Ground States for Potts Model with a Countable Set of Spin Values on a Cayley Tree

G. I. Botirov and M. M. Rahmatullaev

Abstract We consider Potts model, with competing interactions and countable spin values $\Phi = \{0, 1, \dots\}$ on a Cayley tree of order three. We study periodic ground states for this model.

Keywords Potts model · Configuration · Ground state · Weakly periodic ground state · Countable set of spin values

1 Introduction

Each Gibbs measure is associated with a single phase of physical system. As is known, the phase diagram of Gibbs measures for a Hamiltonian is close to the phase diagram of isolated (stable) ground states of this Hamiltonian. At low temperatures, a periodic ground state corresponds to a periodic Gibbs measures, see [1–9]. The problem naturally aries on description of periodic ground states. In [3, 4] for the Ising model with competing interactions, periodic and weakly periodic ground states were studied.

In [1] ground states were described and the Peierls condition for the Potts model is verified. Using a contour argument authors showed the existence of three different Gibbs measures associated with translation invariant ground states.

In [5], (1), [8] studying periodic and weakly periodic ground states for the Potts model with competing interactions on a Cayley tree. In the present paper, we consider Potts model, with competing interactions and a countable set of spin values $\Phi = \{0, 1, \dots\}$ on a Cayley tree of order three. We study periodic ground states.

In [10] the 3-state Potts model with competing binary interactions (with couplings J and J_p) on a Bethe lattice of order two is considered. The set of ground states of

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the one-level model is completely described. The critical temperature of a phase transition is exactly found and the phase diagram is presented.

In [11] an exact phase diagram of the Potts model with next nearest neighbor interactions on the Cayley tree of order two is found.

The paper is organized as follows. In Sect. 2, we recall the main definitions and known facts. In Sect. 3, we study periodic ground states.

2 Main Definitions and Known Facts

Cayley tree. The Cayley tree (Bethe lattice) Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex (see [12]). Let $\Gamma^k = (V, L)$ where V is the set of vertices and L the set of edges. Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them and we denote $l = \langle x, y \rangle$.

On this tree, there is a natural distance to be denoted $d(x; y)$, being the number of nearest neighbor pairs of the minimal path between the vertices x and y (by path one means a collection of nearest neighbor pairs, two consecutive pairs sharing at least a given vertex).

For a fixed $x^0 \in V$, the root, let

$$W_n = \{x \in V : d(x, x^0) = n\}, \quad V_n = \{x \in V : d(x, x^0) \leq n\};$$

be respectively the sphere and the ball of radius n with center at x^0.

It is well-known that there exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k of the free products of $k+1$ cyclic groups of second order with generators $a_1, a_2, \ldots, a_{k+1}$ (see [2, 13]).

3 Configuration Space and the Model

For each $x \in G_k$, let $S(x)$ denote the set of direct successors of x, i.e., if $x \in W_n$ then

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}.$$

For each $x \in G_k$, let $S_1(x)$ denote the set of all neighbors of x, i.e. $S_1(x) = \{y \in G_k : \langle x, y \rangle \in L\}$. The set $S_1(x) \setminus S(x)$ is a singleton. Let $x_\downarrow$ denote the (unique) element of this set.

We consider the models in which the spin takes values in the set $\Phi = \{1, 2, \ldots\}$. A *configuration* σ on the set V is defined as a function $x \in V \to \sigma(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^V$.

Let G_k^* be a subgroup of index $r \geq 1$. Consider the set of right coset $G_k/G_k^* = \{H_1, \ldots, H_r\}$, where G_k^* is a subgroup.

Definition 1 A configuration $\sigma(x)$ is said to be G_k^* - *periodic* if $\sigma(x) = \sigma_i$ for all $x \in H_i$. A G_k-periodic configuration is said to be *translation invariant*.

The period of a periodic configuration is the index of the corresponding subgroup.

Definition 2 A configuration $\sigma(x)$ is said to be G_k^* - *weakly periodic* if $\sigma(x) = \sigma_{ij}$ for all $x \in H_i$ and $x_\downarrow \in H_j$.

The Hamiltonian of the Potts model with competing interactions has the form

$$H(\sigma) = J_1 \sum_{\substack{\langle x,y\rangle: \\ x,y\in V}} \delta_{\sigma(x)\sigma(y)} + J_2 \sum_{\substack{x,y\in V: \\ d(x,y)=2}} \delta_{\sigma(x)\sigma(y)}, \tag{1}$$

where $J_1, J_2 \in \mathbf{R}$ and

$$\delta_{uv} = \begin{cases} 1, & u = v, \\ 0, & u \neq v. \end{cases}$$

4 Ground States

For pair of configurations σ and φ coinciding almost everywhere, i.e., everywhere except at a finite number of points, we consider the relative Hamiltonian $H(\sigma, \varphi)$ of the difference between the energies of the configurations σ and φ, i.e.,

$$H(\sigma,\varphi) = J_1 \sum_{\substack{\langle x,y\rangle, \\ x,y\in V}} (\delta_{\sigma(x)\sigma(y)} - \delta_{\varphi(x)\varphi(y)}) + J_2 \sum_{\substack{x,y\in V: \\ d(x,y)=2}} (\delta_{\sigma(x)\sigma(y)} - \delta_{\varphi(x)\varphi(y)}), \tag{2}$$

where $J = (J_1, J_2) \in \mathbf{R}^2$ is an arbitrary fixed parameter.

