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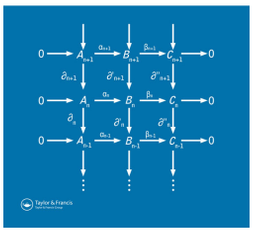

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# Classification of solvable Leibniz algebras with abelian nilradical and $(k-1)$-dimensional extension

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**ABSTRACT**
This work is devoted to the classification of solvable Leibniz algebras with an abelian nilradical. We consider $(k-1)$-dimensional extension of $k$-dimensional abelian algebras and classify all $(2k-1)$-dimensional solvable Leibniz algebras with an abelian nilradical of dimension $k$.

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## 1. Introduction

Leibniz algebras, a "noncommutative version" of Lie algebras, were first introduced in the mid-1960s by Blokh [5] under the name of "$D$-algebras." They appeared again in the 1990s after Loday's work [12], where he introduced calling them Leibniz algebras.

According to the structural theory of Lie algebras, a finite-dimensional Lie algebra can be written as a semidirect sum of its semisimple subalgebra and its solvable radical (Levi's theorem). The semisimple part is a direct sum of simple Lie algebras which were completely classified in the fifties of the last century (see [9]). In the case of Leibniz algebras, there is also an analog to Levi's theorem [3]. Namely, the decomposition of a Leibniz algebra into the semidirect sum of its solvable radical and a semisimple Lie algebra can be obtained. As above, the semisimple part can be composed by simple Lie algebras and the main issue in the classification problem of finite-dimensional complex Leibniz algebras is to study the solvable part. Therefore, the classification of solvable Leibniz algebras is important to construct finite-dimensional Leibniz algebras.

Owing to a result of [13], an approach to the study of solvable Lie algebras through the use of the nilradical was developed in [2, 14, 17], etc. In particular, in [14] solvable Lie algebras with abelian nilradicals are investigated.

The analog of Mubarakzjanov's result has been applied in the Leibniz algebra case in [7], showing the importance of consideration of the nilradical in the case of Leibniz algebras as well. The papers [6, 7, 11, 15, 16] also are devoted to the study of solvable Leibniz algebras by considering their nilradicals.

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It should be noted that any solvable Leibniz algebra $L$ with nilradical $N$ can be written as a direct sum of vector spaces $L = N \oplus Q$, where $Q$ is the complementary vector space to the nilradical. In [1, 4], solvable Leibniz algebras with an abelian nilradical are investigated. It was proven that the maximal dimension of a solvable Leibniz algebra with $k$-dimensional abelian nilradical is $2k$. Additionally, in [1] this maximal case was classified and some results regarding of the classification with one-dimensional extension were presented. In this paper, we give the classification of solvable Leibniz algebras with abelian nilradical and $(k-1)$-dimensional extension.

It should be noted that a solvable Leibniz algebra $L$ with condition $\dim Q = \dim(N/N^2)$ can be classified using the classification of solvable Leibniz algebras with a $k$-dimensional abelian nilradical and a $k$-dimensional complementary vector space.

The natural next step is the classification of solvable Leibniz algebra with condition $\dim Q = \dim(N/N^2) - 1$. In order to perform this classification, the classification of solvable Leibniz algebras with a $k$-dimensional abelian nilradical and a $(k-1)$-dimensional complementary vector space should first be obtained. In the case $k=2$ and $k=3$ we have three- and five-dimensional solvable Leibniz algebras, which were classified in [8] and [10], respectively. In this paper, we consider the case for any $k$, i.e., we classify all $(2k-1)$-dimensional solvable Leibniz algebras with $k$-dimensional abelian nilradical.

Throughout this paper all algebras (vector spaces) considered are finite-dimensional and over the field of complex numbers. Also, in the tables of multiplications of algebras, we give nontrivial products only.

## 2. Preliminaries

This section is devoted to recalling some basic notions and concepts used throughout the paper.

**Definition 2.1.** A $\mathbb{C}$-vector space with a bilinear bracket $(L, [\cdot,\cdot])$ is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y],$$

holds.

Here, we adopt the right Leibniz identity; since the bracket is not skew-symmetric, there exists the version corresponding to the left Leibniz identity

$$[[x,y],z] = [x,[y,z]] - [y,[x,z]].$$

Note that the notions of ideal and subalgebra are defined by the usual way. The sets $Ann_r(L) := \{x \in L : [L,x] = 0\}$ and $Ann_l(L) := \{x \in L : [x,L] = 0\}$ are called the *right* and *left annihilators* of $L$, respectively. It is observed that $Ann_r(L)$ is a two-sided ideal of $L$, and for any $x, y \in L$ the elements $[x,x]$ and $[x,y] + [y,x]$ are always in $Ann_r(L)$.

The set $C(L) := \{x \in L : [x,L] = [L,x] = 0\}$ is called the *center* of $L$.

For a given Leibniz algebra $(L, [\cdot,\cdot])$ the sequences of two-sided ideals is defined recursively as follows:

$$L^1 := L, L^{k+1} := \left[L^k, L\right], k \geq 1, \qquad L^{[1]} := L, L^{[s+1]} := \left[L^{[s]}, L^{[s]}\right], \ s \geq 1.$$

These are said to be the *lower central* and the *derived* series of $L$, respectively.

**Definition 2.2.** A Leibniz algebra $L$ is said to be nilpotent (respectively, solvable), if there exists $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^{[m]} = 0$).

**Definition 2.3.** An ideal of a Leibniz algebra is called nilpotent if it is nilpotent as a subalgebra.

It is well-known that the sum of any two nilpotent ideals is nilpotent. Therefore the maximal nilpotent ideal always exists.

**Definition 2.4.** The maximal nilpotent ideal $N$ of a Leibniz algebra $L$ is said to be the nilradical of the algebra.

**Definition 2.5.** A linear map $d : L \to L$ of a Leibniz algebra $(L, [\cdot,\cdot])$ is said to be a derivation if for all $x, y \in L$, the following condition holds:

$$d([x,y]) = [d(x), y] + [x, d(y)].$$

The set of all derivations of $L$ is denoted by $\mathrm{Der}(L)$ and it is a Lie algebra with respect to the commutator.

For a given element $x$ of a Leibniz algebra $L$, the right multiplication operator $R_x : L \to L$, defined by $R_x(y) = [y, x], y \in L$ is a derivation. In fact, Leibniz algebras are characterized by this property regarding right multiplication operators. (Recall that left Leibniz algebras are characterized by the same property with left multiplication operators.) As in the Lie case, such kind of derivations are said to be inner derivations.

**Definition 2.6.** Let $d_1, d_2, ..., d_n$ be derivations of a Leibniz algebra $L$. The derivations $d_1, d_2, ..., d_n$ are said to be linearly nil-independent if for $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C}$ and a natural number $k$,

$$(\alpha_1 d_1 + \alpha_2 d_2 + \cdots + \alpha_n d_n)^k = 0 \textit{implies } \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Note that in the above definition the power is understood with respect to composition.

Let $L$ be a solvable Leibniz algebra. Then it can be written in the form $L = N \oplus Q$, where $N$ is the nilradical and $Q$ is the complementary vector subspace. The following is a result from [7] on the dimension of $Q$ which we make use of in the paper.

**Theorem 2.7.** *Let $L$ be a solvable Leibniz algebra and $N$ be its nilradical. Then the dimension of $Q$ is not greater than the maximal number of nil-independent derivations of $N$.*

## 3. Main result

We denote by $\mathbf{a_k}$ a $k$-dimensional abelian algebra and by $R(\mathbf{a_k}, s)$ the class of solvable Leibniz algebras with $k$-dimensional abelian nilradical $N$ and $s$-dimensional complementary vector space $Q$.

As above, it has been proven that $s \leq k$ for any algebra from the class $R(\mathbf{a_k}, s)$, and in [1] the classification of such algebras of $R(\mathbf{a_k}, k)$ is given. It is proven that an arbitrary algebra from the family $R(\mathbf{a_k}, k)$ can be decomposed into a direct sum of copies of two-dimensional non-trivial solvable Leibniz algebras.

It is proven that an arbitrary algebra $L$ from the family $R(\mathbf{a_k}, k)$ is

$$L = l_2 \oplus l_2 \oplus \cdots \oplus l_2 \oplus r_2 \oplus r_2 \oplus \cdots \oplus r_2,$$

where $l_2 : [e, x] = e$ and $r_2 : [e, x] = -[x, e] = e$.

