Report on the paper **On isomorphism criterion for a subclass of complex filiform Leibniz algebras** submitted by *I.S. Rakhimov, A.Kh. Khudoyberdiyev, B.A. Omirov, K.A. Mohd Atan* for publication in *International Journal of Algebra and Computation.*

The authors of this work continue the study of the classification of one-dimensional Leibniz central extensions of naturally graded filiform Lie algebras. Specifically, they classify the subclass TLb_{n+1} of complex filiform Leibniz algebras arising from naturally graded filiform Lie algebras. This subclass appears as Leibniz central extensions of linear deformations of the $(n+1)$-dimensional filiform Lie algebra μ_1^n, given by the brackets $[e_0, e_i] = e_{i+1}, i = 0, 1, \ldots, n-1$, in a basis $\{e_0, e_1, \ldots, e_{n-1}, e_n\}$. The difficult is that the class TLb_{n+1} is less suitable to adapted base change. The authors present an algorithm to give a isomorphism criterion for the subclass the subclass TLb_{n+1}. In the end, in an appendix to the work, they give the results applied to a particular case.

This work is a generalization of the results given by two of the authors in Bull. Aust. Math. Soc. 84 (2011), no. 2, 205–224 and Internat. J. Algebra Comput. 21 (2011), no. 5, 715–729.

The structure of the paper is well-developed, the content is mathematically correct, and the authors use similar techniques to the considered in the earlier aforementioned articles. These techniques are effective for such classification, and moreover require a hard work to make all the calculations. I consider that the paper contains mathematically relevant material.

I think, in my opinion, the paper for its mathematical content is of sufficient quality and therefore I recommend its possible publication in *International Journal of Algebra and Computation.*

Some comments and appreciations to the paper that can improve the presentation appear in the original document pdf of the authors, see below.

On isomorphism criterion for a subclass of complex filiform Leibniz algebras

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Abstract

In this paper we present an algorithm to give the isomorphism criterion for a subclass of complex filiform Leibniz algebras arising from naturally graded filiform Lie algebras. This subclass appeared as a Leibniz central extension of a linear deformation of filiform Lie algebra. We give the table of multiplication choosing appropriate adapted basis, identify the elementary base changes and describe behavior of structure constants under these base changes, then combining them the isomorphism criterion is given. The final result of calculations for one particular case also is provided.

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1 Introduction

The class of Lie algebras is one of the important classes of algebras in modern algebra. Intensive and extensive study of Lie algebras led to the appearance of a number of their generalizations like Malcev algebras, Lie superalgebras, binary Lie algebras, Leibniz algebras etc.

The present paper deals with the problem of classification of Leibniz algebras. We study a subclass of the Leibniz algebras naturally gradation of that is a filiform Lie algebra. In fact, there are two sources to get the filiform Leibniz algebras. The first of them is the class of naturally graded filiform Lie algebras and the other one is naturally graded non Lie filiform Leibniz algebras. As for those Leibniz algebras whose naturally gradation is a non Lie filiform Leibniz algebra their classification problem have been studied in [1], [3], [8], [12] and [13]. The classification problem for the class considered in this paper has been initiated in [7], [9], [10] and [11].

Let us be reminded that there already exists a classification of complex filiform Lie algebras up to dimension 11 [4]. However, the methods of classification in this case cannot be applied to ~~more~~ higher dimensional cases. It is observed that in determining of isomorphism classes one of the main problems is to keep track of the behaviors of structure ~~constant~~ under base change. In the present paper we consider this problem for a subclass of ~~finite dimensional~~ complex filiform Leibniz algebras. We give an algorithm for computing the behaviors of the structure constants.

Recall that an algebra L over a field F is called a *Leibniz algebra* if it satisfies the following Leibniz identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \tag{1}$$

where $[\cdot,\cdot]$ denotes the multiplication in L. Leibniz algebras have been introduced by Loday in 1993 (see [6]). Clearly a skew-symmetric Leibniz algebra is a Lie algebra. In this case (1) is just the Jacobi identity.

Let L be a Leibniz algebra. We put:

$$L^1 = L, \ L^{k+1} = [L^k, L], \ k \geq 1.$$

Definition 1.1. *A Leibniz algebra L is said to be nilpotent if there exists an integer $s \in N$, such that*

$$L^1 \supset L^2 \supset \dots \supset L^s = \{0\}.$$

Definition 1.2. *A Leibniz algebra L is said to be filiform if $dimL^i = n - i$, where $n = dimL$ and $2 \leq i \leq n$.*

It is obvious that a filiform Leibniz algebra is nilpotent. We denote the set of all filiform Leibniz algebra structures on an $n-$dimensional vector space by Lb_n.

Let L be a nilpotent Leibniz algebra. Consider $L_i = L^i/L^{i+1}$, $1 \leq i \leq n-1$, and $grL = L_1 \oplus L_2 \oplus \dots \oplus L_{n-1}$. Then $[L_i, L_j] \subseteq L_{i+j}$ and we obtain a graded algebra grL.

Definition 1.3. *A Leibniz algebra L' is said to be naturally graded if L' is isomorphic to grL, for some nilpotent Leibniz algebra L.*

The following theorem from [5] describes the naturally graded filiform Lie algebra structures on an $(n+1)$-dimensional vector space over an infinite field.

Theorem 1.1. *If n is odd then any $(n+1)-$dimensional naturally graded filiform Lie algebra over a field with infinitely many elements is isomorphic to one of the following two non-isomorphic algebras:*

$$\mu_1^n: \ \mu_1^n(e_0, e_i) = e_{i+1}, \quad 1 \leq i \leq n-1,$$

$$\mu_2^n: \begin{cases} \mu_2^n(e_0, e_i) = e_{i+1}, & 1 \leq i \leq n-1, \\ \mu_2^n(e_i, e_{n-i}) = (-1)^i e_n, & 1 \leq i \leq n-1, \end{cases}$$

where $\mu_i^n(\cdot,\cdot)-$ is the composition law of μ_i^n, $i \in \{1,2\}$ and omitted products of the basis vectors are zero.

But if n is even then any $(n+1)-$dimensional naturally graded filiform Lie algebra over a field with infinitely many elements is isomorphic to μ_1^n.

Denote by Δ the set of all pairs (k,r) such that $1 \leq k \leq n-1,\ 2k+1 < r \leq n$ (and in the case when n is odd one assumes that Δ contains the pair $(\frac{n-1}{2}, n)$, as well). For each pair $(k,r) \in \Delta$ one defines $\psi_{k,r}: \mu_1^n \wedge \mu_1^n \to \mu_1^n$ as follows:

$$\psi_{k,r}(e_i, e_j) = -\psi_{k,r}(e_j, e_i) = (-1)^{k-i}\binom{j-k-1}{k-i} e_{i+j+r-2k-1}, \qquad \text{for } \ 1 \leq i \leq k < j \leq n,$$

$$\psi_{k,r}(e_i, e_j) = -\psi_{k,r}(e_j, e_i) = 0, \ \text{~~eslewhere.~~}$$

Let G be an $(n+1)-$dimensional filiform Lie algebra. Then it is isomorphic to the algebra:

$$\mu_1^n + \psi, \quad \text{where} \quad \psi = \sum_{(k,r)\in\Delta} a_{k,r}\psi_{k,r}, \quad \text{with} \quad \psi(\psi(x,y),z) + \psi(\psi(y,z),x) + \psi(\psi(z,x),y) = 0.$$

The following theorem is a result from [3], [14] on filiform Leibniz algebra structures over $(n+1)$-dimensional complex vector space.