Let M be the set of all unit balls with vertices in V. By the *restricted configuration* σ_b we mean the restriction of a configuration σ to a ball $b \in M$. The energy of a configuration σ_b on b is defined by the formula

$$U(\sigma_b) \equiv U(\sigma_b, J) = \frac{1}{2} J_1 \sum_{\substack{\langle x,y\rangle, \\ x,y\in b}} \delta_{\sigma(x)\sigma(y)} + J_2 \sum_{\substack{x,y\in b: \\ d(x,y)=2}} \delta_{\sigma(x)\sigma(y)}, \tag{3}$$

where $J = (J_1, J_2) \in \mathbf{R}^2$.

The following assertion is known (see [2–8])

Lemma 1 *The relative Hamiltonian (2) has the form*

$$H(\sigma,\varphi) = \sum_{b\in M} (U(\sigma_b) - U(\varphi_b)).$$

Note that, in [5] in the case $k = 2$ and $\Phi = \{1, 2, 3\}$ all periodic (in particular translation-invariant) ground states for the Potts model (1) are given. In [9] the set of weakly periodic ground states corresponding to index-two normal divisors of the group representation of the Cayley tree is given. In [8] the sets of periodic and weakly periodic ground states corresponding to normal subgroups of the group representation of the Cayley tree of index 4 are described.

We consider the case $k = 3$. It is easy to see that $U(\sigma_b) \in \{U_1, U_2, \dots, U_{12}\}$ for any σ_b, where

$$U_1 = 2J_1 + 6J_6,\ \ U_2 = \frac{3}{2}J_1 + 3J_2,\ \ U_3 = J_1 + 2J_2,\ \ U_4 = \frac{1}{2}J_1 + 3J_2,$$

$$U_5 = 6J_2,\ \ U_6 = \frac{1}{2}J_1,\ \ U_7 = 3J_2,\ \ U_8 = J_2,$$

$$U_9 = J_1 + J_2,\ \ U_{10} = \frac{1}{2}J_1 + J_2,\ \ U_{11} = 2J_2,\ \ U_{12} = 0.$$

Definition 3 A configuration φ is called a *ground state* of the relative Hamiltonian H if $U(\varphi_b) = \min\{U_1, U_2, \dots, U_{12}\}$ for any $b \in M$.

We set $C_i = \{\sigma_b : U(\sigma_b) = U_i\}$ and $U_i(J) = U(\sigma_b, J)$ if $\sigma_b \in C_i$, $i = 1, 2, \dots, 12$.

If a ground state is a periodic (weakly periodic, translation invariant) configuration then we call it a *periodic (weakly periodic, translation invariant) ground state.*

Let

$$A \subset \{1, 2, \dots, k+1\},\ \ H_A = \{x \in G_k : \sum_{j \in A} w_j(x) \text{ is even}\},$$

$$G_k^{(2)} = \{x \in G_k : |x| \text{ is even}\},\ G_k^{(4)} = H_A \cap G_k^{(2)},$$

where $w_j(x)$ is the number of occurrences of a_j in x and $|x|$ is the length of x, i.e. $|x| = \sum_{j=1}^{k+1} w_j(x)$. Notice that $G_k^{(4)}$ is a normal subgroup of index 4 of G_k.

Then we have

$$G_k^{(4)} = \{x \in G_k : |x| \text{ is even}, \sum_{j \in A} w_j(x) \text{ is even}\}.$$

If $A = \{1, 2, \dots, k+1\}$ then the normal subgroup H_A coincides with the group $G_k^{(2)}$.

For any $i = 1, 2, \dots, 12$ we put

$$A_i = \{J \in \mathbf{R}^2 : U_i = \min\{U_1, U_2, \dots, U_{12}\}\}. \tag{4}$$

Quite cumbersome but not difficult calculations show that

$$A_1 = \{J \in \mathbf{R}^2 : J_1 \le 0, J_2 \le 0\} \cup \{J \in \mathbf{R}^2 : J_1 \le -6J_2, J_2 \ge 0\},$$

$$A_2 = \{J \in \mathbf{R}^2 : J_1 \geq 0, -6J_2 \leq J_1 \leq -4J_2\},$$

$$A_3 = A_4 = A_{10} = \{J \in \mathbf{R}^2 : J_1 = 0, J_2 = 0\},$$

$$A_5 = \{J \in \mathbf{R}^2 : J_1 \geq 0, J_2 \leq 0\},$$

$$A_6 = \{J \in \mathbf{R}^2 : J_2 \geq 0, -2J_2 \leq J_1 \leq 0\},$$

$$A_7 = A_8 = A_{11} = \{J \in \mathbf{R}^2 : J_1 \geq 0, J_2 = 0\},$$

$$A_9 = \{J \in \mathbf{R}^2 : J_2 \leq 0, -4J_2 \leq J_1 \leq -2J_2\},$$

$$A_{12} = \{J \in \mathbf{R}^2 : J_1 \leq 0, J_2 \leq 0\}, \quad \text{and} \quad \mathbf{R}^2 = \bigcup_n A_n.$$