Let $L$ be a Leibniz algebra from the class $R(\mathbf{a_k}, k-1)$. Take a basis $\{e_1, e_2, ..., e_k, x_1, x_2, ..., x_{k-1}\}$ of $L$ such that $\{e_1, e_2, ..., e_k\}$ is a basis of nilradical $N$ and $\{x_1, x_2, ..., x_{k-1}\}$ is a basis of the complementary vector space $Q$. It is known that the right multiplication operators $R_{x_1}, R_{x_2}, ..., R_{x_{k-1}} : N \to N$ are nil-independent derivations [7] and there exists a basis of $N$, for an easier notation, suppose again $\{e_1, e_2, ..., e_k\}$, such that operators $R_{x_1}, R_{x_2}, ..., R_{x_{k-1}}$ simultaneously have the Jordan normal form.

Because of this, observe that:

$$\begin{cases} [e_i, x_j] = a_{i,j}e_i + \beta_{i,j}e_{i+1}, & 1 \le i,j \le k-1, \\ [e_k, x_j] = a_{k,j}e_k & 1 \le j \le k-1, \end{cases}$$

where $a_{i,j}$ are eigenvalues of the operator $R_{x_j}$ and $\beta_{i,j} \in \{0,1\}$.

Since $R_{x_1}, R_{x_2}, \ldots, R_{x_{k-1}}$ are nil-independent we have that

$$rank\begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,k-1} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,k-1} \\ \ldots & \ldots & \ldots & \ldots \\ a_{k-1,1} & a_{k-1,2} & \ldots & a_{k-1,k-1} \\ a_{k,1} & a_{k,2} & \ldots & a_{k,k-1} \end{pmatrix} = k-1.$$

Thus, there exists a minor $M^{k-1}_{i_1,i_2\ldots,i_{k-1}}$ of order $k-1$ which has a non-zero determinant, i.e., there exists $t$ such that $det(M^{k-1}_{1,\ldots,t-1,t+1,\ldots,k}) \neq 0$. Making the change of basis

$$\begin{cases} e'_i = e_i, & 1 \le i \le t-1, \\ e'_i = e_{i+1}, & t \le i \le k-1, \\ e'_k = e_t, \end{cases}$$

we get that

$$\begin{cases} [e_i, x_j] = a_{i,j}e_i + \beta_{i,j}e_{i+1}, & 1 \le i,j \le k-1, i \neq t-1, \\ [e_{t-1}, x_j] = a_{t-1,j}e_{t-1} + \beta_{t-1,j}e_k, & 1 \le j \le k-1, \\ [e_k, x_j] = a_{k,j}e_k + \beta_{k,j}e_t, & 1 \le j \le k-1. \end{cases}$$

It should be noted that operators $R_{x_1}, R_{x_2}, \ldots, R_{x_{k-1}}$ can be considered linearly nil-independent operators on the quotient vector space $\mathbf{a_k}/\langle e_k \rangle$. Since $\dim(\mathbf{a_k}/\langle e_k \rangle) = k-1$ from the result of [1] we obtain that

$$a_{i,i} = 1, \quad 1 \le i \le k-1, \qquad a_{i,j} = 0, \quad 1 \le i,j \le k-1, \quad i \neq j,$$
$$\beta_{i,j} = 0, \quad 1 \le i \le k, \quad 1 \le j \le k-1, \quad i \neq t-1.$$

Let us introduce the following notation:

$$[x_i, e_j] = \sum_{p=1}^{k} \gamma^p_{i,j} e_p, \quad 1 \le i \le k-1, \quad 1 \le j \le k,$$

$$[x_i, x_j] = \sum_{p=1}^{k} \delta^p_{i,j} e_p \quad 1 \le i,j \le k-1,$$

where $\gamma^p_{i,j}, \delta^p_{i,j} \in \mathbb{C}$.

Using the similar algorithms of the proof of Theorem 3.2 in [1], which was given the classification of solvable Leibniz algebras $R(\mathbf{a_k}, k)$, from Leibniz identities and basis changes we obtain that

$$\begin{aligned} &\gamma^p_{i,j} = 0, && 1 \le i,j,p \le k-1, \quad i \neq j \neq p, \\ &\gamma^i_{i,i} \in \{0, -1\}, && 1 \le i \le k-1, \\ &\delta^p_{i,j} = 0, && 1 \le i,j,p \le k-1. \end{aligned}$$

Therefore, the multiplication of the $(2k-1)$-dimensional solvable Leibniz algebras with $k$-dimensional abelian nilradical $N$ has the following form:

$$R(\mathbf{a_k}, k-1) = \begin{cases} [e_i, x_i] = e_i + \beta_{i,i} e_k, & 1 \le i \le k-1, \\ [e_i, x_j] = \beta_{i,j} e_k, & 1 \le i \le k, \quad 1 \le j \le k-1, \quad i \ne j, \\ [x_i, e_i] = \alpha_i e_i + \gamma_{i,i} e_k, & 1 \le i \le k-1, \\ [x_i, e_j] = \gamma_{i,j} e_k, & 1 \le i \le k-1, \quad 1 \le j \le k-1, \quad i \ne j, \\ [x_i, e_k] = \sum_{j=1}^{k} \nu_{i,j} e_j, & 1 \le i \le k-1, \\ [x_i, x_j] = \delta_{i,j} e_k, & 1 \le i,j \le k-1, \end{cases} \tag{3.1}$$

where $\alpha_i \in \{0, -1\}$.

First we investigate the case of $\alpha_i = 0$ for all $1 \le i \le k-1$.

**Theorem 3.1.** *Let $L$ be a Leibniz algebra from the class $R(\mathbf{a_k}, k-1)$ and let $\alpha_i = 0$ for $1 \le i \le k-1$. Then $L$ is isomorphic to one of the following algebras:*

$$L_1(\beta_i): \begin{cases} [e_i, x_i] = e_i, & 1 \le i \le k-1, \\ [e_k, x_i] = \beta_i e_k, & 1 \le i \le k-1, \end{cases} \quad L_2(\beta_i): \begin{cases} [e_i, x_i] = e_i, & 1 \le i \le k-1, \\ [e_k, x_i] = \beta_i e_k, & 1 \le i \le k-1, \\ [x_i, e_k] = -\beta_i e_k, & 1 \le i \le k-1, \end{cases}$$

$$L_3(\beta_i): \begin{cases} [e_1, x_1] = e_1 + \beta_1 e_k, & \\ [e_i, x_i] = e_i, & 2 \le i \le k-1, \\ [e_1, x_i] = \beta_i e_k, & 2 \le i \le k-1, \\ [e_k, x_1] = e_k, & \end{cases} \quad L_4(\nu_i): \begin{cases} [e_i, x_i] = e_i, & 1 \le i \le k-1, \\ [e_k, x_1] = e_k, & \\ [x_1, e_k] = -e_k, & \\ [x_i, e_k] = \nu_i e_1, & 2 \le i \le k-1, \end{cases}$$

$$L_5(\delta_{i,j}): \begin{cases} [e_i, x_i] = e_i, & 1 \le i \le k-1, \\ [x_i, x_j] = \delta_{i,j} e_k, & 1 \le i,j \le k-1. \end{cases}$$

*Proof.* Let $\alpha_i = 0$ for $1 \le i \le k-1$, then the multiplication (3.1) has the form

$$\begin{cases} [e_i, x_i] = e_i + \beta_{i,i} e_k, & 1 \le i \le k-1, \\ [e_i, x_j] = \beta_{i,j} e_k, & 1 \le i \le k, \quad 1 \le j \le k-1, \quad i \ne j, \\ [x_i, e_j] = \gamma_{i,j} e_k, & 1 \le i \le k-1, \quad 1 \le j \le k-1, \\ [x_i, e_k] = \sum_{j=1}^{k} \nu_{i,j} e_j, & 1 \le i \le k-1, \\ [x_i, x_j] = \delta_{i,j} e_k, & 1 \le i,j \le k-1. \end{cases}$$

**Case 1.** Let there exist $i_0 \in \{1, 2, \ldots, k\}$, such that $\beta_{k,i_0} \notin \{0, 1\}$. Without loss of generality, we may assume $i_0 = 1$. Making the change of basis

$$e_1' = e_1 - \frac{\beta_{1,1}}{\beta_{k,1} - 1} e_k, \qquad e_i' = e_i - \frac{\beta_{i,1}}{\beta_{k,1}} e_k, \quad 2 \le i \le k-1,$$

we get

$$\left[e_1', x_1\right] = \left[e_1 - \frac{\beta_{1,1}}{\beta_{k,1}-1}e_k, x_1\right] = e_1 + \beta_{1,1}e_k - \frac{\beta_{1,1}}{\beta_{k,1}-1}\beta_{k,1}e_k = e_1 - \frac{\beta_{1,1}}{\beta_{k,1}-1}e_k = e_1'.$$

$$\left[e_i', x_1\right] = \left[e_i - \frac{\beta_{i,1}}{\beta_{k,1}}e_k, x_1\right] = \beta_{i,1}e_k - \beta_{i,1}e_k = 0, \quad 2 \le i \le k-1.$$

