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ON THE CLASSIFICATION OF COMPLEX LEIBNIZ SUPERALGEBRAS WITH CHARACTERISTIC SEQUENCE $(n-1, 1|m_1, \dots, m_k)$ AND NILINDEX $n+m$

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In this work we investigate the complex Leibniz superalgebras with characteristic sequence $(n-1, 1|m_1, \dots, m_k)$ and with nilindex equal to $n+m$. We prove that such superalgebras with the condition $m_2 \neq 0$ have nilindex less than $n+m$. Therefore the complete classification of Leibniz algebras with characteristic sequence $(n-1, 1|m_1, \dots, m_k)$ and with nilindex equal to $n+m$ is reduced to the classification of filiform Leibniz superalgebras of nilindex equal to $n+m$, which was provided in [3, 7].

Keywords: Lie superalgebras; Leibniz superalgebras; nilpotency; characteristic sequence.

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1. Introduction

During many years the theory of Lie superalgebras has been actively studied by many mathematicians and physicists. A systematic exposition of basic Lie superalgebras theory can be found in [9]. Many works have been devoted to the study of this topic, but unfortunately most of them do not deal with nilpotent Lie superalgebras. In works [4, 5, 7] the problem of the description of some classes of nilpotent Lie superalgebras has studied. It is well known that Lie superalgebras are a generalization of Lie algebras. In the same way, the notion of Leibniz algebras, which were introduced in [10], can be generalized to Leibniz superalgebras.

The elementary properties of Leibniz superalgebras were obtained in [1]. For nilpotent Leibniz superalgebras the description of the case of maximal nilindex (nilpotent Leibniz superalgebras distinguished by the feature of being singly-onegenerated) is not difficult and was done in [1]. However, the next stage (the description of Leibniz superalgebras with dimensions of the even and odd parts equal to n and m, respectively, and of nilindex $n+m$) is a very problematic one. It should be noted that such Lie superalgebras were classified in [7]. Due to the great difficulty of solving in general the problem of description of Leibniz superalgebras of nilindex $n+m$, some restrictions on the characteristic sequence should be added, in particular, since the graded anticommutative identity does not hold in non-Lie Leibniz superalgebras. In the description of the structure of Leibniz superalgebras the crucial task is to prove the existence of a suitable basis (the so-called adapted basis) in which the table of multiplication of the superalgebra has the most convenient form. In the present paper we investigate the Leibniz superalgebras with the characteristic sequence $C(L) = (n-1, 1|m_1, m_2, \dots, m_k)$ and with nilindex equal to $n+m$. Actually, the classification of such superalgebras in the case where $C(L) = (n-1, 1|m)$ was obtained in [3] and the main result of the present work consist of the following fact: Leibniz superalgebras with characteristic sequences equal to $C(L) = (n-1, 1|m_1, m_2, \dots, m_k)$ $(m_2 \neq 0)$ have nilindex less than $n+m$. Therefore, the classification of Leibniz superalgebras of nilindex $n+m$ and with characteristic sequence equal to $(n-1, 1|m_1, m_2, \dots, m_k)$ is reduced to the case where the characteristic sequence is equal to $(n-1, 1|m)$, and this case, as we mentioned above, has already been solved in [3]. In this way we made further step in the solution of the problem of the classification of complex Leibniz superalgebras of nilindex $n+m$.

Throughout this work we shall consider spaces and (super)algebras over the field of complex numbers.

2. Preliminaries

We recall the notions of Lie and Leibniz superalgebras.

Definition 2.1 [9]. A Z_2-graded vector space $G = G_0 \oplus G_1$ is called *a Lie superalgebra* if it is equipped with a product $[-,-]$ which satisfies the following conditions:

(1) $[G_\alpha, G_\beta] \subseteq G_{\alpha+\beta(\bmod 2)}$ for any $\alpha, \beta \in Z_2$,
(2) $[x, y] = -(-1)^{\alpha\beta}[y, x]$, for any $x \in G_\alpha$, $y \in G_\beta$,
(3) $(-1)^{\alpha\gamma}[x, [y, z]] + (-1)^{\alpha\beta}[y, [z, x]] + (-1)^{\beta\gamma}[z, [x, y]] = 0$ — *Jacobi superidentity*, for any $x \in G_\alpha$, $y \in G_\beta$, $z \in G_\gamma, \alpha, \beta, \gamma \in \mathbb{Z}_2$.

Definition 2.2 [1]. A Z_2-graded vector space $L = L_0 \oplus L_1$ is called *a Leibniz superalgebra* if it is equipped with a product $[-,-]$ which satisfies the following conditions:

(1) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta(\bmod 2)}$ for any $\alpha, \beta \in Z_2$,

(2) $[x,[y,z]] = [[x,y],z] - (-1)^{\alpha\beta}[[x,z],y]$ — *Leibniz superidentity*, for any $x \in L$, $y \in L_\alpha$, $z \in L_\beta$.

Evidently, the subspaces G_0 and L_0 are Lie and Leibniz algebras, respectively.

It should be noted that if in a Leibniz superalgebra L the identity:

$$[x,y] = -(-1)^{\alpha\beta}[y,x],$$

holds for any $x \in L_\alpha$ and $y \in L_\beta$, then the Leibniz superidentity can easily be transformed into the Jacobi superidentity. Thus, Leibniz superalgebras are generalizations of both Lie (super)algebras and Leibniz algebras. For examples of Leibniz superalgebras we refer to [1].

The set of Leibniz superalgebras with dimensions of the even part L_0 and the odd part L_1, respectively equal to n and m, shall be denoted by $Leib_{n,m}$.

For a given Leibniz superalgebra L we define a descending central sequence in the following way:

$$L^1 = L, \quad L^{k+1} = [L^k, L^1], \quad k \geq 1.$$

Definition 2.3. A Leibniz superalgebra L is called *nilpotent*, if there exists $s \in \mathbb{N}$ such that $L^s = 0$. The minimal number s with this property is called *index of nilpotency* (*or nilindex*) *of the superalgebra* L.