Theorem 1 *For any class C_i, $i = 1, 2, \dots 12$, and any bounded configuration $\sigma_b \in C_i$, there exists a periodic configuration φ (on the Cayley tree) such that $\varphi_{b'} \in C_i$ for any $b' \in M$ and $\varphi_b = \sigma_b$.*

Proof For an arbitrary class $C_i, i = 1, 2, \dots, 12$, and $\sigma_b \in C_i$, we construct the configuration φ as follows: without loss of generality, we can take the ball centered at $e \in G_3$ (where e is the unit element of G_3) for the ball b, i.e., $b = \{e, a_1, a_2, a_3, a_4\}$.

We consider several cases.

Case C_1. In this case, we have $\sigma(x) = i, i \in \Phi$, for any $x \in b$. The configuration φ hence coincides with the translation-invariant configuration, i.e. $\varphi^{(i)} = \{\varphi^{(i)}(x) = i\}$, where $i \in \Phi$.

Case C_2. This case is considered in [8]. Let $H_{\{i\}}$ be normal subgroup of index two, and $G_k/H_{\{i\}} = \{H_0^{(2)}, H_1^{(2)}\}$ is quotient group, for any $i \in \{1, 2, 3, 4\}$. For any $l, m \in \Phi, l \neq m$, considering the following $H_{\{i\}}-$periodic configuration:

$$\varphi_2^{(lm)}(x) = \begin{cases} l, & \text{if } x \in H_0^{(2)}, \\ m, & \text{if } x \in H_1^{(2)}. \end{cases}$$

In [8] it was proved, that $(\varphi_2^{(lm)})_{b'} \in C_2$ for any $b' \in M$.

Case C_3. Let $H_{\{i,j\}}$, $i, j \in \{1, 2, 3, 4\}$ and $i \neq j$ be a normal subgroup of index two, and $G_k/H_{\{i,j\}} = \{H_0^{(3)}, H_1^{(3)}\}$ be the quotient group. For any $l, m \in \Phi, l \neq m$, consider the following $H_{\{i,j\}}-$ periodic configuration:

$$\varphi_3^{(lm)}(x) = \begin{cases} l, & \text{if } x \in H_0^{(3)}, \\ m, & \text{if } x \in H_1^{(3)}. \end{cases}$$

So we thus obtain a periodic configuration $\varphi_3^{(lm)}$ with period $p = 2$ (equal to the index of the subgroup); then by construction $(\varphi_3^{(lm)})_b = \sigma_b$. Now we shall prove that all restrictions $(\varphi_3^{(lm)})_{b'}$ for any $b' \in M$ of the configuration $\varphi_3^{(lm)}$ belong to C_3.

Let $q_j(x) = |S_1(x) \cap H_j^{(3)}|$, $j = 0, 1$; where $S_1(x) = \{y \in G_k : \langle x, y\rangle\}$, the set of all nearest neighbors of $x \in G_k$. Denote $Q(x) = (q_0(x), q_1(x))$. Clearly $q_0(x)$ (resp. $q_1(x)$) is the number of points y in $S_1(x)$ such that $\varphi_3^{(lm)}(y) = l$ (resp. $\varphi_3^{(lm)}(y) = m$). We note (see Chap. 1 of [2]) that for every $x \in G_k$ there is a permutation π_x of the coordinates of the vector $Q(e)$ (where e as before is the identity of G_k) such that

$$\pi_x Q(e) = Q(x).$$

Moreover $Q(x) = Q(e)$ if $x \in H_0^{(3)}$ and $Q(x) = (q_1(e), q_0(e))$ if $x \in H_1^{(3)}$. Thus for any $b' \in M$ we have (i) if $c_{b'} \in H_0^{(3)}$ (where $c_{b'}$ is the center of b') then $(\varphi_3^{(lm)})_{b'} \in C_3$, (ii) if $c_{b'} \in H_1^{(3)}$, then $(\varphi_3^{(lm)})_{b'} \in C_3$.

Case C_4. Let $H_{\{i,j,r\}}$ be a normal subgroup of index two, and $G_k/H_{\{i,j,r\}} = \{H_0^{(4)}, H_1^{(4)}\}$ is the quotient group, for any $i, j, r \in \{1, 2, 3, 4\}$ and $i \neq j, i \neq r, j \neq r$. For any $l, m \in \Phi, l \neq m$, considering the following $H_{\{i,j,r\}}-$ periodic configuration:

$$\varphi_4^{(lm)}(x) = \begin{cases} l, & \text{if } x \in H_0^{(4)}, \\ m, & \text{if } x \in H_1^{(4)}. \end{cases}$$

We thus obtain a periodic configuration $\varphi_4^{(lm)}$ with period $p = 2$; it is clear that $(\varphi_4^{(lm)})_{b'} \in C_4$ for any $b' \in M$.