Thus, we obtain that $[e_1, x_1] = e_1$ and $[e_i, x_1] = 0$ for $2 \le i \le k-1$. Then using the Leibniz identity for $2 \le i, j (i \ne j) \le n$, we have:

$$0 = [e_i, [x_1, x_j]] = [[e_i, x_1], x_j] - [[e_i, x_j], x_1] = -\beta_{i,j}[e_k, x_1] = -\beta_{i,j}\beta_{k,1}e_k,$$
$$0 = [e_j, [x_1, x_j]] = [[e_j, x_1], x_j] - [[e_j, x_j], x_1] = -\left[e_j + \beta_{j,j}e_k, x_1\right] = -\beta_{j,j}\beta_{k,1}e_k.$$

Hence $\beta_{i,j}\beta_{k,1} = 0$, $\beta_{j,j}\beta_{k,1} = 0$. Since $\beta_{k,1} \ne 0$, we get that

$$\beta_{i,j} = 0, \quad 2 \le i \le k-1, \quad 2 \le j \le k-1.$$

Next we consider

$$0 = [e_1, [x_1, x_j]] = [[e_1, x_1], x_j] - [[e_1, x_j], x_1] = [e_1, x_j] - \left[\beta_{1,j}e_k, x_1\right] = \beta_{1,j}e_k - \beta_{1,j}\beta_{k,1}e_k = \beta_{1,j}(1-\beta_{k,1})e_k.$$

Since $\beta_{k,1} \ne 1$ we get $\beta_{1,j} = 0$ for $2 \le j \le k-1$. Thus, the multiplication has the following form:

$$\begin{cases} [e_i, x_i] = e_i, & 1 \le i \le k-1, \\ [e_k, x_i] = \beta_{k,i}e_k, & 1 \le i \le k-1, \\ [x_i, e_j] = \gamma_{i,j}e_k, & 1 \le i \le k-1, \quad 1 \le j \le k-1, \\ [x_i, e_k] = \sum\limits_{j=1}^{k}\nu_{i,j}e_j, & 1 \le i \le k-1, \\ [x_i, x_j] = \delta_{i,j}e_k, & 1 \le i, j \le k-1. \end{cases}$$

Now we consider the Leibniz identity for the triple of elements $\{x_i, e_j, x_1\}$ for $1 \le i \le k-1$ and $1 \le j \le k-1$. Then

$$0 = [x_i, [e_1, x_1]] - [[x_i, e_1], x_1] + [[x_i, x_1], e_1] = [x_i, e_1] - \gamma_{i,1}[e_k, x_1] = \gamma_{i,1}(1-\beta_{k,1})e_k,$$
$$0 = [x_i, [e_j, x_1]] - [[x_i, e_j], x_1] + [[x_i, x_1], e_j] = -\gamma_{i,j}[e_k, x_1] = -\gamma_{i,j}\beta_{k,1}e_k, \quad 2 \le j \le k-1.$$

Since $\beta_{k,1} \notin \{0, 1\}$, we have $\gamma_{i,j} = 0$ for $1 \le i, j \le k-1$.

Using the Leibniz identity, we have:

$$0 = [x_i, [e_k, x_1]] - [[x_i, e_k], x_1] + [[x_i, x_1], e_k] = \beta_{k,1}[x_i, e_k] - \left[\sum_{j=1}^{k}\nu_{i,j}e_j, x_1\right]$$
$$= \beta_{k,1}\sum_{j=1}^{k}\nu_{i,j}e_j - \nu_{i,1}e_1 - \nu_{i,k}\beta_{k,1}e_k = (\beta_{k,1}-1)\nu_{i,1}e_1 + \beta_{k,1}\sum_{j=2}^{k-1}\nu_{i,j}e_j.$$

Hence $\nu_{i,j} = 0$ for $1 \le i, j \le k-1$.

From

$$0 = [x_i, [x_j, e_k]] - [[x_i, x_j], e_k] + [[x_i, e_k], x_j] = \nu_{j,k}[x_i, e_k] + \nu_{i,k}[e_k, x_j] = \nu_{i,k}(\nu_{j,k} + \beta_{k,j})e_k,$$

we get

$$\nu_{i,k}(\nu_{j,k} + \beta_{k,j}) = 0, \quad 1 \le i, j \le k-1. \tag{3.2}$$

Taking the basis change $x_i' = x_i - \frac{\delta_{i,1}}{\beta_{k,1}} e_k$ for $2 \leq i \leq k-1$, we obtain

$$\left[x_i', x_1\right] = \left[x_i - \frac{\delta_{i,1}}{\beta_{k,1}} e_k, x_1\right] = \delta_{i,1} e_k - \delta_{i,1} e_k = 0, \quad 2 \leq i \leq k-1.$$

Thus, we can assume $\delta_{i,1} = 0$ for $2 \leq i \leq k-1$.

Using the Leibniz identity

$$0 = [x_i, [x_1, x_1]] - [[x_i, x_1], x_1] + [[x_i, x_1], x_1] = \delta_{1,1}[x_i, e_k] = \nu_{i,k}\delta_{1,1} e_k$$

we have

$$\nu_{i,k}\delta_{1,1} = 0, \quad 1 \leq i \leq k-1. \tag{3.3}$$

Another application of the Leibniz identity gives:

$$0 = [x_i, [x_j, x_1]] - [[x_i, x_j], x_1] + [[x_i, x_1], x_j] = -\delta_{i,j}\beta_{k,1} e_k,$$

which implies $\delta_{i,j} = 0$ for $2 \leq i,j \leq k-1$.

From

$$0 = [x_1, [x_j, x_1]] - [[x_1, x_j], x_1] + [[x_1, x_1], x_j] = -\delta_{1,j}[e_k, x_1] + \delta_{1,1}[e_k, x_j] = (-\delta_{1,j}\beta_{k,1} + \delta_{1,1}\beta_{k,j}) e_k,$$

we have $\delta_{1,j} = \frac{\beta_{k,j}}{\beta_{k,1}}\delta_{1,1}$ for $2 \leq j \leq k-1$.

Therefore, we have the following table of multiplications:

$$\begin{cases} [e_i, x_i] = e_i, & 1 \leq i \leq k-1, \\ [e_k, x_i] = \beta_{k,i} e_k, & 1 \leq i \leq k-1, \\ [x_i, e_k] = \nu_{i,k} e_k, & 1 \leq i \leq k-1, \\ [x_1, x_j] = \dfrac{\beta_{k,j}}{\beta_{k,1}}\delta_{1,1} e_k, & 1 \leq j \leq k-1. \end{cases}$$

Let $\nu_{i,k} = 0$ for all $i (1 \leq i \leq k-1)$. Then taking the change $x_1' = x_1 - \frac{\delta_{1,1}}{\beta_{k,1}} e_k$ we have:

$$\left[x_1', x_1'\right] = \left[x_1 - \frac{\delta_{1,1}}{\beta_{k,1}} e_k, x_1 - \frac{\delta_{1,1}}{\beta_{k,1}} e_k\right] = \delta_{1,1} e_k - \delta_{1,1} e_k = 0$$

$$\left[x_1', x_j\right] = \left[x_1 - \frac{\delta_{1,1}}{\beta_{k,1}} e_k, x_j\right] = \frac{\beta_{k,j}}{\beta_{k,1}}\delta_{1,1} e_k - \frac{\delta_{1,1}}{\beta_{k,1}}\beta_{k,j} e_k = 0.$$

Thus, we obtain the algebra $L_1(\beta_i)$, with $\beta_1 \notin \{0,1\}$.

Let there exist $i (1 \leq i \leq k-1)$ such that $\nu_{i,k} \neq 0$. According to the equalities (3.2) and (3.3) we have $\nu_{i,k} = -\beta_{k,i}$ and $\delta_{1,1} = 0$, which implies that $\delta_{1,j} = 0$ for $2 \leq j \leq k-1$. Thus we have the algebra $L_2(\beta_i)$, with $\beta_1 \notin \{0,1\}$.

