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Some irreducible components of the variety of complex $(n+1)$-dimensional Leibniz algebras

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ABSTRACT

In the present paper we indicate some Leibniz algebras whose closures of orbits under the natural action of GL_n form an irreducible component of the variety of complex $(n+1)$-dimensional Leibniz algebras. Moreover, for these algebras we calculate the bases of their second cohomology groups.

1. Introduction

Jean-Louis Loday introduced Leibniz algebras because of considerations in algebraic K-theory [1]. We know that the Lie algebra homology involves the Chevalley–Eilenberg chain complex, which in turns involves exterior powers of the Lie algebra. Loday found that there is a non-antisymmetric generalization where roughly speaking one has the tensor and not the exterior powers of the Lie algebra in the complex; this new complex defines the Leibniz homology of Lie algebras. The Leibniz homology is related to the Hochschild homology in the same way the Lie algebra homology is related to the cyclic homology.

In many cases where the Leibniz algebra involved may depend on the parameters it is useful to know the structure of the set of all Leibniz algebras of a given dimension. The aim of this work is to establish some results from a geometrical point of view in the study of Leibniz algebras. Any Leibniz algebra law is considered as a point of an affine algebraic variety defined by the polynomial equations coming from the Leibniz identity for a given basis. This way provides a description of the difficulties in classification problems referring to the classes of nilpotent and solvable Leibniz algebras. The orbits relative to the action

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of the general linear group correspond to the isomorphism classes of Leibniz algebras and so classification problems (up to isomorphism) can be reduced to the classification of these orbits. An affine algebraic variety is a union of a finite number of irreducible components and the Zariski open orbits provide interesting classes of Leibniz algebras to be classified. The Leibniz algebras of this class are called rigid.

The research of varieties of Lie algebras laws over the field $\mathbb{C}$ of the complex numbers has been extensively studied, establishing various important structural results and properties. On the contrary, the problem for varieties of Leibniz algebras has not been considered in detail. The research of varieties of Lie and Leibniz algebra laws is essentially based on the cohomological study of Leibniz algebras and on deformation theory. Deformations of arbitrary rings and associative algebras, results about rigid Lie algebras and related cohomology questions, were first investigated in 1964 by Gerstenhaber [2]. Later, the notion of deformation was applied to Lie algebras by Nijenhuis and Richardson [3], where they transform the topological problem related to rigidity into a cohomological problem, proving that a Lie algebra $\mathfrak{g}$ is rigid if the second group $H_2(\mathfrak{g}, \mathfrak{g})$ of the Chevalley–Eilenberg cohomology vanishes.

In this paper, we are concerned with the structure of the variety $\mathcal{L}eib_{n+1}$, the variety of the $(n+1)$-dimensional Leibniz algebras, in particular, with answers to the following question: What irreducible components do $\mathcal{L}eib_{n+1}$ fall into? The answers to this question would allow to describe partially the structures of some Leibniz algebras of dimension $(n+1)$. We shall obtain general results on some irreducible components of the variety of finite-dimensional Leibniz algebras and indicate representatives of solvable Leibniz algebras, whose closures of orbits form irreducible components. We hope to develop this line of research in forthcoming works.

The paper is organized as follows. In Section 2 we recall some necessary notions about Leibniz algebras, cohomology and degenerations of Leibniz algebras. In Section 3 we describe derivations of solvable Leibniz algebras whose nilradical is a filiform algebra of type F_n^1 (see below Theorem 2.9), present (1,1)-invariants for various types of solvable algebras and give representatives of irreducible components of the variety $\mathcal{L}eib_{n+1}$ of the $(n+1)$-dimensional Leibniz algebras. Finally, in the last subsection, we give descriptions of the second cohomology group of solvable Leibniz algebras whose nilradical is a filiform algebra of type F_n^1.

Throughout the paper, we denote by L a finite-dimensional Leibniz algebra over a field $\mathbb{F}$ or over the field of the complex numbers $\mathbb{C}$. Moreover, in the multiplication table of a Leibniz algebra the omitted products and in the expansion of 2-cocycles the omitted values are assumed to be zero.

2. Preliminaries

In this section we give necessary definitions on Leibniz algebras, cohomology, degenerations and known results.
We present the definition of the main object of our study.

Definition 2.1 (*[1]*)**.** A Leibniz algebra over a field $\mathbb{F}$ is a vector space L equipped with a bilinear map, called bracket,

$$[-,-] : L \times L \to L$$

satisfying the Leibniz identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y],$$

for all $x, y, z \in L$.

The set $\mathrm{Ann}_r(L) = \{x \in L \mid [y,x] = 0,\ \forall y \in L\}$ is called *the right annihilator of the Leibniz algebra L*. Note that $\mathrm{Ann}_r(L)$ is an ideal of L and for any $x, y \in L$, the elements $[x,x]$, $[x,y]+[y,x] \in \mathrm{Ann}_r(L)$.

2.1. Solvable Leibniz algebras

For a Leibniz algebra L we consider the following *central lower* and *derived series*:

$$L^1 = L, \qquad L^{k+1} = [L^k, L^1], \qquad k \geq 1;$$
$$L^{[1]} = L, \qquad L^{[s+1]} = [L^{[s]}, L^{[s]}], \qquad s \geq 1.$$

Definition 2.2. A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists $n \in \mathbb{N}$ ($m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^{[m]} = 0$). The minimal number n (respectively, m) with such property is said to be the index of nilpotency (respectively, of solvability) of the algebra L.

Obviously, the index of nilpotency of an n-dimensional nilpotent Leibniz algebra is not greater than $n+1$.

Definition 2.3. An n-dimensional Leibniz algebra is said to be null-filiform if $\dim L^i = n+1-i,\ 1 \leq i \leq n+1$.

Remark that a null-filiform Leibniz algebra has maximal index of nilpotency.

Theorem 2.4 ([4]). *An arbitrary n-dimensional null-filiform Leibniz algebra is isomorphic to the algebra:*

$$NF_n: \quad [e_i, e_1] = e_{i+1}, \qquad 1 \le i \le n-1,$$

where $\{e_1, e_2, \dots, e_n\}$ is a basis of the algebra NF_n.

From Theorem 2.4 it is easy to see that a nilpotent Leibniz algebra is null-filiform if and only if it is a one-generated algebra, i.e., an algebra generated by a unique element. Note that this notion has no sense in the Lie algebras case, because they are at least two-generated.

It should be noted that the sum of any two nilpotent (solvable) ideals is nilpotent (solvable).

Definition 2.5. The maximal nilpotent (solvable) ideal of a Leibniz algebra is said to be a nilradical (solvable radical) of the algebra.

Below, we present the description of solvable Leibniz algebras whose nilradical is isomorphic to the algebra NF_n.

Theorem 2.6 ([5]). *Let R be a solvable Leibniz algebra whose nilradical is NF_n. Then there exists a basis $\{e_1, e_2, \dots, e_n, x\}$ of the algebra R such that the multiplication table of R with respect to this basis has the following form:*

$$RNF_n: \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-1, \\ [x, e_1] = -e_1, & \\ [e_i, x] = ie_i, & 1 \le i \le n. \end{cases}$$

Definition 2.7. An n-dimensional Leibniz algebra L is said to be filiform if $\dim L^i = n - i$ for $2 \le i \le n$.

Now let us define a natural graduation for a filiform Leibniz algebra.

Definition 2.8. Given a filiform Leibniz algebra L, put $L_i = L^i/L^{i+1}$, $1 \le i \le n-1$, and $\mathrm{gr}(L) = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1}$. Then $[L_i, L_j] \subseteq L_{i+j}$ and we obtain the graded algebra $\mathrm{gr}(L)$. If $\mathrm{gr}(L)$ and L are isomorphic, then we say that the algebra L is naturally graded.

Thanks to [4] and [6] it is well known that there are three types of naturally graded filiform Leibniz algebras. In fact, the third type encloses the class of naturally graded filiform Lie algebras.

Theorem 2.9. *Any complex n-dimensional naturally graded filiform Leibniz algebra is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$F_n^1: [e_i, e_1] = e_{i+1}, \ 2 \le i \le n-1,$$
$$F_n^2: [e_1, e_1] = e_3, \ [e_i, e_1] = e_{i+1}, \ 3 \le i \le n-1,$$
$$F_n^3(\alpha): \begin{cases} [e_i, e_1] = -[e_1, e_i] = e_{i+1}, & 2 \le i \le n-1, \\ [e_i, e_{n+1-i}] = -[e_{n+1-i}, e_i] = \alpha(-1)^{i+1} e_n, & 2 \le i \le n-1, \end{cases}$$

where $\alpha \in \{0, 1\}$ for even n and $\alpha = 0$ for odd n.

The following theorem decomposes all n-dimensional filiform Leibniz algebras into three families of algebras.

Theorem 2.10 ([4,7]). *Any complex n-dimensional filiform Leibniz algebra admits a basis $\{e_1, e_2, \dots, e_n\}$ such that the table of multiplication of the algebra has one of the following forms:*

$$F_1(\alpha_4, \dots, \alpha_n, \theta) = \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [e_1, e_2] = \theta e_n, & \\ [e_j, e_2] = \alpha_4 e_{j+2} + \alpha_5 e_{j+3} + \cdots + \alpha_{n+2-j} e_n, & 2 \le j \le n-2, \end{cases}$$

$$F_2(\beta_4, \dots, \beta_n, \gamma) = \begin{cases} [e_1, e_1] = e_3, & \\ [e_i, e_1] = e_{i+1}, & 3 \le i \le n-1, \\ [e_1, e_2] = \beta_4 e_4 + \beta_5 e_5 + \cdots + \beta_n e_n, & \\ [e_2, e_2] = \gamma e_n, & \\ [e_j, e_2] = \beta_4 e_{j+2} + \beta_5 e_{j+3} + \cdots + \beta_{n+2-j} e_n, & 3 \le j \le n-2, \end{cases}$$

$$F_3(\theta_1, \theta_2, \theta_3) = \begin{cases} [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [e_1, e_i] = -e_{i+1}, & 3 \le i \le n-1, \\ [e_1, e_1] = \theta_1 e_n, & \\ [e_1, e_2] = -e_3 + \theta_2 e_n, & \\ [e_2, e_2] = \theta_3 e_n, & \\ [e_i, e_j] = -[e_j, e_i] \in \mathrm{span}\langle e_{i+j+1}, e_{i+j+2}, \dots, e_n \rangle, & 2 \le i < j \le n-1, \\ [e_i, e_{n+1-i}] = -[e_{n+1-i}, x_i] = \alpha(-1)^{i+1} x_n, & 2 \le i \le n-1, \end{cases}$$

where $\alpha \in \{0, 1\}$ for even n and $\alpha = 0$ for odd n.

Below, we present the description of solvable Leibniz algebras whose nilradical is isomorphic to the algebra F_n^1.

