Forum Math. **24** (2012), 809–826
DOI 10.1515/FORM.2011.085 **Forum Mathematicum**

On the description of Leibniz superalgebras of nilindex $n + m$

Luisa M. Camacho, Jose R. Gómez, Abror Kh. Khudoyberdiyev
and Bakhrom A. Omirov

Communicated by Karl-Hermann Neeb

Abstract. In this work we investigate complex Leibniz superalgebras with characteristic sequence $(n_1, \dots, n_k|m)$ and nilindex $n+m$, where $n = n_1 + \dots + n_k$, n and m ($m \neq 0$) are dimensions of even and odd parts, respectively. Such superalgebras with the condition $n_1 \geq n - 1$ were completely classified. In the present paper, we prove that in the case $n_1 \leq n-2$ the Leibniz superalgebras have nilindex less than $n + m$. Thus, we obtain the classification of Leibniz superalgebras with characteristic sequence $(n_1, \dots, n_k|m)$ and nilindex $n + m$.

Keywords. Lie superalgebras, Leibniz superalgebras, nilindex, characteristic sequence, naturally gradation.

2010 Mathematics Subject Classification. 17A32, 17B30, 17B70, 17A70.

1 Introduction

The paper is devoted to the study of the nilpotent Leibniz superalgebras. The concept of Leibniz superalgebra and its cohomology were first introduced by Dzhumadil'daev in [6]. Leibniz superalgebras are generalizations of Leibniz algebras [11] and that means that they are the natural generalization of the well-known Lie superalgebras. Several authors study Leibniz superalgebras and its properties [1, 6, 9, 10].

In [8] Lie superalgebras with maximal nilindex were classified. The main properties of such superalgebras are that they are two-generated and their nilindex is equal to $n + m$ (where n and m are the dimensions of the even and odd parts, respectively). In fact, there exists a unique Lie superalgebra of maximal nilindex. The dimensions of the even part of this superalgebra is $n = 2$ and its characteristic sequence is equal to $(1, 1|m)$. This superalgebra is a filiform Lie superalgebra (its

This work is partially supported by the PAI, FQM143 of the Junta de Andalucía (Spain). The last author was supported by grant NATO-Reintegration ref. CBP.EAP.RIG.983169.

characteristic sequence is equal to $(n-1, 1|m)$), see paper [4], where some crucial properties of filiform Lie superalgebras are given.

In the case of Leibniz superalgebras the property of maximal nilindex is equivalent to the property of single-generated superalgebras and they are described in [1]. However, the description of Leibniz superalgebras of nilindex $n + m$ is an open problem and many technical tasks need to be solved. Therefore, they can be studied by applying restrictions on their characteristic sequences. In the present paper we consider Leibniz superalgebras with characteristic sequence $(n_1, \dots, n_k|m)$ and nilindex $n+m$. In the case $n_1 \geq n-1$, such superalgebras have already been classified in [2]–[5]. We need to study the case $n_1 \leq n-2$ now.

In fact, in the previous cases (the cases when $n_1 \geq n-1$, see [3]) we have used some information on the structure of the even part of Leibniz superalgebra and they played a crucial role in that classifications. In the case when $n_1 \leq n-2$ the structure of the even part is unknown, but if we use the properties of natural gradation and the naturally graded basis (so-called adapted basis) of the even part of the superalgebra, then we obtain the results.

All the vector spaces that appear in this paper (and thus, all the algebras) are assumed to be $\mathbb{C}$-vector spaces of finite dimension. The correspondent coefficients with respect to the base of the law of superalgebra products are denoted by $(*)$.

2 Preliminaries

Recall the notion of Leibniz superalgebras.

Definition 2.1. A $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ is called a *Leibniz superalgebra* if it is equipped with a product $[-,-]$ which satisfies the following conditions:

(1) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$,

(2) $[x,[y,z]] = [[x,y],z] - (-1)^{\alpha\beta}[[x,z],y]$ (*Leibniz superidentity*) for all $x \in L$, $y \in L_\alpha$, $z \in L_\beta$ and $\alpha, \beta \in \mathbb{Z}_2$.

Evidently, the even part of the Leibniz superalgebra is a Leibniz algebra.

The vector spaces L_0 and L_1 are said to be the even and odd parts of the superalgebra L, respectively.

Note that if in a Leibniz superalgebra L the identity

$$[x,y] = -(-1)^{\alpha\beta}[y,x]$$

holds for any $x \in L_\alpha$ and $y \in L_\beta$, then the Leibniz superidentity can be transformed into the Jacobi superidentity. Thus, Leibniz superalgebras are a generalization of Lie superalgebras and Leibniz algebras.

We denote by $\mathrm{Leib}_{n,m}$ the set of all Leibniz superalgebras with the dimensions of the even and odd parts, respectively equal to n and m.

For a given Leibniz superalgebra L we define the descending central sequence as follows:

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1.$$

Definition 2.2. A Leibniz superalgebra L is called *nilpotent* if there exists $s \in \mathbb{N}$ such that $L^s = 0$. The smallest number s with this property is called the *nilindex* of the superalgebra L.

The following theorem describes nilpotent Leibniz superalgebras with maximal nilindex.

Theorem 2.3 ([1]). *Let L be a Leibniz superalgebra of* $\mathrm{Leib}_{n,m}$ *with nilindex equal to $n+m+1$. Then L is isomorphic to one of the following non-isomorphic superalgebras:*

$$[e_i, e_1] = e_{i+1}, \qquad 1 \leq i \leq n-1, \ m = 0;$$

$$\begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n+m-1, \\ [e_i, e_2] = 2e_{i+2}, & 1 \leq i \leq n+m-2, \end{cases}$$

(where the products equal to zero are omitted).

Remark 2.4. According to Theorem 2.3 we have that there are two possibilities for n and m, in the case of the non-trivial odd part L_1. The first possibility is $m = n$ if $n+m$ is even, and the second one is $m = n+1$ if $n+m$ is odd. Moreover, it is clear that the Leibniz superalgebra has the maximal nilindex if and only if it is single-generated.

Let $L = L_0 \oplus L_1$ be a nilpotent Leibniz superalgebra. For an arbitrary element $x \in L_0$, we define the operator of right multiplication R_x by $R_x(y) = [y, x]$. This operator is a nilpotent endomorphism of the space L_i, where $i \in \{0, 1\}$. Taking into account the property of complex field we can consider the Jordan form for R_x. Denote by $C_i(x)$ ($i \in \{0, 1\}$) the descending sequence of the Jordan blocks dimensions of R_x. Consider the lexicographical order on the set $C_i(L_0)$.