Case C_5. Let $G_k/G_k^{(2)} = \{H_0^{(5)}, H_1^{(5)}\}$ is quotient group. For any $l, m \in \Phi, l \neq m$, consider the following $G_k^{(2)}-$ periodic configuration:

$$\varphi_5^{(lm)}(x) = \begin{cases} l, & \text{if } x \in H_0^{(5)}, \\ m, & \text{if } x \in H_1^{(5)}. \end{cases}$$

It is easy to see (see [8]) that for each $b' \in M$ we have $(\varphi_5^{(lm)})_{b'} \in C_5$.

Case C_6. Let $G_3^{(6)} = H_i \cap H_j \cap H_r$, for any $i, j, r \in \{1, 2, 3, 4\}, i \neq j, i \neq r, j \neq r$. We note (see [2]) that $G_3^{(6)}$ is a normal index-eight subgroup in G_3, and $G_3/G_3^{(6)} = \{H_0^{(6)}, H_1^{(6)}, \ldots, H_7^{(6)}\}$ is quotient group, where
$H_0^{(6)} = G_3^{(6)} = \{x \in G_3 : w_i(x) \text{ is even}, w_j(x) \text{ is even}, w_r(x) \text{ is even}\}$,
$H_1^{(6)} = \{x \in G_3 : w_i(x) \text{ is even}, w_j(x) \text{ is even}, w_r(x) \text{ is odd}\}$,
$H_2^{(6)} = \{x \in G_3 : w_i(x) \text{ is even}, w_j(x) \text{ is odd}, w_r(x) \text{ is even}\}$,
$H_3^{(6)} = \{x \in G_3 : w_i(x) \text{ is even}, w_j(x) \text{ is odd}, w_r(x) \text{ is odd}\}$,
$H_4^{(6)} = \{x \in G_3 : w_i(x) \text{ is odd}, w_j(x) \text{ is even}, w_r(x) \text{ is even}\}$,
$H_5^{(6)} = \{x \in G_3 : w_i(x) \text{ is odd}, w_j(x) \text{ is even}, w_r(x) \text{ is odd}\}$,
$H_6^{(6)} = \{x \in G_3 : w_i(x) \text{ is odd}, w_j(x) \text{ is odd}, w_r(x) \text{ is even}\}$,
$H_7^{(6)} = \{x \in G_3 : w_i(x) \text{ is odd}, w_j(x) \text{ is odd}, w_r(x) \text{ is odd}\}$.

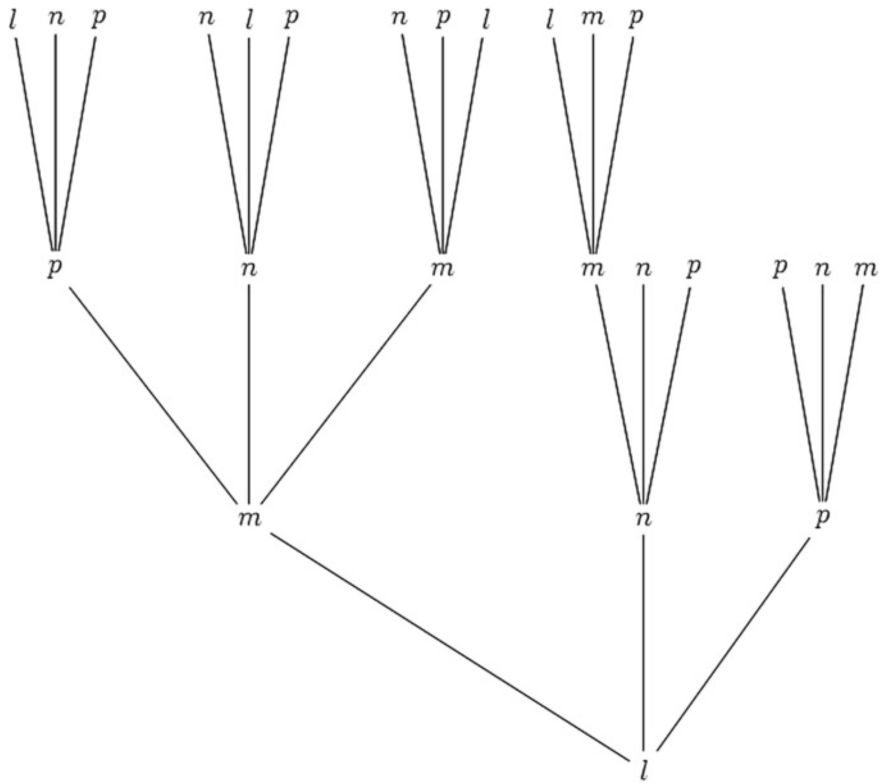

Fig. 1 Representation of the $G_k^{(6)}$ periodic configuration $\varphi_6^{(lmnp)}(x)$ on the Cayley tree of order $k=3$