**Case 2.** Let $\beta_{k,1}, \beta_{k,2}, \ldots, \beta_{k,k-1} \in \{0,1\}$ and $(\beta_{k,1}, \beta_{k,2}, \ldots, \beta_{k,k-1}) \neq (0,0,\ldots,0)$. Without loss of generality, rearrange the basis elements such that the non-zero $\beta_{k,i}$ are the first $s$, where $1 \leq s \leq k-1$. So we have $\beta_{k,j} = 1, 1 \leq j \leq s$ and $\beta_{k,j} = 0, s+1 \leq j \leq k-1$.

In this case the table of multiplication (3.1) is

$$\begin{cases} [e_i, x_i] = e_i + \beta_{i,i} e_k, & 1 \le i \le k-1, \\ [e_i, x_j] = \beta_{i,j} e_k, & 1 \le i \le k-1, \quad 1 \le j \le k-1, \quad i \ne j, \\ [e_k, x_j] = e_k, & 1 \le j \le s, \\ [x_i, e_j] = \gamma_{i,j} e_k, & 1 \le i \le k-1, \quad 1 \le j \le k-1, \\ [x_i, e_k] = \sum\limits_{j=1}^{k} \nu_{i,j} e_j, & 1 \le i \le k-1, \\ [x_i, x_j] = \delta_{i,j} e_k, & 1 \le i,j \le k-1. \end{cases}$$

Changing the basis, let $e'_i = e_i - \beta_{i,1} e_k$ for $2 \le i \le k-1$ we have

$$\left[e'_i, x_1\right] = \left[e_i - \beta_{i,1} e_k, x_1\right] = \beta_{i,1} e_k - \beta_{i,1} e_k = 0, \quad 2 \le i \le k-1.$$

Thus, we may suppose $\beta_{i,1} = 0,\ 2 \le i \le k-1$.

Using the Leibniz identities

$$0 = [e_i, [x_i, x_1]] - [[e_i, x_i], x_1] + [[e_i, x_1], x_i] = -\left[e_i + \beta_{i,i} e_k, x_1\right] = -\beta_{i,i} e_k,$$
$$0 = [e_i, [x_j, x_1]] - [[e_i, x_j], x_1] + [[e_i, x_1], x_j] = -\beta_{i,j} e_k,$$

we have $\beta_{i,j} = 0$ for $2 \le i,j \le k-1$.

Next, from

$$0 = [x_i, [e_j, x_1]] - [[x_i, e_j], x_1] + [[x_i, x_1], e_j] = -\gamma_{i,j}[e_k, x_1] = -\gamma_{i,j} e_k,$$

we obtain $\gamma_{i,j} = 0$ for $1 \le i \le k-1, 2 \le j \le k-1$.

First, consider:

$$0 = [x_1, [e_k, x_1]] - [[x_1, e_k], x_1] + [[x_1, x_1], e_k] = [x_1, e_k] - \left[\sum_{j=1}^{k} \nu_{1,j} e_j, x_1\right] =$$
$$= \sum_{j=1}^{k} \nu_{1,j} e_j - \nu_{1,1}(e_1 + \beta_{1,1} e_k) - \nu_{1,k} e_k = \nu_{1,2} e_2 + \cdots + \nu_{1,k-1} e_{k-1} - \nu_{1,1}\beta_{1,1} e_k.$$

Hence

$$\nu_{1,1}\beta_{1,1} = 0, \qquad \nu_{1,i} = 0, \quad 2 \le i \le k-1.$$

Now consider the Leibniz identity for the triples $\{x_i, e_k, x_i\}$ and $\{x_i, e_k, x_1\}$ with $2 \le i \le s$. We have

$$0 = [x_i, [e_k, x_i]] - [[x_i, e_k], x_i] + [[x_i, x_i], e_k] = \sum_{j=1}^{k} \nu_{i,j} e_j - \left[\sum_{j=1}^{k} \nu_{i,j} e_j, x_i\right] =$$
$$= \sum_{j=1}^{k} \nu_{i,j} e_j - \nu_{i,1}\beta_{1,i} e_k - \nu_{i,i} e_i - \nu_{i,k} e_k = \sum_{j=1, j\ne i}^{k-1} \nu_{i,j} e_j - \nu_{i,1}\beta_{1,i} e_k,$$

which implies

$$\nu_{i,1} = \cdots = \nu_{i,i-1} = \nu_{i,i+1} = \cdots = \nu_{i,k-1} = 0.$$

Next from

$$0 = [x_i, [e_k, x_1]] - [[x_i, e_k], x_1] + [[x_i, x_1], e_k] = [x_i, e_k] - [\nu_{i,i} e_i + \nu_{i,k} e_k, x_1] = \nu_{i,i} e_i,$$

we get $\nu_{i,i} = 0$ for $2 \le i \le s$. Thus, we obtain $[x_i, e_k] = \nu_{i,k} e_k$ for $2 \le i \le s$.

For $s+1 \leq i \leq k-1$, we have

$$0 = [x_i,[e_k,x_i]] - [[x_i,e_k],x_i] + [[x_i,x_i],e_k] = -\left[\sum_{j=1}^{k}\nu_{i,j}e_j, x_i\right] = -\nu_{i,1}\beta_{1,i}e_k - \nu_{i,i}e_i,$$

$$0 = [x_i,[e_k,x_1]] - [[x_i,e_k],x_1] + [[x_i,x_1],e_k] = [x_i,e_k] - \left[\sum_{j=1}^{k}\nu_{i,j}e_j, x_1\right] = \sum_{j=2}^{k-1}\nu_{i,j}e_j - \nu_{i,1}\beta_{1,1}e_k.$$

which implies

$$\nu_{i,1}\beta_{1,i} = 0, \quad \nu_{i,1}\beta_{1,1} = 0, \quad \nu_{i,j} = 0 \quad \text{for} \quad s < i \leq k-1, \quad 2 \leq j \leq k-1.$$

Hence $[x_i,e_k] = \nu_{i,1}e_1 + \nu_{i,k}e_k$ for $s+1 \leq i \leq k-1$.

With the basis change $x_i' = x_i - \delta_{i,1}e_k$ for $2 \leq i \leq k-1$ we have:

$$[x_i', x_1] = [x_i - \delta_{i,1}e_k, x_1] = 0.$$

Then, from

$$0 = [x_i,[x_j,x_1]] - [[x_i,x_j],x_1] + [[x_i,x_1],x_j] = -\delta_{i,j}[e_k,x_1] = -\delta_{i,j}e_k,$$

we obtain $\delta_{i,j} = 0$ for $2 \leq i,j \leq k-1$.

Therefore, the table of multiplications is:

$$\begin{cases} [e_1,x_1] = e_1 + \beta_{1,1}e_k, & \\ [e_i,x_i] = e_i, & 2 \leq i \leq k-1, \\ [e_1,x_j] = \beta_{1,j}e_k, & 2 \leq j \leq k-1, \\ [e_k,x_j] = e_k, & 1 \leq j \leq s, \\ [x_i,e_1] = \gamma_{i,1}e_k, & 1 \leq i \leq k-1, \\ [x_1,e_k] = \nu_{1,1}e_1 + \nu_{1,k}e_k, & \\ [x_i,e_k] = \nu_{i,k}e_k, & 2 \leq i \leq s, \\ [x_i,e_k] = \nu_{i,1}e_1 + \nu_{i,k}e_k, & s+1 \leq i \leq k-1, \\ [x_1,x_j] = \delta_{1,j}e_k, & 1 \leq j \leq k-1, \end{cases}$$

where $\nu_{1,1}\beta_{1,1} = 0$, $\nu_{i,1}\beta_{1,1} = 0$, for $s < i \leq k-1$.