Theorem 2.11 ([8,9]). *An arbitrary $(n+1)$-dimensional solvable Leibniz algebra with nilradical F_n^1 is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$R_1:\begin{cases}[e_i,e_1]=e_{i+1}, & 2\le i\le n-1,\\ [x,e_1]=-e_1-e_2, & \\ [e_1,x]=e_1, & \\ [e_i,x]=(i-1)e_i, & 2\le i\le n,\end{cases}\qquad R_2(\alpha):\begin{cases}[e_i,e_1]=e_{i+1}, & 2\le i\le n-1,\\ [x,e_1]=-e_1, & \\ [e_1,x]=e_1, & \\ [e_i,x]=(i-1+\alpha)e_i, & 2\le i\le n,\end{cases}$$

$$R_3:\begin{cases}[e_i,e_1]=e_{i+1}, & 2\le i\le n-1,\\ [x,e_1]=-e_1, & \\ [e_1,x]=e_1, & \\ [e_i,x]=(i-n)e_i, & 2\le i\le n,\\ [x,x]=e_n, & \end{cases}\qquad R_4:\begin{cases}[e_i,e_1]=e_{i+1}, & 2\le i\le n-1,\\ [x,e_1]=-e_1, & \\ [e_1,x]=e_1+e_n, & \\ [e_i,x]=(i+1-n)e_i, & 2\le i\le n,\\ [x,x]=-e_{n-1}, & \end{cases}$$

$$R_5(\alpha_i)=R_5(\alpha_4,\dots,\alpha_n):\begin{cases}[e_1,e_1]=e_3, & \\ [e_i,e_1]=e_{i+1}, & 2\le i\le n-1,\\ [e_1,x]=e_2+\displaystyle\sum_{i=4}^{n-1}\alpha_i e_i, & \\ [e_i,x]=e_i+\displaystyle\sum_{j=i+2}^{n}\alpha_{j-i+2}e_j, & 2\le i\le n.\end{cases}$$

Moreover, the first non-vanishing parameter $\{\alpha_4,\dots,\alpha_n\}$ in the algebra $R_5(\alpha_4,\dots,\alpha_n)$ can be scaled to 1.

Remark 2.12. Notice that seven isomorphism classes were presented in [8, Theorem 4.2]. Nevertheless, this classification can be improved by changing the number of parameters of $\alpha_4,\dots,\alpha_{n-1}$ to $\alpha_4,\dots,\alpha_n$. Namely, the algebra $R_6(\alpha_4,\dots,\alpha_{n-1})$ in the list of [8, Theorem 4.2] is isomorphic to the algebra $R_5(\alpha_4,\dots,\alpha_{n-1},0)$ via the isomorphism $x'=x,\ e_1'=e_1,\ e_2'=e_2+e_n,\ e_i'=e_i,\ 3\le i\le n$.

Moreover, the family $R_7(\alpha_4,\dots,\alpha_{n-1})$ in the list of [8, Theorem 4.2] is a partial case of the family $R_5(\alpha_i)=R_5(\alpha_4,\dots,\alpha_n)$ in Theorem 2.11. Thus, the final list of $(n+1)$-dimensional solvable Leibniz algebras with nilradical F_n^1 should be as in Theorem 2.11 (see also [9]).

Due to the work [10] we conclude that there is no $(n+1)$-dimensional solvable Leibniz algebra whose nilradical is an algebra from the family $F_1(\alpha_4,\dots,\alpha_n,\theta)$ except the algebra F_n^1. Moreover, any $(n+1)$-dimensional solvable Leibniz algebra whose nilradical is an algebra from the family $F_3(\theta_1,\theta_2,\theta_3)$ is a Lie algebra. Concerning the second family, from [10] we know that there exists a unique solvable Leibniz algebra of dimension $(n+1)$ when the nilradical is one of the following:

$$L_1=F_2(0,0,\dots,0,1),\quad L_2^{\beta_{\frac{n+3}{2}}}=F_2^1(0,0,\dots,0,\beta_{\frac{n+3}{2}},0,\dots,0,1),\quad n \text{ is odd},$$
$$L_3^j=F_2^j(0,0,\dots,0,\underbrace{1}_{\beta_j=1},0,\dots,0,0),\quad 4\le j\le n.$$

In particular, any $(n+1)$-dimensional solvable Leibniz algebra whose nilradical is either L_1, $L_2^{\beta_{\frac{n+3}{2}}}$ or L_3^j is isomorphic, respectively, to the algebra with the following table of multiplication:

$$R(L_1):\begin{cases}[e_1,e_1]=e_3, & & [e_2,e_2]=e_n,\\ [e_i,e_1]=e_{i+1}, & 3\le i\le n-1, & [x,e_2]=-\dfrac{n-1}{2}e_2,\\ [x,e_1]=-e_1, & & \\ [e_1,x]=e_1, & & \\ [e_2,x]=\dfrac{n-1}{2}e_2, & & \\ [e_i,x]=ie_i, & 3\le i\le n, & \end{cases}$$

$$R(L_2^{\beta_{\frac{n+3}{2}}}):\begin{cases}[e_1,e_1]=e_3, & & [e_1,e_2]=\beta_{\frac{n+3}{2}}e_{\frac{n+3}{2}}, & \\ [e_i,e_1]=e_{i+1}, & 3\le i\le n-1, & [e_2,e_2]=e_n, & \\ [x,e_1]=-e_1, & & [e_i,e_2]=\beta_{\frac{n+3}{2}}e_{\frac{n-1+2i}{2}}, & 3\le i\le\dfrac{n+1}{2},\\ [e_1,x]=e_1, & & [x,e_2]=-\dfrac{n-1}{2}e_2-\beta_{\frac{n+3}{2}}e_{\frac{n+1}{2}}, & \\ [e_2,x]=\dfrac{n-1}{2}e_2, & & & \\ [e_i,x]=(i-1)e_i, & 3\le i\le n, & & \end{cases}$$

$$R(L_3^j):\begin{cases}[e_1,e_1]=e_3, & & [e_1,e_2]=e_j, & \\ [e_i,e_1]=e_{i+1}, & 3\le i\le n-1, & [e_i,e_2]=e_{j+i-2}, & 3\le i\le n+2-j,\\ [x,e_1]=-e_1, & & [x,e_2]=-(j-2)e_2-e_{j-1}, & \\ [e_1,x]=e_1, & & & \\ [e_2,x]=(j-2)e_2, & & & \\ [e_i,x]=(i-1)e_i, & 3\le i\le n. & & \end{cases}$$

For acquaintance with the definition of cohomology group of Leibniz algebras and its applications to the description of the variety of Leibniz algebras (similar to Lie algebras case) we refer the reader to the papers [1–3,11–13]. Here just recall that the second cohomology group of a Leibniz algebra L with coefficients in a representation M is the quotient space

$$HL^2(L,M)=ZL^2(L,M)/BL^2(L,M),$$

where the 2-cocycles $\varphi\in ZL^2(L,M)$ and the 2-coboundaries $f\in BL^2(L,M)$ are defined as follows

$$(d^2\varphi)(x,y,z)=[x,\varphi(y,z)]-[\varphi(x,y),z]+[\varphi(x,z),y]+\varphi(x,[y,z])-\varphi([x,y],z)+\varphi([x,z],y)=0 \tag{2.1}$$

and

$$f(x,y)=[d(x),y]+[x,d(y)]-d([x,y]) \text{ for some linear map } d. \tag{2.2}$$

The following proposition summarizes results regarding derivations and the second cohomology group for the algebras RFN_n and R_1 from the works [14] and [15].

Proposition 2.13.

$$\begin{array}{ll}\dim \mathrm{Der}(RFN_n)=2, & \dim \mathrm{Der}(R_1)=2,\\ \dim BL^2(RFN_n,RFN_n)=(n+1)^2-2, & \dim BL^2(R_1,R_1)=(n+1)^2-2,\\ \dim ZL^2(RFN_n,RFN_n)=(n+1)^2-2, & \dim ZL^2(R_1,R_1)=(n+1)^2-2,\\ \dim HL^2(RFN_n,RFN_n)=0, & \dim HL^2(R_1,R_1)=0.\end{array}$$

2.2. Degeneration of Leibniz algebras

The bilinear maps $V\times V\to V$ form a vector space $\mathrm{Hom}(V\otimes V,V)$ of dimension $\dim(V)^3$, which can be viewed with its natural structure of affine algebraic variety over the field $\mathbb{F}$. An n-dimensional Leibniz algebra L of the variety $\mathcal{L}eib_n$ may be considered as an element $\lambda(L)$ via the linear map $\lambda: L\otimes L\to L$ satisfying Leibniz identity.

The group $\mathrm{GL}_n(F)$ naturally acts on $\mathcal{L}eib_n$ via change of basis, i.e.,

$$(g*\lambda)(x,y)=g\Big(\lambda\big(g^{-1}(x),g^{-1}(y)\big)\Big),\quad g\in \mathrm{GL}_n(F),\ \lambda\in\mathcal{L}eib_n.$$

The orbits Orb$(-)$ under this action are the isomorphism classes of algebras.

Note that solvable Leibniz algebras of the same dimension also form an invariant subvariety of the variety of Leibniz algebras under the mentioned action.

Definition 2.14. It is said that an algebra λ degenerates to an algebra μ, if Orb(μ) lies in the Zariski closure of Orb(λ) (denoted by $\overline{\mathrm{Orb}(\lambda)}$). We denote this by $\lambda\to\mu$.

It is remarkable that Orb(NF_n) and Orb(RNF_n) are open sets in the subvariety of n-dimensional nilpotent Leibniz algebras [4] and the variety of $(n+1)$-dimensional Leibniz algebras [14], respectively (they are so-called *rigid* algebras).

In the case when the ground field is the field of the complex numbers $\mathbb{C}$, we give an equivalent definition of degeneration.

Definition 2.15. Let $g:(0,1]\to \mathrm{GL}_n(V)$ be a continuous mapping. We construct a parameterized family of Leibniz algebras $g_t=(V,[-,-]_t)$, $t\in(0,1]$ isomorphic to L. For each t the new Leibniz bracket $[-,-]_t$ on V is defined via the old one as follows: $[x,y]_t=g_t[g_t^{-1}(x),g_t^{-1}(y)]$, $\forall x,y\in V$. If for any $x,y\in V$ there exists the limit

$$\lim_{t\to+0}[x,y]_t=\lim_{t\to+0}g_t[g_t^{-1}(x),g_t^{-1}(y)]=:[x,y]_0,$$

then $[-,-]_0$ is a well-defined Leibniz bracket. The Leibniz algebra $L_0=(V,[-,-]_0)$ is called a degeneration of the algebra L.

For given Leibniz algebras $\lambda,\mu\in\mathcal{L}eib_{n+1}$ sometimes it is quite difficult to establish the existence of degeneration $\lambda\to\mu$. It is helpful to obtain some necessary invariant conditions for the existence of a degeneration. The complete list of the invariant conditions can be found in the works [16–19]. Here we give some of them which we shall use.

We denote by Der(λ), λ^m, Lie(λ) the space of derivations, powers and maximal Lie subalgebra of the algebra λ, respectively.

Let $\lambda \to \mu$ be a nontrivial degeneration. Then the following inequalities hold:

$$\dim \mathrm{Der}(\lambda) < \dim \mathrm{Der}(\mu), \quad \dim \lambda^m \geq \dim \mu^m \text{ for any } m \in \mathbb{N}, \quad \dim \mathrm{Lie}(\lambda) \leq \dim \mathrm{Lie}(\mu). \tag{2.3}$$

Further we shall use (i, j)-invariant. This invariant was given for Lie algebras in [16] and it is also applicable for Leibniz algebras.