Definition 2.4. The set $\mathcal{R}(L) = \{x \in L \mid [y,x] = 0 \text{ for any } y \in L\}$ is called *the right annihilator of a superalgebra* L.

Using the Leibniz superidentity it is not difficult to see that $\mathcal{R}(L)$ is an ideal of the superalgebra L. Moreover, element of the form $[a,b] + (-1)^{\alpha\beta}[b,a]$ belong to $\mathcal{R}(L)$, where $a \in L_\alpha$, $b \in L_\beta$.

The following theorem describes nilpotent Leibniz superalgebras with maximal nilindex.

Theorem 2.1 [1]. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with nilindex equal to $n+m+1$. Then L is isomorphic to one of the following two non-isomorphic superalgebras*:

$$[e_i, e_1] = e_{i+1}, \ 1 \leq i \leq n-1; \quad \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n+m-1, \\ [e_i, e_2] = 2e_{i+2}, & 1 \leq i \leq n+m-2, \end{cases}$$

where omitted products are equal to zero and $\{e_1, e_2, \ldots, e_n\}$ is the basis of the superalgebra L.

Remark 2.1. From the description of Theorem 2.1 we have that if the odd part L_1 of the superalgebra L is non-trivial, then either m or $m+1$ and the table of multiplication of the second superalgebra in a graded basis $\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$

can be written in the following form:

$$\begin{aligned}
&[y_j, x_1] = y_{j+1},\ 1 \le j \le m-1, \quad [x_1, y_1] = \frac{1}{2} y_2, \\
&[x_i, y_1] = \frac{1}{2} y_i, 2 \le i \le m, \quad [y_1, y_1] = x_1, \\
&[y_j, y_1] = x_{j+1}, \quad 2 \le j \le n-1, \\
&[x_i, x_1] = x_{i+1}, \quad 1 \le i \le n-1.
\end{aligned}$$

Let $L = L_0 \oplus L_1$ be a nilpotent Leibniz superalgebra. For an arbitrary element $x \in L_0$, the operator of right multiplication R_x (defined as $R_x : y \to [y, x]$) is a nilpotent endomorphism of the space L_i, where $i \in \{0, 1\}$. Denote by $C_i(x)$ ($i \in \{0, 1\}$) the descending sequence of the dimensions of Jordan blocks of the operator R_x. Consider the lexicographical order on the set $C_i(L_0)$.

Definition 2.5. The sequence

$$C(L) = \left(\max_{x \in L_0 \setminus [L_0, L_0]} C_0(x) \middle| \max_{\widetilde{x} \in L_0 \setminus [L_0, L_0]} C_1(\widetilde{x}) \right)$$

is said to be *the characteristic sequence of the Leibniz superalgebra L*.

Similarly to [5] (Corollary 3.0.1) it can be proved that the characteristic sequence is invariant under isomorphisms.

Further we need the following definition.

Definition 2.6. A Leibniz algebra L of dimension n is said to be *filiform* if $\dim L^i = n - i$ for $2 \le i \le n$.

Lemma 2.1 [2]. *Let L be an n-dimensional Leibniz algebra. Then the following statements are equivalent*:

(a) $C(L) = (n-1, 1)$;
(b) *L is a filiform Leibniz algebra*;
(c) *$L^{n-1} \neq 0$ and $L^n = 0$.*

Let L be a Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence equal to $(n-1, 1|m_1, m_2, \ldots, m_k)$, (where $m_1 + m_2 + \cdots + m_k = m$). Since in [7] the Leibniz superalgebras with characteristic sequence and nilindex equal to $(n-1, n|m)$ and $n+m$, respectively, have already been obtained, we shall henceforth reduce our investigation to the case where $m_2 \neq 0$.

From Lemma 2.1 we can conclude that the even part L_0 of L is a filiform Leibniz algebra. Due to the description of filiform Leibniz algebras in [6, 8, 11] we can obtain the existence of an adapted basis in superalgebra with $C(L) = (n-1, 1|m_1, m_2, \ldots, m_k)$, according to the following theorem:

Theorem 2.2. *Let $L = L_0 \oplus L_1$ be a superalgebra from $Leib_{n,m}$ with characteristic sequence equal to $(n-1, 1|m_1, m_2, \ldots, m_k)$. Then there exists a basis*

$\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$ *of* L, *in which the multiplication satisfies one of the following three conditions*:

(a) $[x_1, x_1] = x_3,$
$[x_i, x_1] = x_{i+1},\ 2 \leq i \leq n-1,$
$[y_j, x] = y_{j+1},\ j \notin \{m_1, m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_k\},$
$[y_j, x] = 0, j \in \{m_1, m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_k\},$
for some $x \in L_0 \backslash L_0^2,$
$[x_1, x_2] = \alpha_4 x_4 + \alpha_5 x_5 + \cdots + \alpha_{n-1} x_{n-1} + \theta x_n,$
$[x_j, x_2] = \alpha_4 x_{j+2} + \alpha_5 x_{j+3} + \cdots + \alpha_{n+2-j} x_n,\ 2 \leq j \leq n-2,$
where the omitted products in L_0 *are equal to zero*;

(b) $[x_1, x_1] = x_3,$
$[x_i, x_1] = x_{i+1},\ 3 \leq i \leq n-1,$
$[y_j, x] = y_{j+1},\ j \notin \{m_1, m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_k\},$
$[y_j, x] = 0, j \in \{m_1, m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_k\},$
for some $x \in L_0 \backslash L_0^2,$
$[x_1, x_2] = \beta_4 x_4 + \beta_5 x_5 + \cdots + \beta_n x_n,$
$[x_2, x_2] = \gamma x_n,$
$[x_j, x_2] = \beta_4 x_{j+2} + \beta_5 x_{j+3} + \cdots + \beta_{n+2-j} x_n,\ 3 \leq j \leq n-2,$
where the omitted products in L_0 *are equal to zero*;