Definition 2.5. A sequence

$$C(L) = \left(\max_{x \in L_0 \setminus L_0^2} C_0(x) \,\middle|\, \max_{\widetilde{x} \in L_0 \setminus L_0^2} C_1(\widetilde{x}) \right)$$

is said to be the *characteristic sequence* of the Leibniz superalgebra L.

As in [7, Corollary 3.0.1] it can be proved that the characteristic sequence is invariant under isomorphism.

Since the Leibniz superalgebras from $\mathrm{Leib}_{n,m}$ with characteristic sequence equal to $(n_1, \dots, n_k|m)$ and nilindex $n + m$ have already been classified for the case $n_1 \geq n - 1$, henceforth we shall reduce our investigation to the case $n_1 \leq n - 2$.

From Definition 2.5 we conclude that a Leibniz algebra L_0 has characteristic sequence $(n_1, \dots, n_k)$. Let $s \in \mathbb{N}$ be a nilindex of the Leibniz algebra L_0. Since $n_1 \leq n - 2$, we have $s \leq n - 1$ and the Leibniz algebra L_0 has at least two generators (the elements which belong to the set $L_0 \setminus L_0^2$).

For the completeness of the statement below we present the classifications offered in [2], [5] and [8].

$\mathrm{Leib}_{1,m}$:

$$\Big\{[y_i, x_1] = y_{i+1}, \quad 1 \leq i \leq m-1.$$

$\mathrm{Leib}_{n,1}$:

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \leq i \leq n-1, \\ [y_1, y_1] = \alpha x_n, & \alpha = \{0,\ 1\}. \end{cases}$$

$\mathrm{Leib}_{2,2}$:

$$\begin{cases} [y_1, x_1] = y_2, \\ [x_1, y_1] = \frac{1}{2} y_2, \\ [x_2, y_1] = y_2, \\ [y_1, x_2] = 2y_2, \\ [y_1, y_1] = x_2, \end{cases} \qquad \begin{cases} [y_1, x_1] = y_2, \\ [x_2, y_1] = y_2, \\ [y_1, x_2] = 2y_2, \\ [y_1, y_1] = x_2. \end{cases}$$

$\mathrm{Leib}_{2,m}$, m is odd:

$$\begin{cases} [x_1, x_1] = x_2, & m \geq 3, \\ [y_i, x_1] = y_{i+1}, & 1 \leq i \leq m-1, \\ [x_1, y_i] = -y_{i+1}, & 1 \leq i \leq m-1, \\ [y_i, y_{m+1-i}] = (-1)^{j+1} x_2, & 1 \leq i \leq \frac{m+1}{2}, \end{cases}$$

$$\begin{cases} [y_i, x_1] = [x_1, y_i] = -y_{i+1}, & 1 \leq i \leq m-1, \\ [y_{m+1-i}, y_i] = (-1)^{j+1} x_2, & 1 \leq i \leq \frac{m+1}{2}. \end{cases}$$

In order to present the classification of Leibniz superalgebras with characteristic sequence $(n-1, 1|m)$, $n \geq 3$, and nilindex $n+m$, we need to introduce the following families of superalgebras:

Leib$_{n,n-1}$.

• $L(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta)$:

$$\begin{cases} [x_1, x_1] = x_3, & \\ [x_i, x_1] = x_{i+1}, & 2 \leq i \leq n-1, \\ [y_j, x_1] = y_{j+1}, & 1 \leq j \leq n-2, \\ [x_1, y_1] = \frac{1}{2} y_2, & \\ [x_i, y_1] = \frac{1}{2} y_i, & 2 \leq i \leq n-1, \\ [y_1, y_1] = x_1, & \\ [y_j, y_1] = x_{j+1}, & 2 \leq j \leq n-1, \\ [x_1, x_2] = \alpha_4 x_4 + \alpha_5 x_5 + \cdots + \alpha_{n-1} x_{n-1} + \theta x_n, & \\ [x_j, x_2] = \alpha_4 x_{j+2} + \alpha_5 x_{j+3} + \cdots + \alpha_{n+2-j} x_n, & 2 \leq j \leq n-2, \\ [y_1, x_2] = \alpha_4 y_3 + \alpha_5 y_4 + \cdots + \alpha_{n-1} y_{n-2} + \theta y_{n-1}, & \\ [y_j, x_2] = \alpha_4 y_{j+2} + \alpha_5 y_{j+3} + \cdots + \alpha_{n+1-j} y_{n-1}, & 2 \leq j \leq n-3. \end{cases}$$

• $G(\beta_4, \beta_5, \ldots, \beta_n, \gamma)$:

$$\begin{cases} [x_1, x_1] = x_3, & \\ [x_i, x_1] = x_{i+1}, & 3 \leq i \leq n-1, \\ [y_j, x_1] = y_{j+1}, & 1 \leq j \leq n-2, \\ [x_1, x_2] = \beta_4 x_4 + \beta_5 x_5 + \cdots + \beta_n x_n, & \\ [x_2, x_2] = \gamma x_n, & \\ [x_j, x_2] = \beta_4 x_{j+2} + \beta_5 x_{j+3} + \cdots + \beta_{n+2-j} x_n, & 3 \leq j \leq n-2, \\ [y_1, y_1] = x_1, & \\ [y_j, y_1] = x_{j+1}, & 2 \leq j \leq n-1, \\ [x_1, y_1] = \frac{1}{2} y_2, & \\ [x_i, y_1] = \frac{1}{2} y_i, & 3 \leq i \leq n-1, \\ [y_j, x_2] = \beta_4 y_{j+2} + \beta_5 y_{j+3} + \cdots + \beta_{n+1-j} y_{n-1}, & 1 \leq j \leq n-3. \end{cases}$$

$\mathbf{Leib}_{n,n}$.