Let $\lambda \in \mathcal{L}eib_{n+1}$ with structure constants $(\gamma_{i,j}^k)$, and (i, j) be a pair of positive integers such that

$$c_{i,j} = \frac{\mathrm{tr}(\mathrm{ad}\, x)^i \cdot \mathrm{tr}(\mathrm{ad}\, y)^j}{\mathrm{tr}\big((\mathrm{ad}\, x)^i \circ (\mathrm{ad}\, y)^j\big)}$$

is independent of the elements x, y of the Leibniz algebra λ. Then $c_{i,j}(\lambda) = c_{j,i}(\lambda)$ is a quotient of two polynomials in $\mathbb{C}[\gamma_{i,j}^k]$. If neither of these polynomials is zero, we call $c_{i,j} \in \mathbb{C}[\gamma_{i,j}^k]$ an (i, j)-**invariant** of λ. Suppose $\lambda \in \mathcal{L}eib_{n+1}$ has an (i, j)-invariant $c_{i,j}$. Then all $\mu \in \overline{\mathrm{Orb}(\lambda)}$ have the same (i, j)-invariant.

Denote by $\mathcal{LR}_n(N)$ the set of all n-dimensional solvable Leibniz algebras whose nilradical is N. In the paper [20] it is proved that a given degeneration between two solvable Leibniz algebras implies some restriction on their nilradicals. In particular, in the case of equality of dimensions of nilradicals the existence of a degeneration between solvable Leibniz algebras implies the existence of a degeneration between their nilradicals.

Proposition 2.16. *Let R_1, R_2 be n-dimensional solvable Leibniz algebras and $R_1 \in \mathcal{LR}_n(N_1)$, $R_2 \in \mathcal{LR}_n(N_2)$. If $R_1 \to R_2$, then $\dim N_2 \geq \dim N_1$. Moreover, if $\dim N_1 = \dim N_2$ and $R_1 \to R_2$, then $N_1 \to N_2$.*

3. Some irreducible components of the variety $\mathcal{L}eib_{n+1}$

In this section we present some irreducible components of the variety $\mathcal{L}eib_{n+1}$ in terms of closures of orbits of some Leibniz algebras. The following equality $\dim \mathrm{Orb}(\lambda) = (n + 1)^2 - \dim \mathrm{Der}(\lambda)$ gives us the dimensional relation between orbits and derivations of an algebra λ.

Let L be an $(n+1)$-dimensional solvable Leibniz algebra whose nilradical is the filiform algebra F_n^1. The proposition below describes derivations of the algebras from $R_2(\alpha) - R_5(\alpha_i)$ in the list of Theorem 2.11.

Proposition 3.1. *The bases of the spaces of the derivations of the algebras $R_2(\alpha)$– $R_5(\alpha_i)$ are the following:*

$$\mathrm{Der}(R_2(\alpha)): \quad \begin{cases} d_1(e_1) = e_1, & d_1(e_i) = (i-2)e_i, \quad 3 \leq i \leq n, \\ d_2(e_i) = e_i, & \qquad 2 \leq i \leq n, \qquad \alpha \neq 2-n;\ 1-n, \\ d_3(x) = -e_1, & d_3(e_i) = e_{i+1}, \quad 2 \leq i \leq n-1; \end{cases}$$

$$\mathrm{Der}(R_2(2-n)): \quad \begin{cases} d_1(e_1) = e_1, & d_1(e_i) = (i-2)e_i, \quad 3 \leq i \leq n, \\ d_2(e_i) = e_i, & \qquad 2 \leq i \leq n, \\ d_3(x) = -e_1, & d_3(e_i) = e_{i+1}, \quad 2 \leq i \leq n-1, \\ d_4(x) = -e_{n-1}, & d_4(e_1) = e_n; \end{cases}$$

$$\mathrm{Der}(R_2(1-n)): \quad \begin{cases} d_1(e_1) = e_1, & d_1(e_i) = (i-2)e_i, \quad 3 \leq i \leq n, \\ d_2(e_i) = e_i, & \qquad 2 \leq i \leq n, \\ d_3(x) = -e_1, & d_3(e_i) = e_{i+1}, \quad 2 \leq i \leq n-1, \\ d_4(x) = e_n; \end{cases}$$

$$\mathrm{Der}(R_3): \quad \begin{cases} d_1(e_1) = e_1, & d_1(e_i) = (i-n)e_i, \quad 2 \leq i \leq n-1, \\ d_2(x) = e_1, & d_2(e_i) = -e_{i+1}, \quad 2 \leq i \leq n-1, \\ d_3(x) = e_n; \end{cases}$$

$$\mathrm{Der}(R_4): \quad \begin{cases} d_1(e_1) = e_1, & d_1(e_i) = (i+1-n)e_i, \quad 2 \leq i \leq n, \\ d_2(x) = -e_1, & d_2(e_i) = e_{i+1}, \quad 2 \leq i \leq n-1, \\ d_3(e_1) = e_n, & d_3(x) = -e_{n-1}; \end{cases}$$

$$\mathrm{Der}(R_5(0)): \quad \begin{cases} d_1(e_1) = e_1, & d_1(e_i) = (i-1)e_i, \quad 2 \leq i \leq n, \\ d_2(e_1) = e_2, & d_2(e_i) = e_i, \quad 2 \leq i \leq n, \\ d_j(e_1) = e_j, & d_j(e_i) = e_{i+j-2}, \quad 3 \leq j \leq n,\ 2 \leq i \leq n-j+2; \end{cases}$$

$$\mathrm{Der}(R_5(\alpha_i)): \quad \begin{cases} d_1(e_1) = e_2, & d_2(e_i) = e_2 + \alpha_n e_n, \quad d_1(e_i) = e_i, \quad 3 \leq i \leq n, \\ d_j(e_1) = e_{j+1}, & d_j(e_i) = e_{i+j-1}, \quad 2 \leq j \leq n-1, \quad 2 \leq i \leq n-j+1, \end{cases}$$

where in the case of $R_5(\alpha_i)$ one of the parameters α_i is nonzero.

Proof. Let d be a derivation of $R_2(\alpha)$.

We put

$$d(x) = a_{0,0}x + \sum_{j=1}^{n} a_{0,j}e_j, \qquad d(e_1) = a_{1,0}x + \sum_{j=1}^{n} a_{1,j}e_j, \qquad d(e_2) = a_{2,0}x + \sum_{j=1}^{n} a_{2,j}e_j.$$

From the conditions

$$d([x,x]) = [d(x),x] + [x,d(x)], \quad d([e_1,e_1]) = [d(e_1),e_1] + [e_1,d(e_1)], \quad d([e_2,e_2]) = [d(e_2),e_2] + [e_2,d(e_2)],$$

we obtain that

$$\begin{aligned} a_{0,j}(j-1+\alpha) &= 0, & & 2 \le j \le n, \\ a_{2,1} = a_{2,0}(1+\alpha) &= 0, \quad a_{1,j} = 0, & & 2 \le j \le n-1. \end{aligned}$$

By using the property of derivation

$$d([e_1,x]) = [d(e_1),x] + [e_1,d(x)], \quad d([x,e_1]) = [d(x),e_1] + [x,d(e_1)], \quad d([e_2,x]) = [d(e_2),x] + [e_2,d(x)],$$

we obtain that

$$\begin{aligned} a_{0,0} = a_{1,0} &= 0, & (n-2+\alpha)a_{1,n} &= 0, \\ a_{0,n-1} &= -a_{1,n}, & a_{0,j} &= 0, \quad 2 \le j \le n-2, \\ a_{0,1} &= -a_{2,3}, & a_{2,j} &= 0, \quad 4 \le j \le n. \end{aligned}$$

From the property

$$d(e_i) = d([e_{i-1},e_1]) = [d(e_{i-1}),e_1] + [e_{i-1},d(e_1)],$$

inductively we have

$$d(e_i) = ((i-2)a_{1,1} + a_{2,2})e_i + a_{2,3}e_{i+1}, \qquad 3 \le i \le n.$$

Moreover, the property $0 = d([e_3,e_2]) = [d(e_3),e_2] + [e_3,d(e_2)] = (2+\alpha)a_{2,0}$, implies $(2+\alpha)a_{2,0} = 0$. Taking into account $(1+\alpha)a_{2,0} = 0$, we deduce $a_{2,0} = 0$.

Therefore, we derive that any derivation of the algebra $R_2(\alpha)$ has the form

$$\begin{aligned} d(e_1) &= a_{1,1}e_1 + a_{1,n}e_n, \\ d(e_2) &= a_{2,2}e_2 + a_{2,3}e_3, \\ d(x) &= -a_{2,3}e_1 - a_{1,n}e_{n-1} + a_{0,n}e_n, \\ d(e_i) &= ((i-2)a_{1,1} + a_{2,2})e_i + a_{2,3}e_{i+1}, \qquad 3 \le i \le n, \end{aligned}$$

with the conditions

$$(n-2+\alpha)a_{1,n} = 0, \qquad (n-1+\alpha)a_{0,n} = 0.$$

From these conditions it is not difficult to obtain a basis of the derivations in the cases of $\alpha = 2-n$, $\alpha = 1-n$ and $\alpha \neq 2-n; 1-n$.

The proof of the proposition for the algebras R_3, R_4 and $R_5(\alpha_i)$ is carried out by analogous direct computations. □

Due to equality (2.2) defining the space BL^2 we have

Corollary 3.2.

$$\dim BL^2\big(R_2(\alpha), R_2(\alpha)\big) = \begin{cases} (n+1)^2 - 4, & \alpha = 2-n \text{ or } 1-n, \\ (n+1)^2 - 3, & \alpha \neq 2-n,\ 1-n; \end{cases}$$

$$\dim BL^2(R_3, R_3) = (n+1)^2 - 3;$$

$$\dim BL^2(R_4, R_4) = (n+1)^2 - 3;$$

$$\dim BL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big) = \begin{cases} n^2 + n + 1, & \alpha_i = 0 \ \text{ for all } i, \\ n^2 + n + 2, & \alpha_i \neq 0 \ \text{ for some } i. \end{cases}$$

Proof. From Proposition 3.1 we have that

$$\dim(\mathrm{Der}(R_2(\alpha))) = \begin{cases} 4, & \alpha = 2-n \text{ or } 1-n, \\ 3, & \alpha \neq 2-n,\ 1-n; \end{cases}$$

$$\dim(\mathrm{Der}(R_3)) = 4,$$

$$\dim(\mathrm{Der}(R_4)) = 3,$$
$$\dim(\mathrm{Der}(R_5(\alpha_i))) = \begin{cases} n, & \alpha_i = 0 \ \text{ for all } i, \\ n-1, & \alpha_i \neq 0 \ \text{ for some } i. \end{cases}$$

We obtain that $\dim BL^2(L, L) = (\dim L)^2 - \dim \mathrm{Der}(L)$, using the fact that $BL^2(L, L) = \{f(x, y) \mid f(x, y) = [d(x), y] + [x, d(y)] - d([x, y])$ for some linear map $d\}$. Since $\dim(L) = n + 1$, we complete the proof. □

The following result presents values of $(1, 1)$-invariants for the algebras RNF_n, R_1–$R_5(\alpha_i)$, $R(L_1)$, $R(L_2^{\beta_{\frac{n+3}{2}}})$, $R(L_3^j)$.