For given σ_b, we have

$$\sigma_b(x)=\begin{cases} l, & \text{if } x\in H_0^{(6)}\cup H_7^{(6)}\cap b,\\ m, & \text{if } x\in H_1^{(6)}\cup H_6^{(6)}\cap b,\\ n, & \text{if } x\in H_2^{(6)}\cup H_5^{(6)}\cap b,\\ p, & \text{if } x\in H_3^{(6)}\cup H_4^{(6)}\cap b.\end{cases}$$

For any $l,m,n,p\in\Phi, l\neq m, l\neq n, l\neq p, m\neq n, m\neq p, n\neq p$, continue the bounded configuration $\sigma_b\in C_6$ to the entire lattice Γ^3 (which is denoted by $\varphi_6^{(lmnp)}$, (see Fig. 1)) as

$$\varphi_6^{(lmnp)}(x)=\begin{cases} l, & \text{if } x\in H_0^{(6)}\cup H_7^{(6)},\\ m, & \text{if } x\in H_1^{(6)}\cup H_6^{(6)},\\ n, & \text{if } x\in H_2^{(6)}\cup H_5^{(6)},\\ p, & \text{if } x\in H_3^{(6)}\cup H_4^{(6)}.\end{cases}$$

Case C_7. Let $G_k/G_k^{(4)} = \{H_0^{(7)}, H_1^{(7)}, H_2^{(7)}, H_3^{(7)}\}$ be quotient group. In [8] it was proved that for $G_k^{(4)}-$ periodic configurations

$$\varphi_7^{(lmn)}(x) = \begin{cases} l, & \text{if } x \in H_0^{(7)} \cap H_1^{(7)}, \\ m, & \text{if } x \in H_2^{(7)}, \\ n, & \text{if } x \in H_3^{(7)}, \end{cases}$$

and

$$\psi_7^{(lmn)}(x) = \begin{cases} l, & \text{if } x \in H_0^{(7)}, \\ m, & \text{if } x \in H_1^{(7)}, \\ n, & \text{if } x \in H_2^{(7)} \cap H_3^{(7)}, \end{cases}$$

one has: $(\varphi_7^{(lmn)})_{b'} \in C_7$, $(\psi_7^{(lmn)})_{b'} \in C_7$ for all $l, m, n \in \Phi, l \neq m, l \neq n, m \neq n$ and for any $b' \in M$.

In [9] it was proved that $H_{\{1,2,3\}}-$ weakly periodic configurations

$$\xi_7^{(lmn)}(x) = \begin{cases} l, & \text{if } x_{\downarrow} \in H_0, \ x \in H_0 \\ m, & \text{if } x_{\downarrow} \in H_0, \ x \in H_1 \\ n, & \text{if } x_{\downarrow} \in H_1, \ x \in H_0 \\ l, & \text{if } x_{\downarrow} \in H_1, \ x \in H_1, \end{cases}$$

satisfy the following: $(\xi_7^{(lmn)})_{b'} \in C_7$ for all $l, m, n \in \Phi, l \neq m, l \neq n, m \neq n$ and for any $b' \in M$.

Case C_8. Let $G_3^{(8)} = H_{\{i,j\}} \cap H_{\{k\}} \cap H_{\{r\}},\ i, j, k, r \in \{1, 2, 3, 4\}, i \neq j, i \neq k, i \neq r, j \neq k, j \neq r, k \neq r$. We note (see [2]) that $G_3^{(8)}$ is a normal index-eight subgroup in G_3, and $G_3/G_3^{(8)} = \{H_0^{(8)}, H_1^{(8)}, \ldots, H_7^{(8)}\}$ is quotient group, where
$H_0^{(8)} = G_3^{(8)} = \{x \in G_3 : w_i(x) + w_j(x) \text{ is even}, w_k(x) \text{ is even}, w_r(x) \text{ is even}\}$,
$H_1^{(8)} = \{x \in G_3 : w_i(x) + w_j(x) \text{ is even}, w_k(x) \text{ is even}, w_r(x) \text{ is odd}\}$,
$H_2^{(8)} = \{x \in G_3 : w_i(x) + w_j(x) \text{ is even}, w_k(x) \text{ is odd}, w_r(x) \text{ is even}\}$,
$H_3^{(8)} = \{x \in G_3 : w_i(x) + w_j(x) \text{ is even}, w_k(x) \text{ is odd}, w_r(x) \text{ is odd}\}$,
$H_4^{(8)} = \{x \in G_3 : w_i(x) + w_j(x) \text{ is odd}, w_k(x) \text{ is even}, w_r(x) \text{ is even}\}$,
$H_5^{(8)} = \{x \in G_3 : w_i(x) + w_j(x) \text{ is odd}, w_k(x) \text{ is even}, w_r(x) \text{ is odd}\}$,
$H_6^{(8)} = \{x \in G_3 : w_i(x) + w_j(x) \text{ is odd}, w_k(x) \text{ is odd}, w_r(x) \text{ is even}\}$,
$H_7^{(8)} = \{x \in G_3 : w_i(x) + w_j(x) \text{ is odd}, w_k(x) \text{ is odd}, w_r(x) \text{ is odd}\}$.