Theorem 1.2. *Any $(n+1)-$dimensional complex filiform Leibniz algebra admits a basis $\{e_0, e_1, ..., e_n\}$ called adapted, such that the table of multiplication of the algebra has one of the following forms:*

$$FLb_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0, e_1] = \alpha_3 e_3 + \alpha_4 e_4 + ... + \alpha_{n-1}e_{n-1} + \theta e_n, & \\ [e_j, e_1] = \alpha_3 e_{j+2} + \alpha_4 e_{j+3} + ... + \alpha_{n+1-j}e_n, & 1 \leq j \leq n-2, \\ \quad \alpha_3, \alpha_4, ..., \alpha_n, \theta \in \mathbb{C}. & \end{cases}$$

$$SLb_{n+1} = \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_0, e_1] = \beta_3 e_3 + \beta_4 e_4 + ... + \beta_n e_n, & \\ [e_1, e_1] = \gamma e_n, & \\ [e_j, e_1] = \beta_3 e_{j+2} + \beta_4 e_{j+3} + ... + \beta_{n+1-j}e_n, & 2 \leq j \leq n-2, \\ \quad \beta_3, \beta_4, ..., \beta_n, \gamma \in \mathbb{C}. & \end{cases}$$

$$TLb_{n+1} = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0, e_i] = -e_{i+1}, & 2 \leq i \leq n-1, \\ [e_0, e_0] = b_{0,0}e_n, & \\ [e_0, e_1] = -e_2 + b_{0,1}e_n, & \\ [e_1, e_1] = b_{1,1}e_n, & \\ [e_i, e_j] = a_{i,j}^1 e_{i+j+1} + \cdots + a_{i,j}^{n-(i+j+1)}e_{n-1} + b_{i,j}e_n, & 1 \leq i < j \leq n-2, \\ [e_i, e_j] = -[e_j, e_i], & 1 \leq i \leq j \leq n-1, \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = (-1)^i b_{i,n-i}e_n & 1 \leq i \leq n-1, \end{cases}$$

where $b_{i,n-i} = b \in \{0;1\}$, whenever $1 \le i \le n-1$, and $b = 0$ for even n and the structure constants $a_{i,j}^k, b_{i,j}$ are under the condition that the algebra to be a filiform Leibniz algebra.

Note that the classes FLb_n and SLb_n have been treated before in [3], [8], [12] and [13]. The present paper deals with a subclass of TLb_n.

The outline of the paper is as follows. Sections 1 and 2 give a brief introduction to a subclass of Leibniz algebras that we will study in this paper. The main result of the paper is in Section 3. In this section we first give the table of multiplication of the subclass of TLb_n which is supposed to be studied. Here we prove a few lemmas and propositions on behavior of the structure constants under adapted base change. The final result is given as Theorem 3.1 at the end of the paper. The calculations for one particular case are given in Appendix section.

2 Adapted base change and Simplifications in TLb_n

The center of a Leibniz algebra L is defined as $C(L) = \{x \in L \mid [x, L] = [L, x] = 0, \forall x \in L\}$.

Note that the center of an algebra L from TLb_{n+1} is ~~one dimensional~~ and spanned by the basis element e_n. Moreover, quotient algebra $L/C(L)$ is the $n-$dimensional filiform Lie algebra with the composition law

$$\mu = \mu_1^{n-1} + \psi, \quad \text{where} \quad \psi = \sum_{(k,r)\in\Delta} a_{k,r}\psi_{k,r}. \tag{2}$$

Therefore, the class TLb_{n+1} can be considered as a one-dimensional Leibniz central extension of the Lie algebra μ. This approach has been initiated in [10], [11]. In [10] Leibniz central extension of the simplest filiform Lie algebra, whereas in [11] the Leibniz central extension of its linear deformation ~~have~~ been considered. This paper is a generalization of the results given in [10], [11], where we choose as a generating Lie (co)cycle ψ with non trivial coefficient $a_{1,r}$.

Definition 2.1. *Let $\{e_0, e_1, \dots, e_n\}$ be an adapted basis of L. Then a linear transformation $f : L \to L$ is said to be adapted if the basis $\{f(e_0), f(e_1), \dots, f(e_n)\}$ is adapted.*

The set of all adapted transformations of algebras from TLb_n is denoted by G_{ad}.

To simplify notations we denote by

$$L(\mathbf{a}) = L(b_{0,0}, b_{0,1}, b_{1,1}, a_{i,j}^1, \dots, a_{i,j}^{n-(i+j+1)}, b_{i,j}),$$

an algebra from TLb_{n+1}, with the structure constants $b_{0,0}, b_{0,1}, b_{1,1}, a_{i,j}^1, \dots, a_{i,j}^{n-(i+j+1)}, b_{i,j}$.

Let $L(\mathbf{a}') = L(b'_{0,0}, b'_{0,1}, b'_{1,1}, a_{i,j}^{1\,\prime}, \dots, a_{i,j}^{n-(i+j+1)\,\prime}, b'_{i,j})$ and f be an adapted transformation sending $L(\mathbf{a})$ to $L(\mathbf{a}')$.

Since a filiform Leibniz algebra is 2-generated an adapted transformation can be taken as follows:

$$f(e_0) = A_0e_0 + A_1e_1 + \dots + A_ne_n, \quad f(e_1) = B_0e_0 + B_1e_1 + \dots + B_ne_n,$$

with $f(e_{i+1}) = [f(e_i), f(e_0)]$, where $(A_0B_1 - A_1B_0)(A_0 + A_1b) \neq 0$. ~~Here is~~ a result from [9] on changing of $b_{0,0}, b_{0,1}$ and $b_{1,1}$ under f.

Lemma 2.1. *For $b'_{0,0}, b'_{0,1}$ and $b'_{1,1}$ one has:*

$$b'_{0,0} = \frac{A_0^2b_{0,0} + A_0A_1b_{0,1} + A_1^2b_{1,1}}{A_0^{n-2}B_1(A_0 + A_1b)}, \quad b'_{0,1} = \frac{A_0b_{0,1} + 2A_1b_{1,1}}{A_0^{n-2}(A_0 + A_1b)}, \quad b'_{1,1} = \frac{B_1b_{1,1}}{A_0^{n-2}(A_0 + A_1b)} \text{ and } B_0 = 0.$$

Definition 2.2. *The following types of adapted transformations of $L \in TLb_{n+1}$ are said to be elementary:*

$$\textit{first type} - \quad \tau(a,b,c) = \begin{cases} \tau(e_0) = ae_0 + be_1, & \\ \tau(e_1) = ce_1, & ac \neq 0 \\ \tau(e_{i+1}) = [\tau(e_i), \tau(e_0)], & 1 \le i \le n-1, \end{cases}$$

$$\textit{second type} - \ \sigma(a,k)=\begin{cases}\sigma(e_0)=e_0+ae_k, & 2\le k\le n\\ \sigma(e_1)=e_1, & \\ \sigma(e_{i+1})=[\sigma(e_i),\sigma(e_0)], & 1\le i\le n-1,\end{cases}$$

$$\textit{third type} - \ \varphi(c,k)=\begin{cases}\varphi(e_0)=e_0, & \\ \varphi(e_1)=e_1+ce_k, & 2\le k\le n\\ \varphi(e_{i+1})=[\varphi(e_i),\varphi(e_0)], & 1\le i\le n-1\end{cases}$$

where $a,b,c\in\mathbb{C}$.

The proposition below is handled easily.