**Case 2.1.** Let $s \geq 2$, then using the Leibniz identities

$$0 = [e_1,[x_1,x_2]] - [[e_1,x_1],x_2] + [[e_1,x_2],x_1] = -[e_1 + \beta_{1,1}e_k, x_2] + \beta_{1,2}[e_k,x_1] = -\beta_{1,1}e_k,$$
$$0 = [e_1,[x_i,x_2]] - [[e_1,x_i],x_2] + [[e_1,x_2],x_i] = -\beta_{1,i}[e_k,x_2] + \beta_{1,2}[e_k,x_i] = (-\beta_{1,i} + \beta_{1,2})e_k, \quad 2 \leq i \leq s,$$
$$0 = [e_1,[x_j,x_2] - [[e_1,x_j],x_2] + [[e_1,x_2],x_j] = -\beta_{1,j}[e_k,x_2] + \beta_{1,2}[e_k,x_j] = -\beta_{1,j}e_k, \quad s+1 \leq j \leq k-1,$$
$$0 = [x_1,[e_k,x_2] - [[x_1,e_k],x_2] + [[x_1,x_2],e_k] = [x_1,e_k] - [\nu_{1,1}e_1 + \nu_{1,k}e_k, x_2] = \nu_{1,1}e_1 - \nu_{1,1}\beta_{1,2}e_k,$$
$$0 = [x_i,[e_k,x_2]] - [[x_i,e_k],x_2] + [[x_i,x_2],e_k] = [x_i,e_k] - [\nu_{i,1}e_1 + \nu_{i,k}e_k, x_2] = \nu_{i,1}e_1 - \nu_{i,1}\beta_{1,2}e_k, \quad s+1 \leq i \leq k-1,$$

we obtain

$$\beta_{1,1} = 0, \quad \beta_{1,i} = \beta_{1,2}, \qquad 2 \leq i \leq s, \qquad \beta_{1,j} = 0, \quad s+1 \leq j \leq k-1,$$
$$\nu_{1,1} = 0, \qquad \nu_{i,1} = 0, \qquad s+1 \leq i \leq k-1.$$

Making the basis change $e_1' = e_1 - \beta_{1,2}e_k$ we obtain $[e_1,x_i] = 0$, $2 \leq i \leq s$, and from the Leibniz identity

$$0 = [x_i,[e_1,x_2]] - [[x_i,e_1],x_2] + [[x_i,x_2],e_1] = -\gamma_{i,1}[e_k,x_2] = -\gamma_{i,1}e_k, \quad 1 \leq i \leq k-1,$$

we get

$$\gamma_{i,1}=0,\quad 1\le i\le k-1.$$

Thus we have:

$$\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_j]=e_k, & 1\le j\le s,\\ [x_i,e_k]=\nu_{i,k}e_k, & 1\le i\le k-1,\\ [x_1,x_j]=\delta_{1,j}e_k, & 1\le j\le k-1.\end{cases}$$

**Case 2.1.1.** Let $e_k\in Ann_r(L)$, then $\nu_{i,k}=0$ with $1\le i\le k-1$. Making the change $x_1'=x_1-\delta_{1,1}e_k$, we may assume that $\delta_{1,1}=0$. Then from the Leibniz identity

$$0=[x_1,[x_j,x_1]]-[[x_1,x_j],x_1]-[[x_1,x_1],x_j]=-\delta_{1,j}[e_k,x_1]=-\delta_{1,j}e_k,\quad 2\le j\le k-1,$$

we get $\delta_{1,j}=0$ for $2\le j\le k-1$ and we obtain the algebra

$$L_1(\beta_i),\quad \text{with } \beta_i=1 \text{ for } 1\le i\le s \quad\text{and}\quad \beta_i=0 \text{ for } s+1\le i\le k-1.$$

**Case 2.1.2.** Let $e_k\notin Ann_r(L)$. Since the following elements

$$\begin{aligned}&[x_1,x_i]+[x_i,x_1]=\delta_{1,i}e_k, && [x_1,x_1]=\delta_{1,1}e_k,\\ &[x_i,e_k]+[e_k,x_i]=(\nu_{i,k}+1)e_k, && 1\le i\le s,\\ &[x_i,e_k]+[e_k,x_i]=\nu_{i,k}e_k, && s+1\le i\le k-1,\end{aligned}$$

belong to the right annihilator, we deduce

$$\delta_{1,i}=0,\quad 1\le i\le k-1,\qquad \nu_{i,k}=-1,\quad 1\le i\le s,\qquad \nu_{i,k}=0,\quad s+1\le i\le k-1.$$

Thus, in this case we obtain the algebra

$$L_2(\beta_i),\quad \text{with } \beta_i=1 \text{ for } 1\le i\le s \quad\text{and}\quad \beta_i=0 \text{ for } s+1\le i\le k-1.$$

**Case 2.2.** Let $s=1$, then we have the following table of multiplication

$$\begin{cases}[e_1,x_1]=e_1+\beta_{1,1}e_k, & \\ [e_i,x_i]=e_i, & 2\le i\le k-1,\\ [e_1,x_j]=\beta_{1,j}e_k, & 2\le j\le k-1,\\ [e_k,x_1]=e_k, & \\ [x_i,e_1]=\gamma_{i,1}e_k, & 1\le i\le k-1,\\ [x_i,e_k]=\nu_{i,1}e_1+\nu_{i,k}e_k, & 1\le i\le k-1,\\ [x_1,x_j]=\delta_{1,j}e_k, & 1\le j\le k-1,\end{cases}$$

where $\nu_{1,1}\beta_{1,1}=0$, $\nu_{i,1}\beta_{1,1}=0$, for $s<i\le k-1$.

**Case 2.2.1.** Let $e_k\in Ann_r(L)$. Then $\nu_{i,1}=\nu_{i,k}=0$ with $1\le i\le k-1$. Next, $[x_1,e_1]+[e_1,x_1]=e_1+(\beta_{1,1}+\gamma_{1,1})e_k\in Ann_r(L)$. Since $e_k\in Ann_r(L)$ we have $e_1\in Ann_r(L)$. Hence $\gamma_{i,1}=0$.

Performing a basis change $x_1'=x_1-\delta_{1,1}e_k$ we get $[x_1',x_1']=[x_1-\delta_{1,1}e_k,x_1-\delta_{1,1}e_k]=0$. Thus, we may assume $\delta_{1,1}=0$. Using the Leibniz identity:

$$0=[x_1,[x_j,x_1]]-[[x_1,x_j],x_1]+[[x_1,x_1],x_j]=\delta_{1,j}e_k,$$

we derive $\delta_{1,j}=0$ for $2\le j\le k-1$, from which we obtain the algebra $L_3(\beta_i)$.

**Case 2.2.2.** Let $e_k \notin Ann_r(L)$. Then since the following elements

$$[e_1, x_j] + [x_j, e_1] = (\beta_{1,j} + \gamma_{j,1})e_k, \quad [x_1, x_1] = \delta_{1,1}e_k, \quad [x_1, x_j] + [x_j, x_1] = \delta_{1,j}e_k$$

belong to the right annihilator, we deduce

$$\gamma_{j,1} = -\beta_{1,j} \quad 2 \le j \le k-1, \qquad \delta_{1,j} = 0, \quad 1 \le j \le k-1.$$

From the Leibniz identities

$$0 = [x_i, [e_1, x_1]] - [[x_i, e_1], x_1] + [[x_i, x_1], e_1] = -\beta_{1,i}e_k + \beta_{1,1}(\nu_{i,1}e_1 + \nu_{i,k}e_k) + \beta_{1,i}e_k = \beta_{1,1}(\nu_{i,1}e_1 + \nu_{i,k}e_k),$$
$$0 = [x_i, [e_1, x_j]] - [[x_i, e_1], x_j] + [[x_i, x_j], e_1] = \beta_{1,j}(\nu_{i,1}e_1 + \nu_{i,k}e_k), \quad 2 \le j \le k-1,$$

we obtain

$$\beta_{1,j}\nu_{i,1} = 0, \quad \beta_{1,j}\nu_{i,k} = 0, \quad 1 \le i \le k-1, 1 \le j \le k-1.$$

Suppose there exists a $j \in \{1, 2, \ldots, k-1\}$, such that $\beta_{1,j} \neq 0$. Then $\nu_{i,1} = \nu_{i,k} = 0$ for $1 \le i \le k-1$ which implies that $e_k \in Ann_r(L)$. This contradicts the assumption that $e_k \notin Ann_r(L)$. Therefore, $\beta_{1,j} = 0$ for $1 \le j \le k-1$. Thus we have the multiplication:

$$\begin{cases} [e_i, x_i] = e_i, & 1 \le i \le k-1, \\ [e_k, x_1] = e_k, & \\ [x_1, e_1] = \gamma_{1,1}e_k, & \\ [x_i, e_k] = \nu_{i,1}e_1 + \nu_{i,k}e_k, & 1 \le i \le k-1. \end{cases}$$

Using the Leibniz identity, for $2 \le i \le k-1$, we get

$$0 = [x_i, [x_i, e_k]] - [[x_i, x_i], e_k] + [[x_i, e_k], x_i] = [x_i, \nu_{i,1}e_1 + \nu_{i,k}e_k] + [\nu_{i,1}e_1 + \nu_{i,k}e_k, x_i] = \nu_{i,k}(\nu_{i,1}e_1 + \nu_{i,k}e_k).$$

This implies that $\nu_{i,k} = 0$ for $2 \le i \le k$.

