1-(i+1)x_2-E_1,p}e_p\right]$$
$$= -\sum_{p=1}^{n} pB_{x_1-(i+1)x_2-e_1,p}e_p + (i+1)\sum_{p=2}^{n_1+1} B_{x_1-(i+1)x_2-e_1,p}e_p$$
$$- A_{x_1-(i+1)x_2-e_1,1}x_1 + \sum_{j=2}^{k+1}\sum_{p=n_1+\dots+n_{j-2}+2}^{n_1+\dots+n_{j-1}} B_{x_1-(i+1)x_2-e_1,p}e_{p+1}.$$

Comparing the coefficients at the basis elements $e_2, e_3, \dots e_{i+1}$, we get

$$\begin{aligned}
&(i-1)B_{x_1-(i+1)x_2-e_1,2} = (i-1)\beta_2,\\
&(i-2)B_{x_1-(i+1)x_2-e_1,3} + B_{x_1-(i+1)x_2-e_1,2} = (i-2)\beta_3 + \delta_{1,2},\\
&(i-3)B_{x_1-(i+1)x_2-e_1,4} + B_{x_1-(i+1)x_2-e_1,3} = (i-3)\beta_4 + \delta_{1,3},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&B_{x_1-(i+1)x_2-e_1,i} + B_{x_1-(i+1)x_2-e_1,i-1} = \beta_i + \delta_{1,i-1},\\
&B_{x_1-(i+1)x_2-e_1,i} = \delta_{1,i}.
\end{aligned} \tag{4}$$

By induction hypothesis we have $\delta_{1,2} = \beta_2$, $\delta_{1,3} = \beta_3$, $\delta_{1,i-1} = \beta_{i-1}$ and from (4) we obtain $\delta_{1,i} = \beta_i$.

In a similar way considering $\Delta(x_1 - (n_1 + \cdots + n_{j-2} + i + 1)x_j - e_1)$ for $3 \le j \le k + 1$, $2 \le i \le n_{j-1}$ we get

$$\delta_{1,i} = \beta_1, \quad n_1 + \cdots + n_{j-2} + 2 \le i \le n_1 + \cdots + n_{j-1},\ 2 \le j \le k+1.$$

From $\Delta(x_1 - (n_1 + \cdots + n_{j-2} + 3)x_j - e_{n_1+\cdots+n_{j-2}+2})$ for $2 \le j \le k+1$, we obtain

$$\delta_{n_1+\cdots+n_{j-2}+2,1} = \beta_1, \quad 2 \le j \le k+1.$$

Therefore, we derive that for the element

$$z = y + \gamma_{1,1}x_1 + \sum_{j=2}^{k+1}(\gamma_{n_1+\cdots+n_{j-2}+2,3} + (n_1 + \cdots + n_{j-2} + 2)(\gamma_{n_1+\cdots+n_{j-2}+2,1} - \gamma_{1,1}))x_j$$

the following equalities hold:

$$\Delta(x_j) = [x_j, z], \quad \Delta(e_1) = [e_1, z], \quad \Delta(e_{n_1+\cdots+n_{j-2}+2}) = [e_{n_1+\cdots+n_{j-2}+2}, z],$$
$$2 \le j \le k+1.$$

■

From Propositions 4.1 and 4.2, we have that for any local derivation Δ there exists an element $z \in L_{k+1}(N)$ such that $\Delta(x) = [z, x]$ for each generator of $x \in L_{k+1}(N)$.

Thus, if we put $z = \sum_{p=1}^{k+1} \gamma_p e_p + \sum_{p=1}^{n} \beta_p e_p$, then we have

$$\Delta(x_1) = -\sum_{p=1}^{n} p\beta_p e_p, \quad \Delta(x_j) = -\sum_{p=n_1+\cdots+n_{j-2}+2}^{n_1+\cdots+n_{j-1}+1} \beta_p e_p, \quad 2 \le j \le k+1,$$

$$\Delta(e_1) = \gamma_1 e_1 - \sum_{j=2}^{k+1} \sum_{p=n_1+\cdots+n_{j-2}+2}^{n_1+\cdots+n_{j-1}} \beta_p e_{p+1}, \tag{5}$$

$$\Delta(e_{n_1+\cdots+n_{j-2}+2}) = \big((n_1 + \cdots + n_{j-2} + 2)\gamma_1 + \gamma_j\big)\, e_{n_1+\cdots+n_{j-2}+2} + \beta_1 e_{n_1+\cdots+n_{j-2}+3},$$
$$2 \le j \le k+1.$$