Proposition 2.1. *Let f be an adapted transformation of $L\in TLb_{n+1}$. Then it is represented as the composition*

$$f=\varphi(B_n,n)\circ\cdots\circ\varphi(B_2,2)\circ\sigma(A_n,n)\circ\cdots\circ\sigma(A_2,2)\circ\tau(A_0,A_1,B_1).$$

3 Main result

In this section we deal with a subclass of TLb_{n+1}. One considers the Lie cocycle ψ in (2) (see ~~section~~ 2) as $\psi=-a_{1,r}\psi_{1,r}$ with $a_{1,r}\neq 0$. We denote this subclass by $T_{1,r}$ and specify it as follows:

$$T_{1,r}=\{L\in TLb_{n+1}\mid L/<e_n>\cong\mu_1^{n-1}-a_{1,r}\psi_{1,r}\}.$$

The classification problem for TLb_n in general case seems to be complicated. We hope that the solution to the classification problem of $T_{1,r}$ will give hints toward solving the problem in the general case. Here we have succeeded to keep track of the behavior of the structure constants under the base change. We create an isomorphism criterion and according to that in any fixed dimensional case complete lists of isomorphism classes can be given.

An easy observation shows that under the adapted basis the class $T_{1,r}$ is written as follows:

$$T_{1,r}=\begin{cases}[e_i,e_0]=e_{i+1}, & 1\le i\le n-1, & \\ [e_0,e_i]=-e_{i+1}, & 2\le i\le n-1, & \\ [e_0,e_0]=b_{0,0}e_n, & & \\ [e_0,e_1]=-e_2+b_{0,1}e_n, & & \\ [e_1,e_1]=b_{1,1}e_n, & & \\ [e_i,e_1]=-[e_1,e_i]=a_{1,r}e_{i+r-2}+b_{i,1}e_n, & 2\le i\le n-r+1, & 4\le r\le n-1,\\ [e_i,e_1]=-[e_1,e_i]=b_{i,1}e_n, & n-r+2\le i\le n-2, & \\ [e_i,e_j]=-[e_j,e_i]=b_{i,j}e_n, & 2\le i\le n-2, & 2\le j\le n-i-1,\\ [e_i,e_{n-i}]=-[e_{n-i},e_i]=(-1)^ib_{i,n-i}e_n, & 1\le i\le n-1, & \\ \textit{where } b_{i,n-i}=b, 1\le i\le n-1, & & \\ b\in\{0,1\} \textit{ for odd } n, \textit{ and } b=0 \textit{ for even } n. & & \end{cases}$$

Let us be reminded that the structure constants in the table above are not free. They are under the condition that the algebras from $T_{1,r}$ to be filiform Leibniz algebras. To find the interrelations between the structure constants sequentially we apply the Leibniz identity to the adapted basis of the algebra from $T_{1,r}$.

Consider the Leibniz identity

$$[e_0,[e_1,e_k]]=[[e_0,e_1],e_k]-[[e_0,e_k],e_1].$$

It is observed that

$$[[e_0,e_1],e_k]-[[e_0,e_k],e_1]=\begin{cases}a_{1,r}e_{k+r-1}-(b_{1,k+1}+b_{2,k})e_n, & 2\le k\le n-r,\\ -(b_{1,k+1}+b_{2,k})e_n, & n-r+1\le k\le n-3,\\ 0, & n-2\le k\le n.\end{cases}$$

and

$$[e_0,[e_1,e_k]] = \begin{cases} a_{1,r}e_{k+r-1}, & 2\leq k\leq n-r+1,\\ 0, & n-r+2\leq k\leq n.\end{cases}$$

Comparing the ~~coefficient~~ we obtain

$$\begin{cases} b_{2,k}=-b_{1,k+1}, & 2\leq k\leq n-3, \quad k\neq n-r+2,\\ b_{2,n-r+1}=-a_{1,r}-b_{1,n-r+2}. \end{cases}$$

Similarly considering the Leibniz identity for e_0, e_j and e_k at $2\leq j<k\leq n-j-1$ we get

$$b_{j+1,k}=-b_{j,k+1}.$$

According to the fact that $b_{i,i}=0$, one obtains

$$0=b_{i,i}=-b_{i-1,i+1}=b_{i-2,i+2}=\cdots=(-1)^{i-2}b_{2,2i-2} \tag{3}$$

If $n+1-r$ is even then putting $2i-2=n+1-r$ we get $0=b_{2,2i-2}=b_{2,n+1-r}=-a_{1,r}-b_{1,n+2-r}$ and $b_{1,n+2-r}=-a_{1,r}$. Generally,

$$b_{1,n+2-r}=-a_{1,r} \ \text{ and } \ b_{i+t,t}=\begin{cases} 0, & i=2s_0,\\ (-1)^{s_0}b_{i+t-s_0,t+s_0}, & i=2s_0+1,\\ 0, & i=n-2t+3-r \ \text{ and } \ t\neq 1.\end{cases}$$

Let us consider the case when $n+1-r$ is odd. Then $n+1-r\neq 2i-2$ for any values of i and applying (3) we derive

$$b_{i+t,t}=\begin{cases} 0, & i=2s_0,\\ (-1)^{s_0}b_{i+t-s_0,t+s_0}, & i=2s_0+1,\\ (-1)^t(b_{1,n-r+2}+a_{1,r}), & i=n-2t+3-r \ \text{ and } \ t\neq 1.\end{cases}$$

The Leibniz identity

$$[e_1,[e_i,e_j]]=[[e_1,e_i],e_j]-[[e_1,e_j],e_i] \text{ for } 2\leq i<j\leq n-r+1$$

gives

$$0=[e_1,[e_i,e_j]]=[[e_1,e_i],e_j]-[[e_1,e_j],e_i]=(-a_{1,r}e_{i+r-2}+b_{1,i}e_n)e_j+(-a_{1,r}e_{j+r-2}+b_{1,j}e_n)e_i$$

$$=a_{1,r}(b_{j,i+r-2}-b_{i,j+r-2})e_n=\begin{cases} 2a_{1,r}b_{i,j+r-2}e_n, & \text{if } i+j \text{ is odd, } r \text{ is even,}\\ 0, & \text{otherwise.}\end{cases}$$

Since $a_{1,r}\neq 0$, we have

$$b_{i,j+r-2}=0, \quad i+j \text{ is odd, } r \text{ is even, } 2\leq i<j\leq n-r+1.$$

Due to the observations above we conclude that the expressions for $b_{i,j}$ in the case of odd r can be rewritten as follows.

Lemma 3.1. *Let r be odd. Then*

a) for even n

$$\begin{aligned} &b_{i,j}=0, && i+j - \text{even},\ (i,j)\neq(1,n+r-2),\\ &b_{1,n+2-r}=-a_{1,r}, &&\\ &b_{i,j}=(-1)^{i-1}b_{1,i+j-1}, && i+j-\text{odd},\end{aligned}$$

b) for odd n

$$\begin{array}{ll} b_{i,j}=0, & i+j-\mathit{even} \\ b_{i,j}=(-1)^{i-1}b_{1,i+j-1}, & i+j-\mathit{odd\ and}\ i+j\neq n+3-r.\ \cancel{)} \\ b_{i,n+3-r-i}=(-1)^{i-1}(b_{1,n+2-r}+a_{1,r}), & 2\leq i\leq n+1-r. \end{array}$$

The proof of the following lemma is similar to that of Lemma 3.1.