Proposition 3.3.

$$c_{1,1}(RNF_n) = \frac{(1+2+3+\cdots+n)^2}{1+2^2+3^2+\cdots+n^2} = \frac{3n(n+1)}{2(2n+1)},$$
$$c_{1,1}(R_1) = \frac{(1+1+2+3+\cdots+n-1)^2}{1+1+2^2+3^2+\cdots+(n-1)^2} = \frac{3(n^2-n+2)^2}{2(2n^3-3n^2+n+6)},$$
$$c_{1,1}(R(L_1)) = \frac{(1+2+3+\cdots+n-1+\frac{n-1}{2})^2}{1+2^2+3^2+\cdots+(n-1)^2+(\frac{n-1}{2})^2} = \frac{3(n^2-1)}{4n-3},$$
$$c_{1,1}(R(L_2^{\beta_{\frac{n+3}{2}}})) = \frac{(1+2+3+\cdots+n-1+\frac{n-1}{2})^2}{1+2^2+3^2+\cdots+(n-1)^2+(\frac{n-1}{2})^2} = \frac{3(n^2-1)}{4n-3},$$
$$c_{1,1}(R(L_3^j)) = \frac{(1+2+3+\cdots+n-1+j-2)^2}{1+2^2+3^2+\cdots+(n-1)^2+(j-2)^2} = \frac{3(n^2-n+2j-4)^2}{2(2n^3-3n^2+n+6j^2-24j+24)},$$
$$c_{1,1}(R_2(\alpha)) = \frac{(1+(1+\alpha)+(2+\alpha)+\cdots+(n-1+\alpha))^2}{1^2+(1+\alpha)^2+(2+\alpha)^2+\cdots+(n-1+\alpha)^2}$$
$$= \frac{3(2-2\alpha-n+2\alpha n+n^2)^2}{4n^3+(12\alpha-6)n^2+(2-12\alpha+12\alpha^2)n-12\alpha^2+12},$$
$$c_{1,1}(R_3) = \frac{(1+(2-n)+(3-n)+\cdots+(-1))^2}{1^2+(2-n)^2+(3-n)^2+\cdots+(-1)^2} = \frac{3n(n-3)^2}{2(2n^2-9n+13)},$$
$$c_{1,1}(R_4) = \frac{(1+(3-n)+(4-n)+\cdots+(-1)+0+1)^2}{1^2+(3-n)^2+(4-n)^2+\cdots+(-1)^2+0^2+1^2} = \frac{3(n^2-5n+2)^2}{2(2n^3-15n^2+37n-18)},$$
$$c_{1,1}(R_5(\alpha_i)) = n-1.$$

Proof. Consider $\mathrm{ad}\, x$ for the algebra $RNF_n : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n-1, \\ [x, e_1] = -e_1, & \\ [e_i, x] = ie_i, & 1 \le i \le n. \end{cases}$

It is easy to see that the matrix representation of $\mathrm{ad}\, x$ has the form

$$\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & n & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

Therefore,

$$c_{1,1}(RNF_n) = \frac{\mathrm{tr}(\mathrm{ad}\, x)\cdot \mathrm{tr}(\mathrm{ad}\, x)}{\mathrm{tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, x)} = \frac{(1+2+3+\cdots+n)^2}{1+2^2+3^2+\cdots+n^2} = \frac{3n(n+1)}{2(2n+1)}.$$

The proof of the proposition for the rest of the algebras also is carried out by applying similar arguments. □

Remark 3.4. From Proposition 3.3 it is easy to see that

$$c_{1,1}(R_2(1)) = c_{1,1}(RNF_n), \qquad c_{1,1}(R_2(0)) = c_{1,1}(R_1),$$
$$c_{1,1}(R_2(1-n)) = c_{1,1}(R_3), \qquad c_{1,1}(R_2(2-n)) = c_{1,1}(R_4).$$

Below, we give some degenerations.

Example 3.5. There exist the following degenerations

$$\begin{array}{llllll} F_n^1 \to F_n^2 & \text{via} & g_t(e_1) = e_1 - t^{-1}e_2, & g_t(e_2) = t^{-1}e_2, & g_t(e_i) = e_i, & 3 \le i \le n; \\ RNF_n \to R_2(1) & \text{via} & g_t(x) = x, & g_t(e_1) = t^{-1}e_1, & g_t(e_i) = t^{-i+2}e_i, & 2 \le i \le n; \\ R_1 \to R_2(0) & \text{via} & g_t(x) = x, & g_t(e_1) = e_1, & g_t(e_i) = te_i, & 2 \le i \le n; \\ R_3 \to R_2(1-n) & \text{via} & g_t(x) = x, & g_t(e_1) = e_1, & g_t(e_i) = te_i, & 2 \le i \le n; \\ R_4 \to R_2(2-n) & \text{via} & g_t(x) = x, & g_t(e_1) = e_1, & g_t(e_i) = te_i, & 2 \le i \le n; \\ R_5(\alpha_i) \to R_5(0) & \text{via} & g_t(x) = x, & g_t(e_1) = te_1, & g_t(e_i) = t^{i-1}e_i, & 2 \le i \le n. \end{array}$$

Now we present representatives of irreducible components of the variety $\mathcal{L}eib_{n+1}$.

Theorem 3.6. *$\overline{\mathrm{Orb}(R_3)}$, $\overline{\mathrm{Orb}(R_4)}$ and $\overline{\cup_{\alpha\in K}\mathrm{Orb}(R_2(\alpha))}$, where $K = \mathbb{C}\setminus\{0, 1, 1-n, 2-n\}$, are irreducible components of $\mathcal{L}eib_{n+1}$.*

Proof. Firstly, we will prove the assertion of the theorem for the algebra R_3, that is, we will prove the non-existence of degeneration from any algebra $X \in \mathcal{L}eib_{n+1}$ to the algebra R_3. Let us assume the contrary, i.e., $X \to R_3$, then from Proposition 3.1 and inequalities (2.3) we conclude that $\dim(\mathrm{Der}\,X) < \dim(\mathrm{Der}\,R_3) = 3$. Actually, the algebra X is a solvable Leibniz algebra. Indeed, if X is not solvable then by Levi's Theorem [21] it decomposes into a semidirect sum of a semi-simple Lie algebra S and a solvable radical $\mathrm{Rad}(X)$. Since in a semi-simple Lie algebra (which, clearly, has dimension greater than or equal to 3) an operator $\mathrm{ad}(x)$ for any $x \in S$ is a derivation of the algebra X, we obtain $\dim(\mathrm{Der}\,X) \ge 3$. Therefore, the algebra X is solvable.

Taking into account that $\dim R_3^2 = n$ and solvability of X from inequalities (2.3), we derive that $\dim X^2 = n$. Since the square of a solvable algebra belongs to the nilradical, the solvable algebra X has nilradical of dimension n. Therefore, the nilradical of X degenerates to F_n^1 (because nilradical of the algebra R_3 is F_n^1). Again applying inequality (2.3) we derive that the nilradical of X is one of the following algebras:

$$NF_n, \qquad F_n^1(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta), \qquad F_2(\beta_4, \ldots, \beta_n, \gamma), \qquad F_3(\theta_1, \theta_2, \theta_3).$$

The non-trivial degeneration $F_n^1 \to F_n^2$ from Example 3.5 implies the non-existence of degeneration from the algebra F_n^2 to the algebra F_n^1.

From arguments above we conclude that the possibilities for the solvable algebra X are the following:

$$RNF_n, \qquad R_1 - R_5(\alpha_i), \qquad R(L_1), \qquad R(L_2^{\beta_{\frac{n+3}{2}}}), \qquad R(L_3^j).$$

By comparing $c_{1,1}$ invariants from Proposition 3.3, dimensions of the spaces of derivations from Propositions 2.13, 3.1 and applying inequalities (2.3), we conclude that none of the above algebras degenerates to the algebra R_3. Hence, $\overline{\mathrm{Orb}(R_3)}$ is an irreducible component of the variety $\mathcal{L}eib_{n+1}$.

The assertion of theorem for the algebra R_4 and the family of algebras $R_2(\alpha)$, with $\alpha \in K$, is proved applying the same arguments as used for the algebra R_3 and degenerations from Example 3.5. □

Let us assume that $X \to R_5(\alpha_4, \ldots, \alpha_n)$. Since $\dim \mathrm{Der}(R_5(\alpha_4, \ldots, \alpha_n)) > 3$ then from dimensions arguments as they were used for the algebra R_3 we cannot assert the solvability of X. Nevertheless applying similar arguments as in Theorem 3.6 we obtain the following result.

Proposition 3.7. *$\overline{\cup_{\alpha\in\mathbb{C}^*}\mathrm{Orb}(R_5(\alpha_4, \ldots, \alpha_n))}$ forms an irreducible component in the subvariety of $(n+1)$-dimensional solvable Leibniz algebras of the variety $\mathcal{L}eib_{n+1}$.*

3.1. Second cohomology of the algebras R_2–R_5

In this subsection we give descriptions of the second cohomology group of the algebras R_2–R_5 by presenting their bases. In fact, we find bases of the space BL^2 and ZL^2 for these algebras. These descriptions can be applied in the study of infinitesimal deformations and extensions of mentioned algebras (see works [3,22–26] and references therein).

In the next theorem we present the general form of 2-cocycles for the algebra R_3.

Theorem 3.8. *An arbitrary $\varphi \in ZL^2(R_3, R_3)$ has the following form:*

$$\varphi(e_1, e_1) = \sum_{i=3}^{n} a_{1,i}e_i,$$

$$\varphi(e_i, e_1) = a_{i,0}x + \sum_{s=1}^{n} a_{i,s}e_s, \quad 2 \le i \le n-1,$$

$$\varphi(e_n, e_1) = a_{n-1,0}e_1 - \sum_{i=3}^{n-1}\left(\sum_{t=2}^{i-1} a_{n+t-i,t}\right)e_i + a_{n,n}e_n,$$

$$
\begin{aligned}
&\varphi(e_1, e_2) = b_{1,1}e_1,\\
&\varphi(e_i, e_2) = (i-n)b_{1,1}e_i + b_{2,3}e_{i+1}, \quad 2 \le i \le n,\\
&\varphi(e_1, e_i) = -a_{i-1,0}e_1, \ 3 \le i \le n,\\
&\varphi(e_i, e_3) = (n-i)a_{2,0}e_i - (a_{2,1} + b_{1,1})e_{i+1}, \quad 2 \le i \le n-1,\\
&\varphi(e_i, e_j) = (n-i)a_{j-1,0}e_i + (a_{j-2,0} - a_{j-1,1})e_{i+1}, \ 2 \le i \le n-1, \quad 4 \le j \le n,\\
&\varphi(e_1, x) = \frac{2}{(n-2)(n-1)} \sum_{t=2}^{n} a_{t,t}x + c_{1,1}e_1 + \sum_{i=2}^{n-1}(i-n-1)a_{1,i+1}e_i + c_{1,n}e_n,\\
&\varphi(e_2, x) = (n-2)b_{1,1}x + (n-1)b_{2,3}e_1 + \sum_{i=2}^{n} c_{2,i}e_i,\\
&\varphi(e_3, x) = (3-n)a_{2,0}x + (2-n)(a_{2,1} + b_{1,1})e_1 + \Big(a_{2,2} - \frac{2}{n-1}\sum_{t=2}^{n} a_{t,t}\Big)e_2 + (c_{2,2} + c_{1,1})e_3\\
&\qquad + \sum_{i=4}^{n-1}(c_{2,i-1} + (3-i)a_{2,i})e_i + (c_{2,n-1} + (3-n)a_{2,n} - a_{2,0})e_n,\\
&\varphi(e_i, x) = (i-n)a_{i-1,0}x + (n-i+1)(a_{i-2,0} - a_{i-1,1})e_1 + \sum_{s=2}^{i-2}(i-s)\sum_{t=2}^{s} a_{i+t-s-1,t}e_s\\
&\qquad + \Big(\sum_{t=2}^{i-1} a_{t,t} + \frac{(i+1-2n)(i-2)}{(n-2)(n-1)}\sum_{t=2}^{n} a_{t,t}\Big)e_{i-1} + (c_{2,2} + (i-2)c_{1,1})e_i\\
&\qquad + \sum_{s=i+1}^{n-1}\Big(c_{2,s-i+2} + (i-s)\sum_{t=2}^{i-1} a_{t,s-i+t+1}\Big)e_s + \Big(c_{2,n-i+2} - a_{i-1,0} + (i-n)\sum_{t=2}^{i-1} a_{t,n-i+t+1}\Big)e_n, \quad 4 \le i \le n,\\
&\varphi(x, e_1) = -\frac{2}{(n-2)(n-1)}\sum_{t=2}^{n} a_{t,t}x - c_{1,1}e_1 + a_{1,3}e_2 + \sum_{i=3}^{n} d_{1,i}e_i,\\
&\varphi(x, e_2) = -b_{2,3}e_1 + b_{1,1}e_n,\\
&\varphi(x, e_3) = (a_{2,1} + b_{1,1})e_1 - a_{2,0}e_n,\\
&\varphi(x, e_i) = (a_{i-1,1} - a_{i-2,0})e_1 - a_{i-1,0}e_n, \quad 4 \le i \le n,\\
&\varphi(x, x) = a_{n-1,0}x + (a_{n-1,1} - a_{n-2,0})e_1 + \sum_{i=2}^{n-2}\Big((n-i)(a_{1,i+2} - d_{1,i+1}) + \sum_{s=2}^{i} a_{n-i+s-1,s}\Big)e_i\\
&\qquad - (2c_{1,0} + d_{1,n} + c_{1,n} + a_{n,n})e_{n-1} + \beta_n e_n.
\end{aligned}
$$