(c) $[x_i, x_1] = x_{i+1},\ 2 \leq i \leq n-1,$
$[x_1, x_i] = -x_{i+1},\ 3 \leq i \leq n-1,$
$[y_j, x] = y_{j+1},\ j \notin \{m_1, m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_k\},$
$[y_j, x] = 0, j \in \{m_1, m_1 + m_2, \ldots, m_1 + m_2 + \cdots + m_k\},$
for some $x \in L_0 \backslash L_0^2,$
$[x_1, x_1] = \theta_1 x_n,$
$[x_1, x_2] = -x_3 + \theta_2 x_n,$
$[x_2, x_2] = \theta_3 x_n,$
$[x_i, x_j] = -[x_j, x_i] \in lin\langle x_{i+j+1}, x_{i+j+2}, \ldots, x_n\rangle,\ 2 \leq i < j \leq n-2.$

3. On the Classification of Leibniz Superalgebras with Characteristic Sequence $(n-1, 1|m_1, m_2, \ldots, m_k)$ and Nilindex $n+m$ $(m_2 \neq 0)$.

Let L satisfy to the conditions of Theorem 2.2 and let $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$ be the adapted basis of L. It is not difficult to see that if L has nilindex equal to $n+m$, then the superalgebra L has two generators (due to Theorem 2.1 we have a description of singly-generated Leibniz superalgebras, which have nilindex $n+m+1$) and $\dim L^i = n+m-i$ for $2 \leq i \leq n+m$. It should be noted that the filiform Leibniz algebra L_0 has also two generators, x_1 and x_2.

Lemma 3.1. *In three the classes of superalgebras of Theorem* 2.2 *instead of the element* x *one can choose the element* x_1.

Proof. Without loss of generality we can assume that x has the form: $x = A_1x_1 + A_2x_2$, where $(A_1, A_2) \neq (0, 0)$.

Let us consider the first class of superalgebras of Theorem 2.2 and investigate three cases.

Case 1. Let $A_1(A_1 + A_2) \neq 0$. Then applying the following change of basis:

$$x_1' = A_1x_1 + A_2x_2, \quad x_2' = (A_1 + A_2)x_2 + A_2(\theta - \alpha_n)x_{n-1},$$
$$x_i' = [x_{i-1}', x_1'], \quad 3 \le i \le n, \quad y_j' = y_j, \quad 1 \le j \le m,$$

we obtain that the first four multiplications in class (a) do not change.

Case 2. Let $A_1 = 0$. Let us make a change of basis as follows:

$$x_1' = x_1 + aA_2x_2, \quad \text{where } a(1 + aA_2) \neq 0,$$
$$x_2' = (1 + A_2)x_2 + aA_2(\theta - \alpha_n)x_{n-1}, \quad x_i' = [x_{i-1}', x_1'], \ 3 \le i \le n,$$
$$y_j' = y_j, \quad j \in \{1, m_1 + 1, m_1 + m_2 + 1, \dots, m_1 + m_2 + \dots + m_{k-1} + 1\},$$
$$y_j' = [y_{j-1}', x_1'], \quad j \notin \{1, m_1 + 1, m_1 + m_2 + 1, \dots, m_1 + m_2 + \dots + m_{k-1} + 1\}.$$

If we choose a sufficiently big value of the parameter a then we obtain that the first four multiplications in the class (a) also do not change. Indeed, the first three multiplications do not change by the construction and the products $[y_j', x_1']$ for $j \in \{m_1, m_1 + m_2, \dots, m_1 + m_2 + \dots + m_k\}$ are equal to zero, because otherwise we easily can get a contradiction with the characteristic sequence or nilpotence conditions.

Case 3. Let $A_1 \neq 0$ and $A_1 = -A_2$. Then taking the following transformation of basis:

$$x_1' = A_1x_1 - A_1x_2 + ax_2, \quad x_2' = ax_2 + (a - A_1)(\theta - \alpha_n)x_{n-1}, \ (a \neq 0),$$
$$x_i' = [x_{i-1}', x_1'], \quad 3 \le i \le n, \ y_j' = y_j,$$
$$j \in \{1, m_1 + 1, m_1 + m_2 + 1, \dots, m_1 + m_2 + \dots + m_{k-1} + 1\},$$
$$y_j' = [y_{j-1}', x_1'], \quad j \notin \{1, m_1 + 1, m_1 + m_2 + 1, \dots, m_1 + m_2 + \dots + m_{k-1} + 1\},$$

it is not difficult to check that for sufficiently small values of the parameter a the first four multiplications in the class (a) are preserved.

Thus, we have shown that in the first case of superalgebras of Theorem 2.2 instead of element x one can choose element x_1.

Let us consider the class (b) of Theorem 2.2.

If $A_1 \neq 0$, then applying a transformation of basis of the form:

$$x_1' = A_1x_1 + A_2x_2, \quad x_2' = x_2 - \frac{A_2\gamma}{A_1}x_{n-1},$$
$$x_3' = [x_1', x_1'], \quad x_i' = [x_{i-1}', x_1'], \quad 4 \le i \le n, \quad y_j' = y_j, \qquad 1 \le j \le m,$$

we obtain that the first four multiplications do not change.

If $A_1 = 0$, then the following change of basis:

$$x_1' = x_1 + aA_2x_2, \ (a \neq 0), \quad x_2' = x_2 - aA_2\gamma x_{n-1},$$

$$x_3' = [x_1', x_1'], \quad x_i' = [x_{i-1}', x_1'], \ 4 \leq i \leq n,$$

$$y_j' = y_j, \quad j \in \{1, m_1+1, m_1+m_2+1, \dots, m_1+m_2+\dots+m_{k-1}+1\},$$

$$y_j' = [y_{j-1}', x_1'], \quad j \notin \{1, m_1+1, m_1+m_2+1, \dots, m_1+m_2+\dots+m_{k-1}+1\},$$

with a sufficiently big value of the parameter a allow to conclude that first four multiplications in class (b) do not change.