Lemma 3.2. *Let r be even. Then*

a) for even n we have

$$b_{i,j}=0, \quad i+j-\mathit{even},$$

- *$i+j$ is odd and $i+j\neq n+3-r$,*

$$\begin{array}{ll} b_{i,j}=(-1)^{i-1}b_{1,i+j-1}, & i+j\leq r+1, \\ b_{i,j}=0, & r+3\leq i+j\leq n, \end{array}$$

- *for $i+j=n+3-r$,*
  - *if $r\leq\frac{n}{2}$, then*

$$b_{1,n+2-r}=-a_{1,r}, \quad b_{i,n+3-r-i}=0, \quad 2\leq i\leq n+1-r,$$

  - *if $r\geq\frac{n+2}{2}$, then*

$$b_{i,n+3-r-i}=(-1)^{i-1}(b_{1,n+2-r}+a_{1,r}), \quad 2\leq i\leq n+1-r$$

b) for odd n

$$\begin{array}{ll} b_{i,j}=0, & i+j-\mathit{even},\ (i,j)\neq(1,n+r-2), \\ b_{1,n+2-r}=-a_{1,r}, & \\ b_{i,j}=(-1)^{i-1}b_{1,i+j-1}, & i+j-\mathit{odd},\ i+j\leq r+1. \\ b_{i,j}=0, & i+j-\mathit{odd},\ r+3\leq i+j\leq n. \end{array}$$

Thus, from Lemmas 3.1 and 3.2 we conclude that all the structure constants $b_{i,j}$ are expressed

via	$a_{1,r},\ b_{1,2i},$	$1\leq i\leq\frac{n-2}{2},\ i\neq\frac{n+2-r}{2},$	if r is odd and n is even;
via	$a_{1,r},\ b_{1,2i},\quad b_{1,n-1},$	$1\leq i\leq\frac{n-3}{2},$	if both r and n ~~is~~ odd;
via	$a_{1,r},\ b_{1,2i},$	$1\leq i\leq\frac{r}{2},\quad 4\leq r\leq\frac{n}{2},$	if both r and n ~~is~~ even;
via	$a_{1,r},\ b_{1,2i},\quad b_{1,n+2-r},$	$1\leq i\leq\frac{r}{2},\quad\frac{n+2}{2}\leq r\leq n-4,$	if both r and n ~~is~~ even;
and via	$a_{1,r},\ b_{1,2i},$	$1\leq i\leq\frac{r}{2},\ 4\leq r\leq n-4,$	if r is even and n is odd.

In the rest, where $r=n-3,n-2,n-1$ the situation is as follows:

If n is even then the structure constants $b_{i,j}$ are expressed via

$$a_{1,r},\ b_{1,2i}, \quad 1\leq i\leq\frac{n-2}{2}, \quad i\neq\frac{n+2-r}{2};$$

If n is odd they are written via

$$a_{1,r},\ b_{1,2i},\ 1\leq i\leq\frac{n-3}{2}.$$

Thus, we enhance the table of multiplication of $T_{1,r}$.

Let us now study the actions of adapted transformations on $T_{1,r}$. First consider the action of τ on $T_{1,r}$.

$$\tau(A_0, A_1, B_1) = \begin{cases} \tau(e_0) = A_0e_0 + A_1e_1, & \\ \tau(e_1) = B_1e_1, & \cancel{A_1B_1} \neq 0 \\ \tau(e_{i+1}) = [\tau(e_i), \tau(e_0)], & 1 \leq i \leq n-1. \end{cases}$$

It is easy to see that $\tau(e_2) = A_0B_1e_2 + A_1B_1b_{1,1}e_n$, but for the action of the τ to $e_i,\ 3 \leq i \leq n$ we have to consider a few cases. In the following proposition we present expressions for $\tau(e_i),\ 3 \leq i \leq n$ and track of changes of the structure constants.

Proposition 3.1. *Under the first type elementary base change we have*

$$a'_{1,r} = \frac{B_1}{A_0^{r-2}} a_{1,r},$$

a) for $4 \leq r \leq \frac{n+2}{2}$

$$A_1 = 0, \quad b'_{1,2} = \frac{B_1}{A_0^{n-2}} b_{1,2}.$$

$$\tau(e_i) = A_0^{i-1}B_1e_i, \quad 3 \leq i \leq n.$$

b) for $r = \frac{n+3}{2}$ *(n odd)*

$$b'_{1,2} = \frac{A_0^2B_1b_{1,2} - A_1B_1a_{1,r}\big(A_0b_{1,r-1} - (r-3)A_0a_{1,r} + (r-3)A_1a_{1,r}b_{1,n-1}\big)}{A_0^{n-1}(A_0 - A_1b_{1,n-1})}, \quad \textit{if } r \textit{ is odd},$$

$$b'_{1,2} = \frac{A_0^2B_1b_{1,2} + (r-3)A_1B_1a_{1,r}^2(A_0 - A_1b_{1,n-1})}{A_0^{n-1}(A_0 - A_1b_{1,n-1})} \quad \textit{if } r \textit{ is even},$$

$$\begin{aligned} \tau(e_i) &= A_0^{i-1}B_1e_i + (i-2)A_0^{i-2}A_1B_1a_{1,r}e_{i+r-3} - A_0^{i-3}A_1B_1(A_0b_{1,i-1} \\ &\qquad + (i-3)A_1a_{1,r}b_{1,i+r-4})e_n, \quad 3 \leq i \leq r-1, \\ \tau(e_r) &= A_0^{r-1}B_1e_r + A_0^{r-3}A_1B_1[(r-3)A_0a_{1,r} - A_0b_{1,r-1} - (r-3)A_1a_{1,r}b_{1,n-1}]e_n, \\ \tau(e_i) &= A_0^{i-1}B_1e_i - A_0^{i-2}A_1B_1b_{1,i-1}e_n,\ \ r+1 \leq i \leq n. \end{aligned}$$

c) for $\frac{n+4}{2} \leq r \leq n-1$

$$\begin{array}{lll} b'_{1,2} = \frac{B_1(A_0b_{1,2} - A_1a_{1,r}b_{1,r-1})}{A_0^{n-1}}, & r \textit{ is odd}, & n \textit{ is even}, \\ b'_{1,2} = \frac{B_1(A_0b_{1,2} - A_1a_{1,r}b_{1,r-1})}{A_0^{n-2}(A_0 + A_1b_{1,n-1})}, & r \textit{ is odd}, & n \textit{ is odd}, \\ b'_{1,2} = \frac{B_1}{A_0^{n-2}}b_{1,2}, & r \textit{ is even}, & n \textit{ is even}, \\ b'_{1,2} = \frac{B_1b_{1,2}}{A_0^{n-3}(A_0 - A_1b_{1,n-1})}, & r \textit{ is even}, & n \textit{ is odd}. \end{array}$$

$$\begin{aligned} \tau(e_i) &= A_0^{i-1}B_1e_i + (i-2)A_0^{i-2}A_1B_1a_{1,r}e_{i+r-3} - A_0^{i-3}A_1B_1(A_0b_{1,i-1} \\ &\qquad + (i-3)A_1a_{1,r}b_{1,i+r-4})e_n, \quad 3 \leq i \leq r-1, \\ \tau(e_i) &= A_0^{i-1}B_1e_i - A_0^{i-2}A_1B_1b_{1,i-1}e_n, \quad r \leq i \leq n. \end{aligned}$$

Proof. Since

$$\tau(e_2) = A_0B_1e_2 + A_1B_1b_{1,1}e_n,$$

by induction we obtain

$$\tau(e_i) = A_0^{i-1}B_1e_i + (i-2)A_0^{i-2}A_1B_1a_{1,r}e_{i+r-3} + \cdots + (*)e_n, \quad 3 \leq i \leq n-r+2,$$

$$\tau(e_{n-r+3}) = A_0^{n-r+2}B_1e_{n-r+3} + [(n-r)A_0^{n-r+1}A_1B_1a_{1,r} - A_0^{n-r+1}A_1B_1b_{1,n-r+2}$$

$$-(n-r)A_0^{n-r}A_1^2B_1a_{1,r}b_{1,n-1}]e_n,$$

$$\tau(e_i) = A_0^{i-1}B_1e_i - A_0^{i-2}A_1B_1b_{1,i-1}e_n, \quad n-r+4 \le i \le n.$$