- $M(\alpha_4, \alpha_5, \dots, \alpha_n, \theta, \tau)$:

$$\begin{cases} [x_1, x_1] = x_3, & \\ [x_i, x_1] = x_{i+1}, & 2 \le i \le n-1, \\ [y_j, x_1] = y_{j+1}, & 1 \le j \le n-1, \\ [x_1, y_1] = \frac{1}{2} y_2, & \\ [x_i, y_1] = \frac{1}{2} y_i, & 2 \le i \le n, \\ [y_1, y_1] = x_1, & \\ [y_j, y_1] = x_{j+1}, & 2 \le j \le n-1, \\ [x_1, x_2] = \alpha_4 x_4 + \alpha_5 x_5 + \dots + \alpha_{n-1} x_{n-1} + \theta x_n, & \\ [x_2, x_2] = \gamma_4 x_4, & \\ [x_j, x_2] = \alpha_4 x_{j+2} + \alpha_5 x_{j+3} + \dots + \alpha_{n+2-j} x_n, & 3 \le j \le n-2, \\ [y_1, x_2] = \alpha_4 y_3 + \alpha_5 y_4 + \dots + \alpha_{n-1} y_{n-2} + \theta y_{n-1} + \tau y_n, & \\ [y_2, x_2] = \alpha_4 y_4 + \alpha_5 y_4 + \dots + \alpha_{n-1} y_{n-1} + \theta y_n, & \\ [y_j, x_2] = \alpha_4 y_{j+2} + \alpha_5 y_{j+3} + \dots + \alpha_{n+2-j} y_n, & 3 \le j \le n-2. \end{cases}$$

- $H(\beta_4, \beta_5, \dots, \beta_n, \delta, \gamma)$:

$$\begin{cases} [x_1, x_1] = x_3, & \\ [x_i, x_1] = x_{i+1}, & 3 \le i \le n-1, \\ [y_j, x_1] = y_{j+1}, & 1 \le j \le n-2, \\ [x_1, x_2] = \beta_4 x_4 + \beta_5 x_5 + \dots + \beta_n x_n, & \\ [x_2, x_2] = \gamma x_n, & \\ [x_j, x_2] = \beta_4 x_{j+2} + \beta_5 x_{j+3} + \dots + \beta_{n+2-j} x_n, & 3 \le j \le n-2, \\ [y_1, y_1] = x_1, & \\ [y_j, y_1] = x_{j+1}, & 2 \le j \le n-1, \\ [x_1, y_1] = \frac{1}{2} y_2, & \\ [x_i, y_1] = \frac{1}{2} y_i, & 3 \le i \le n-1, \\ [y_1, x_2] = \beta_4 y_3 + \beta_5 y_4 + \dots + \beta_n y_{n-1} + \delta y_n, & \\ [y_j, x_2] = \beta_4 y_{j+2} + \beta_5 y_{j+3} + \dots + \beta_{n+2-j} y_n, & 2 \le j \le n-2. \end{cases}$$

Let us also introduce the following operators which act on k-dimensional vectors:

$$V_{j,k}^m(\alpha_1,\alpha_2,\dots,\alpha_k) = (0,0,\dots,\overset{j-1}{0},1,S_{m,j}^{j+1}\alpha_{j+1},S_{m,j}^{j+2}\alpha_{j+2},\dots S_{m,j}^{k-1}\alpha_{k-1},S_{m,j}^{k}\alpha_k),$$

$$V_{k+1,k}^m(\alpha_1,\alpha_2,\dots,\alpha_k) = (0,0,\dots,0),$$

$$W_{s,k}^m(0,0,\dots,\overset{j-1}{0},\overset{j}{1},S_{m,j}^{j+1}\alpha_{j+1},S_{m,j}^{j+2}\alpha_{j+2},\dots,S_{m,j}^{k}\alpha_k,\gamma) = (0,0,\dots,\overset{j}{1},0,\dots,\overset{s+j}{1},S_{m,s}^{s+1}\alpha_{s+j+1}, S_{m,s}^{s+2}\alpha_{s+j+2},\dots,S_{m,s}^{k-j}\alpha_k,S_{m,s}^{k+6-2j}\gamma),$$

$$W_{k+1-j,k}^m(0,0,\dots,\overset{j-1}{0},\overset{j}{1},S_{m,j}^{j+1}\alpha_{j+1},S_{m,j}^{j+2}\alpha_{j+2},\dots,S_{m,j}^{k}\alpha_k,\gamma) = (0,0,\dots,\overset{j}{1},0,\dots,1),$$

$$W_{k+2-j,k}^m(0,0,\dots,\overset{j-1}{0},\overset{j}{1},S_{m,j}^{j+1}\alpha_{j+1},S_{m,j}^{j+2}\alpha_{j+2},\dots,S_{m,j}^{k}\alpha_k,\gamma) = (0,0,\dots,\overset{j}{1},0,\dots,0),$$

where $k \in N$, $1 \le j \le k$, $1 \le s \le k-j$, $S_{m,t} = \cos\frac{2\pi m}{t} + i\sin\frac{2\pi m}{t}$ $(m = 0,1,\dots,t-1)$.

Below we present the complete list of the pairwise non-isomorphic Leibniz superalgebras with characteristic sequence equal to $(n-1,1|m)$ and nilindex $n+m$ (see [2]):

$$L\big(V_{j,n-3}(\alpha_4,\alpha_5,\dots,\alpha_n),S_{m,j}^{n-3}\theta\big),\quad 1\le j\le n-3,$$

$$L(0,0,\dots,0,1),\ L(0,0,\dots,0),\ G(0,0,\dots,0,1),\ G(0,0,\dots,0),$$

$$G\big(W_{s,n-2}\big(V_{j,n-3}(\beta_4,\beta_5,\dots,\beta_n),\gamma\big)\big),\quad 1\le j\le n-3,\ 1\le s\le n-j,$$

$$M\big(V_{j,n-2}(\alpha_4,\alpha_5,\dots,\alpha_n),S_{m,j}^{n-3}\theta\big),\quad 1\le j\le n-2,$$

$$M(0,0,\dots,0,1),\ M(0,0,\dots,0),\ H(0,0,\dots,0,1),\ H(0,0,\dots,0),$$

$$H\big(W_{s,n-1}\big(V_{j,n-2}(\beta_4,\beta_5,\dots,\beta_n),\gamma\big)\big),\quad 1\le j\le n-2,\ 1\le s\le n+1-j,$$

where the omitted products are equal to zero.