Proof. We set

$$
\begin{cases}
\varphi(e_i, e_1) = a_{i,0}x + \sum_{s=1}^{n} a_{i,s}e_s, \ 1 \le i \le n, & \varphi(e_1, x) = c_{1,0}x + \sum_{i=1}^{n} c_{1,i}e_i,\\
\varphi(e_1, e_2) = b_{1,0}x + \sum_{i=1}^{n} b_{1,i}e_i, & \varphi(e_2, x) = c_{2,0}x + \sum_{i=1}^{n} c_{2,i}e_i,\\
\varphi(e_2, e_2) = b_{2,0}x + \sum_{i=1}^{n} b_{2,i}e_i, & \varphi(x, e_1) = d_{1,0}x + \sum_{i=1}^{n} d_{1,i}e_i,\\
\varphi(x, x) = \beta_0 x + \sum_{i=1}^{n} \beta_i e_i, & \varphi(x, e_2) = d_{2,0}x + \sum_{i=1}^{n} d_{2,i}e_i.
\end{cases}
$$

By applying equality (2.1) for the triple (e_i, e_1, e_1), we obtain $[e_i, \varphi(e_1, e_1)] = 0$. Hence, $\varphi(e_1, e_1) = \sum_{i=2}^{n} a_{1,i}e_i$. Similarly, the equation $(d^2\varphi)(e_i, e_j, e_k) = 0$, for $2 \le j, k \le n$, leads to $[e_i, \varphi(e_j, e_k)] = 0$. Consequently, we have $\varphi(e_j, e_k) \in \operatorname{span}\langle e_2, e_3, \ldots, e_n\rangle$.

Considering $(d^2\varphi)(e_1, e_1, e_2) = 0$, we derive

$$\varphi(e_1, e_2) = b_{1,0}x + b_{1,1}e_1 + b_{1,n}e_n.$$

Moreover, from $(d^2\varphi)(e_i, e_1, e_2) = 0$ with $2 \le i \le n-1$, we deduce

$$\varphi(e_{i+1}, e_2) = [e_i, \varphi(e_1, e_2)] + [\varphi(e_i, e_2), e_1],$$

which inductively gets

$$\varphi(e_i, e_2) = b_{1,0}\frac{(i-2)(i+1-2n)}{2}e_i + \big((i-2)b_{1,1} + b_{2,2}\big)e_i + \sum_{s=i+1}^{n} b_{2,s-i+2}e_s, \quad 2 \le i \le n.$$

Similarly, we have

$$\begin{array}{lll} (d^2\varphi)(e_n, e_1, e_2) = 0 & \Rightarrow & b_{1,0} = 0, \\ (d^2\varphi)(e_1, e_2, x) = 0 & \Rightarrow & b_{1,n} = 0, \quad c_{2,0} = (n-2)b_{1,1}, \\ (d^2\varphi)(e_2, e_2, x) = 0 & \Rightarrow & b_{2,2} = (2-n)b_{1,1}, \quad c_{2,1} = (n-1)b_{1,3}, \quad b_{2,i} = 0,\ 4 \le i \le n. \end{array}$$

Now we consider $(d^2\varphi)(e_i, e_j, e_1) = 0$ with $2 \le j \le n-1$. Then we have

$$\varphi(e_i, e_{j+1}) = [\varphi(e_i, e_j), e_1] - [e_i, \varphi(e_j, e_1)] - \varphi([e_i, e_1], e_j).$$

Applying the induction on j for any i and the equality above we obtain the following:

$$\begin{array}{ll} \varphi(e_1, e_j) = -a_{j-1,0}e_1, & 3 \le j \le n, \\ \varphi(e_i, e_3) = (n-i)a_{2,0}e_i - (a_{2,1} + b_{1,1})e_{i+1}, & 2 \le i \le n-1, \\ \varphi(e_i, e_j) = (n-i)a_{j-1,0}e_i + (a_{j-2,0} - a_{j-1,1})e_{i+1}, & 2 \le i \le n-1,\ 4 \le j \le n. \end{array}$$

On the other hand, the condition $(d^2\varphi)(e_i, e_n, e_1) = 0$ implies

$$a_{n,0} = 0, \quad a_{n,1} = a_{n-1,0}.$$

We consider equality (2.1) for the triple (e_1, x, e_1), then we get

$$d_{1,0} = -c_{1,0}, \quad a_{1,2} = 0, \quad c_{1,i} = (i-1-n)a_{1,i+1},\ 2 \le i \le n-1.$$

Thus, we have

$$\varphi(e_1, x) = c_{1,0}x + c_{1,1}e_1 + \sum_{i=2}^{n-1}(i-n-1)a_{1,i+1}e_i + c_{1,n}e_n,$$

$$\varphi(e_2, x) = (n-2)b_{1,1}x + (n-1)b_{2,3}e_1 + \sum_{i=2}^{n} c_{2,i}e_i$$

and $[e_1, \varphi(e_i, x)] = (i-n)a_{i-1,0}e_1$ for $3 \le i \le n$.

From the equality $(d^2\varphi)(e_i, e_1, x) = 0,\ 2 \le i \le n-1$, we have

$$\begin{aligned} \varphi(e_{i+1}, x) &= [\varphi(e_i, x), e_1] + [e_i, \varphi(e_1, x)] - [\varphi(e_i, e_1), x] + \varphi(e_i, [e_1, x]) + \varphi([e_i, x], e_1) \\ &= [\varphi(e_i, x), e_1] + (i+1-n)x + c_{1,0}e_i + c_{1,1}e_{i+1} + (i-n)a_{i,1}e_1 + \sum_{s=2}^{n}(i+1-s)a_{i,s}e_s. \end{aligned}$$

Hence, we obtain inductively that

$$\begin{aligned} \varphi(e_3, x) &= (3-n)a_{2,0}x + (2-n)(a_{2,1} + b_{1,1})e_1 + (a_{2,2} + (2-n)c_{1,0})e_2 + (c_{2,2} + c_{1,1})e_3 \\ &\quad + \sum_{i=4}^{n-1}(c_{2,i-1} + (3-i)a_{2,i})e_i + (c_{2,n-1} + (3-n)a_{2,n} - a_{2,0})e_n, \\ \varphi(e_i, x) &= (i-n)a_{i-1,0}x + (n-i+1)(a_{i-2,0} - a_{i-1,1})e_1 + \sum_{s=2}^{i-2}(i-s)\sum_{t=2}^{s} a_{i+t-s-1,t}e_s \\ &\quad + \left(\frac{(i+1-2n)(i-2)}{2}c_{1,0} + \sum_{t=2}^{i-1} a_{t,t}\right)e_{i-1} + (c_{2,2} + (i-2)c_{1,1})e_i \\ &\quad + \sum_{s=i+1}^{n-1}\left(c_{2,s-i+2} + (i-s)\sum_{t=2}^{i-1} a_{t,s-i+t+1}\right)e_s + \left(c_{2,n-i+2} - a_{i-1,0} + (i-n)\sum_{t=2}^{i-1} a_{t,n-i+t+1}\right)e_n, \end{aligned}$$

where $4 \le i \le n$.

Moreover, the condition $(d^2\varphi)(e_n, e_1, x) = 0$ implies $[\varphi(e_n, x), e_1] - [\varphi(e_n, e_1), x] + \varphi(e_n, [e_1, x]) = 0$, which derives

$$(n-1)a_{n,2}e_2 + \sum_{s=3}^{n-1}(n-s+1)\Big(a_{n,s} + \sum_{t=2}^{s-1} a_{n+t-s,t}\Big)e_s + \Big(-\frac{(n-1)(n-2)}{2}c_{1,0} + \sum_{t=2}^{n} a_{t,t}\Big)e_n = 0.$$

Thus, we get

$$a_{n,2} = 0, \quad c_{1,0} = \frac{2}{(n-1)(n-2)}\sum_{t=2}^{n} a_{t,t}, \quad a_{n,s} = -\sum_{t=2}^{s-1} a_{n+t-s,t},\ 3 \le s \le n-1.$$

Considering equality (2.1) for the following triples $(e_2, x, e_1),\ (e_1, x, e_2),\ (x, e_1, e_2),\ (e_2, x, e_2)$ we obtain:

$$d_{1,1} = -c_{1,1}, \quad d_{2,0} = 0, \quad d_{2,s} = 0,\ 2 \le s \le n-1, \quad d_{2,1} = -b_{2,3}.$$

From $(d^2\varphi)(x, e_i, e_1) = 0$ with $2 \le i \le n-1$, we have

$$\varphi(x, e_{i+1}) = [\varphi(x, e_i), e_1] - [x, \varphi(e_i, e_1)] + \varphi(e_1, e_i),$$

which inductively implies

$$\varphi(x, e_3) = (a_{2,1} + b_{1,1})e_1 - a_{2,0}e_n,$$
$$\varphi(x, e_i) = (a_{i-1,1} - a_{i-2,0})e_1 - a_{i-1,0}e_n, \qquad 4 \le i \le n.$$

Finally, the equalities $(d^2\varphi)(e_1, x, x) = (d^2\varphi)(x, e_1, x) = (d^2\varphi)(x, e_2, x) = 0$ imply

$$d_{1,2} = a_{1,3}, \qquad d_{2,n} = b_{1,1},$$
$$\beta_0 = a_{n-1,0}, \qquad \beta_1 = a_{n-1,1} - a_{n-2,0},$$
$$\beta_{n-1} = -d_{1,n} - c_{1,n} - a_{n,n} - \frac{4}{(n-1)(n-2)}\sum_{t=2}^{n} a_{t,t},$$
$$\beta_i = (n-i)(a_{1,i+2} - d_{1,i+i}) + \sum_{s=2}^{i} a_{n-i+s-1}, \quad 2 \le i \le n-2,$$

which complete the proof of the theorem. □

Corollary 3.9. $\dim ZL^2(R_3, R_3) = (n+1)^2 - 2$ *and* $\dim HL^2(R_3, R_3) = 1$.