Let us consider the product

$$[\tau(e_2),\tau(e_1)] = a'_{1,r}\tau(e_r) - b'_{1,2}\tau(e_n) = a'_{1,r}\tau(e_r) - b'_{1,2}(A_0^{n-1}B_1 - A_0^{n-2}A_1B_1b_{1,n-1})e_n.$$

On the other hand we have

$$[\tau(e_2),\tau(e_1)] = [A_0B_1e_2 + A_1B_1b_{1,1}e_n, B_1e_1] = A_0B_1^2a_{1,r}e_r - A_0B_1^2b_{1,2}e_n.$$

Therefore, comparing these two we obtain the equation

$$a'_{1,r}\tau(e_r) - b'_{1,2}(A_0^{n-1}B_1 + A_0^{n-2}A_1B_1b_{n-1,1})e_n = A_0B_1^2a_{1,r}e_r - A_0B_1^2b_{1,2}e_n. \tag{4}$$

Case 1.1. Let r be odd and n be even. Then $b_{1,n-1}=0$.

Consider the equation (4) with $4 \le r \le n-r+2$. Then we have

$$a'_{1,r}(A_0^{r-1}B_1e_r + (r-2)A_0^{r-2}A_1B_1a_{1,r}e_{2r-3} + \dots + (*)e_n) - b'_{1,2}A_0^{n-1}B_1e_n = A_0B_1^2a_{1,r}e_r - A_0B_1^2b_{1,2}e_n,$$

from which we get

$$a'_{1,r} = \frac{B_1}{A_0^{r-2}}a_{1,r}, \quad A_1 = 0, \quad b'_{1,2} = \frac{B_1}{A_0^{n-2}}b_{1,2}.$$

Since r is odd and n is even we have $r \ne n-r+3$.

Let us consider the equation (4) with $n-r+4 \le r \le n-1$. Then

$$a'_{1,r}(A_0^{r-1}B_1e_r - A_0^{r-2}A_1B_1b_{1,r-1}e_n) - b'_{1,2}A_0^{n-1}B_1e_n = A_0B_1^2a_{1,r}e_r - A_0B_1^2b_{1,2}e_n.$$

Comparing the coefficients at the basis element we deduce

$$a'_{1,r} = \frac{B_1}{A_0^{r-2}}a_{1,r}, \quad b'_{1,2} = \frac{B_1}{A_0^{n-1}}(A_0b_{1,2} - A_1a_{1,r}b_{1,r-1}).$$

Case 1.2. Let both r and n be odd.

The equation (4) with $4 \le r \le n-r+1$ has the form

$$a'_{1,r}(A_0^{r-1}B_1e_r + (r-2)A_0^{r-2}A_1B_1a_{1,r}e_{2r-3} + \dots + (*)e_n) - b'_{1,2}(A_0^{n-1}B_1 - A_0^{n-2}A_1B_1b_{1,n-1})e_n$$
$$= A_0B_1^2a_{1,r}e_r - A_0B_1^2b_{1,2}e_n.$$

This implies

$$a'_{1,r} = \frac{B_1}{A_0^{r-2}}a_{1,r}, \quad A_1 = 0, \quad b'_{1,2} = \frac{B_1}{A_0^{n-2}}b_{1,2}.$$

Due to the conditions for r and n we have $r \ne n-r+2$.

If $r = n-r+3$, then we obtain

$$a'_{1,r}(A_0^{r-1}B_1e_r + [(r-3)A_0^{r-2}A_1B_1a_{1,r} - A_0^{r-2}A_1B_1b_{1,r-1} - (r-3)A_0^{r-3}A_1^2B_1a_{1,r}b_{1,n-1}]e_n)$$

$$-b'_{1,2}(A_0^{n-1}B_1 - A_0^{n-2}A_1B_1b_{1,n-1})e_n = A_0B_1^2a_{1,r}e_r - A_0B_1^2b_{1,2}e_n.$$

Comparing the coefficients at the basis elements we get

$$a'_{1,r} = \frac{B_1}{A_0^{r-2}}a_{1,r}, \quad b'_{1,2} = \frac{A_0^2B_1b_{1,2} - A_1B_1a_{1,r}\big(A_0b_{1,r-1} - (r-3)A_0a_{1,r} + (r-3)A_1a_{1,r}b_{1,n-1}\big)}{A_0^{n-1}(A_0 - A_1b_{1,n-1})}.$$

The equality (4) with $n-r+4 \le r \le n-1$ has the form

$$a'_{1,r}(A_0^{r-1}B_1e_r - A_0^{r-2}A_1B_1b_{1,r-1}e_n) - b'_{1,2}(A_0^{n-1}B_1 - A_0^{n-2}A_1B_1b_{1,n-1})e_n = A_0B_1^2a_{1,r}e_r - A_0B_1^2b_{1,2}e_n.$$

Hence

$$a'_{1,r} = \frac{B_1}{A_0^{r-2}} a_{1,r}, \quad b'_{1,2} = \frac{A_0 B_1 b_{1,2} - A_1 B_1 a_{1,r} b_{1,r-1}}{A_0^{n-2}(A_0 - A_1 b_{1,n-1})}.$$

Case 2.1. Let both r and n be even. Then due to Lemma 3.2 we have $b_{1,n-1} = b_{1,r-1} = 0$. Similar to the previous cases we get

$$a'_{1,r} = \frac{B_1}{A_0^{r-2}} a_{1,r}, \quad A_1 = 0, \quad b'_{1,2} = \frac{B_1}{A_0^{n-2}} b_{1,2}, \quad 4 \leq r \leq n-r+2,$$

The condition for r and n implies $r \neq n-r+3$. Analogously, we obtain

$$a'_{1,r} = \frac{B_1}{A_0^{r-2}} a_{1,r}, \quad b'_{1,2} = \frac{B_1}{A_0^{n-2}} b_{1,2}, \quad \text{where} \quad n-r+4 \leq r \leq n-1.$$

Case 2.2. Let now r be even and n be odd. Then thanks to Lemma 3.2 we have $b_{1,r-1} = 0$. The equality (4) with $4 \leq r \leq n-r+1$ gives

$$a'_{1,r} = \frac{B_1}{A_0^{r-2}} a_{1,r}, \quad A_1 = 0, \quad b'_{1,2} = \frac{B_1}{A_0^{n-2}} b_{1,2}.$$

It is observed that $r \neq n-r+2$.
For the subcase with $r = n-r+3$ we get

$$a'_{1,r} = \frac{B_1}{A_0^{r-2}} a_{1,r}, \quad b'_{1,2} = \frac{A_0^2 B_1 b_{1,2} + (r-3) A_1 B_1 a_{1,r}^2 (A_0 - A_1 b_{1,n-1})}{A_0^{n-1}(A_0 - A_1 b_{1,n-1})}.$$

Similarly we obtain

$$a'_{1,r} = \frac{B_1}{A_0^{r-2}} a_{1,r}, \quad b'_{1,2} = \frac{B_1 b_{1,2}}{A_0^{n-3}(A_0 - A_1 b_{1,n-1})}, \quad n-r+4 \leq r \leq n-1.$$