In this case, for any $l, m, n, p \in \Phi, l \neq m, l \neq n, l \neq p, m \neq n, m \neq p, n \neq p$ we define the configuration $\varphi_8^{(lmnp)}$ as

$$\varphi_8^{(lmnp)}(x) = \begin{cases} l, & \text{if } x \in H_0^{(8)} \cup H_7^{(8)}, \\ m, & \text{if } x \in H_1^{(8)} \cup H_6^{(8)}, \\ n, & \text{if } x \in H_2^{(8)} \cup H_5^{(8)}, \\ p, & \text{if } x \in H_3^{(8)} \cup H_4^{(8)}. \end{cases}$$

We thus obtain a periodic configuration $\varphi_8^{(lmnp)}$ with the period $p = 8$ such that $(\varphi_8^{(lmnp)})_b = \sigma_b$, $(\varphi_8^{(lmnp)})_{b'} \in C_8$ for any $b' \in M$.

Case C_9. We consider a normal subgroup $\mathscr{H}_0 \in G_3$ (see [2]) of infinite index constructed as follows. Let the mapping $\pi_0 : \{a_1, a_2, a_3, a_4\} \to \{e, a_1, a_2\}$ be defined by

$$\pi_0(a_i) = \begin{cases} a_i, \text{ if } i = 1, 2 \\ e, \text{ if } i \neq 1, 2. \end{cases}$$

Consider

$$f_0(x) = f_0(a_{i_1}a_{i_2}\dots a_{i_m}) = \pi_0(a_{i_1})\pi_0(a_{i_2})\dots\pi_0(a_{i_m}).$$

Then it is easy to see that f_0 is a homomorphism and hence $\mathscr{H}_0 = \{x \in G_3 : f_0(x) = e\}$ is a normal subgroup of infinity index.

Now we consider the factor group

$$G_3/\mathscr{H}_0 = \{\mathscr{H}_0, \mathscr{H}_0(a_1), \mathscr{H}_0(a_2), \mathscr{H}_0(a_1a_2), \mathscr{H}_0(a_2a_1), ...\},$$

where $\mathscr{H}_0(y) = \{x \in G_3 : f_0(x) = y\}$. We introduce the notations

$$\mathscr{H}_n = \mathscr{H}_0(\underbrace{a_1a_2...}_{n}), \mathscr{H}_{-n} = \mathscr{H}_0(\underbrace{a_2a_1...}_{n}).$$

In this notation, the factor group can be represented as

$$G_3/\mathscr{H}_0 = \{..., \mathscr{H}_{-2}, \mathscr{H}_{-2}, \mathscr{H}_0, \mathscr{H}_1, \mathscr{H}_2, ...\}.$$

It is known (see [6]), that for $x \in \mathscr{H}_n$ we have $|S_1(x) \cap \mathscr{H}_{n-1}| = 1$, $|S_1(x) \cap \mathscr{H}_n| = k - 1$, $|S_1(x) \cap \mathscr{H}_{n+1}| = 1$.

Consider the following configuration

$$\varphi_9^{(lm)}(x) = \begin{cases} 2nl, & \text{if } x \in \mathscr{H}_n, n \neq 0, \\ 0, & \text{if } x \in \mathscr{H}_0, \\ (2n-1)m, & \text{if } x \in \mathscr{H}_{-n}, n \neq 0, \end{cases}$$

where $l, m \in \Phi, l \neq m, n = 1, 2, 3\dots$

We thus obtain a periodic configuration $\varphi_9^{(lm)}$ with the infinity period, such that $(\varphi_9^{(lm)})_b = \sigma_b \in C_9$, and $(\varphi_9^{(lm)})_{b'} \in C_9$ for any $b' \in M$.

Case C_{10}. Let S_3 be the group of third-order permutations. We choose $\pi_0, \pi_1, \pi_2 \in S_3$ as

$$\pi_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \pi_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \pi_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}. \quad (5)$$

It is easily seen that $\pi_0 = \pi_1^2 = \pi_2^2$.