Definition 2.6. For a given Leibniz algebra A with nilindex s we put $\mathrm{gr}(A)_i = A^i/A^{i+1}$ for $1 \le i \le s-1$, and $\mathrm{gr}(A) = \mathrm{gr}(A)_1 \oplus \mathrm{gr}(A)_2 \oplus \cdots \oplus \mathrm{gr}(A)_{s-1}$. Then $[\mathrm{gr}(A)_i, \mathrm{gr}(A)_j] \subseteq \mathrm{gr}(A)_{i+j}$ and we obtain the graded algebra $\mathrm{gr}(A)$. The gradation constructed in this way is called a *natural gradation* and if the Leibniz algebra A is isomorphic to $\mathrm{gr}(A)$, we say that A is a *naturally graded Leibniz algebra.*

Further we shall consider the basis of the even part of L which corresponds with the natural gradation, that is, $\{x_1, \dots, x_{t_1}\} = L_0 \setminus L_0^2$, $\{x_{t_1+1}, \dots, x_{t_2}\} = L_0^2 \setminus L_0^3, \dots, \{x_{t_{s-2}+1}, \dots, x_n\} = L_0^{s-1}$.

Since the second part of the characteristic sequence of a Leibniz superalgebra L is equal to m, there exists a nilpotent endomorphism R_x ($x \in L_0 \setminus L_0^2$) of the space L_1 such that its Jordan form consists of one Jordan block. Therefore, we can assume the existence of an adapted basis $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$ such that

$$[y_j, x_1] = y_{j+1}, \quad 1 \le j \le m-1. \tag{2.1}$$

3 The main result

Let L be a Leibniz superalgebra with characteristic sequence $(n_1, \dots, n_k|m)$, $n_1 \le n-2$, and let $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n\}$ be the adapted basis of L. In this section we shall prove that the nilindex of such a superalgebra is less than $n+m$. According to Theorem 2.3 we have a description of single-generated Leibniz superalgebras which have nilindex $n+m+1$. If the number of generators is greater than two, then evidently the superalgebra has nilindex less than $n+m$. Therefore, we should consider the case of two generators.

Note that the case where both generators lie in the even part is not possible (since $m \neq 0$). Equality (2.1) implies that the basic elements $y_2, y_3, \dots, y_m$ cannot be generators. Therefore, the first generator belongs to L_0 and the second one lies in L_1. Moreover, without loss of generality we can suppose that y_1 is a generator. Let us find the generator of Leibniz superalgebra L which lies in L_0.

Lemma 3.1. *Let $L = L_0 \oplus L_1$ be a two generated Leibniz superalgebra from* $\mathrm{Leib}_{n,m}$ *with characteristic sequence equal to $(n_1, \dots, n_k|m)$. Then x_1 and y_1 can be chosen as generators of L.*

Proof. As we mentioned above, y_1 can be chosen as first generator of L. If $x_1 \in L \setminus L^2$, then the assertion of the lemma is evident. If $x_1 \in L^2$, then there exists some i_0 ($2 \le i_0 \le t_1$) such that $x_{i_0} \in L \setminus L^2$. Put $x_1' = Ax_1 + x_{i_0}$; then x_1' is a generator of the superalgebra L (since $x_1' \in L \setminus L^2$). Moreover, making the following transformation of the basis of L_1,

$$y_1' = y_1, \quad y_j' = [y_{j-1}', x_1'], \quad 2 \le j \le m,$$

and taking sufficiently big the value of the parameter A we preserve equality (2.1). Thus, in the basis $\{x_1', x_2, \dots, x_n, y_1', y_2', \dots, y_m'\}$ of the L the elements x_1' and y_1' are generators. □

Due to Lemma 3.1 we shall suppose that $\{x_1, y_1\}$ are generators of the Leibniz superalgebra L.

Let us introduce the notations

$$[x_i, y_1] = \sum_{j=2}^{m} \alpha_{i,j} y_j, \ \ 1 \le i \le n, \quad [y_i, y_1] = \sum_{j=2}^{n} \beta_{i,j} x_j, \ \ 1 \le i \le m. \quad (3.1)$$

Since x_1 and y_1 are generators of the Leibniz superalgebra L, we have

$$L = \{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m\},$$
$$L^2 = \{x_2, x_3, \dots, x_n, y_2, y_3, \dots, y_m\}.$$

If we consider the next power of L, then from the multiplication (2.1) we obviously get $\{y_3, \dots, y_m\} \subseteq L^3$. However, we do not have information about the position of the element y_2.

Theorem 3.2. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from* $\mathrm{Leib}_{n,m}$ *with characteristic sequence equal to $(n_1, \dots, n_k|m)$ and let $y_2 \notin L^3$ be. Then L has nilindex less than $n+m$.*

Proof. Let us assume the opposite, i.e., the nilindex of the superalgebra L is equal to $n+m$. Then the condition $y_2 \notin L^3$ deduces $\{x_2, x_3, \dots, x_n\} \subseteq L^3$. Therefore,

$$L^3 = \{x_2, x_3, \dots, x_n, y_3, \dots, y_m\}.$$

Let $s \in \mathbb{N}$ be a number such that $x_2 \in L^s \setminus L^{s+1}$, that is,

$$L^s = \{x_2, x_3, \dots, x_n, y_s, \dots, y_m\}, \quad s \ge 3,$$
$$L^{s+1} = \{x_3, x_4, \dots, x_n, y_s, \dots, y_m\}.$$

It means that x_2 can only be obtained from the product $[y_{s-1}, y_1]$ and thereby $\beta_{s-1,2} \neq 0$.

Similarly, we assume that k is a number by which $x_3 \in L^{s+k} \setminus L^{s+k+1}$. Then for the powers of the superalgebra L we have the following:

$$\begin{aligned} L^{s+k} &= \{x_3, x_4, \dots, x_n, y_{s+k-1}, \dots, y_m\}, \quad k \geq 1, \\ L^{s+k+1} &= \{x_4, \dots, x_n, y_{s+k-1}, \dots, y_m\}. \end{aligned}$$

Let us suppose $k = 1$. Then

$$L^{s+2} = \{x_4, \dots, x_n, y_s, \dots, y_m\}.$$

Since $x_3 \notin L^{s+2}$ and the vector space L^{s+1} is generated by multiplying the space L^s to the elements x_1 and y_1 on the right side (because of the Leibniz superidentity), it follows that x_3 is obtained by the product $[x_2, x_1]$, i.e. $[x_2, x_1] = ax_3 + \sum_{i\geq 4}(*)x_i$ with $a \neq 0$. Making the change of the basic element x_3 as $x_3' = ax_3 + \sum_{i\geq 4}(*)x_i$ we can conclude that $[x_2, x_1] = x_3$.