The analysis of Cases 1.1 - 2.2 and the use of induction lead to the following laws for calculation of $\tau(e_i)$:

a) for $4 \leq r \leq \frac{n+2}{2}$

$$\tau(e_i) = A_0^{i-1} B_1 e_i, \quad 3 \leq i \leq n.$$

b) for $r = \frac{n+3}{2}$ (n odd)

$$\tau(e_i) = A_0^{i-1} B_1 e_i + (i-2) A_0^{i-2} A_1 B_1 a_{1,r} e_{i+r-3} - A_0^{i-3} A_1 B_1 (A_0 b_{1,i-1} + (i-3) A_1 a_{1,r} b_{1,i+r-4}) e_n, \quad 3 \leq i \leq r-1,$$

$$\tau(e_r) = A_0^{r-1} B_1 e_r + A_0^{r-3} A_1 B_1 [(r-3) A_0 a_{1,r} - A_0 b_{1,r-1} - (r-3) A_1 a_{1,r} b_{1,n-1}] e_n,$$

$$\tau(e_i) = A_0^{i-1} B_1 e_i - A_0^{i-2} A_1 B_1 b_{1,i-1} e_n, \ r+1 \leq i \leq n.$$

c) for $\frac{n+4}{2} \leq r \leq n-1$

$$\tau(e_i) = A_0^{i-1} B_1 e_i + (i-2) A_0^{i-2} A_1 B_1 a_{1,r} e_{i+r-3} - A_0^{i-3} A_1 B_1 (A_0 b_{1,i-1} + (i-3) A_1 a_{1,r} b_{1,i+r-4}) e_n, \quad 3 \leq i \leq r-1,$$

$$\tau(e_i) = A_0^{i-1} B_1 e_i - A_0^{i-2} A_1 B_1 b_{1,i-1} e_n, \quad r \leq i \leq n.$$

$\square$

By virtue of Lemmas 3.1 and 3.2 we can express the structure constants $b'_{i,j}$ via $b'_{1,j}$ and $a_{1,r}$. Therefore, we shall restrict the study to the change of the structure constants $b_{1,j}$.

Proposition 3.2.

1. *If* $4 \leq r \leq n-r+2$, *then*
$$b'_{1,i} = \frac{B_1}{A_0^{n-i}} b_{1,i}, \quad 2 \leq i \leq n-2.$$

2. *If* $n-r+3 \leq r \leq n-1$, *then*

a)
$$b'_{1,i} = \frac{B_1(A_0 b_{1,i} + (i-3)A_1 a_{1,r} b_{1,i+r-3})}{A_0^{n-i}(A_0 - A_1 b_{1,n-1})}, \quad \text{for } 3 \leq i \leq n-r+1,$$

b)
$$b'_{1,i} = \frac{B_1 b_{1,i}}{A_0^{n-i-1}(A_0 - A_1 b_{1,n-1})}, \quad \text{for } n-r+2 \leq i \leq n-2.$$

Proof. Consider the products
$$[\tau(e_1), \tau(e_i)] = -a'_{1,r}\tau(e_{r+i-2}) + b'_{1,i}\tau(e_n), \ 3 \leq i \leq n-r+1,$$

Also we have
$$[\tau(e_1), \tau(e_i)] = b'_{1,i}\tau(e_n), \ n-r+2 \leq i \leq n-2.$$

Then due to Proposition 3.1 we get three possible cases.

Case 1). Let $4 \leq r \leq n-r+2$. Then from part a) of Proposition 3.1 and on substituting the expression for $\tau(e_i) = A_0^{i-1}B_1 e_i, \ 3 \leq i \leq n$ and comparing coefficients we get
$$b'_{1,i} = \frac{B_1}{A_0^{n-i}} b_{1,i}, \quad 2 \leq i \leq n-r+1.$$

Case 2). Let $n-r+3 = r$. Then applying Proposition 3.1 again we have
$$[\tau(e_1), \tau(e_i)] = -a'_{1,r}\tau(e_{r+i-2}) + b'_{1,i}\tau(e_n)$$
$$= -a'_{1,r}A_0^{r+i-3}B_1 e_{r+i-2} - a'_{1,r}A_0^{r+i-4}A_1 B_1 b_{r+i-3,1}e_n + b'_{1,i}(A_0^{n-1}B_1 + A_0^{n-2}A_1 B_1 b_{n-1,1})e_n.$$

On the other hand one has
$$[\tau(e_1), \tau(e_i)] = [B_1 e_1, A_0^{i-1}B_1 e_i + (i-2)A_0^{i-2}A_1 B_1 a_{1,r} e_{i+r-3} + A_0^{i-3}A_1 B_1 (A_0 b_{i-1,1}$$
$$+(i-3)A_1 a_{1,r} b_{i+r-4,1})e_n] = -A_0^{i-1}B_1^2 a_{1,r} e_{r+i-2} + A_0^{i-1}B_1^2 b_{1,i} e_n + (i-2)A_0^{i-2}A_1 B_1^2 a_{1,r} b_{1,r+i-3} e_n.$$

Therefore,
$$b'_{1,i} = \frac{B_1(A_0 b_{1,i} + (i-3)A_1 a_{1,r} b_{1,i+r-3})}{A_0^{n-i}(A_0 - A_1 b_{1,n-1})}, \quad 3 \leq i \leq n-r+1.$$

For $n-r+2 \leq i \leq n-2$ we consider the products
$$[\tau(e_1), \tau(e_i)] = b'_{1,i}\tau(e_n) = b'_{1,i}A_0^{n-2}B_1(A_0 - A_1 b_{1,n-1})e_n.$$

and
$$[\tau(e_1), \tau(e_i)] = [B_1 e_1, A_0^{i-1}B_1 e_i + A_0^{i-2}A_1 B_1 b_{i-1,1} e_n] = -A_0^{i-1}B_1^2 b_{1,i} e_n.$$

Comparing the coefficients of the respective basis vectors we obtain
$$b'_{1,i} = \frac{B_1 b_{1,i}}{A_0^{n-i-1}(A_0 - A_1 b_{1,n-1})}, \quad n-r+2 \leq i \leq n-2.$$

Case 3). Let $n-r+4 \leq r \leq n-1$. Then applying similar arguments as above we obtain

$$b'_{1,i} = \frac{B_1(A_0 b_{1,i} + (i-3)A_1 a_{1,r} b_{1,i+r-3})}{A_0^{n-i}(A_0 - A_1 b_{1,n-1})}, \quad 3 \leq i \leq n-r+1,$$

$$b'_{1,i} = \frac{B_1 b_{1,i}}{A_0^{n-i-1}(A_0 - A_1 b_{1,n-1})}, \quad n-r+2 \leq i \leq n-2.$$

Since the results of the cases 2 and 3 coincide we combine them together. $\square$

Since $b_{1,2i+1} = 0$ and applying Lemmas 3.1 and 3.2 we obtain

$$\begin{array}{lll}
b'_{1,2i} = \frac{B_1}{A_0^{n-2i}} b_{1,2i}, & 1 \leq i \leq \left[\frac{n-2}{2}\right], \ 4 \leq r \leq n-r+2, \\
b'_{1,2i} = \frac{B_1(A_0 b_{1,2i} + (2i-3)A_1 a_{1,r} b_{1,2i+r-3})}{A_0^{n-2i}(A_0 - A_1 b_{1,n-1})}, & 1 \leq i \leq \left[\frac{n-r+1}{2}\right], \ n-r+3 \leq r \leq n-1, \\
b'_{1,2i} = \frac{B_1 b_{1,2i}}{A_0^{n-2i-1}(A_0 - A_1 b_{1,n-1})}, & \left[\frac{n-r+1}{2}\right] + 1 \leq i \leq \left[\frac{n-2}{2}\right], \ n-r+3 \leq r \leq n-1.
\end{array} \quad (5)$$

Consider now the behavior of the structure constants under the second type elementary transformations.