We also have

$$0 = [x_1, [x_1, e_1]] - [[x_1, x_1], e_1] + [[x_1, e_1], x_1] = \gamma_{1,1}\nu_{1,1}e_1 + \gamma_{1,1}(1 + \nu_{1,k})e_k,$$
$$0 = [x_1, [x_1, e_k]] - [[x_1, x_1], e_k] + [[x_1, e_k], x_1] = \nu_{1,1}(\nu_{1,k} + 1)e_1 + (\nu_{1,1}\gamma_{1,1} + \nu_{1,k}(\nu_{1,k} + 1))e_k,$$
$$0 = [x_1, [x_i, e_k]] - [[x_1, x_i], e_k] + [[x_1, e_k], x_i] = \nu_{i,1}\gamma_{1,1}e_k, \quad \text{for} \quad 2 \le i \le k,$$
$$0 = [x_i, [x_1, e_k]] - [[x_i, x_1], e_k] + [[x_i, e_k], x_1] = \nu_{i,1}(\nu_{i,k} + 1)e_1.$$

Thus,

$$\begin{aligned} &\gamma_{1,1}\nu_{1,1} = 0, && \gamma_{1,1}(1 + \nu_{1,k}) = 0, \\ &\nu_{1,1}(\nu_{1,k} + 1) = 0, && \nu_{1,k}(\nu_{1,k} + 1) = 0, \\ &\nu_{i,1}\gamma_{1,1} = 0, && \nu_{i,1}(\nu_{1,k} + 1) = 0, \quad 2 \le i \le k-1. \end{aligned}$$

If $\nu_{1,k} = 0$, then $\gamma_{1,1} = \nu_{i,1} = 0$ for $1 \le i \le k-1$, which implies that $e_k \in Ann_r(L)$. This is a contradiction with assumption that $e_k \notin Ann_r(L)$. Thus, $\nu_{1,k} = -1$.

If $\gamma_{1,1} \neq 0$, then $\nu_{i,1} = 0$ for $1 \le i \le k-1$. The table of multiplication (3.1) multiplication is as follows:

$$\begin{cases} [e_i, x_i] = e_i, & 1 \le i \le k-1, \\ [e_k, x_1] = e_k, & \\ [x_1, e_1] = \gamma_{1,1}e_k, & \\ [x_1, e_k] = -e_k. & \end{cases}$$

Making the basis change $e_1' = e_1 + \gamma_{1,1}e_k$ we have the algebra

$$L_1(\beta_i), \quad \text{with } \beta_1 = 1 \quad \text{and} \quad \beta_i = 0 \text{ for } 2 \le i \le k-1.$$

If $\gamma_{1,1} = 0$, then we obtain the algebra $L_4(\nu_i)$.

**Case 3.** Let $\beta_{k,1} = \beta_{k,2} = \cdots = \beta_{k,k-1} = 0$. Then the multiplication is:

$$\begin{cases} [e_i, x_i] = e_i + \beta_{i,i}e_k, & 1 \le i \le k-1, \\ [e_i, x_j] = \beta_{i,j}e_k, & 1 \le i,j \le k-1, i \ne j, \\ [x_i, e_j] = \gamma_{i,j}e_k, & 1 \le i,j \le k-1, \\ [x_i, e_k] = \sum\limits_{j=1}^{k} \nu_{i,j}e_j, & 1 \le i \le k-1, \\ [x_i, x_j] = \delta_{i,j}e_k, & 1 \le i,j \le k-1. \end{cases}$$

From the Leibniz identity

$$0 = [e_i, [x_j, x_i]] - [[e_i, x_j], x_i] + [[e_i, x_i], x_j] = \beta_{i,j}e_k,$$

we have

$$\beta_{i,j} = 0, \quad 1 \le i,j \le k-1, \quad i \ne j.$$

Then making the change $e_i' = e_i + \beta_{i,i}e_k$ for $1 \le i \le k-1$, we get

$$[e_i', x_i] = [e_i + \beta_{i,i}e_k, x_i] = e_i'.$$

Thus, we may suppose $\beta_{i,i} = 0$ for $1 \le i \le k-1$.

Notice that

$$0 = [x_i, [e_k, x_j]] - [[x_i, e_k], x_j] + [[x_i, x_j], e_k] = -\nu_{i,j}e_j$$

implies that $\nu_{i,j} = 0$ for $1 \le i,j \le k-1$. This gives $[x_i, e_k] = \nu_{i,k}e_k$ for $1 \le i \le k-1$.

Using the Leibniz identity, we have:

$$0 = [x_i, [e_j, x_j]] - [[x_i, e_j], x_j] + [[x_i, x_j], e_j] = \gamma_{i,j}e_k,$$
$$0 = [x_i, [x_i, e_k]] - [[x_i, x_i], e_k] + [[x_i, e_k], x_i] = \nu_{i,k}^2 e_k,$$

which implies that $\gamma_{i,j} = \nu_{i,k} = 0$ for $1 \le i,j \le k-1$.

Our final multiplication is:

$$L_5(\delta_{i,j}): \begin{cases} [e_i, x_i] = e_i, & 1 \le i \le k-1, \\ [x_i, x_j] = \delta_{i,j}e_k, & 1 \le i,j \le k-1. \end{cases}$$

□

Now we give the description of solvable algebras $R(\mathbf{a_k}, k-1)$ in the case of $\alpha_i = -1$ for $1 \le i \le k-1$.

**Theorem 3.2.** *Let $L$ be a solvable Leibniz algebra from the class $R(\mathbf{a_k}, k-1)$ and $\alpha_i = -1$ for $1 \le i \le k-1$. Then $L$ is isomorphic to one of the following algebras:*

$$L_6(\beta_j):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_j]=\beta_j e_k, & 1\le j\le k-1,\\ [x_i,e_i]=-e_i, & 1\le i\le k-1,\end{cases}\qquad L_7(\beta_j):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_j]=\beta_j e_k, & 1\le j\le k-1,\\ [x_i,e_i]=-e_i, & 1\le i\le k-1,\\ [x_j,e_k]=-\beta_j e_k, & 1\le j\le k-1,\end{cases}$$

$$L_8(\gamma_i):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_1]=e_k,\\ [x_i,e_i]=-e_i, & 1\le i\le k-1,\\ [x_i,e_1]=\gamma_i e_k, & 2\le i\le k-1,\end{cases}\qquad L_9:\begin{cases}[e_1,x_1]=e_1+\beta_1 e_k,\\ [e_i,x_i]=e_i, & 2\le i\le k-1,\\ [e_1,x_i]=\beta_i e_k, & 2\le i\le k-1,\\ [e_k,x_1]=e_k,\\ [x_1,e_1]=-e_1-\beta_1 e_k,\\ [x_i,e_i]=-e_i, & 2\le i\le k-1,\\ [x_i,e_1]=-\beta_i e_k, & 2\le i\le k-1,\\ [x_1,e_k]=-e_k.\end{cases}$$

$$L_{10}(\delta_{i,j}):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [x_i,e_i]=-e_i, & 1\le i\le k-1,\\ [x_i,x_j]=\delta_{i,j}e_k, & 1\le i,j\le k-1.\end{cases}$$

*Proof.* Let $\alpha_i=-1$ for $1\le i\le k-1$, then the multiplication (3.1) has the form

$$\begin{cases}[e_i,x_i]=e_i+\beta_{i,i}e_k, & 1\le i\le k-1,\\ [e_i,x_j]=\beta_{i,j}e_k, & 1\le i\le k, \quad 1\le j\le k-1, \quad i\ne j,\\ [x_i,e_i]=-e_i+\gamma_{i,i}e_k, & 1\le i\le k-1,\\ [x_i,e_j]=\gamma_{i,j}e_k, & 1\le i\le k-1, \quad 1\le j\le k-1, \quad i\ne j,\\ [x_i,e_k]=\sum\limits_{j=1}^{k}\nu_{i,j}e_j, & 1\le i\le k-1,\\ [x_i,x_j]=\delta_{i,j}e_k, & 1\le i,j\le k-1,\end{cases}$$