Let us define the products $[y_{s-j}, y_j]$ for $1 \leq j \leq s-1$.

Applying the Leibniz superidentity and j induction we prove

$$[y_{s-j}, y_j] = (-1)^{j+1}\beta_{s-1,2}x_2 + \sum_{i\geq 3}(*)x_i. \tag{3.2}$$

The equality (3.2) is true by notation (3.1) for $j = 1$. Let us suppose that equality (3.2) holds for $j = t$. Then we have for $j = t+1$

$$\begin{aligned} [y_{s-t-1}, y_{t+1}] &= [y_{s-t-1}, [y_t, x_1]] \\ &= [[y_{s-t-1}, y_t], x_1] - [[y_{s-t-1}, x_1], y_t] \\ &= -[y_{s-t}, y_t] + \left[\sum_{i\geq 2}(*)x_i, x_1\right] \\ &= -(-1)^{t+1}\beta_{s-1,2}x_2 + \sum_{i\geq 3}(*)x_i + \left[\sum_{i\geq 2}(*)x_i, x_1\right] \\ &= (-1)^{t+2}\beta_{s-1,2}x_2 + \sum_{i\geq 3}(*)x_i. \end{aligned}$$

It should be noted that the coefficients of the basic elements x_2 and x_3 are equal to zero in the decomposition of $[y_s, y_1]$ and $[y_1, y_s]$.

Let us define the products $[y_{s+1-j}, y_j]$ for $2 \leq j \leq s$. In fact, if $j = 2$, then

$$\begin{aligned} [y_{s-1}, y_2] &= [y_{s-1}, [y_1, x_1]] = [[y_{s-1}, y_1], x_1] - [y_s, y_1] \\ &= \left[\sum_{i=2}^{n}\beta_{s-1,i}x_i, x_1\right] - \sum_{i=4}^{n}\beta_{s,i}x_i = \beta_{s-1,2}x_3 + \sum_{i\geq 4}(*)x_i. \end{aligned}$$

Inductively, applying the above arguments for $j \geq 3$ and using equality (3.2) we obtain

$$[y_{s+1-j}, y_j] = (-1)^j (j-1)\beta_{s-1,2}x_3 + \sum_{i\geq 4}(*)x_i, \quad 2 \leq j \leq s.$$

In particular, $[y_1, y_s] = (-1)^s(s-1)\beta_{s-1,2}x_3 + \sum_{i\geq 4}(*)x_i$. On the other hand (as we have mentioned above), the coefficient of the basic element x_3 is equal to zero in the decomposition of $[y_1, y_s]$. Therefore, $\beta_{s-1,2} = 0$, which contradicts the condition $\beta_{s-1,2} \neq 0$. Thus, our assumption $k = 1$ is not possible. Hence, $k \geq 2$ and we have

$$L^{s+2} = \{x_3, \dots, x_n, y_{s+1}, \dots, y_m\}.$$

Since $y_s \notin L^{s+2}$, it follows that

$$\alpha_{2,s} \neq 0, \quad \alpha_{2,j} = 0 \quad \text{for } j < s,$$
$$\alpha_{i,j} = 0 \quad \text{for any } i \geq 3, \ j < s+1.$$

Consider the product

$$[[y_{s-1}, y_1], y_1] = \frac{1}{2}[y_{s-1}, [y_1, y_1]] = \frac{1}{2}\left[y_{s-1}, \sum_{i=2}^{n} \beta_{1,i}x_i\right].$$

The element y_{s-1} belongs to L^{s-1} and the elements $x_2, x_3, \dots, x_n$ lie in L^3. Hence, $\frac{1}{2}[y_{s-1}, \sum_{i=2}^{n} \beta_{1,i}x_i] \in L^{s+2}$. We get $[[y_{s-1}, y_1], y_1] = \sum_{j\geq s+1}(*)y_j$, so $L^{s+2} = \{x_3, \dots, x_n, y_{s+1}, \dots, y_m\}$. On the other hand,

$$\begin{aligned}
[[y_{s-1}, y_1], y_1] &= \left[\sum_{i=2}^{n} \beta_{s-1,i}x_i, y_1\right] = \sum_{i=2}^{n} \beta_{s-1,i}[x_i, y_1] \\
&= \sum_{i=2}^{n} \beta_{s-1,i} \sum_{j\geq s} \alpha_{i,j}y_j \\
&= \beta_{s-1,2}\alpha_{2,s}y_s + \sum_{j\geq s+1}(*)y_j.
\end{aligned}$$

Comparing the coefficients of the basic elements we obtain $\beta_{s-1,2}\alpha_{2,s} = 0$, which contradicts the conditions $\beta_{s-1,2} \neq 0$ and $\alpha_{2,s} \neq 0$.

Thus, we get a contradiction by assuming that the superalgebra L has nilindex equal to $n+m$ and therefore the assertion of the theorem holds. □

The investigation of the case $y_2 \in L^3$ depends on the structure of the Leibniz algebra L_0. So, we present some remarks on naturally graded nilpotent Leibniz algebras.

Let $A = \mathbb{C}\{z_1, z_2, \dots, z_n\}$ be an n-dimensional nilpotent Leibniz algebra of nilindex p $(p < n)$. Note that the algebra A is not single-generated.

Let us consider the case when $\mathrm{gr}(A)$ is a non-Lie Leibniz algebra.

Lemma 3.3. *Let* $\mathrm{gr}(A)$ *be a naturally graded non-Lie Leibniz algebra. Then* $\dim((\mathrm{gr}(A)_1) + \dim(\mathrm{gr}(A)_2) \geq 4$.

Proof. The construction of $\mathrm{gr}(A)$ implies that every subspace $\mathrm{gr}(A)_i$ for $1 \leq i \leq p-1$ is not empty. Obviously, $\dim(\mathrm{gr}(A)_1) \geq 2$ (otherwise $p = n+1$). If the dimension of the subspace $\mathrm{gr}(A)_1$ is greater than two, then the statement of the lemma is true. If $\dim(\mathrm{gr}(A)_1) = 2$ and $\dim(\mathrm{gr}(A)_2) = 2$, then the assertion of the lemma is evident.

Let us suppose that $\dim(\mathrm{gr}(A)_1) = 2$ and $\dim(\mathrm{gr}(A)_2) = 1$. Then taking into account the condition $p < n$ we conclude that there exists some t $(t > 2)$ such that $\dim(\mathrm{gr}(A)_t) \geq 2$ (otherwise the nilindex is equal to n).