Proposition 3.3. *The actions of the second type elementary base changes are expressed as follows*

$$\begin{array}{ll}
b'_{1,2i} = b_{1,2i} - A_k a_{1,r}(b_{1,k+2i+r-4} + b_{k,2i+r-3}) + A_k^2 a_{1,r}^2 b_{k,k+2i+2r-7}, & 1 \leq i \leq \left[\frac{n}{2}\right] + 3 - k - r. \\
b'_{1,2i} = b_{1,2i} - A_k a_{1,r}(b_{1,k+2i+r-4} + b_{k,2i+r-3}), & \left[\frac{n}{2}\right] + 4 - k - r \leq i \leq \left[\frac{n+3-k-r}{2}\right],
\end{array}$$

where $b_{k,2i+r-3}$ and $b_{k,k+2i+2r-7}$ under the conditions of Lemmas 3.1 and 3.2.

Proof. Indeed,

$$\sigma(A_k, k) = \begin{cases} \sigma(e_0) = e_0 + A_k e_k, & 2 \leq k \leq n \\ \sigma(e_1) = e_1, \\ \sigma(e_{i+1}) = [\sigma(e_i), \sigma(e_0)], & 1 \leq i \leq n-1, \end{cases}$$

and by induction it is not difficult to see that

$$\sigma(e_i) = \begin{cases} e_2 - A_k a_{1,r} e_{k+r-2} + b_{k,1} e_n) & i = 2, \\ e_i - A_k a_{1,r} e_{k+i+r-4} + (A_k b_{i-1,k} - A_k^2 a_{1,r} b_{k+i+r-5,k}) e_n, & 3 \leq i \leq n+4-2k-r, \\ e_i - A_k a_{1,r} e_{k+i+r-4} + A_k b_{i-1,k} e_n, & n+5-2k-r \leq i \leq n+4-k-r, \\ e_i + A_k b_{i-1,k} e_n, & n+5-k-r \leq i \leq n-k, \\ e_i, & n-k+1 \leq i \leq n. \end{cases}$$

Let us consider the product

$$[\sigma(e_1), \sigma(e_i)] = \begin{cases} -a'_{1,r}\sigma(e_{r+i-2}) + b'_{1,i}\sigma(e_n) & 3 \leq i \leq n-r+1, \\ b'_{1,i}\sigma(e_n), & n-r+2 \leq i \leq n-1. \end{cases}$$

$$= \begin{cases} b'_{1,i} e_n - a'_{1,r}\big(e_{r+i-2} - A_k a_{1,r} e_{k+2r+i-6}\big) - \\ \qquad -a'_{1,r}\big(A_k b_{r+i-3,k} - A_k^2 a_{1,r} b_{k+2r-i-7,k}\big) e_n & 2 \leq i \leq n+6-2k-2r, \\ b'_{1,i} e_n - a'_{1,r}\big(e_{r+i-2} - A_k a_{1,r} e_{k+2r+i-6} + A_k b_{r+i-3,k} e_n\big) & n+7-2k-2r \leq i \leq n+5-k-2r, \\ b'_{1,i} e_n - a'_{1,r}\big(e_{r+i-2} + A_k b_{r+i-3,k} e_n\big) & n+6-k-2r \leq i \leq n+3-k-r, \\ b'_{1,i} e_n - a'_{1,r} e_{r+i-2} & n+4-k-r \leq i \leq n+1-r, \\ b'_{1,i} e_n & n+2-r \leq i \leq n-1. \end{cases}$$

On the other hand one has

$$[\sigma(e_1), \sigma(e_i)] = \begin{cases} [e_1, e_i - A_k a_{1,r} e_{k+r-4+i}] & 2 \leq i \leq n+3-k-r, \\ [e_1, e_i] & n+4-k-r \leq i \leq n-1, \end{cases}$$

$$= \begin{cases} -a_{1,r}e_{r+i-2} + b_{1,i}e_n - A_k a_{1,r}(-a_{1,r}e_{k+2r+i-6} + b_{1,k+r+i-4}e_n) & 2 \le i \le n+5-k-2r, \\ -a_{1,r}e_{r+i-2} + b_{1,i}e_n - A_k a_{1,r} b_{1,k+r+i-4}e_n & n+6-k-2r \le i \le n+3-k-r, \\ -a_{1,r}e_{r+i-2} + b_{1,i}e_n & n+4-k-r \le i \le n+1-r, \\ b_{1,i}e_n & n+2-r \le i \le n-1, \end{cases}$$

Therefore we derive

$$\begin{array}{ll} a'_{1,r} = a_{1,r}, & \\ b'_{1,i} = b_{1,i} - A_k a_{1,r}(b_{1,k+i+r-4} + b_{k,i+r-3}) + A_k^2 a_{1,r}^2 b_{k,k+i+2r-7}, & 2 \le i \le n+6-2k-2r. \\ b'_{1,i} = b_{1,i} - A_k a_{1,r}(b_{1,k+i+r-4} + b_{k,i+r-3}), & n+7-2k-2r \le i \le n+3-k-r, \\ b'_{1,i} = b_{1,i}, & n+4-k-r \le i \le n-1, \end{array}$$

Using the conditions of Lemmas 3.1 and 3.2 we obtain $b'_{1,2i+1} = 0,\ 2i+1 \ne n+2-r$, which completes the proof of the proposition. □

Applying consistently the second elementary transformation $\sigma(A_k, k)$ for $2 \le k \le n$ we obtain recurrent formula

$$b_{1,2i}^{\{k\}} = b_{1,2i}^{\{k-1\}} - A_k a_{1,r}(b_{1,k+2i+r-4}^{\{k-1\}} + (-1)^{\{k-1\}} b_{k,2i+r-3}^{\{k-1\}}) + (-1)^{k-1} A_k^2 a_{1,r}^2 b_{k,k+2i+2r-7}^{\{k-1\}}, \tag{6}$$

where $2 \le k \le n$ and $b_{1,t}^{\{k-1\}}$ with $t \ge n-1$ are assumed equal to zero and $b_{1,i}^{\{1\}} = b_{1,i}$. Moreover $b_{i,j}^{\{k-1\}}$ satisfy the conditions of Lemmas 3.1 and 3.2 as $b_{i,j}$.

Let us now consider the action of the third type elementary transformations on $T_{1,r}$.

Proposition 3.4. *The action of the third type elementary base change on the structure constants of algebras from $T_{1,r}$ is expressible as follows*

$$\begin{array}{ll} b'_{1,2i} = b_{1,2i} + B_k(b_{k,2i} + b_{1,k+2i-1}) + B_k^2 b_{k,k+2i-1}, & 1 \le i \le \left[\frac{n+1}{2}\right] - k \\ b'_{1,2i} = b_{1,2i} + B_k(b_{k,2i} + b_{1,k+2i-1}), & \left[\frac{n+1}{2}\right] - k + 1 \le i \le \left[\frac{n-k}{2}\right], \end{array}$$

Proof. Consider

$$\varphi(B_k, k) = \begin{cases} \varphi(e_0) = e_0, & \\ \varphi(e_1) = e_1 + B_k e_k, & 2 \le k \le n, \\ \varphi(e_{i+1}) = [\varphi(e_i), \varphi(e_0)], & 1 \le i \le n-1. \end{cases}$$

Then inductively we find

$$\varphi(e_i) = e_i + B_k e_{k+i-1}, 2 \le i \le n+1-k,$$
$$\varphi(e_i) = e_i,\ n+2-k \le i \le n.$$