Now we present a basis of $HL^2(R_3, R_3)$.

Proposition 3.10. *The equivalence class* $\overline{\xi}$ *forms a basis of* $HL^2(R_3, R_3)$*, where*

$$\xi : \begin{cases} \xi(e_1, x) = e_1, \\ \xi(e_i, x) = (i-2)e_i, \quad 3 \le i \le n, \\ \xi(x, e_1) = -e_1. \end{cases}$$

Proof. In order to find a basis of $HL^2(R_3, R_3)$ we need to describe linear independent elements which lie in $ZL^2(R_3, R_3)$ and do not lie in $BL^2(R_3, R_3)$. For achieving this purpose we will find a basis of 2-cocycles and 2-coboundaries.

Since an arbitrary element of $ZL^2(R_3, R_3)$ has the form of Theorem 3.8 we shall use this description.

Note that there are parameters $(a_{i,j}, b_{1,1}, b_{2,3}, c_{1,1}, c_{1,n}, \beta_n, c_{2,k}, d_{1,s})$ in the general form of elements $ZL^2(R_3, R_3)$. One of the natural basis of the space ZL^2 is a basis whose basis elements are obtained by the instrumentality of these parameters. For the fixed pair (i, j) we denote by $\chi(a_{i,j})$ a cocycle which has $a_{i,j} = 1$ and all other parameters are equal to zero. Define such type of notation for other parameters.

Set

$$\varphi_{i,j} = \chi(a_{i,j}), \quad \psi_1 = \chi(b_{1,1}), \quad \psi_2 = \chi(b_{2,3}), \quad \psi_3 = \chi(c_{1,1}),$$
$$\psi_4 = \chi(c_{1,n}), \quad \psi_5 = \chi(\beta_n), \quad \eta_k = \chi(c_{2,k}), \quad \rho_s = \chi(d_{1,s}),$$

where $1 \le i \le n, 0 \le j \le n, 2 \le k \le n, 3 \le s \le n$, and

$$(i, j) \notin \{(1, 0), (1, 1), (1, 2), (n, 0), (n, 1), \dots, (n, n-1)\}.$$

In order to find a basis of $BL^2(R_3, R_3)$ we consider the endomorphisms $f_{j,k} : R_3 \to R_3$ defined as follows

$$\begin{aligned} f_{i,j}(e_i) &= e_j, && 1 \le i, j \le n, \\ f_{i,n+1}(e_i) &= x, && 1 \le i \le n, \\ f_{n+1,j}(x) &= e_j, && 1 \le j \le n, \\ f_{n+1,n+1}(x) &= x, \end{aligned}$$

where in the expansion of the endomorphisms the omitted values are assumed to be zero.

Consider

$$g_{i,j}(x, y) = [f_{i,j}(x), y] + [x, f_{i,j}(y)] - f_{i,j}([x, y]).$$

Note that $g_{i,j} \in BL^2(R_3, R_3)$ and now we separate a basis from these elements. Since the dimension of the space $\mathrm{Der}(R_3)$ is equal to 3, to take a basis we should exclude three elements $g_{i,j}$. The description of $\mathrm{Der}(R_3)$ allows us to release the elements $g_{1,1}, g_{2,3}$ and $g_{n+1,n}$.

By direct computation we obtain

$$g_{j,k} : \begin{cases} g_{1,2} = -\varphi_{1,3}, \\ g_{1,i} = -\rho_i - \varphi_{1,i+1}, & 3 \le i \le n-1, \\ g_{1,n} = \psi_4 - \rho_n, \\ g_{1,n+1} = \sum\limits_{k=2}^{n-1} (n-k)\varphi_{k,k} - \psi_4 - \rho_n, \\ g_{2,1} = -\psi_2, \\ g_{2,i} = -\varphi_{2,i+1} + (2-i)\eta_i, & 2 \le i \le n-1,\ i \ne 3, \\ g_{2,n} = (2-n)\eta_n, \\ g_{2,n+1} = \varphi_{2,1} - \psi_1 - \eta_n, \\ g_{i,k} = \varphi_{i-1,k} - \varphi_{i,k+1}, & 3 \le i \le n-1,\ 1 \le k \le n-1, \\ g_{i,n} = \varphi_{i-1,n}, & 3 \le i \le n-1, \\ g_{i,n+1} = \varphi_{i-1,0} + \varphi_{i,1}, & 3 \le i \le n-1, \\ g_{n,1} = \varphi_{n-1,1}, \\ g_{n,i} = \varphi_{n-1,i}, & 2 \le i \le n-2, \\ g_{n,n-1} = \varphi_{n-1,n-1} - \varphi_{n,n}, \\ g_{n,n} = \varphi_{n-1,n} + \psi_5, \\ g_{n,n+1} = \varphi_{n-1,0}, \\ g_{n+1,1} = -\eta_3, \\ g_{n+1,i} = -\rho_{i+1}, & 2 \le i \le n-1, \\ g_{n+1,n+1} = \psi_3 - 2\psi_5 + (n-2)\eta_2. \end{cases}$$

From these equalities, it is not difficult to check that ψ_3 and η_2 do not belong to $BL^2(R_3, R_3)$, but $\psi_3 + (n-2)\eta_2 \in BL^2(R_3, R_3)$. Thus, we can take the equivalence class of ψ_3 as a basis of $ZL^2(R_3, R_3)$. $\square$

Theorem 3.11. *Let* $\dim(R_4) \ge 5$. *An arbitrary* $\varphi \in ZL^2(R_4, R_4)$ *has the following form:*

$$\begin{aligned} &\varphi(e_1, e_1) = \sum_{i=3}^{n} a_{1,i} e_i, \\ &\varphi(e_i, e_1) = a_{i,0} x + \sum_{s=1}^{n} a_{i,s} e_s,\ 2 \le i \le n-1, \\ &\varphi(e_n, e_1) = a_{n-1,0} e_1 - \sum_{i=3}^{n-1} \left(\sum_{t=2}^{i-1} a_{n+t-i,t} \right) e_i - \left(\sum_{t=2}^{n-1} a_{t,t} + \frac{(n-4)(n-1)}{2} d_{1,0} \right) e_n, \\ &\varphi(e_1, e_2) = b_{1,1} e_1 + b_{1,1} e_n, \\ &\varphi(e_i, e_2) = (i-n+1) b_{1,1} e_i + b_{2,3} e_{i+1}, \quad 2 \le i \le n, \\ &\varphi(e_1, e_i) = -a_{i-1,0} e_1 - a_{i-1,0} e_n, \quad 3 \le i \le n, \\ &\varphi(e_i, e_3) = (n-i-1) a_{2,0} e_i - (a_{2,1} + b_{1,1}) e_{i+1}, \quad 2 \le i \le n-1, \\ &\varphi(e_n, e_j) = -a_{j-1,0} e_n, \quad 3 \le j \le n, \\ &\varphi(e_i, e_j) = (n-i-1) a_{j-1,0} e_i + (a_{j-2,0} - a_{j-1,1}) e_{i+1}, \quad 2 \le i \le n-1, \quad 4 \le j \le n, \end{aligned}$$

$$\varphi(e_1,x)=(a_{n-1,0}-d_{1,0})x+(a_{n-1,1}-a_{n-2,0}-d_{1,1})e_1+\sum_{i=2}^{n-2}\Big(\sum_{t=2}^{i}a_{n+t-i-1,t}-(n-i)a_{1,i+1}\Big)e_i$$
$$+\Big(\sum_{t=2}^{n-1}a_{t,t}-a_{1,n}+\frac{(n-2)(n-3)}{2}d_{1,0}\Big)e_{n-1}+c_{1,n}e_n,$$
$$\varphi(e_2,x)=(n-3)b_{1,1}x+(n-2)b_{2,3}e_1+\sum_{i=2}^{n}c_{2,i}e_i,$$
$$\varphi(e_3,x)=(4-n)a_{2,0}x+(3-n)(a_{2,1}+b_{1,1})e_1+(a_{2,2}+(n-3)d_{1,0})e_2+(c_{2,2}-d_{1,1})e_3$$
$$+\sum_{i=4}^{n-2}(c_{2,i-1}+(3-i)a_{2,i})e_i+(c_{2,n-2}+(4-n)a_{2,n-1}+a_{2,0})e_{n-1}+(c_{2,n-1}+(3-n)a_{2,n}-a_{2,1})e_n,$$
$$\varphi(e_i,x)=(i-n+1)a_{i-1,0}x+(n-i)(a_{i-2,0}-a_{i-1,1})e_1$$
$$+\sum_{s=2}^{i-2}(i-s)\sum_{t=2}^{s}a_{i+t-s-1,t}e_s+\Big(\sum_{t=2}^{i-1}a_{t,t}+\frac{(2n-i-3)(i-2)}{2}d_{1,0}\Big)e_{i-1}$$
$$+(c_{2,2}+(2-i)d_{1,1})e_i+\sum_{s=i+1}^{n-2}\Big(c_{2,s-i+2}+(i-s)\sum_{t=2}^{i-1}a_{t,s-i+t+1}\Big)e_s$$
$$+(c_{2,n-i+1}+a_{i-1,0}+(i-n+1)\sum_{t=2}^{i-1}a_{t,n-i+t})e_{n-1}$$
$$+\Big(c_{2,n-i+2}+a_{i-2,0}-a_{i-1,1}+(i-n)\sum_{t=2}^{i-1}a_{t,n-i+t+1}\Big)e_n,\quad 4\le i\le n-2,$$
$$\varphi(e_{n-1},x)=(a_{n-3,0}-a_{n-2,1})e_1+\sum_{s=2}^{n-3}(n-1-s)\sum_{t=2}^{s}a_{n+t-s-2,t}e_s$$
$$+\Big(\sum_{t=2}^{n-2}a_{t,t}+\frac{(n-2)(n-3)}{2}d_{1,0}\Big)e_{n-2}+(c_{2,2}+a_{n-2,0}-(n-3)d_{1,1})e_{n-1}$$
$$+\Big(c_{2,3}+a_{n-3,0}-a_{n-2,1}-\sum_{t=2}^{n-2}a_{t,t+2}\Big)e_n,$$
$$\varphi(e_n,x)=a_{n-1,0}x+\sum_{s=2}^{n-2}(n-s)\sum_{t=2}^{s}a_{n+t-s-1,t}e_s+\Big(\sum_{t=2}^{n-1}a_{t,t}+\frac{(n-2)(n-3)}{2}d_{1,0}+a_{n-1,0}\Big)e_{n-1}$$
$$+(c_{2,2}+a_{n-2,0}-a_{n-1,1}-(n-2)d_{1,1})e_n,$$
$$\varphi(x,e_1)=d_{1,0}x+d_{1,1}e_1+a_{1,3}e_2+\sum_{i=3}^{n}d_{1,i}e_i,$$
$$\varphi(x,e_2)=-b_{2,3}e_1-b_{1,1}e_{n-1},$$
$$\varphi(x,e_3)=(a_{2,1}+b_{1,1})e_1+a_{2,0}e_{n-1},$$
$$\varphi(x,e_i)=(a_{i-1,1}-a_{i-2,0})e_1+a_{i-1,0}e_{n-1},\quad 4\le i\le n,$$
$$\varphi(x,x)=-a_{n-2,0}x+(a_{n-3,0}-a_{n-2,1})e_1+\sum_{i=2}^{n-3}\Big((i-n+1)(d_{1,i+1}-a_{1,i+2})-\sum_{t=2}^{i}a_{n-i+t-2,t}\Big)e_i$$
$$+\Big(a_{1,n}-d_{1,n-1}-\frac{n^2-5n+10}{2}d_{1,0}-\sum_{t=2}^{n-2}a_{t,t}\Big)e_{n-2}+(a_{n-1,n}+d_{1,1}-c_{1,n})e_{n-1}+\beta_ne_n.$$