Now consider the class (c) of Theorem 2.2.

If $A_1 \neq 0$, then applying a transformation of basis of the form:

$$x_1' = A_1x_1 + A_2x_2, \quad x_2' = x_2, \quad x_i' = [x_{i-1}', x_1'], \ 3 \leq i \leq n, \quad y_j' = y_j, \ 1 \leq j \leq m,$$

we obtain that the first four multiplications are preserved.

If $A_1 = 0$, then take the transformation of basis:

$$x_1' = x_1 + aA_2x_2, \ (a \neq 0), \quad x_2' = x_2, \quad x_i' = [x_{i-1}', x_1'], \ 3 \leq i \leq n,$$

$$y_j' = y_j, \quad j \in \{1, m_1+1, m_1+m_2+1, \dots, m_1+m_2+\dots+m_{k-1}+1\},$$

$$y_j' = [y_{j-1}', x_1'], \quad j \notin \{1, m_1+1, m_1+m_2+1, \dots, m_1+m_2+\dots+m_{k-1}+1\}.$$

Then choosing a sufficiently big value of the parameter a allow us to conclude that the first four products in the case (c) of Theorem 2.2 do not change.

Thus, we have proven that in the three classes of superalgebras of Theorem 2.2 instead of the element x we can choose element x_1. $\square$

Since the superalgebra $L = L_0 \oplus L_1$ has two generators the possible cases are as follow: both generators lie in L_0; one generator lies in L_0 and the another one lies in L_1; both generators lie in L_1.

We shall not consider the case where both generators lie in the even part (since $m_2 \neq 0$). Firstly we consider the second possible, i.e. the case where one of the generators lies in L_0 and the another one lies in L_1. It is easy to see that there exist some $m_j, 0 \leq j \leq k-1$ (here $m_0 = 0$), such that $y_{m_1+m_2+\dots+m_j+1} \notin L^2$. By a shifting of basic elements one can assume that $m_j = m_0$, i.e. the basic element $y_1 \notin L^2$. Of course, by this choice the condition $m_1 \geq m_2 \geq \dots \geq m_k$, is broken, but we shall not use this condition in our study further. Thus, as generators we can choose the elements $A_1x_1 + A_2x_2$ and y_1.

Let us introduce the notations

$$[x_i, y_1] = \sum_{j=2}^{m} \alpha_{i,j}y_j, \ 1 \leq i \leq n, \quad [y_s, y_1] = \sum_{t=1}^{n} \beta_{s,t}x_t,$$

$$[y_p, x_2] = \sum_{q=2}^{m} \gamma_{p,q}y_q, \ 1 \leq s, p \leq m.$$

Theorem 3.1. *Let L be a Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence $(n-1,1|m_1,m_2,\dots,m_k)$, where $m_1 \geq 2$, $n \geq 4$. Let the elements $A_1x_1 + A_2x_2, y_1$ be generators and $x_1 \in L^2$. Then L has a nilindex less than $n+m$.*

Proof. Since $x_1 \in L^2$, then $x_2 \notin L^2$ and therefore as a generator of the L which lies in L_0 we can choose x_2. Let us assume the contrary, i.e. the nilindex of the superalgebra L is equal to $n+m$. Then we have

$$L = \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m\}, \quad L^2 = \{x_1, x_3, \dots, x_n, y_2, y_3, \dots, y_m\}.$$

Since x_1 is generator in the filiform Leibniz algebra L_0 then it should lie in linear span of products $[y_s, y_1]$, $1 \leq s \leq m$. Therefore, there exists some s $(1 \leq s \leq m)$ such that $\beta_{s,1} \neq 0$.

Denote by $y_{m_1+m_2+\dots+m_{t_0}+1}$ the basic element which is the earliest generated among the elements $\{y_{m_1+1}, y_{m_1+m_2+1}, \dots, y_{m_1+m_2+\dots+m_{k-1}+1}\}$, i.e. $y_{m_1+m_2+\dots+m_{t_0}+1}$ is the basic element which first is absent among the elements $\{y_{m_1+1}, y_{m_1+m_2+1}, \dots, y_{m_1+m_2+\dots+m_{k-1}+1}\}$ in the descending central sequence. Therefore, the element $y_{m_1+m_2+\dots+m_{t_0}+1}$ is generated by products of elements of the form either $[x_i, y_1], 2 \leq i \leq n$ or $[y_j, x_2], 1 \leq j \leq m_1$.

Let us show that $x_1 \notin L^3$. Indeed, if x_1 lies in L^3, then it should be generated by the product $[y_{m_1+m_2+\cdots m_{t_0}+1}, y_1]$. From the nilpotence condition we have that x_1 is generated by the products $[[x_2, y_1], y_1]$, $[[y_1, x_2], y_1]$ (since the basic elements $x_3, x_4, \dots, x_n$ and $y_2, y_3, \dots, y_{m_1}$ are obtained by the products involving x_1). Hence for generating the element $y_{m_1+m_2+\dots+m_{t_0}+1}$ it is enough to consider the cases where $i = 2$, $j = 1$.

From the equalities

$$[[x_2, y_1], y_1] = \frac{1}{2}[x_2, [y_1, y_1]] = \frac{1}{2}\left[x_2, \sum_{s=1, s\neq 2}^{n} \beta_{1,s}x_s\right] = \sum_{t\geq 3}(*)x_t$$

(where by the symbol $(*)$ we denote the coefficients of the basic elements x_t), we have that the element x_1 is not present in the decomposition of the product $[[x_2, y_1], y_1]$. If x_1 is generated from the product $[[y_1, x_2], y_1]$, then the expression $[[y_1, x_2], y_1] + [[x_2, y_1], y_1]$ lies in $\mathcal{R}(L)$ and x_1 appear in its decomposition. Using the table of multiplication in the algebra L_0 from Theorem 2.2 we establish that multiplying the expression $[[y_1, x_2], y_1] + [[x_2, y_1], y_1]$ on the right side by the element x_1 sufficiently many times we obtain $x_n \in \mathcal{R}(L)$. Then repeating this procedure finally we obtain $x_1 \in \mathcal{R}(L)$. Thus, we obtain a contradiction, because $x_1 \notin \mathcal{R}(L)$. Therefore, $x_1 \notin L^3$.