We consider the map $u : \{a_1, a_2, a_3, a_4\} \to \{\pi_1, \pi_2\}$

$$u(a_i) = \begin{cases} \pi_1, \ i = 1, 2; \\ \pi_2, \ i = 3, 4 \end{cases} \tag{6}$$

and assume that the function $f : G_3 \to S_3$ is defined as

$$f(x) = f(a_{i_1} a_{i_2} \dots a_{i_n}) = u(a_{i_1}) \dots u(a_{i_n}).$$

Let

$$\pi_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \pi_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \pi_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}.$$

We note (see [14]) that $H_{10} = \{x \in G_3 : f(x) = \pi_0\}$ is a normal index-six subgroup. Let $G_3/H_{10} = \{\aleph_0, \dots, \aleph_5\}$ be the quotient group, where

$$\aleph_i = \{x \in G_3 : f(x) = \pi_i\}, i = \overline{0, 5}.$$

In this case, we define the configuration

$$\varphi_{10}^{(l,m,n)}(x) = \begin{cases} l, & x \in \aleph_0 \cup \aleph_5, \\ m, & x \in \aleph_1 \cup \aleph_4, \\ n, & x \in \aleph_2 \cup \aleph_3, \end{cases}$$

where $l, m, n \in \Phi, l \neq m, l \neq n, m \neq n$.

We thus obtain a periodic configuration $\varphi_{10}^{(l,m,n)}$ with the period six, such that $(\varphi_{10}^{(l,m,n)})_b = \sigma_b \in C_{10}$, $(\varphi_{10}^{(l,m,n)})_{b'} \in C_{10}$ for any $b' \in M$.

Case C_{11}. Let S_3 be the group of third-order permutations. It is easily seen that $\pi_0 = \pi_1^2 = \pi_5^2$.

We consider the map $u : \{a_1, a_2, a_3, a_4\} \to \{\pi_1, \pi_5\}$

$$u(a_i) = \begin{cases} \pi_5, \ i = 1, 2; \\ \pi_1, \ i = 3, 4, \end{cases} \tag{7}$$

and assume that the function $f : G_3 \to S_3$ is defined as

$$f(x) = f(a_{i_1} a_{i_2} \dots a_{i_n}) = u(a_{i_1}) \dots u(a_{i_n}).$$

We note (see [14]) that $H_{11} = \{x \in G_3 : f(x) = \pi_0\}$ is a normal index-six subgroup. Let $G_3/H_{11} = \{\aleph_0, \dots, \aleph_5\}$ be the quotient group, where

$$\aleph_i = \{x \in G_3 : f(x) = \pi_i\}, i = \overline{0, 5}.$$

In this case, we define the configuration

$$\varphi_{11}^{(l,m,n)}(x) = \begin{cases} l, & x \in \aleph_0 \cup \aleph_2, \\ m, & x \in \aleph_4 \cup \aleph_5, \\ n, & x \in \aleph_1 \cup \aleph_3, \end{cases}$$

where $l, m, n \in \Phi, l \neq m, l \neq n, m \neq n$.

We thus obtain a periodic configuration $\varphi_{11}^{(l,m,n)}$ with the period six, such that $(\varphi_{11}^{(l,m,n)})_b = \sigma_b \in C_{11}$, $(\varphi_{11}^{(l,m,n)})_{b'} \in C_{11}$ for any $b' \in M$.

Case C_{12}. Let $\mathcal{U} = \{(a_1a_2)^n \in G_3 : n \in \mathbf{Z}\}$. It is easy to see, that $\mathcal{U}$ is subgroup of the group G_3. Consider the set of right cosets $G_3/\mathcal{U} = \{\mathcal{U}, \mathcal{U}a_1, \ldots, \mathcal{U}a_{k+1}, \mathcal{U}a_1a_2, ...\}$ of $\mathcal{U}$ in G_3. We introduce the notations

$$H_0 = \mathcal{U}, H_1 = \mathcal{U}a_1, ..., H_{k+1} = \mathcal{U}a_{k+1}, H_{k+2} = \mathcal{U}a_1a_2, ...$$

In this notation, the set of right coset can be represented as

$$G_3/\mathcal{U} = \{H_0, H_1, \ldots H_{k+1}, H_{k+2}, ...\}.$$

Consider the following configuration: $\varphi_{12}^l(x) = l + i$, if $x \in H_i$ for all $i = 0, 1, 2, ...$ and for any $l \in \Phi$.
Let $x \in H_n$, then $\varphi_{12}^l(x) = l + n$ and if $H_n = \mathcal{U}a_{j_1}a_{j_2}...a_{j_n}$, then for all $y \in S_1(x)$ we have $y \in \mathcal{U}a_{j_1}a_{j_2}...a_{j_n}a_t$, $t = 1, 2, 3, 4$. By construction of configuration we have $\varphi_{12}^l(y) \neq \varphi_{12}^l(x)$ and $\varphi_{12}^l(y_1) \neq \varphi_{12}^l(y_2)$ for all $y, y_1, y_2 \in S_1(x)$, $y_1 \neq y_2$.

We thus obtain a $\mathcal{U}-$periodic configuration φ_{12}^l with the infinity period, such that $(\varphi_{12}^l)_{b'} \in C_{12}$ for any $b' \in M$.