Similar to that of Proposition 3.3 we derive

$$\begin{array}{ll} b'_{1,i} = b_{1,i} + B_k(b_{k,i} + b_{1,k+i-1}) + B_k^2 b_{k,k+i-1}, & 2 \le i \le n-2k+1, \\ b'_{1,i} = b_{1,i} + B_k(b_{k,i} + b_{1,k+i-1}), & n-2k+2 \le i \le n-k, \\ b'_{1,i} = b_{1,i}, & n-k+1 \le i \le n. \end{array}$$

Using the conditions of Lemmas 3.1 and 3.2 we obtain $b'_{1,2i+1} = 0,\ 2i+1 \ne n+2-r$, which completes the proof of the proposition. □

Applying consistently such type of transformations for $k = 2, 3, \dots, n$ we obtain a recurrent formula:

$$b_{1,2i}^{\{k\}} = b_{1,2i}^{\{k-1\}} + B_k(b_{1,2i+k-1}^{\{k-1\}} + (-1)^{k-1} b_{k,2i}^{\{k-1\}}) + (-1)^{k-1} B_k^2 b_{k,2i+k-1}^{\{k-1\}}, \tag{7}$$

where $2 \le k \le n,\ b_{1,t}^{\{k-1\}} = 0$ with $t \ge n-1$ and $b_{1,2i}^{\{1\}} = b_{1,2i}$. Here $b_{i,j}^{\{k-1\}}$ satisfy the conditions of Lemmas 3.1 and 3.2 as $b_{i,j}$.

Therefore from Lemma 2.1 and formulas (5), (6), (7) we obtain the final result as follows.

Theorem 3.1. *The action of the adapted transformation*

$$f=\varphi(B_n,n)\circ\dots\circ\varphi(B_2,2)\circ\sigma(A_n,n)\circ\dots\circ\sigma(A_2,2)\circ\tau(A_0,A_1,B_1),$$

on structure constants of an algebra from $T_{1,r}$ is expressible as follows

$$\begin{array}{ll}
a_{1,r}^{\{1\}}=\frac{B_1}{A_0^{r-2}}a_{1,r}, & \\
b_{0,0}^{\{1\}}=\frac{A_0^2b_{0,0}+A_0A_1b_{0,1}+A_1^2b_{1,1}}{A_0^{n-2}B_1(A_0+A_1b)}, & \\
b_{0,1}^{\{1\}}=\frac{A_0b_{0,1}+2A_1b_{1,1}}{A_0^{n-2}(A_0+A_1b)}, & \\
b_{1,1}^{\{1\}}=\frac{B_1b_{1,1}}{A_0^{n-2}(A_0+A_1b)} & \\
b_{1,2i}^{\{1\}}=\frac{B_1}{A_0^{n-2i}}b_{1,2i}, & 1\le i\le\left[\frac{n-2}{2}\right],\ 4\le r\le n-r+2,\\
b_{1,2i}^{\{1\}}=\frac{B_1(A_0b_{1,2i}+(2i-3)A_1a_{1,r}b_{1,2i+r-3})}{A_0^{n-2i}(A_0-A_1b_{1,n-1})}, & 1\le i\le\left[\frac{n-r+1}{2}\right],\ n-r+3\le r\le n-1,\\
b_{1,2i}^{\{1\}}=\frac{B_1b_{1,2i}}{A_0^{n-2i-1}(A_0-A_1b_{1,n-1})}, & \left[\frac{n-r+1}{2}\right]+1\le i\le\left[\frac{n-2}{2}\right],\ n-r+3\le r\le n-1.
\end{array}$$

$$\begin{array}{ll}
b_{1,2i}^{\{k\}}=b_{1,2i}^{\{k-1\}}-A_ka_{1,r}(b_{1,k+2i+r-4}^{\{k-1\}}+(-1)^{k-1}b_{k,2i+r-3}^{\{k-1\}})+(-1)^{k-1}A_k^2a_{1,r}^2b_{k,k+2i+2r-7}^{\{k-1\}}, & 2\le k\le n,\\
b_{1,2i}^{\{n-1+k\}}=b_{1,2i}^{\{n-2+k\}}+B_k(b_{1,2i+k-1}^{\{n-2+k\}}+(-1)^{k-1}b_{k,2i}^{\{n-2+k\}})+(-1)^{k-1}B_k^2b_{k,2i+k-1}^{\{n-2+k\}}, & 2\le k\le n,
\end{array}$$

where $b_{1,t}^{\{k-1\}}=0$ with $t\ge n-1$. Here $b_{i,j}^{\{k\}}$ satisfy the conditions of Lemmas 3.1 and 3.2 as $b_{i,j}$.

4 Appendix

Consider $dim\ L=11$ and $r=6$ case. Then from Lemma 3.2 we obtain the following table of multiplications

$$T_{1,6}=\left\{\begin{array}{lll}
[e_i,e_0]=e_{i+1}, & 1\le i\le 10, & \\
[e_0,e_i]=-e_{i+1}, & 2\le i\le 10, & \\
[e_0,e_0]=b_{0,0}e_n, & & \\
[e_0,e_1]=-e_2+b_{0,1}e_{11}, & & \\
[e_1,e_1]=b_{1,1}e_{11}, & & \\
[e_1,e_{2s}]=-[e_{2s},e_1]=-a_{1,6}e_{2s+4}+b_{1,2s}e_{11}, & 1\le s\le 3, & \\
[e_1,e_{2s+1}]=-[e_{2s+1},e_1]=-a_{1,6}e_{2s+5}, & 1\le s\le 3, & \\
[e_i,e_j]=-[e_j,e_i]=(-1)^{i-1}b_{1,i+j-1}e_n, & 2\le i\le 5, & 2\le j\le 7-i,\\
 & i+j\text{ is odd}, &
\end{array}\right.$$

Theorem 3.1 gives

$$b_{0,0}^1=\frac{A_0^2b_{0,0}+A_0A_1b_{0,1}+A_1^2b_{1,1}}{A_0^9B_1(A_0+A_1b)},\quad b_{0,1}^1=\frac{A_0b_{0,1}+2A_1b_{1,1}}{A_0^9(A_0+A_1b)},\quad b_{1,1}^1=\frac{B_1b_{1,1}}{A_0^9(A_0+A_1b)},$$

$$\begin{array}{llll}
\tau(A_0,A_1,B_1): & a'_{1,6}=\frac{B_1}{A_0^4}a_{1,6}, & b_{1,2i}^1=\frac{B_1}{A_0^{11-2i}}b_{1,2i}, & 1\le i\le 3,\\
\sigma(A_3,3): & b_{1,2}^2=b_{1,2}^1-A_3(a'_{1,6})^2, & b_{1,4}^2=b_{1,4}^1, & b_{1,6}^2=b_{1,6}^1,\\
\varphi(B_2,2): & b_{1,2}^3=b_{1,2}^2-B_2^2b_{1,4}^2, & b_{1,4}^3=b_{1,4}^2-B_2^2b_{1,6}^2, & b_{1,6}^3=b_{1,6}^2,\\
\varphi(B_3,3): & b_{1,2}^4=b_{1,2}^3+2B_3b_{1,4}^3+B_3^2b_{1,6}^3, & b_{1,4}^4=b_{1,4}^3+B_3b_{1,6}^3, & b_{1,6}^4=b_{1,6}^3,\\
\varphi(B_5,5): & b_{1,2}^5=b_{1,2}^4+2B_5b_{1,6}^4, & b_{1,4}^5=b_{1,4}^4, & b_{1,6}^5=b_{1,6}^4.
\end{array}$$

where the omitted $\sigma(A_i,i)$ and $\varphi(B_j,j)$ do not effect the structure constants.

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Here, and in all the references leave a space between the initial of the first name and last name

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