Theorem 4.3: *Any local derivation on the solvable Lie algebra $L_{k+1}(N)$ is a derivation.*

Proof: First we prove the Theorem for case $k = 1$, i.e. the characteristic sequence of the model nilradical N is $(n, 1)$. Then we have the basis $\{x_1, x_2, e_1, e_2, \ldots, e_{n+1}\}$ of $L_2(N)$ such that

$$[e_i, e_1] = e_{i+1}, \quad 2 \le i \le n,$$
$$[e_i, x_1] = ie_i, \quad 1 \le i \le n+1,$$
$$[e_i, x_2] = e_i, \quad 2 \le i \le n+1.$$

Let Δ be a local derivation of $L_2(N)$. By Propositions 4.1 and 4.2 we have

$$\Delta(e_1) = \gamma_1 e_1 - \sum_{k=2}^{n} \beta_k e_{k+1}, \qquad \Delta(x_1) = \sum_{k=1}^{n+1} k\beta_k e_k,$$

$$\Delta(e_2) = (2\gamma_1 + \gamma_2)e_2 + \beta_1 e_3, \qquad \Delta(x_2) = -\sum_{k=2}^{n+1} \beta_k e_k,$$

$$\Delta(e_i) = (i\gamma_{i,1} + \gamma_{i,2})e_i + \delta_{i,1}e_{i+1}, \qquad 3 \le i \le n,$$

$$\Delta(e_{n+1}) = ((n+1)\gamma_{n+1,1} + \gamma_{n+1,2})e_{n+1}.$$

Consider

$$\Delta(x_1 - (j+1)x_2 + e_j) = -\beta_1 e_1 + (j\gamma_{j,1} + \gamma_{j,2})e_j + \delta_{j,1}e_{j+1} + \sum_{k=2}^{n}(j+1-k)\beta_k e_k.$$

On the other hand

$$\begin{aligned}\Delta(x_1 - (j+1)x_2 + e_j) &= \left[x_1 - (j+1)x_2 + e_j, A_{j,1}x_1 + A_{j,2}x_2 + \sum_{k=1}^{n+1} B_{j,k}e_k\right] \\ &= -B_{j,1}e_1 + (jA_{j,1} + A_{j,2} + B_{j,j-1})e_j + B_{j,1}e_{j+1} \\ &\quad + \sum_{k=2}^{n+1}(j+1-k)B_{j,k}e_k.\end{aligned}$$

Comparing the coefficients at the basis elements e_1 and e_{j+1} we have $B_{j,1} = \delta_{j,1}$ and $B_{j,1} = \beta_1$, which implies

$$\delta_{j,1} = \beta_1, \quad 3 \le j \le n.$$

Now consider

$$\begin{aligned}&\Delta\left(x_1 - (j+1)x_2 + e_1 - (j-1)e_2 + \frac{1}{(j-2)!}e_{j+1}\right) \\ &= (\gamma_1 - \beta_1)e_1 - (j-1)(2\gamma_1 + \gamma_2 - \beta_2)e_2 \\ &\quad - \big((j-1)\beta_1 + \beta_2 - (j-2)\beta_3\big)e_3 - \left(\beta_j - \frac{1}{(j-2)!}((j+1)\gamma_{j+1,1} + \gamma_{i+1,2})\right)e_{j+1} \\ &\quad - \sum_{k=4, k\neq j+1}^{n+1}(\beta_{k-1} - (j+1-k)\beta_k)e_k.\end{aligned}$$