Let t_0 $(t_0 > 2)$ be the smallest number with the condition $\dim(\mathrm{gr}(A)_{t_0}) \geq 2$. Then

$$\begin{aligned}
\mathrm{gr}(A)_1 &= \{\overline{z}_1, \overline{z}_2\},\\
\mathrm{gr}(A)_2 &= \{\overline{z}_3\},\\
&\vdots\\
\mathrm{gr}(A)_{t_0-1} &= \{\overline{z}_{t_0}\},\\
\mathrm{gr}(A)_{t_0} &= \{\overline{z}_{t_0+1}, \overline{z}_{t_0+2}\}.
\end{aligned}$$

As in [12] we obtain that

$$\begin{cases}
[\overline{z}_1, \overline{z}_1] = \alpha_1 \overline{z}_3,\\
[\overline{z}_2, \overline{z}_1] = \alpha_2 \overline{z}_3,\\
[\overline{z}_1, \overline{z}_2] = \alpha_3 \overline{z}_3,\\
[\overline{z}_2, \overline{z}_2] = \alpha_4 \overline{z}_3,\\
[\overline{z}_i, \overline{z}_1] = \overline{z}_{i+1}, \quad 3 \leq i \leq t_0.
\end{cases}$$

Since $\mathrm{gr}(A)$ is a non-Lie Leibniz algebra, there exists an element of $\mathrm{gr}(A)_1$ such that its square is not zero. It is not difficult to see that $\overline{z}_3, \dots, \overline{z}_{t_0+1}$ belong to the right annihilator $\mathfrak{R}(\mathrm{gr}(A))$, which is defined as

$$\mathfrak{R}(\mathrm{gr}(A)) = \{\overline{z} \in \mathrm{gr}(A) \mid [\overline{y}, \overline{z}] = 0 \text{ for any } \overline{y} \in \mathrm{gr}(A)\}.$$

Moreover, one can assume $[\overline{z}_{t_0}, \overline{z}_2] = \overline{z}_{t_0+2}$. On the other hand,

$$\begin{aligned}[\overline{z}_{t_0}, \overline{z}_2] &= [[\overline{z}_{t_0-1}, \overline{z}_1], \overline{z}_2] = [\overline{z}_{t_0-1}, [\overline{z}_1, \overline{z}_2]] + [[\overline{z}_{t_0-1}, \overline{z}_2], \overline{z}_1] \\ &= [\overline{z}_{t_0-1}, \alpha_3 \overline{z}_3] + [\beta \overline{z}_{t_0}, \overline{z}_1] = \beta \overline{z}_{t_0+1}.\end{aligned}$$

The obtained equality $\overline{z}_{t_0+2} = \beta_{t_0-1,2} \overline{z}_{t_0+1}$ derives a contradiction, which leads to the assertion of the lemma. □

From Lemma 3.3 the next corollary follows.

Corollary 3.4. *Let A be a Leibniz algebra satisfying the condition of Lemma 3.3. Then* $\dim(A^3) \leq n-4$.

The following proposition shows a restriction on the nilindex of the superalgebra with the condition $\dim(L_0^3) \leq n-4$.

Proposition 3.5. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from* $\mathrm{Leib}_{n,m}$ *with characteristic sequence $(n_1, \dots, n_k|m)$ and* $\dim(L_0^3) \leq n-4$. *Then L has nilindex less than $n+m$.*

Proof. Let us assume the opposite, i.e., the nilindex of the superalgebra L is equal to $n+m$. According to Theorem 3.2 we need to consider the case when y_2 belongs to L^3, which leads to $x_2 \notin L^3$. Thus, we have

$$L^3 = \{x_3, x_4, \dots, x_n, y_2, y_3, \dots, y_m\}.$$

From the condition $\dim(L_0^3) \leq n-4$ it follows that there exist at least two basic elements that do not belong to L_0^3. Without loss of generality, one can assume $x_3, x_4 \notin L_0^3$.

Let s be a natural number such that $x_3 \in L^{s+1} \setminus L^{s+2}$. Then we have

$$\begin{aligned}L^{s+1} &= \{x_3, x_4, \dots, x_n, y_s, y_{s+1}, \dots, y_m\}, \quad s \geq 2 \text{ and } \beta_{s-1,3} \neq 0, \qquad (3.3)\\ L^{s+2} &= \{x_4, \dots, x_n, y_s, y_{s+1}, \dots, y_m\}.\end{aligned}$$

Let us suppose $x_3 \notin L_0^2$. Then we have that x_3 cannot be obtained by the products $[x_i, x_1]$ with $2 \leq i \leq n$. Therefore, it is generated by the products $[y_j, y_1]$ for $2 \leq j \leq m$, which implies $s \geq 3$ and $\alpha_{2,2} \neq 0$.

If $s = 3$, then $\beta_{2,3} \neq 0$.

Consider the chain of equalities

$$[[x_2, y_1], y_1] = \left[\sum_{j=2}^{m} \alpha_{2,j} y_j, y_1\right] = \sum_{j=2}^{m} \alpha_{2,j} [y_j, y_1] = \alpha_{2,2}\beta_{2,3} x_3 + \sum_{i \geq 4} (*) x_i.$$

On the other hand,

$$[[x_2, y_1], y_1] = \frac{1}{2}[x_2, [y_1, y_1]] = \frac{1}{2}\left[x_2, \sum_{i=2}^{n} \beta_{1,i} x_i\right] = \frac{1}{2}\sum_{i=2}^{n} \beta_{1,i}[x_2, x_i]$$
$$= \sum_{i\geq 4}(*)x_i.$$

Comparing the coefficients with the corresponding basic elements we get a contradiction with $\beta_{2,3} = 0$, see (3.3). Thus, $s \geq 4$.

Since $y_2 \in L^3$ and $s \geq 4$, we have $y_{s-2} \in L^{s-1}$, which implies $[y_{s-2}, y_2] \in L^{s+2} = \{x_4, \dots, x_n, y_s, y_{s+1}, \dots, y_m\}$. The coefficient of the basic element x_3 is equal to zero in the decomposition of $[y_{s-2}, y_2]$. On the other hand,

$$[y_{s-2}, y_2] = [y_{s-2}, [y_1, x_1]] = [[y_{s-2}, y_1], x_1] - [[y_{s-2}, x_1], y_1]$$
$$= \left[\sum_{i=2}^{n} \beta_{s-2,i} x_i, x_1\right] - [y_{s-1}, y_1]$$
$$= -\beta_{s-1,3}x_3 + \sum_{i\geq 4}(*)x_i.$$

Hence, $\beta_{s-1,3} = 0$, which is a contradiction. Therefore, we have $x_3 \in L_0^2 \setminus L_0^3$.