Let us consider the cases (a) and (b). The condition $x_1 \notin L^3$ leads to $\beta_{1,1} \neq 0$. Since in the cases (a) and (b) the elements $x_i, 3 \leq i \leq n$ lie in $\mathcal{R}(L)$, then we can put $x_1' = \beta_{1,1}x_1 + \beta_{1,3}x_3 + \cdots + \beta_{1,n}x_n$ and suppose that $[y_1, y_1] = x_1$.

Consider the subsuperalgebra generated by y_1 ($\langle y_1\rangle$). Then it is easy to check that $\langle y_1\rangle = \{x_1, x_3, \dots, x_n, y_1, y_2, \dots, y_{m_1}\}$. Since this subsuperalgebra is single-generated then from Remark 2.1 we have

$$[y_j, x_1] = y_{j+1},\ \ 1 \le j \le m_1 - 1, \quad [x_1, y_1] = \frac{1}{2}y_2,$$
$$[x_i, y_1] = \frac{1}{2}y_i,\ \ 3 \le i \le m_1, \quad [y_1, y_1] = x_1,$$
$$[y_j, y_1] = x_{j+1},\ \ 2 \le j \le n-1,$$

where if $n + m_1$ is even then $n = m_1$ and if $n + m_1$ is odd then $n = m_1 + 1$.

From above products we have $[y_1, x_1] + [x_1, y_1] = \frac{3}{2}y_2$ which yields $y_i \in \mathcal{R}(L)$ for $2 \le i \le m_1$.

Using the fact that in the cases (a) and (b) the product $[x_1, x_2]$ belongs to $\mathcal{R}(L)$ and the following equalities hold

$$\begin{aligned}[y_i, x_2] &= [[y_{i-1}, x_1], x_2] = [y_{i-1}, [x_1, x_2]] + [[y_{i-1}, x_2], x_1] = [[y_{i-1}, x_2], x_1]\\ &= \gamma_{1,2}y_{i+1} + (*)y_{i+2} + \cdots + (*)y_{m_1} + (*)y_{m_1+2} + \cdots\\ &\quad + (*)y_{m_1+m_2} + (*)y_{m_1+m_2+2} + \cdots + (*)y_m,\end{aligned}$$

for $2 \le i \le m_1$, we can conclude that $y_{m_1+m_2+\cdots+m_{t_0}+1} \notin L^3$, i.e. $\dim L^3 < n + m - 3$. Hence we obtain a contradiction with the assumption that the nilindex is equal to $n + m$.

Now, consider superalgebras from the class (c) of Theorem 2.2. We have

$$\begin{aligned}L^2 &= \{x_1, x_3, \dots, x_n, y_2, y_3, \dots, y_m\},\\ L^3 &= \{x_3, \dots, x_n, y_2, y_3, \dots, y_m\}.\end{aligned}$$

Consider the equalities

$$[y_1, x_3] = [y_1, [x_2, x_1]] = [[y_1, x_2], x_1] - [[y_1, x_1], x_2] = \sum_{p=2}^{m-2} \gamma_{1,p}y_{p+1} - [y_2, x_2].$$

From the nilpotence condition of the superalgebra L it follow that in the decomposition of the product $[y_2, x_2]$ the basic element y_2 does not participate, i.e. $[y_2, x_2] = \sum\limits_{i=3}^{m} \gamma_{2,i}y_i$. Therefore

$$[y_1, x_3] = (\gamma_{1,2} - \gamma_{2,3})y_3 + (\gamma_{1,3} - \gamma_{2,4})y_4 + \cdots + (\gamma_{1,m-2} - \gamma_{2,m-1})y_{m-1} - \gamma_{2,m}y_m.$$

In a similar way we obtain

$$[y_i, x_3] = (\gamma_{i,i+1} - \gamma_{i+1,i+2})y_{i+2} + \cdots + (\gamma_{i,m-2} - \gamma_{i+1,m-1})y_{m-1} - \gamma_{i+1,m}y_m,$$
$$2 \le i \le m-3.$$

Applying the above arguments for $[y_i, x_j],\ 4 \le j \le n$ we get that $[y_i, x_j] = \sum_{j\ge i+2}(*)y_j$ for $4 \le j \le n$. Therefore without loss of generality one can assume that the expression $\beta_{1,1}x_1+\beta_{1,3}x_3+\cdots+\beta_{1,n}x_n$ can be replaced by x_1, i.e. we can suppose $[y_1,y_1]=x_1$. Consider the equalities

$$[x_1,y_1]=[[y_1,y_1],y_1]=\frac{1}{2}[y_1,[y_1,y_1]]=\frac{1}{2}[y_1,x_1]=\frac{1}{2}y_2.$$

Since $[x_1,y_1]+[y_1,x_1]=\frac{3}{2}y_2\in\mathcal{R}(L)$ and $y_j=[y_j,x_1]$ for $1\le j\le m_1-1$, then $y_3,y_4,\ldots,y_{m_1}\in\mathcal{R}(L)$.