Let there exist $i_0\in\{1,2,\ldots,k\}$, such that $\beta_{k,i_0}\notin\{0,1\}$. Without loss of generality, we may assume $i_0=1$. Making the change of basis

$$e_1'=e_1-\frac{\beta_{1,1}}{\beta_{k,1}-1}e_k,\qquad e_i'=e_i-\frac{\beta_{i,1}}{\beta_{k,1}}e_k,\quad 2\le i\le k-1,$$

we get $[e_1',x_1]=e_1'$ and $[e_i',x_1]=0$ for $2\le i\le k-1$. Thus, we may suppose

$$\beta_{i,1}=0,\quad 1\le i\le k-1.$$

Consider the Leibniz identities

$$0=[e_1,[x_1,x_j]]=[[e_1,x_1],x_j]-[[e_1,x_j],x_1]=[e_1,x_j]-\left[\beta_{1,j}e_k,x_1\right]=\beta_{1,j}e_k-\beta_{1,j}\beta_{k,1}e_k=\beta_{1,j}(1-\beta_{k,1})e_k,$$
$$0=[e_i,[x_1,x_j]]=[[e_i,x_1],x_j]-[[e_i,x_j],x_1]=-\beta_{i,j}[e_k,x_1]=-\beta_{i,j}\beta_{k,1}e_k,\quad 2\le i,j(i\ne j)\le n,$$
$$0=[e_j,[x_1,x_j]]=[[e_j,x_1],x_j]-[[e_j,x_j],x_1]=-\left[e_j+\beta_{j,j}e_k,x_1\right]=-\beta_{j,j}\beta_{k,1}e_k,\quad 2\le j\le n.$$

Since $\beta_{k,1}\notin\{0,1\}$, we get that

$$\beta_{1,j}=0, 2\le j\le k-1,$$
$$\beta_{i,j}=0, 2\le i\le k-1,$$

Now we consider the Leibniz identity for the triple of elements $\{x_i, e_j, x_1\}$. Then

$$0 = [x_1,[e_1,x_1]] - [[x_1,e_1],x_1] + [[x_1,x_1],e_1] = [x_1,e_1] - [-e_1 + \gamma_{1,1}e_k, x_1] = \gamma_{1,1}(1-\beta_{k,1})e_k,$$
$$0 = [x_i,[e_1,x_1]] - [[x_i,e_1],x_1] + [[x_i,x_1],e_1] = [x_i,e_1] - \gamma_{i,1}[e_k,x_1] = \gamma_{i,1}(1-\beta_{k,1})e_k, \quad 2 \le i \le k-1,$$
$$0 = [x_i,[e_i,x_1]] - [[x_i,e_i],x_1] + [[x_i,x_1],e_i] = [e_i - \gamma_{i,i}e_k, x_1] = -\gamma_{i,i}\beta_{k,1}e_k, \quad 2 \le i \le k-1.$$
$$0 = [x_i,[e_j,x_1]] - [[x_i,e_j],x_1] + [[x_i,x_1],e_j] = -\gamma_{i,j}[e_k,x_1] = -\gamma_{i,j}\beta_{k,1}e_k, \quad 1 \le i \le k-1, 2 \le j \le k-1.$$

Since $\beta_{k,1} \notin \{0,1\}$, we have

$$\gamma_{i,j} = 0, \quad 1 \le i,j \le k-1.$$

Using the Leibniz identity, we have:

$$0 = [x_i,[e_k,x_1]] - [[x_i,e_k],x_1] + [[x_i,x_1],e_k] = \beta_{k,1}[x_i,e_k] - \left[\sum_{j=1}^{k}\nu_{i,j}e_j, x_1\right]$$
$$= \beta_{k,1}\sum_{j=1}^{k}\nu_{i,j}e_j - \nu_{i,1}e_1 - \nu_{i,k}\beta_{k,1}e_k = (\beta_{k,1}-1)\nu_{i,1}e_1 + \beta_{k,1}\sum_{j=2}^{k-1}\nu_{i,j}e_j.$$

Hence

$$\nu_{i,j} = 0, \quad 1 \le i,j \le k-1.$$

From

$$0 = [x_i,[x_j,e_k]] - [[x_i,x_j],e_k] + [[x_i,e_k],x_j] = \nu_{j,k}[x_i,e_k] + \nu_{i,k}[e_k,x_j] = \nu_{i,k}(\nu_{j,k} + \beta_{k,j})e_k,$$

we get

$$\nu_{i,k}(\nu_{j,k} + \beta_{k,j}) = 0, \quad 1 \le i,j \le k-1. \tag{3.4}$$

Taking the basis change $x_i' = x_i - \frac{\delta_{i,1}}{\beta_{k,1}}e_k$ for $2 \le i \le k-1$, we obtain

$$[x_i', x_1] = \left[x_i - \frac{\delta_{i,1}}{\beta_{k,1}}e_k, x_1\right] = \delta_{i,1}e_k - \delta_{i,1}e_k = 0, \quad 2 \le i \le k-1.$$

Thus, we can assume

$$\delta_{i,1} = 0, \quad 2 \le i \le k-1.$$

Using the Leibniz identities

$$0 = [x_i,[x_1,x_1]] - [[x_i,x_1],x_1] + [[x_i,x_1],x_1] = \delta_{1,1}[x_i,e_k] = \nu_{i,k}\delta_{1,1}e_k$$
$$0 = [x_1,[x_j,x_1]] - [[x_1,x_j],x_1] + [[x_1,x_1],x_j] = -\delta_{1,j}[e_k,x_1] + \delta_{1,1}[e_k,x_j] = (-\delta_{1,j}\beta_{k,1} + \delta_{1,1}\beta_{k,j})e_k,$$
$$0 = [x_i,[x_j,x_1]] - [[x_i,x_j],x_1] + [[x_i,x_1],x_j] = -\delta_{i,j}\beta_{k,1}e_k,$$

we have

$$\begin{aligned} &\nu_{i,k}\delta_{1,1} = 0, && 1 \le i \le k-1,\\ &\delta_{1,j} = \frac{\beta_{k,j}}{\beta_{k,1}}\delta_{1,1}, && 2 \le j \le k-1,\\ &\delta_{i,j} = 0, && \le i,j \le k-1. \end{aligned} \tag{3.5}$$

Let $\nu_{i,k}=0$ for all $i(1\leq i\leq k-1)$. Then taking the change $x_1'=x_1-\frac{\delta_{1,1}}{\beta_{k,1}}e_k$ we have:

$$\left[x_1',x_1'\right]=\left[x_1-\frac{\delta_{1,1}}{\beta_{k,1}}e_k,x_1-\frac{\delta_{1,1}}{\beta_{k,1}}e_k\right]=\delta_{1,1}e_k-\delta_{1,1}e_k=0$$

$$\left[x_1',x_j\right]=\left[x_1-\frac{\delta_{1,1}}{\beta_{k,1}}e_k,x_j\right]=\frac{\beta_{k,j}}{\beta_{k,1}}\delta_{1,1}e_k-\frac{\delta_{1,1}}{\beta_{k,1}}\beta_{k,j}e_k=0.$$

Thus, we obtain the algebra $L_6(\beta_i)$, with $\beta_1\notin\{0,1\}$.

Let there exist $i(1\leq i\leq k-1)$ such that $\nu_{i,k}\neq 0$. According to the equalities (3.4) and (3.5) we have $\nu_{i,k}=-\beta_{k,i}$ and $\delta_{1,1}=0$, which implies that $\delta_{1,j}=0$ for $2\leq j\leq k-1$. Thus we have the algebra $L_7(\beta_i)$ with $\beta_1\notin\{0,1\}$.

Considering the case $\beta_{k,1},\beta_{k,2},\ldots,\beta_{k,k-1}\in\{0,1\}$, similarly to the proof of the Theorem 3.1 we obtain the algebras $L_6(\beta_i)$, $L_7(\beta_i)$ with $\beta_1\in\{0,1\}$, $L_8(\gamma_i)$, $L_9$ and $L_{10}(\delta_{i,j})$. □

Now we give the description of solvable Leibniz algebras from the class $R(\mathbf{a_k},k-1)$ in the general case. Let there exist $i_0$ and $j_0$ such that $\alpha_{i_0}=-1$ and $\alpha_{j_0}=0$. Without loss of generality we can assume that $\alpha_1=\cdots=\alpha_{t-1}=-1$ and $\alpha_t=\cdots=\alpha_{k-1}=0$.