We set $B = A_1 \cap A_2$, $B_0 = A_1 \cap A_5$, $B_1 = A_2 \cap A_9$, $B_2 = A_9 \cap A_6$, $B_3 = A_6 \cap A_{12}$, $\widetilde{A_1} = A_1 \setminus (B \cup B_0)$, $\widetilde{A_2} = A_2 \setminus (B_0 \cup B_1)$, $\widetilde{A_5} = A_5 \setminus (B_0 \cup A_7)$, $\widetilde{A_6} = A_6 \setminus (B_2 \cup B_3)$, $\widetilde{A_9} = A_9 \setminus (B_1 \cup B_2)$ and $\widetilde{A_{12}} = A_{12} \setminus (B_3 \cup A_7)$. Let $GS(H)$ be the set of all ground states, and let $GS_p(H)$ be the set of all periodic ground states.

Remark 1 (1) Note that,

(i) If $q \geq 3$ then the ground states $\sigma(x)$, $\varphi_2^{(lm)}$, $\varphi_3^{(lm)}$, $\varphi_4^{(lm)}$, $\varphi_5^{(lm)}$, $\varphi_7^{(lmp)}$, $\psi_7^{(lmn)}$, $\xi_7^{(lmn)}$, $\varphi_{10}^{(lmn)}$, $\varphi_{11}^{(lmn)}$ described in Theorem 1 are ground states for the q state Potts model on the Cayley tree of order three.

(ii) If $q \geq 4$ then the ground states $\varphi_6^{(lmnp)}$, $\varphi_8^{(lmnp)}$ (described in Theorem 1) are ground states for the q state Potts model on the Cayley tree of order three.

(iii) The ground states $\varphi_9^{(lm)}$, φ_{12}^l are ground states only for the Potts model with countable set of spin values on the Cayley tree of order three.

(2) In this paper we considered the case $k = 3$. If one considers the case $k \geq 4$, the ground states described in Theorem 1 may not be ground state, and this class of ground states may be extended. Besides the set A_i in (4) will be different.

Theorem 2 *A. If* $J = (0, 0)$*, then* $GS(H) = \Omega$.

B. 1. If $J \in \widetilde{A_1}$*, then* $GS_p(H) = \{\varphi^{(i)} : i \in \Phi\}$.
2. If $J \in \widetilde{A_2}$*, then* $GS_p(H) = \{\varphi_2^{(lm)} : l, m \in \Phi, l \neq m\}$.

3. If $J \in \widetilde{A_5}$*, then* $GS_p(H) = \{\varphi_5^{(lm)} : l, m \in \Phi, l \neq m\}$.
4. If $J \in \widetilde{A_6}$*, then* $GS_p(H) = \{\varphi_6^{(lmnp)} : l, m, n, p \in \Phi, l \neq m, l \neq n, l \neq p, m \neq n, m \neq p, n \neq p\}$.
5. If $J \in \widetilde{A_9}$*, then* $GS_p(H) = \{\varphi_9^{(lm)} : l, m \in \Phi, l \neq m\}$.
6. If $J \in \widetilde{A_{12}}$*, then* $GS_p(H) = \{\varphi_{12}^l : l \in \Phi\}$.
C. 1. If $J \in B \setminus \{(0,0)\}$*, then* $GS_p(H) = \{\varphi^{(i)}, \varphi_2^{(lm)} : i, l, m \in \Phi, l \neq m\}$.
2. If $J \in B_0 \setminus \{(0,0)\}$*, then* $GS_p(H) = \{\varphi^{(i)}, \varphi_5^{(lm)} : i, l, m \in \Phi, l \neq m\}$.
3. If $J \in B_1 \setminus \{(0,0)\}$*, then* $GS_p(H) = \{\varphi_2^{(lm)}, \varphi_9^{(lm)} : i, l, m \in \Phi, l \neq m\}$.
4. If $J \in B_2 \setminus \{(0,0)\}$ *then* $GS_p(H) = \{\varphi_6^{(lmnp)}, \varphi_9^{(lm)} : l, m, n, p \in \Phi, l \neq m, l \neq n, l \neq p, m \neq n, m \neq p, n \neq p\}$.
5. If $J \in B_3 \setminus \{(0,0)\}$ *then* $GS_p(H) = \{\varphi_6^{(lmnp)}, \varphi_{12}^l : l, m, n, p \in \Phi, l \neq m, l \neq n, l \neq p, m \neq n, m \neq p, n \neq p\}$.
6. If $J \in A_8$*, then periodic configuration* $\varphi_5^{(lm)}, \xi_7^{(lmn)}(x), \psi_7^{(lmn)}(x), \varphi_8^{(lmnp)}(x), \varphi_{12}^l$ *are periodic ground states, and weakly periodic configuration* $\xi_7^{(lmn)}(x)$ *is weakly periodic ground state, where* $l, m, n, p \in \Phi, l \neq m, l \neq n, l \neq p, m \neq n, m \neq p, n \neq p$.

Proof Case A is trivial. In cases B and C, for a given configuration σ_b for which the energy $U(\sigma_b)$ is minimal, we can use Theorem 1 to construct the periodic configurations.

Remark 2 Since the set Φ is countable, it follows that the periodic and weakly periodic ground states described in Theorem 2 are countable.

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