On the other hand

$$\Delta\left(x_1 - (j+1)x_2 + e_1 - (j-1)e_2 + \frac{1}{(j-2)!}e_{j+1}\right)$$

$$\left[x_1 - (j+1)x_2 + e_1 - (j-1)e_2 + \frac{1}{(j-2)!}e_{j+1}, A_{j,1}x_1 + A_{j,2}x_2 + \sum_{k=1}^{n+1} B_{j,k}e_k\right]$$

$$= (A_{j,1} - B_{j,1})e_1 - (j-1)(2A_{j,1} + A_{j,2} - B_{j,2})e_2$$
$$- \big((j-1)B_{j,1} + B_{j,2} - (j-2)B_{j,3}\big)e_3$$
$$\left(B_{j,k} - \frac{1}{(j-2)!}((j+1)A_{j,1} + A_{j,2})\right) e_{j+1}$$
$$- \sum_{k=4, k\neq j+1}^{n+1} (B_{j,k-1} - (j+1-k)B_{j,k})e_k.$$

Comparing the coefficients at the basis elements of $e_1, e_2, \dots e_{j+1}$, we have

$$\begin{aligned}
&A_{j,1} - B_{j,1} = \gamma_1 - \beta_1,\\
&2A_{j,2} + A_{j,2} - B_{j,2} = 2\gamma_1 + \gamma_2 - \beta_2,\\
&(j-1)B_{j,1} + B_{j,2} - (j-2)B_{j,3} = (j-1)\beta_1 + \beta_2 - (j-2)\beta_3,\\
&B_{j,k-1} - (j+1-k)B_{j,k} = \beta_{k-1} - (j+1-k)\beta_k, \qquad 4 \le k \le j,\\
&B_{j,k} - \frac{1}{(j-2)!}\big((j+1)A_{j,1} + A_{j,2}\big) = \beta_k - \frac{1}{(j-2)!}\big((j+1)\gamma_{j+1,1} + \gamma_{i+1,2}\big).
\end{aligned}$$

From this system of equations considering $(j-1)[1] + [2] + [3] + \sum_{k=4}^{j+1} \frac{(j-2)!}{(j+1-k)!}[K]$ we have that

$$(j+1)\gamma_{j+1,1} + \gamma_{i+1,2} = (j+1)\gamma_1 + \gamma_2,$$

where $[K]$ is the k-th equation of the previous system.

Therefore, we obtain $\Delta(y) = [y, \gamma_1 x_1 + \gamma_2 x_2 + \sum_{k=1}^{n+1} \beta_k e_k]$ for any $y \in L$. Hence, Δ is a derivation.

Now we are able to prove the Theorem for the general case. Let Δ be a local derivation on $L_{k+1}(N)$, then by Propositions 4.1 and 4.2 we have

$$\Delta(x_1) = -\sum_{p=1}^{n} p\beta_p e_p, \quad \Delta(x_j) = -\sum_{p=n_1+\dots+n_{j-2}+2}^{n_1+\dots+n_{j-1}+1} \beta_p e_p, \quad 2 \le j \le k+1, \tag{6}$$

$$\Delta(e_1) = \gamma_1 e_1 - \sum_{j=2}^{k+1} \sum_{p=n_1+\dots+n_{j-2}+2}^{n_1+\dots+n_{j-1}} \beta_p e_{p+1},$$

$$\Delta(e_{n_1+\dots+n_{j-2}+2}) = \big((n_1 + \dots + n_{j-2} + 2)\gamma_1 + \gamma_j\big) e_{n_1+\dots+n_{j-2}+2} + \beta_1 e_{n_1+\dots+n_{j-2}+3},$$
$$2 \le j \le k+1,$$
$$\Delta(e_i) = (i\gamma_{i,1} + \gamma_{i,j})e_i + \delta_{i,1}e_{i+1}, \quad n_1 + \dots + n_{j-2}$$
$$+ 3 \le i \le n_1 + \dots + n_{j-1}, 2 \le j \le k-1,$$
$$\Delta(e_i) = (i\gamma_{i,1} + \gamma_{i,j})e_i, \quad i = n_1 + \dots + n_{j-1} + 1,\ 2 \le j \le k-1.$$

To prove the theorem we have to show

$$\delta_{i,1} = \beta_i, \quad n_1 + \dots + n_{j-2} + 3 \le i \le n_1 + \dots + n_{j-1},\ 2 \le j \le k-1, \tag{7}$$

and

$$i\gamma_{i,1} + \gamma_{i,j} = i\gamma_1 + \gamma_j, \quad n_1 + \cdots + n_{j-2} + 3 \leq i \leq n_1 + \cdots + n_{j-1},\ 2 \leq j \leq k. \quad (8)$$