Using induction and the following chain of equalities

$$\begin{aligned}
&[y_2,y_1]=[[y_1,x_1],y_1]=[y_1,[x_1,y_1]]+[[y_1,y_1],x_1]=\theta_1x_n,\\
&[y_i,y_1]=[[y_{i-1},x_1],y_1]=[y_{i-1},[x_1,y_1]]+[[y_{i-1},y_1],x_1]=[[y_{i-1},y_1],x_1],\\
&[x_j,y_1]=[[x_{j-1},x_1],y_1]=[x_{j-1},[x_1,y_1]]+[[x_{j-1},y_1],x_1]=[[x_{j-1},y_1],x_1],
\end{aligned}$$

we establish that

$$[y_i,y_1]=0\quad\text{for } 3\le i\le m_1$$

and

$$\begin{aligned}
[x_i,y_1]=\alpha_{2,2}y_i&+\cdots+\alpha_{2,m_1+2-i}y_{m_1}+\alpha_{2,m_1+1}y_{m_1+i-1}+\cdots\\
&+\alpha_{2,m_1+m_2+2-i}y_{m_1+m_2}+\alpha_{2,m_1+m_2+1}y_{m_1+m_2+i-1}+\cdots\\
&+\alpha_{2,m+2-i}y_m,\quad 3\le j\le n.
\end{aligned}$$

The obtained products lead to $y_2\notin L^4$ and that the basic element $y_{m_1+m_2+\cdots+m_{t_0}+1}$ is generated by the products $[y_j,x_2],\ 2\le j\le m_1$.

Since $y_j\in\mathcal{R}(L)$ for $2\le j\le m_1$ and the other basic elements $y_{m_1+1},\ldots,y_m$ are generated by products of the form $[[y_j,x_2],\ldots,x_2]$, where $2\le j\le m_1$, then $\{y_2,y_3,\ldots,y_m\}\in\mathcal{R}(L)$. Since $[[y_j,x_2],y_1]=[y_j,[x_2,y_1]]+[[y_j,y_1],x_2]=0$ and

$$\begin{aligned}
[[[y_j,x_2],\ldots,x_2],y_1]&=[[[y_j,x_2],\ldots],[x_2,y_1]]+[[[[y_j,x_2],\ldots],y_1],x_2]\\
&=[[[[y_j,x_2],\ldots],y_1],x_2]=\cdots=0,
\end{aligned}$$

then we easily obtain that

$$[y_{m_1+m_2+\cdots+m_t+1},y_1]=0,\quad\text{for } 1\le t\le k-1.$$

Inductively we get

$$[y_j,y_1]=0\quad\text{for } 2\le j\le m,$$

and that x_3 does not lie in L^4. Thus, we obtain that

$$L^4\subseteq\{x_4,\ldots,x_n,y_3,\ldots,y_m\}.$$

Hence $\dim L^4<n+m-4$, but this contradict the condition that the nilindex is equal to $n+m$. Thus, in the three classes of Theorem 2.2 we obtain a contradiction with the assumption that the superalgebra L has nilindex equal to $n+m$ and therefore the assertion of the theorem is proved. $\square$

From Theorem 3.1 we can assume that x_1 and y_1 are generators of the superalgebra L.

Theorem 3.2. *Let L be a Leibniz superalgebra from $Leib_{n,m}$ with characteristic sequence equal to $(n-1, 1|m_1, m_2, \ldots, m_k)$ and let $\{x_1, y_1\}$ be generators of L. Then the superalgebra L has nilindex less than $n+m$.*

Proof. Let L be a superalgebra satisfying the conditions of the theorem. Then

$$L^2 = \{x_2, x_3, \ldots, x_n, y_2, y_3, \ldots, y_m\}.$$

Since $y_{m_1+\cdots+m_t+1} \in L^2$ for any t $(1 \leq t \leq k-1)$, we can conclude that $(\alpha_{1,m_1+\cdots+m_t+1}, \alpha_{2,m_1+\cdots+m_t+1}, \ldots, \alpha_{n,m_1+\cdots+m_t+1}) \neq (0, 0, \ldots, 0)$ for any t $(1 \leq t \leq k-1)$. But this means that for any t $(1 \leq t \leq k-1)$ the basic element $y_{m_1+\cdots+m_t+1}$ is generated by the products $[x_i, y_1]$, $1 \leq i \leq n$.

As in the proof of Theorem 3.1 denote by $y_{m_1+\cdots+m_{t_0}+1}$ the basic element which first is absent among the elements

$$\{y_{m_1+1}, y_{m_1+m_2+1}, \ldots, y_{m_1+m_2+\cdots+m_{k-1}+1}\}$$

in descending lower sequence. Then $(\alpha_{1,m_1+\cdots+m_{t_0}+1}, \alpha_{2,m_1+\cdots+m_{t_0}+1}, \ldots, \alpha_{n,m_1+\cdots+m_{t_0}+1}) \neq (0, 0, \ldots, 0)$. Let f be the natural number such that $\alpha_{f,m_1+\cdots+m_{t_0}+1} \neq 0$ and $\alpha_{k,m_1+\cdots+m_{t_0}+1} = 0$ for $f \leq k \leq n$.

We shall prove that f. Let us suppose the opposite, i.e. $f < n$. Then for the powers of descending lower sequences we have the following:

$$L^s = \{x_f, \ldots, x_n, y_r, \ldots, y_{m_1}, y_{m_1+1}, \ldots, y_{m_1+\cdots+m_{t_0}}, y_{m_1+\cdots+m_{t_0}+1}, \ldots, y_m\},$$

$$L^{s+1} = \{x_{f+1}, \ldots, x_n, y_r, \ldots, y_{m_1}, y_{m_1+1}, \ldots, y_{m_1+\cdots+m_{t_0}}, y_{m_1+\cdots+m_{t_0}+1}, \ldots, y_m\},$$

$$L^{s+2} = \{x_{f+1}, \ldots, x_n, y_r, \ldots, y_{m_1}, y_{m_1+1}, \ldots, y_{m_1+\cdots+m_{t_0}}, y_{m_1+\cdots+m_{t_0}+2}, \ldots, y_m\}.$$

From these we have that the elements $\{y_r, \ldots, y_{m_1}, y_{m_1+1}, \ldots, y_{m_1+\cdots+m_{t_0}}, y_{m_1+\cdots+m_{t_0}+2}, \ldots, y_m\}$ are obtained from the products $[x_i, y_1]$, $f+1 \leq i \leq n$.