**Theorem 3.3.** *Let $L$ be a solvable Leibniz algebra from the class $R(\mathbf{a_k},k-1)$ and let $\alpha_1=\cdots=\alpha_{t-1}=-1$ and $\alpha_t=\cdots=\alpha_{k-1}=0$. Then $L$ is isomorphic to one of the following algebras:*

$$M_{1,t}(\beta_1,\beta_2,\ldots,\beta_{k-1}):\begin{cases}[e_i,x_i]=e_i, & 1\leq i\leq k-1,\\ [e_k,x_i]=\beta_ie_k, & 1\leq i\leq k-1,\\ [x_i,e_i]=-e_i, & 1\leq i\leq t-1,\end{cases}$$

$$M_{2,t}(\beta_1,\beta_2,\ldots,\beta_{k-1}):\begin{cases}[e_i,x_i]=e_i, & 1\leq i\leq k-1,\\ [e_k,x_i]=\beta_ie_k, & 1\leq i\leq k-1,\\ [x_i,e_k]=-\beta_ie_k, & 1\leq i\leq k-1,\\ [x_i,e_i]=-e_i, & 1\leq i\leq t-1,\end{cases}$$

$$M_{3,t}(\beta_1,\beta_2,\ldots,\beta_{k-1}):\begin{cases}[e_t,x_t]=e_t+\beta_te_k,\\ [e_i,x_i]=e_i, & 1\leq i\leq k-1,\\ [e_t,x_i]=\beta_ie_k, & 1\leq i\leq k-1,\\ [e_k,x_t]=e_k,\\ [x_i,e_i]=-e_i, & 1\leq i\leq t-1,\end{cases}$$

$$M_{4,t}(\beta_1,\beta_2,\ldots,\beta_{k-1}):\begin{cases}[e_1,x_1]=e_1+\beta_1e_k,\\ [e_i,x_i]=e_i, & 2\leq i\leq k-1,\\ [e_1,x_i]=\beta_ie_k, & 2\leq i\leq k-1,\\ [e_k,x_1]=e_k,\\ [x_1,e_1]=-e_1-\beta_1e_k,\\ [x_i,e_i]=-e_i, & 2\leq i\leq t-1,\\ [x_i,e_1]=-\beta_ie_k, & 2\leq i\leq k-1,\\ [x_1,e_k]=-e_k,\end{cases}$$

$$M_{5,t}(\gamma_2,\gamma_3,\ldots,\gamma_{k-1}):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_1]=e_k,\\ [x_i,e_i]=-e_i, & 1\le i\le t-1,\\ [x_i,e_1]=\gamma_i e_k, & 2\le i\le k-1,\end{cases}$$

$$M_{6,t}(\nu_1,\nu_2,\ldots,\nu_{k-1}):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [e_k,x_t]=e_k,\\ [x_i,e_i]=-e_i, & 1\le i\le t-1,\\ [x_t,e_k]=-e_k,\\ [x_i,e_k]=\nu_i e_t, & 1\le i\le k-1,\end{cases}$$

$$M_{7,t}(\delta_{i,j}):\begin{cases}[e_i,x_i]=e_i, & 1\le i\le k-1,\\ [x_i,e_i]=-e_i, & 1\le i\le t-1,\\ [x_i,x_j]=\delta_{i,j}e_k, & 1\le i,j\le k-1.\end{cases}$$

*Proof.* The proof is similar to the proof of the Theorem 3.1. □

In the following theorem we give the classification of $(2k-1)$-dimensional solvable Leibniz algebras with $k$-dimensional abelian nilradical.

**Theorem 3.4.** *Let $L$ be a $(2k-1)$-dimensional solvable Leibniz algebra with $k$-dimensional abelian nilradical. Then $L$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$M_{1,t}(\beta_1,\beta_2,\ldots,\beta_{k-1})$,
$M_{2,t}(\beta_1,\ldots,\beta_{t-1},0,\ldots,0)$,
$M_{3,t}(1,\beta_2,\ldots,\beta_{t-1},\beta_t,\beta_{t+1},\beta_{t+2}\ldots,\beta_{k-1})$,
$M_{3,t}(0,0,\ldots,0,1,\beta_{t+1},\beta_{t+2}\ldots,\beta_{k-1})$,
$M_{3,t}(0,0,\ldots,0,0,1,\beta_{t+2}\ldots,\beta_{k-1})$,
$M_{7,t}(\delta_{i,j})$.
$M_{4,t}(1,\beta_2,\beta_3,\ldots,\beta_{t-1},\beta_t,\beta_{t+1},\ldots,\beta_{k-1})$,
$M_{4,t}(0,1,\beta_3,\ldots,\beta_{t-1},\beta_t,\beta_{t+1},\ldots,\beta_{k-1})$,
$M_{4,t}(0,0,0,\ldots,0,1,\beta_{t+1},\ldots,\beta_{k-1})$,
$M_{5,t}(1,\gamma_3,\ldots,\gamma_{t-1},\gamma_t,\ldots,\gamma_{k-1})$,
$M_{5,t}(0,0,\ldots,0,1,\gamma_{t+1},\ldots,\gamma_{k-1})$,

*where at least one of the parameters $\delta_{i,j}$ is non-zero and this non-zero parameter can be scaled to 1.*

*Proof.* From Theorem 3.3 we have the list of solvable Leibniz algebras from the class $R(\mathbf{a_k},k-1)$. It is obvious that the class $M_{1,t}(\beta_i)$ gives us pairwise non-isomorphic algebras for any parameters $\beta_i\in\mathbb{C}$. Moreover, in the case $t=1$ we get the algebra $L_1(\beta_i.)$

In the class of algebras $M_{2,t}(\beta_i)$, at least one of the parameters $\beta_i$ is non-zero. Otherwise we obtain the algebra $M_{1,t}(0,0,\ldots,0)$. Moreover if $\beta_j\neq 0$ with $j\ge t$, then making the change

$$e_t'=e_k,\quad e_j'=e_t,\quad e_k'=e_j,$$

$$x_t'=\frac{1}{\beta_j}x_j,\quad x_j'=x_t-\frac{\beta_t}{\beta_j}x_j,\quad x_i'=x_i-\frac{\beta_i}{\beta_j}x_j,\quad 1\le i\le k-1,$$

we obtain that

$$M_{2,t}(\beta_1,\ldots,\beta_j,\ldots\beta_{k-1})\cong M_{1,t+1}\left(-\frac{\beta_1}{\beta_j},\ldots,\frac{1}{\beta_j},\ldots,-\frac{\beta_{k-1}}{\beta_j}\right).$$

Thus, we have the class $M_{2,t}(\beta_1, ..., \beta_{t-1}, 0, ..., 0)$. It is not difficult to check that two algebras from this class are non-isomorphic.

In the class of algebras $M_{3,t}(\beta_1, \beta_2, ..., \beta_{k-1})$ also at least one of the parameters $\beta_i$ is non-zero. Moreover, if $\beta_j \neq 0$ with $1 \leq j \leq t-1$, then without lost of generality we may suppose $\beta_1 \neq 0$ and making the change $e'_k = \beta_1 e_k$ we may assume $\beta_1 = 1$. In the case of $\beta_j = 0$ for $1 \leq j \leq t-1$ and $\beta_t \neq 0$, the parameter $\beta_t$ can be scaled to 1. If $\beta_j = 0$ for $1 \leq j \leq t$, then without lost of generality we may assume $\beta_{t+1} \neq 0$ and making the change $e'_k = \beta_{t+1} e_k$ we obtain $\beta_{t+1} = 1$.

Thus,

$$M_{3,t}(1, \beta_2, ..., \beta_{t-1}, \beta_t, \beta_{t+1}, \beta_{t+2}..., \beta_{k-1})$$
$$M_{3,t}(0, 0, ..., 0, 1, \beta_{t+1}, \beta_{t+2}..., \beta_{k-1})$$
$$M_{3,t}(0, 0, ..., 0, 0, 1, \beta_{t+2}..., \beta_{k-1}).$$

are non-isomorphic algebras.

Analyzing the class of algebras $M_{4,t}(\beta_1, \beta_2, ..., \beta_{k-1})$ and $M_{5,t}(\gamma_1, \gamma_2, ..., \gamma_{k-1})$ similarly we obtain following non-isomorphic algebras

$$M_{4,t}(1, \beta_2, \beta_3, ..., \beta_{t-1}, \beta_t, \beta_{t+1}, ..., \beta_{k-1}),$$
$$M_{4,t}(0, 1, \beta_3, ..., \beta_{t-1}, \beta_t, \beta_{t+1}, ..., \beta_{k-1}),$$
$$M_{4,t}(0, 0, 0, ..., 0, 1, \beta_{t+1}, ..., \beta_{k-1}),$$

$$M_{5,t}(1, \gamma_3, ..., \gamma_{t-1}, \gamma_t, ..., \gamma_{k-1}),$$
$$M_{5,t}(0, 0, ..., 0, 1, \gamma_{t+1}, ..., \gamma_{k-1}).$$

In the class of $M_{6,t}(\nu_1, \nu_2, ..., \nu_{k-1})$ making the change

$$e'_1 = e_k, \quad e'_t = e_1, \quad e'_k = e_t, \qquad x'_1 = x_t, \quad x'_t = x_1,$$

we get that $M_{6,t}(\nu_1, \nu_2, ..., \nu_{k-1}) \cong M_{5,t+1}(\nu_1, \nu_2, ..., \nu_{k-1})$. □

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