The condition $x_4 \notin L_0^3$ deduces that x_4 cannot be obtained by the products $[x_i, x_1]$ with $3 \leq i \leq n$. Therefore, it is generated by the products $[y_j, y_1]$ for $s \leq j \leq m$. Hence, $L^{s+3} = \{x_4, \dots, x_n, y_{s+1}, \dots, y_m\}$ and $y_s \in L^{s+2} \setminus L^{s+3}$, which implies $\alpha_{3,s} \neq 0$.

We repeat the above argumentations for the product $[[x_3, y_1], y_1]$ and we get $\alpha_{3,s}\beta_{s,4} = 0$. It implies $\beta_{s,4} = 0$. Thus, we conclude that $x_4 \in L^{s+4}$ and

$$L^{s+4} = \{x_4, \dots, x_n, y_{s+2}, \dots, y_m\}.$$

Let k ($4 \leq k \leq m-s+2$) be a natural number such that $x_4 \in L^{s+k} \setminus L^{s+k+1}$. Then by properties of the descending central sequences we have

$$L^{s+k-2} = \{x_4, \dots, x_n, y_{s+k-4}, \dots, y_m\},$$
$$L^{s+k-1} = \{x_4, \dots, x_n, y_{s+k-3}, \dots, y_m\},$$
$$L^{s+k} = \{x_4, \dots, x_n, y_{s+k-2}, \dots, y_m\},$$
$$L^{s+k+1} = \{x_5, \dots, x_n, y_{s+k-2}, \dots, y_m\}.$$

It is easy to see that $\beta_{s+k-3,4} \neq 0$ in the decomposition $[y_{s+k-3}, y_1] = \sum_{i=4}^{n} \beta_{s+k-3,i} x_i$.

Consider the equalities

$$[y_{s+k-4}, y_2] = [y_{s+k-4}, [y_1, x_1]] = [[y_{s+k-4}, y_1], x_1] - [[y_{s+k-4}, x_1], y_1]$$
$$= \left[\sum_{i=3}^{n} \beta_{s+k-3,i} x_i, x_1\right] - [y_{s+k-3}, y_1]$$
$$= -\beta_{s+k-3,4} x_4 + \sum_{i \geq 5} (*) x_i.$$

Since $y_{s+k-4} \in L^{s+k-2}$, $y_2 \in L^3$ and $\beta_{s+k-3,4} \neq 0$, the element x_4 should lie in L^{s+k+1}, but it contradicts $L^{s+k+1} = \{x_5, \dots, x_n, y_{s+k-2}, \dots, y_m\}$. Thus, the superalgebra L has nilindex less than $n + m$. □

From Proposition 3.5 we conclude that the Leibniz superalgebra $L = L_0 \oplus L_1$ with characteristic sequence $(n_1, \dots, n_k|m)$ and nilindex $n+m$ can appear only if $\dim(L_0^3) \geq n-3$. Taking into account the condition $n_1 \leq n-2$ and the properties of the naturally graded subspaces $\mathrm{gr}(L_0)_1$ and $\mathrm{gr}(L_0)_2$ we get $\dim(L_0^3) = n - 3$. Then

$$\mathrm{gr}(L_0)_1 = \{\overline{x}_1, \overline{x}_2\}, \ \mathrm{gr}(L_0)_2 = \{\overline{x}_3\}.$$

Therefore, by Corollary 3.4, the naturally graded Leibniz algebra $\mathrm{gr}(L_0)$ is a Lie algebra, i.e., the following multiplication rules are true:

$$\begin{cases} [\overline{x}_1, \overline{x}_1] = 0, \\ [\overline{x}_2, \overline{x}_1] = \overline{x}_3, \\ [\overline{x}_1, \overline{x}_2] = -\overline{x}_3, \\ [\overline{x}_2, \overline{x}_2] = 0. \end{cases}$$

We have the following products of the Leibniz algebras L_0, with the basis $\{x_1, x_2, x_3, \dots, x_n\}$:

$$\begin{cases} [x_1, x_1] = \gamma_{1,4} x_4 + \gamma_{1,5} x_5 + \cdots + \gamma_{1,n} x_n, \\ [x_2, x_1] = x_3, \\ [x_1, x_2] = -x_3 + \gamma_{2,4} x_4 + \gamma_{2,5} x_5 + \cdots + \gamma_{2,n} x_n, \\ [x_2, x_2] = \gamma_{3,4} x_4 + \gamma_{3,5} x_5 + \cdots + \gamma_{3,n} x_n, \end{cases}$$

which have been obtained by the extension of the products of $\mathrm{gr}(L_0)$.

Proposition 3.6. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from* $\mathrm{Leib}_{n,m}$ *with characteristic sequence* $(n_1, \dots, n_k|m)$ *and* $\dim(L_0^3) = n - 3$. *Then L has nilindex less than $n + m$.*

Proof. Let us suppose the opposite, i.e., the nilindex of the superalgebra L is equal to $n+m$. Then by Theorem 3.2 we can assume $x_2 \notin L^3$. Hence,

$$L^2 = \{x_2, x_3, \ldots, x_n, y_2, y_3, \ldots, y_m\},$$
$$L^3 = \{x_3, x_4, \ldots, x_n, y_2, y_3, \ldots, y_m\}.$$

If $y_2 \in L^4$, then it should be generated from the products $[x_i, y_1]$ for $3 \leq i \leq n$, but the elements x_i $(3 \leq i \leq n)$ are in L_0^2. Therefore, they are generated by linear combinations of the products of the elements from L_0. The equalities

$$\begin{aligned}[][[x_i, x_j], y_1] &= [x_i, [x_j, y_1]] + [[x_i, y_1], x_j] \\ &= \left[x_i, \sum_{t=2}^{m} \alpha_{j,t} y_t\right] + \left[\sum_{t=2}^{m} \alpha_{i,t} y_t, x_i\right] \\ &= \sum_{t \geq 3} (*) y_t\end{aligned}$$

show that the element y_2 cannot be obtained by the products $[x_i, y_1]$ for $3 \leq i \leq n$, i.e. $y_2 \notin L^4$. Thus, we have

$$L^4 = \{x_3, x_4, \ldots, x_n, y_3, \ldots, y_m\}.$$

A simple analysis of the descending central sequences L^3 and L^4 derives $\alpha_{2,2} \neq 0$.