The elements $\{y_r, \ldots, y_{m_1}, y_{m_1+1}, \ldots, y_{m_1+\cdots+m_{t_0}}, y_{m_1+\cdots+m_{t_0}+2}, \ldots, y_m\}$ belong to L^{s+3} (because $\{x_{f+1}, \ldots, x_m\} \in L^{s+2}$) and hence $x_{f+1} \notin L^{s+3}$. Therefore in the decomposition

$$[y_{m_1+\cdots+m_{t_0}+1}, y_1] = \beta_{m_1+\cdots+m_{t_0}+1,f+1}x_{f+1} + \cdots + \beta_{m_1+\cdots+m_{t_0}+1,n}x_n$$

we have $\beta_{m_1+\cdots+m_{t_0}+1,f+1} \neq 0$.

Consider the equalities

$$\begin{aligned}[x_f, [y_1, y_1]] = 2[[x_f, y_1], y_1] &= 2[\alpha_{f,r}y_r + \alpha_{f,r+1}y_{r+1} + \cdots + \alpha_{f,m_1+\cdots+m_{t_0}+1} \\ &\quad \times y_{m_1+\cdots+m_{t_0}+1} + \cdots + \alpha_{f,m}y_m, y_1] \\ &= 2\alpha_{f,m_1+\cdots+m_{t_0}+1}\beta_{m_1+\cdots+m_{t_0}+1,f+1}x_{f+1} + \sum_{i\geq f+2}(*)x_i.\end{aligned}$$

On the other hand

$$[x_f, [y_1, y_1]] = [x_f, \beta_{1,2}x_2 + \beta_{1,3}x_3 + \cdots + \beta_{1,n}x_n] = \sum_{j \geq f+2} (*)x_j.$$

Comparing the coefficients of the basic elements we obtain

$$\alpha_{f,m_1+\cdots+m_{t_0}+1}\beta_{m_1+\cdots+m_{t_0}+1,f+1} = 0,$$

which contradicts the conditions: $\alpha_{f,m_1+\cdots+m_{t_0}+1} \neq 0, \beta_{m_1+\cdots+m_{t_0}+1,f+1} \neq 0$. Thus, we get a contradiction with the assumption that $f < n$. Now we shall study the case where $f = n$, i.e. $\alpha_{n,m_1+\cdots+m_{t_0}+1} \neq 0$. In this case for some natural number p we have

$$L^p = \{y_{m_1+1}, \ldots, y_{m_1+\cdots+m_{t_0}}, y_{m_1+\cdots+m_{t_0}+1}, \ldots, y_m\}.$$

It is clear that if $k \geq 3$, then $\dim L^{p+1} - \dim L^p \geq 2$ and we have a contradiction with the nilindex condition. If $k = 2$ then the vector space generated by the elements $\langle y_{m_1+1}, \ldots, y_{m_1+m_2}\rangle$ forms an ideal of the superalgebra L. The quotient superalgebra $\overline{L} = L/\langle y_{m_1+1}, \ldots, y_{m_1+m_2}\rangle$ is also two generated and $C(\overline{L}) = (n-1, 1|m_1)$. Now applying Lemma 3.4 from [3] we get a contradiction, which completes the proof of the theorem. $\square$

Let us investigate the case where both generators lie in odd part of the superalgebra L. The following theorem clears up the situation in this case.

Theorem 3.3. *Let $L = L_0 \oplus L_1$ be a superalgebra from $Leib_{n,m}$ with characteristic sequence equal to $(n-1, 1|m_1, m_2, \ldots, m_k)$, where $m_1 \geq 2, n \geq 3$ and let both generators lie in L_1. Then the superalgebra L has nilindex less than $n + m$.*

Proof. Since both generators of the superalgebra L lie in L_1, they are linear combinations of the elements $\{y_1, y_{m_1+1}, \ldots, y_{m_1+m_2+\cdots+m_{k-1}+1}\}$. Without loss of generality we may assume that y_1 and y_{m_1+1} are generators.

Let $L^{2t} = \{x_i, x_{i+1}, \ldots, x_n, y_j, \ldots, y_m\}$ for some natural number t and let z be an arbitrary element such that $z \in L^{2t} \backslash L^{2t+1}$. Then z is generated by the products of even an number of generators. Hence $z \in L_0$ and $L^{2t+1} = \{x_{i+1}, \ldots, x_n, y_j, \ldots, y_m\}$. In a similar way, having $L^{2t+1} = \{x_{i+1}, \ldots, x_n, y_j, \ldots, y_m\}$ we obtain $L^{2t+2} = \{x_{i+1}, \ldots, x_n, y_{j+1}, \ldots, y_m\}$.

From the above arguments we conclude that $n = m - 1$ or $n = m - 2$.

Let us consider powers of L:

$$\begin{aligned}
L^2 &= \{x_1, x_2, \ldots, x_n, y_2, y_3, \ldots, y_{m_1}, y_{m_1+2}, \ldots, y_m\},\\
L^3 &= \{A_1x_1 + A_2x_2, x_3, \ldots, x_n, y_2, y_3, \ldots, y_{m_1}, y_{m_1+2}, \ldots, y_m\},\\
L^4 &\supseteq \{A_1x_1 + A_2x_2, x_3, \ldots, x_n, y_3, y_4, \ldots, y_{m_1}, y_{m_1+3}, \ldots, y_m\}.
\end{aligned}$$

Applying the above arguments we get that an element from the set $\{y_3, y_4, \dots, y_{m_1}, y_{m_1+3}, \dots, y_m\}$ disappears in L^4. If necessary then by a shifting of basic elements we can suppose that $y_2 \notin L^4$ without loss of the generality. Then

$$L^4 = \{A_1x_1 + A_2x_2, x_3, \dots, x_n, y_3, y_4, \dots, y_{m_1}, y_{m_1+2}, \dots, y_m\},$$
$$L^5 = \{x_3, \dots, x_n, y_3, y_4, \dots, y_{m_1}, y_{m_1+2}, \dots, y_m\}.$$