Proof. The proof of this theorem is carried out by applying similar arguments as in the proof of Theorem 3.8. □

Remark 3.12. It should be noted that in the case $\dim(R_4) = 4$, an arbitrary 2-cocycle for the algebra R_4 is different from the description of Theorem 3.11 and it has the form:

$$\begin{aligned}
\varphi(e_1, e_1) &= a_{1,3}e_3,\\
\varphi(e_2, e_1) &= a_{2,0}x + \sum_{s=1}^{3} a_{2,s}e_s,\\
\varphi(e_3, e_1) &= a_{2,0}e_1 - (a_{2,2} - d_{1,0})e_3,\\
\varphi(e_1, e_2) &= b_{1,1}e_1 + b_{1,1}e_3,\\
\varphi(e_2, e_2) &= b_{2,3}e_3,\\
\varphi(e_3, e_2) &= b_{1,1}e_3,\\
\varphi(e_1, e_3) &= -a_{2,0}e_1 - a_{2,0}e_3,\\
\varphi(e_2, e_3) &= -(a_{2,1} + b_{1,1})e_3,\\
\varphi(e_3, e_3) &= -a_{2,0}e_3,\\
\varphi(e_1, x) &= (a_{2,0} - d_{1,0})x + (a_{2,1} + b_{1,1} - d_{1,1})e_1 + (a_{2,2} - a_{1,3})e_2 - c_{1,3}e_3,\\
\varphi(e_2, x) &= b_{2,3}e_1 + c_{2,2}e_2 + c_{2,3}e_3,\\
\varphi(e_3, x) &= a_{2,0}x + (a_{2,2} + a_{2,0})e_2 - (a_{2,1} + c_{2,2} - d_{1,1})e_3,\\
\varphi(x, e_1) &= d_{1,0}x + d_{1,1}e_1 + (a_{1,3} - 2d_{1,0})e_2 + d_{1,3}e_3,\\
\varphi(x, e_2) &= -b_{2,3}e_1 - b_{1,1}e_2,\\
\varphi(x, e_3) &= (a_{2,1} + b_{1,1})e_1 + a_{2,0}e_2,\\
\varphi(x, x) &= b_{1,1}x + b_{2,3}e_1 + (a_{2,3} - c_{1,3} + d_{1,1})e_2 + \beta_3 e_3.
\end{aligned}$$

Corollary 3.13. $\dim ZL^2(R_4, R_4) = (n+1)^2 - 2$, $\dim HL^2(R_4, R_4) = 1$ *and the equivalence class* $\overline{\rho}$ *forms a basis of* $HL^2(R_4, R_4)$, *where*

$$\rho : \begin{cases} \rho(e_1, x) = e_1,\\ \rho(e_i, x) = (i-2)e_i, \ 3 \le i \le n,\\ \rho(x, e_1) = -e_1,\\ \rho(x, x) = -e_{n-1}. \end{cases}$$

Theorem 3.14. *An arbitrary* $\varphi \in ZL^2\big(R_2(\alpha), R_2(\alpha)\big)$ *has the following form:*

$$\begin{aligned}
\varphi(e_1, e_1) &= \sum_{i=2}^{n} a_{1,i}e_i,\\
\varphi(e_i, e_1) &= \sum_{s=1}^{n} a_{i,s}e_s + a_{i,0}x, \ 2 \le i \le n-1,\\
\varphi(e_n, e_1) &= a_{n-1,0}e_1 - \sum_{i=3}^{n-1}\Big(\sum_{t=2}^{i-1} a_{n+t-i,t}\Big)e_i + \Big(\frac{(n-1)(2\alpha+n)}{2}d_{1,0} - \sum_{t=2}^{n-1} a_{t,t}\Big)e_n,\\
\varphi(e_1, e_2) &= b_{1,1}e_1 + b_{1,n}e_n,\\
\varphi(e_i, e_2) &= \big((i-2)b_{1,1} + b_{2,2}\big)e_i + b_{2,3}e_{i+1}, \quad 2 \le i \le n,\\
\varphi(e_1, e_i) &= -a_{i-1,0}e_1, \ 3 \le i \le n,\\
\varphi(e_i, e_3) &= -(\alpha + i - 1)a_{2,0}e_i - (a_{2,1} + b_{1,1})e_{i+1}, \quad 2 \le i \le n,\\
\varphi(e_i, e_j) &= -(\alpha + i - 1)a_{j-1,0}e_i + (a_{j-2,0} - a_{j-1,1})e_{i+1}, \quad 2 \le i \le n, \quad 4 \le j \le n,\\
\varphi(e_1, x) &= -d_{1,1}e_1 + \sum_{i=2}^{n-1}(\alpha + i - 2)a_{1,i+1}e_i + c_{1,n}e_n - d_{1,0}x,\\
\varphi(e_2, x) &= -\alpha b_{2,3}e_1 + \sum_{i=2}^{n} c_{2,i}e_i - (\alpha+1)b_{1,1}x,\\
\varphi(e_3, x) &= (\alpha+1)(a_{2,1} + b_{1,1})e_1 + (a_{2,2} - (\alpha+1)d_{1,0})e_2 + (c_{2,2} - d_{1,1})e_3\\
&\quad + \sum_{i=4}^{n}(c_{2,i-1} + (3-i)a_{2,i})e_i + (\alpha+2)a_{2,0}x,
\end{aligned}$$

$$\varphi(e_i,x)=(\alpha+i-2)(a_{i-1,1}-a_{i-2,0})e_1+\sum_{s=2}^{i-2}(i-s)\sum_{t=2}^{s}a_{i+t-s-1,t}e_s$$
$$+\left(\sum_{t=2}^{i-1}a_{t,t}-\frac{(2\alpha+i-1)(i-2)}{2}d_{1,0}\right)e_{i-1}+(c_{2,2}-(i-2)d_{1,1})e_i$$
$$+\sum_{s=i+1}^{n}\left(c_{2,s-i+2}+(i-s)\sum_{t=2}^{i-1}a_{t,s-i+t+1}\right)e_s+(\alpha+i-1)a_{i-1,0}x,\quad 4\le i\le n,$$
$$\varphi(x,e_1)=\sum_{i=1}^{n}d_{1,i}e_i+d_{1,0}x,$$
$$\varphi(x,e_2)=-b_{2,3}e_1-b_{1,n}e_{n-1},$$
$$\varphi(x,e_3)=(a_{2,1}+b_{1,1})e_1,$$
$$\varphi(x,e_i)=(a_{i-1,1}-a_{i-2,0})e_1,\quad 4\le i\le n,$$
$$\varphi(x,x)=\sum_{i=2}^{n-2}(\alpha+i-1)(d_{1,i+1}-a_{1,i+2})e_i+\big((\alpha+n-2)d_{1,n}-c_{1,n}\big)e_{n-1}+\beta_ne_n$$

with restrictions

$$\begin{cases}(\alpha-1)a_{1,2}=0, & (\alpha+1)b_{1,n}=0, \quad (\alpha+1)(b_{2,2}-(\alpha+1)b_{1,1})=0,\\ (n-3)b_{1,n}=0, & \alpha(d_{1,2}-a_{1,3})=0.\end{cases}\tag{3.1}$$

Proof. The proof of this proposition is carried out by applying similar arguments as in the proof of Theorem 3.8. □

From the equalities (3.1) it implies that if $n\neq 3$, then $b_{1,n}=0$. Thus, we distinguish the cases $n=3$ and $n>3$. Moreover, the general form of infinitesimal deformations also depends on the value of α. Therefore, we have

Corollary 3.15.

$$\dim ZL^2\big(R_2(\alpha),R_2(\alpha)\big)=\begin{cases}(n+1)^2-1, & \alpha=0;\pm1,\\ (n+1)^2-2, & \alpha\neq0;\pm1,\end{cases}\quad \textit{for } n>3;$$
$$\dim ZL^2\big(R_2(\alpha),R_2(\alpha)\big)=\begin{cases}15, & \alpha=0;1,\\ 16, & \alpha=-1,\\ 14, & \alpha\neq0;\pm1,\end{cases}\quad \textit{for } n=3.$$

Corollary 3.16.

$$\dim HL^2\big(R_2(\alpha),R_2(\alpha)\big)=\begin{cases}2, & \alpha=0;\pm1;1-n;2-n,\\ 1, & \alpha\neq0;\pm1;1-n;2-n,\end{cases}\quad \textit{for } n>3;$$
$$\dim HL^2\big(R_2(\alpha),R_2(\alpha)\big)=\begin{cases}4, & \alpha=-1,\\ 2, & \alpha=0;1;-2,\\ 1, & \alpha\neq0;\pm1;-2,\end{cases}\quad \textit{for } n=3.$$

In the following proposition similarly to the proof of Proposition 3.10 we find a basis of $HL^2\big(R_2(\alpha),R_2(\alpha)\big)$.

Proposition 3.17. *The basis of* $HL^2\big(R_2(\alpha),R_2(\alpha)\big)$ *consists of the following equivalence classes*

$$\begin{cases}\overline{\rho},\overline{\psi_1} & \alpha=0;1;1-n;2-n,\\ \overline{\psi_1},\overline{\psi_2}, & \alpha=-1,\\ \overline{\rho}, & \alpha\neq0;\pm1;1-n;2-n,\end{cases}\quad \textit{for } n>3;$$
$$\begin{cases}\overline{\rho},\overline{\psi_1} & \alpha=0;-1;-2,\\ \overline{\psi_1},\overline{\psi_2},\overline{\psi_3},\overline{\psi_4}, & \alpha=-1,\\ \overline{\rho}, & \alpha\neq0;\pm1;-2,\end{cases}\quad \textit{for } n=3,$$

where

$$\rho:\begin{cases}\rho(e_1,x)=e_1,\\ \rho(e_i,x)=(i-2)e_i, \quad 3\le i\le n,\\ \rho(x,e_1)=-e_1,\end{cases}$$
$$\psi_1:\psi_1(e_i,x)=e_i,\quad 2\le i\le n,\qquad \psi_2:\psi_2(e_i,e_2)=e_i,\quad 2\le i\le n,$$
$$\psi_3:\begin{cases}\psi_3(e_1,x)=e_n,\\ \psi_3(x,x)=-e_{n-1},\end{cases}\qquad \psi_4:\begin{cases}\psi_4(e_1,e_2)=e_n,\\ \psi_4(x,e_2)=-e_{n-1}.\end{cases}$$