Let s be a natural number such that $x_3 \in L^{s+1} \setminus L^{s+2}$, i.e.

$$\begin{aligned} L^s &= \{x_3, x_4, \ldots, x_n, y_{s-1}, y_s, \ldots, y_m\}, \quad s \geq 3, \\ L^{s+1} &= \{x_3, x_4, \ldots, x_n, y_s, y_{s+1}, \ldots, y_m\}, \\ L^{s+2} &= \{x_4, \ldots, x_n, y_s, y_{s+1}, \ldots, y_m\} \quad \text{and} \quad \beta_{s-1,3} \neq 0. \end{aligned}$$

If $s = 3$, then $\beta_{2,3} \neq 0$ and we can write the product

$$\begin{aligned}[][[x_2, y_1], y_1] &= \left[\sum_{j=2}^{m} \alpha_{2,j} y_j, y_1\right] = \sum_{j=2}^{m} \alpha_{2,j} [y_j, y_1] \\ &= \alpha_{2,2}\beta_{2,3} x_3 + \sum_{i \geq 4} (*) x_4.\end{aligned}$$

On the other hand,

$$[[x_2, y_1], y_1] = \frac{1}{2}[x_2, [y_1, y_1]] = \frac{1}{2}\left[x_2, \sum_{i=2}^{n} \beta_{1,i} x_i\right] = \sum_{i \geq 4} (*) x_i.$$

Comparing the coefficients with the corresponding basic elements we get the equality $\alpha_{2,2}\beta_{2,3} = 0$, which contradicts the supposition $s = 3$.

If $s \geq 4$, then we consider the chain of equalities

$$\begin{aligned}
[y_{s-2}, y_2] &= [y_{s-2}, [y_1, x_1]] = [[y_{s-2}, y_1], x_1] - [[y_{s-2}, x_1], y_1] \\
&= \left[\sum_{i=3}^{n} \beta_{s-2,i} x_i, x_1\right] - [y_{s-1}, y_1] \\
&= -\beta_{s-1,3}x_3 + \sum_{i\geq 4}(*)x_i.
\end{aligned}$$

Since $y_{s-2} \in L^{s-1}$ and $y_2 \in L^3$, we get $x_3 \in L^{s+2} = \{x_4, \dots, x_n, y_{s-1}, \dots, y_m\}$. This is a contradiction with the assumption that the nilindex of L is equal to $n+m$. □

To conclude, taking into account Theorem 3.2 and Propositions 3.5 and 3.6, we have the following result.

Theorem 3.7. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from* $\mathrm{Leib}_{n,m}$ *with characteristic sequence equal to $(n_1, \dots, n_k|m)$. Then the nilindex of L is less than $n+m$.*

Bibliography

[1] S. Albeverio, Sh. A. Ayupov and B. A. Omirov, On nilpotent and simple Leibniz algebras, *Comm. Algebra* **33** (2005), no. 1, 159–172.

[2] Sh. A. Ayupov, A. Kh. Khudoyberdiyev and B. A. Omirov, The classification of filiform Leibniz superalgebras of nilindex $n+m$, *Acta Math. Sinica (english series)* **25** (2009), no. 2, 171–190.

[3] Sh. A. Ayupov and B. A. Omirov, On some classes of nilpotent Leibniz algebras, *Sib. Math. J.* **42** (2001), no. 1, 18–29.

[4] M. Bordemann, J. R. Gómez, Yu. Khakimdjanov and R. M. Navarro, Some deformations of nilpotent Lie superalgebras, *J. Geom. Phys.* **57** (2007), 1391–1403.

[5] L. M. Camacho, J. R. Gómez, B. A. Omirov and A. Kh. Khudoyberdiyev, On complex Leibniz superalgebras of nilindex $n+m$, submitted to *Algebras and Representation Theory*, arXiv:math/0812.2156.

[6] A. S. Dzhumadil'daev, Cohomologies of colour Leibniz algebras: pre-simplicial approach, in: *Lie Theory and Its Applications in Physics III* (Proceedings of the Third International Workshop, Clausthal, 1999), pp. 124–135, edited by H.-D. Doebner, V. K. Dobrev and J. Hilgert, World Scientific, Singapore, 2000.

[7] M. Gilg, *Super-algèbres de Lie nilpotentes*, PhD thesis, University of Haute Alsace, 2000.

[8] J. R. Gómez, Yu. Khakimdjanov and R. M. Navarro, Some problems concerning to nilpotent Lie superalgebras, *J. Geom. Phys.* **51** (2004), no. 4, 473–486.

[9] D. Liu and N. Hu, Leibniz superalgebras and central extensions, *J. Alg. Appl.* **5** (2006), no. 6, 765–780.

[10] M. Livernet, Rational homotopy of Leibniz algebras, *Manuscripta Math.* **96** (1998), 295–315.

[11] J.-L. Loday, Une version non commutative des algèbres de Lie: les algèbres de Leibniz, *Ens. Math.* **39** (1993), 269–293.

[12] M. Vergne, Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes, *Bull. Soc. Math. France.* **98** (1970), 81–116.

Received November 23, 2009; revised July 5, 2010.

Author information

Luisa M. Camacho, Departamento Matemática Aplicada I, Universidad de Sevilla,
Avda. Reina Mercedes, s/n. 41012 Sevilla, Spain.
E-mail: lcamacho@us.es

Jose R. Gómez, Departamento Matemática Aplicada I, Universidad de Sevilla,
Avda. Reina Mercedes, s/n. 41012 Sevilla, Spain.
E-mail: jrgomez@us.es

Abror Kh. Khudoyberdiyev, Institute of Mathematics and Information Technologies of
Academy of Uzbekistan, 29, F. Hodjaev srt., 100125, Tashkent, Uzbekistan.
E-mail: khabror@mail.ru

Bakhrom A. Omirov, Institute of Mathematics and Information Technologies of
Academy of Uzbekistan, 29, F. Hodjaev srt., 100125, Tashkent, Uzbekistan.
E-mail: omirovb@mail.ru