Similar to the case $k = 1$, considering $\Delta(x_1 - (n_1 + \cdots + n_{j-2} + s + 1)x_j + e_{n_1+\cdots+n_{j-2}+s})$ we obtain the equality (7) and analysing

$$\Delta\left(x_1 - (n_1 + \cdots + n_{j-2} + s + 1)x_j + e_1 - (s-1)e_{n_1+\cdots+n_{j-2}+s} + \frac{1}{(s-2)!}e_{n_1+\cdots+n_{j-2}+s+1}\right)$$

for $2 \leq s \leq n_j - 1$ we get the equality (8). ■

4.1. Non-model nilradical case

In this subsection, we give some examples of local derivations of solvable Lie algebras with maximal dimension of complementary vector space. Solvable algebras exhibiting this property shall be called maximal solvable Lie algebras with nilradical N, i.e. a solvable Lie algebra L with nilradical N is said to be maximal, if there is no algebra M with nilradical N such that $dim(M) > dim(L)$.

In the first example, we consider solvable Lie algebra with non-model nilradical and dimension of complementary vector space equal to the number of generators of the nilradical.

Example 4.4: Let N be 8-dimensional nilpotent algebra with multiplication

$$[e_2, e_1] = e_4, [e_4, e_1] = e_5, [e_5, e_1] = e_6,$$
$$[e_3, e_2] = e_7, [e_7, e_1] = e_8, [e_4, e_3] = -e_8.$$

It is obvious that this is a non-model nilpotent algebra with the characteristic sequence $(4, 3, 1)$ and the number of generators is equal to 3. Moreover, the maximal solvable Lie algebra with nilradical N is 11-dimensional and has the multiplication:

$$L: \begin{cases} [e_2, e_1] = e_4, \quad [e_4, e_1] = e_5, \quad [e_5, e_1] = e_6, \quad [e_3, e_2] = e_7, \quad [e_7, e_1] = e_8, \\ [e_4, e_3] = -e_8, \\ [e_1, x_1] = e_1, \quad [e_4, x_1] = e_4, \quad [e_5, x_1] = 2e_5, \quad [e_6, x_1] = 3e_6, \quad [e_8, x_1] = e_8, \\ [e_2, x_2] = e_2, \quad [e_4, x_2] = e_4, \quad [e_5, x_2] = e_5, \quad [e_6, x_2] = e_6, \quad [e_7, x_2] = e_7, \\ [e_8, x_2] = e_8, \\ [e_3, x_3] = e_3, \quad [e_7, x_3] = e_7, \quad [e_8, x_3] = e_8. \end{cases}$$

Any local derivation on L is a derivation, moreover it is an inner derivation, i.e.

$$LocDer(L) = Der(L) = ad(L).$$

Now we consider an example with the dimension of complementary vector space of solvable Lie algebra less than number of generators of the nilradical.

Example 4.5: Let L be a maximal solvable Lie algebra with five-dimensional Heisenberg nilradical

$$[e_2, e_1] = e_5, \quad [e_4, e_3] = e_5.$$

Then $dim(L) = 8$ and L has the multiplication:

$$L : \begin{cases} [e_2, e_1] = e_5, & [e_4, e_3] = e_5, \\ [e_1, x_1] = e_1, & [e_2, x_1] = e_2, \quad [e_3, x_1] = e_3, \quad [e_4, x_1] = e_4, \quad [e_5, x_1] = 2e_5, \\ [e_1, x_2] = e_2, & [e_2, x_2] = e_2, \quad [e_3, x_2] = 2e_3, \quad [e_5, x_2] = 2e_5, \\ [e_1, x_3] = 2e_1, & [e_3, x_3] = e_3, \quad [e_4, x_3] = e_4, \quad [e_5, x_3] = 2e_5. \end{cases}$$

Any local derivation of L is a derivation (moreover, inner derivation).

Now we formulate the following conjecture

Conjecture 4.6: *Let L be a maximal solvable Lie algebra with nilradical N. Then any local derivation on L is a derivation.*

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