Theorem 3.18. *An arbitrary 2-cocycle of* $ZL^2\big(R_5(\alpha_4,\alpha_5,\dots,\alpha_n), R_5(\alpha_4,\alpha_5,\dots,\alpha_n)\big)$ *has the following form:*

$$\varphi(e_1,e_1)=a_{2,0}x+a_{2,1}e_1+(a_{2,2}-\alpha_n a_{n,2})e_2+\sum_{i=3}^{n}a_{1,i}e_i,$$

$$\varphi(e_i,e_1)=a_{i,0}x+\sum_{s=1}^{n}a_{i,s}e_s,\quad 2\le i\le n-1,$$

$$\varphi(e_n,e_1)=\sum_{i=2}^{n}a_{n,i}e_i,$$

$$\varphi(e_1,e_2)=\Big(-c_0+\sum_{j=4}^{n}\alpha_j a_{j-1,0}\Big)e_2+\Big(-c_1+\sum_{j=4}^{n}\alpha_j a_{j-1,1}\Big)e_3+\sum_{i=4}^{n-1}\alpha_i\Big(-c_0+\sum_{j=4}^{n}\alpha_j a_{j-1,0}\Big)e_i,$$

$$\varphi(e_i,e_2)=\Big(-c_0+\sum_{j=4}^{n}\alpha_j a_{j-1,0}\Big)e_i+\Big(-c_1+\sum_{j=4}^{n}\alpha_j a_{j-1,1}\Big)e_{i+1}+\sum_{k=i+2}^{n}\alpha_i\Big(-c_0+\sum_{j=4}^{n}\alpha_j a_{j-1,0}\Big)e_k,\quad 2\le i\le n,$$

$$\varphi(e_1,e_j)=-a_{j-1,0}e_2-a_{j-1,1}e_3-a_{j-1,0}\sum_{k=4}^{n-1}\alpha_k e_k,\quad 3\le j\le n,$$

$$\varphi(e_i,e_j)=-a_{j-1,0}e_i-a_{j-1,1}e_{i+1}-a_{j-1,0}\sum_{k=i+2}^{n}\alpha_{k-i+2}e_k,\quad 2\le i\le n,\quad 3\le j\le n,$$

$$\varphi(x,e_1)=-d_2e_1+\sum_{i=2}^{n}d_ie_i,$$

$$\begin{aligned}\varphi(e_1,x)&=(c_0-\alpha_n a_{n-1,0})x+(c_1-d_2-\alpha_n a_{n-1,1})e_1\\&+(c_2+a_{1,3}-a_{2,3}+d_2+\alpha_n a_{n-1,1}+\alpha_n a_{n,3})e_2+\big(c_3+a_{1,4}-a_{2,4}+\alpha_n(a_{n,3}-\alpha_4 a_{n,2})\big)e_3\\&+\sum_{i=4}^{n-1}(c_i+a_{1,i+1}-a_{2,i+1}+\alpha_n(a_{n,i}-\alpha_{i+1}a_{n,2})+\sum_{j=4}^{i}\alpha_j(a_{1,i-j+3}-a_{2,i-j+3}))e_i+c_{n+1}e_n,\end{aligned}$$

$$\varphi(e_2,x)=c_0x+\sum_{i=1}^{n}c_ie_i,$$

$$\begin{aligned}\varphi(e_3,x)&=\Big(a_{2,0}+\sum_{j=4}^{n}\alpha_j a_{j,0}\Big)x+(a_{2,1}+\sum_{j=4}^{n-1}\alpha_j a_{j,1})e_1+\Big(-a_{2,1}+\sum_{j=4}^{n}\alpha_j a_{j,2}\Big)e_2\\&+\Big(c_1+c_2+\sum_{j=4}^{n}\alpha_j a_{j,3}\Big)e_3+\sum_{k=4}^{n}\Big(c_{k-1}-a_{2,1}\alpha_k-\sum_{j=2}^{k-2}\alpha_{k-j+2}a_{2,j}+\sum_{j=4}^{n}\alpha_j a_{j,k}\Big)e_k+a_{2,1}\alpha_n e_n,\end{aligned}$$

$$\begin{aligned}\varphi(e_{i+1},x)&=\Big(a_{i,0}+\sum_{j=i+2}^{n}\alpha_{j-i+2}a_{j,0}\Big)x+\Big(a_{i,1}+\sum_{j=i+2}^{n-1}\alpha_{j-i+2}a_{j,1}\Big)e_1+\Big(-a_{i,1}+\sum_{j=i+2}^{n}\alpha_{j-i+2}a_{j,2}\Big)e_2\\&+\sum_{k=3}^{i-1}\Big(\sum_{j=i+1}^{n-1}\alpha_{j-i+k}a_{j,1}+\sum_{s=2}^{k-1}\sum_{j=i+1}^{n}\alpha_{j+k-i-s+2}a_{j,s}+\sum_{j=i+2}^{n}\alpha_{j-i+2}a_{j,k}\Big)e_k\\&+\Big(\sum_{j=i+1}^{n-1}\alpha_j a_{j,1}+\sum_{s=2}^{i-1}\sum_{j=i+1}^{n}\alpha_{j-s+2}a_{j,s}+\sum_{j=i+2}^{n}\alpha_{j-i+2}a_{j,i}\Big)e_i\\&+\Big(c_1+c_2-(i-2)d_2-\sum_{j=3}^{i}\alpha_{j+1}(a_{j,1}+a_{j,2})+\sum_{s=3}^{i}\sum_{j=i+1}^{n}\alpha_{j-s+3}a_{j,s}+\sum_{j=i+2}^{n}\alpha_{j-i+2}a_{j,i+1}\Big)e_{i+1}\\&+\sum_{k=i+2}^{n}\Big(c_{k+1-i}+\sum_{j=2}^{i}\alpha_{k-i+j}a_{j,1}-\sum_{s=2}^{i}\sum_{j=2}^{k-s}\alpha_{k+4-j-s}a_{i-s+2,j}+\sum_{s=2}^{i}\sum_{j=s+2}^{n}\alpha_{j-s+2}a_{j,k-i+s}\Big)e_k\\&+\alpha_n a_{i,1}e_n,\end{aligned}$$

$$\varphi(x,x)=d_3e_2+d_4e_3+\sum_{i=4}^{n-2}\Big(d_{i+1}+\sum_{j=4}^{i}\alpha_j d_{i+3-j}\Big)e_i+\Big(d_n+\sum_{j=4}^{n}\alpha_j d_{n+2-j}\Big)e_{n-1}+\beta e_n.$$

Proof. The proof of this proposition is carried out by applying similar arguments as in the proof of Theorem 3.8. □

Corollary 3.19.

$$\dim ZL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big) = n^2 + 3n - 3,$$

$$\dim HL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big) = \begin{cases} 2n-4, & \alpha_i = 0 \text{ for all } i,\\ 2n-5, & \alpha_i \neq 0 \text{ for some } i.\end{cases}$$

Let us introduce the notations

$$\rho : \begin{cases} \rho(e_1, x) = e_1 - e_2,\\ \rho(e_i, x) = (i-3)e_i, & 4 \le i \le n,\\ \rho(x, e_1) = e_1 - e_2,\end{cases}$$

$$\psi_k\,(4 \le k \le n-1) : \begin{cases} \psi_k(e_1, x) = e_k,\\ \psi_k(e_i, x) = e_{k+i-2}, & 2 \le i \le n-k+2,\end{cases}$$

$$\psi_n : \big\{\psi_n(e_2, x) = e_n,$$

$$\varphi_{n,2} : \begin{cases} \varphi_{n,2}(e_n, e_1) = e_2,\\ \varphi_{n,2}(e_1, x) = -\alpha_n \sum\limits_{j=3}^{n-1} \alpha_{j+1} e_j,\\ \varphi_{n,2}(e_i, x) = \sum\limits_{j=2}^{i-1} \alpha_{n+j+1-i} e_j, & 3 \le i \le n-1,\\ \varphi_{n,2}(e_n, x) = \sum\limits_{j=3}^{n-1} \alpha_{j+1} e_j,\end{cases}$$

$$\varphi_{n,k}\,(3 \le k \le n-1) : \begin{cases} \varphi_{n,k}(e_n, e_1) = e_k,\\ \varphi_{n,k}(e_1, x) = -\alpha_n e_k,\\ \varphi_{n,k}(e_i, x) = \sum\limits_{j=k}^{i+k-3} \alpha_{n+j+3-i-k} e_j, & 3 \le i \le n+2-k,\\ \varphi_{n,k}(e_i, x) = \sum\limits_{j=k}^{n} \alpha_{n+j+3-i-k} e_j, & n+3-k \le i \le n-1,\\ \varphi_{n,k}(e_n, x) = \sum\limits_{j=k+1}^{n} \alpha_{j+3-k} e_j.\end{cases}$$

Proposition 3.20. *The equivalence classes $\overline{\rho}$, $\overline{\psi_k}$ $(4 \le k \le n)$ and $\overline{\varphi_{n,k}}$ $(2 \le k \le n-1)$ form a basis of $HL^2\big(R_5(0), R_5(0)\big)$. The basis of $HL^2\big(R_5(\alpha), R_5(\alpha)\big)$ with $\alpha = (\alpha_4, \ldots, \alpha_n) \neq (0, \ldots, 0)$, is also the same except one cocycle $\overline{\psi_k}$ with $\alpha_k \neq 0$.*

Proof. Since there are parameters $(a_{i,j}, c_k, \beta, d_s)$ in the general form of 2-cocycles for the algebra $R_5(\alpha_4, \ldots, \alpha_n)$, we consider the natural basis of the space ZL^2 whose basis elements are obtained by the instrumentality of these parameters.

Similarly as in the proof of Proposition 3.10, we denote by $\chi(a_{i,j})$ the cocycle which satisfies $a_{i,j} = 1$ and all other parameters are equal to zero. We define such type of notation for other parameters by notations

$$\varphi_{i,j} = \chi(a_{i,j}), \quad \psi_k = \chi(c_k), \quad \rho_s = \chi(d_s), \quad \eta = \chi(\beta),$$

where $1 \le i \le n$, $0 \le j \le n$, $0 \le k \le n+1$, $2 \le s \le n$ and

$$(i,j) \notin \{(1,0), (1,1), (1,2)\}.$$

To define the basis of $BL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big)$ we consider the endomorphisms $f_{j,k} : R_5(\alpha_i) \to R_5(\alpha_i)$ defined as follows

$$\begin{aligned} f_{i,j}(e_i) &= e_j, & 1 \le i,j \le n,\\ f_{i,n+1}(e_i) &= x, & 1 \le i \le n,\\ f_{n+1,j}(x) &= e_j, & 1 \le j \le n,\\ f_{n+1,n+1}(x) &= x, \end{aligned}$$

where in the expansion of the endomorphisms the omitted values are assumed to be zero.

Consider

$$g_{i,j}(x,y) = [f_{i,j}(x), y] + [x, f_{i,j}(y)] - f_{i,j}([x,y]).$$

Note that $g_{i,j} \in BL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big)$ and by direct computation we express $g_{i,j}$ via the elements $\varphi_{i,j}$, ψ_k, ρ_s and η.

In the case of $\alpha_4 = \alpha_5 = \cdots = \alpha_n = 0$ we obtain that any linear combination of elements ρ_2, ψ_k, $4 \leq k \leq n$ and $\varphi_{n,j}$, $2 \leq j \leq n-1$ does not belong to $BL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big)$. Therefore, the equivalence classes of these elements form a basis of $HL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big)$.

However, in the case of $(\alpha_4, \alpha_5, \ldots, \alpha_n) \neq (0, 0, \ldots, 0)$ we obtain that

$$2\alpha_4\psi_4 + 3\alpha_5\psi_5 + \cdots + (n-2)\alpha_n\psi_n \in BL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big).$$

Hence, in this case we get that the basis of $HL^2\big(R_5(\alpha_i), R_5(\alpha_i)\big)$ also consists of $\overline{\rho_2}$, $\overline{\psi_k}$ $(4 \leq k \leq n)$ and $\overline{\varphi_{n,k}}$ $(2 \leq k \leq n-1)$, except one cocycle $\overline{\psi_k}$ with $\alpha_k \neq 0$. $\square$

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