From these restrictions on the powers of L in the following products

$$\begin{aligned}
[y_1, y_1] &= \beta_{1,1}x_1 + \beta_{1,2}x_2 + \cdots + \beta_{1,n}x_n,\\
[y_2, y_1] &= \beta_{2,2}(A_1x_1 + A_2x_2) + \beta_{2,3}x_3 + \cdots + \beta_{2,n}x_n,\\
[y_1, y_{m_1+1}] &= \gamma_{1,1}x_1 + \gamma_{1,2}x_2 + \cdots + \gamma_{1,n}x_n,\\
[y_2, y_{m_1+1}] &= \gamma_{2,2}(A_1x_1 + A_2x_2) + \gamma_{2,3}x_3 + \cdots + \gamma_{2,n}x_n,
\end{aligned}$$

we obtain the condition $(\beta_{2,2}, \gamma_{2,2}) \neq (0,0)$.

Let us introduce the notations

$$[x_1, y_1] = \alpha_{1,2}y_2 + \alpha_{1,3}y_3 + \cdots + \alpha_{1,m_1}y_{m_1} + \alpha_{1,m_1+2}y_{m_1+2} + \cdots + \alpha_{1,m}y_m,$$

$$[x_2, y_1] = \alpha_{2,2}y_2 + \alpha_{2,3}y_3 + \cdots + \alpha_{2,m_1}y_{m_1} + \alpha_{2,m_1+2}y_{m_1+2} + \cdots + \alpha_{2,m}y_m.$$

Consider the equalities

$$\begin{aligned}
[x_1, [y_1, y_1]] = 2[[x_1, y_1], y_1] &= 2[\alpha_{1,2}y_2 + \alpha_{1,3}y_3 + \cdots + \alpha_{1,m_1}y_{m_1}\\
&\quad + \alpha_{1,m_1+2}y_{m_1+2} + \cdots + \alpha_{1,m}y_m, y_1]\\
&= 2\alpha_{1,2}\beta_{2,2}(A_1x_1 + A_2x_2) + \sum_{i\geq 3}(*)x_i.
\end{aligned}$$

On the other hand

$$[x_1, [y_1, y_1]] = [x_1, \beta_{1,1}x_1 + \beta_{1,2}x_2 + \cdots + \beta_{1,n}x_n] = \sum_{j\geq 3}(*)x_j.$$

Comparing the coefficients of the basic elements in these equations we obtain

$$\alpha_{1,2}\beta_{2,2} = 0.$$

Consider the product

$$\begin{aligned}
[y_1, [y_1, x_1]] &= [[y_1, y_1], x_1] - [[y_1, x_1], y_1]\\
&= [\beta_{1,1}x_1 + \beta_{1,2}x_2 + \cdots + \beta_{1,n}x_n, x_1] - [y_2, y_1]\\
&= -\beta_{2,2}(A_1x_1 + A_2x_2) + \sum_{s\geq 3}(*)x_s.
\end{aligned}$$

Since $[y_1, [y_1, x_1]] = [y_1, y_2]$ then $[y_1, y_2] = -\beta_{2,2}(A_1x_1 + A_2x_2) + \sum_{s\geq 3}(*)x_s$.

From the following the chain of equalities

$$\begin{aligned}[y_1,[x_1,y_1]] &= [[y_1,x_1],y_1] - [[y_1,y_1],x_1] \\ &= [y_2,y_1] - [\beta_{1,1}x_1 + \beta_{1,2}x_2 + \cdots + \beta_{1,n}x_n, x_1] \\ &= \beta_{2,2}(A_1x_1 + A_2x_2) + \sum_{t\geq 3}(*)x_t\end{aligned}$$

and

$$\begin{aligned}[y_1,[x_1,y_1]] &= [y_1, \alpha_{1,2}y_2 + \alpha_{1,3}y_3 + \cdots + \alpha_{1,m_1}y_{m_1} \\ &\quad + \alpha_{1,m_1+2}y_{m_1+2} + \cdots + \alpha_{1,m}y_m] \\ &= -\alpha_{1,2}\beta_{2,2}(A_1x_1 + A_2x_2) + \sum_{p\geq 3}(*)x_p\end{aligned}$$

we obtain the restriction $\beta_{2,2} = -\alpha_{1,2}\beta_{2,2}$.

Taking into account the condition $\alpha_{1,2}\beta_{2,2} = 0$ we get $\beta_{2,2} = 0$. Consider

$$\begin{aligned}[y_1,[y_{m_1+1},x_1]] &= [[y_1,y_{m_1+1}],x_1] - [[y_1,x_1],y_{m_1+1}] = [\gamma_{1,1}x_1 + \gamma_{1,2}x_2 + \cdots \\ &\quad + \gamma_{1,n}x_n, x_1] - [y_2, y_{m_1+1}] = -\gamma_{2,2}(A_1x_1 + A_2x_2) + \sum_{q\geq 3}(*)x_q.\end{aligned}$$

On the other hand we have $[y_1,[y_{m_1+1},x_1]] = \sum_{l\geq 3}(*)x_l$. Comparing the coefficients of the basic elements we get $\gamma_{2,2} = 0$, which contradicts the condition $(\beta_{2,2},\gamma_{2,2}) \neq (0,0)$. Hence, we have $L^3 = \{x_3,\ldots,x_n,y_2,y_3,\ldots,y_{m_1},y_{m_1+2},y_{m_1+3},\ldots,y_m\}$, i.e. $A_1x_1 + A_2x_2 \notin L^3$. Therefore the nilindex of the superalgebra L is less than $n+m$. □

The investigation of the cases where L is a Leibniz superalgebra with characteristic sequence $(n-1,1|m_1,m_2,\ldots,m_k)$, where either $m_1 < 2$ or $n < 4$, give us the same result. Considering these cases consist is a simple routine work, mainly repeating the above technique, hence we omit details for these cases.

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