УДК 512.554

Some remarks on nilpotent Leibniz superalgebras.

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Abstract

В данной работе исследуются супералгебры Лейбница, обладающие характеристической последовательностю равной $(n_1, \dots, n_k|m_1, \dots m_s)$ с условиеми $n_1 \leq n-2$ и $m_1 \leq m-1$, где $n_1+\dots+n_k = n,\ m_1+\dots+m_s = m$ и n, m размерности четной и нечетной частей, соответственно. В частности, мы доказываем что такие супералгебры имеют нильиндекс меньше чем $n+m$.

Ushbu maqolada xarakteristik ketma-ketligi $(n_1, \dots, n_k|m_1, \dots m_s)$ uchun $n_1 \leq n-2$ va $m_1 \leq m-1$ shartlar o'rinli bo'ladigan Leibniz superalgebralari o'rganilgan. Bu yerda n, m mos ravishda superalgebra juft va toq qismlarining o'lchamlari, xamda $n_1+\dots+n_k = n,\ m_1+\dots+m_s = m$. Xususan, bunday superalgebralarining nilindeksi $n+m$ dan kichik ekanligi isbotlangan.

It is well known that the Leibniz superalgebras are generalization of Lie superalgebras and on the other hand they naturally generalize the Leibniz algebras. The notion of Leibniz superalgebras were firstly introduced in [1]. It is known that the class of filiform algebras, which have the nilindex equal to the dimension of the algebra, is a very important for to study of nilpotent Lie and Leibniz algebras. The filiform Leibniz algebras are investigated in [2] and it is proved that the condition on the characteristic sequence to be equal to $C(L) = (n-1, 1)$ is necessary and sufficient for the Leibniz algebras to be filiform. In the work [6] the Lie superalgebras with nilindex equal to n + m were classified, and it was established that such superalgebra exist only in the cases of $n = 2$, m is odd and characteristic sequence $C(L) = (1, 1|m)$.

In the present paper we investigate the Leibniz superalgebras, with even part having the nilindex less than n. We proved that such Leibniz superalgebras have the nilindex less than $n+m$. Since the superalgebra with characteristic sequence $(n_1, \dots, n_k|m_1, \dots, m_s)$ for $n_1 \leq n-2$ and $m_1 = m$ have been already investigated in the work [5], we consider the case where $m_1 \leq m-1$.

Throughout this work we shall consider spaces and (super)algebras over the field of complex numbers.

Recall the notion of Leibniz superalgebras.

Definition 1. A $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ is called *a Leibniz superalgebra* if it is equipped with a product $[-,-]$ which satisfies the following conditions:

1. $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta (mod\ 2)}$,

2. $[x, [y, z]] = [[x, y], z] - (-1)^{\alpha\beta}[[x, z], y]-$ Leibniz superidentity,

for all $x \in L$, $y \in L_\alpha$, $z \in L_\beta$ and $\alpha, \beta \in \mathbb{Z}_2$.

The vector spaces L_0 and L_1 are said to be the even and odd parts of the superalgebra L, respectively. Evidently, the even part of a Leibniz superalgebra is a Leibniz algebra.

The set of all Leibniz superalgebras with the dimensions of the even and odd parts, respectively equal to n and m, we denote by $Leib_{n,m}$.

For a given Leibniz superalgebra L we define the descending central sequence as follows:

$$L^1 = L, \quad L^{k+1} = [L^k, L], \quad k \geq 1.$$

Definition 2. A Leibniz superalgebra L is called *nilpotent*, if there exists $s \in \mathbb{N}$ such that $L^s = 0$. The minimal number s with this property is called nilindex of the superalgebra L.

Let $L = L_0 \oplus L_1$ be a nilpotent Leibniz superalgebra. For an arbitrary element $x \in L_0$, the operator of right multiplication $R_x : L \to L$ (defined as $R_x(y) = [y, x]$) is a nilpotent endomorphism of the space L_i, where $i \in \{0, 1\}$. Taking into account the property of complex endomorphisms we can consider the Jordan form for R_x. For operator R_x denote by $C_i(x)$ $(i \in \{0, 1\})$ the descending sequence of its Jordan blocks dimensions. Consider the lexicographical order on the set $C_i(L_0)$.

Definition 3. The sequence

$$C(L) = \left(\max_{x \in L_0 \setminus L_0^2} C_0(x) \,\middle|\, \max_{\widetilde{x} \in L_0 \setminus L_0^2} C_1(\widetilde{x}) \right)$$

is said to be the characteristic sequence of the Leibniz superalgebra L.

From the Definition 3 we have that a Leibniz algebra L_0 has characteristic sequence $(n_1, \dots, n_k)$. Let $l \in \mathbb{N}$ be the nilindex of the Leibniz algebra L_0. Since $l \leq n - 1$ we have that the Leibniz algebra L_0 has at least two generators (the elements which belong to the set $L_0 \setminus L_0^2$).

For a given Leibniz algebra A of the nilindex l we put $gr(A)_i = A^i / A^{i+1}, \quad 1 \leq i \leq l - 1$ and $gr(A) = gr(A)_1 \oplus gr(A)_2 \oplus \dots \oplus gr(A)_{l-1}$. Then $[gr(A)_i, gr(A)_j] \subseteq gr(A)_{i+j}$ and we obtain the graded algebra $gr(A)$.

Definition 4. The gradation constructed in this way is called the *natural gradation* and if a Leibniz algebra G is isomorphic to $gr(A)$ we say that the algebra G is *naturally graded Leibniz algebra.*

Since the second part of the characteristic sequence of the Leibniz superalgebra L is equal to $(m_1, \dots, m_s)$ then by the definition of the characteristic sequence there exists a nilpotent endomorphism R_x $(x \in L_0 \setminus L_0^2)$ of the space L_1 such that its Jordan form consists of s Jordan blocks. Therefore, we can assume the existence of an adapted basis $\{y_1, y_2, \dots, y_m\}$ of the subspace L_1, such that

$$\begin{cases} [y_j, x] = y_{j+1}, & j \notin \{m_1, m_1 + m_2, \dots, m_1 + m_2 + \dots + m_s\}, \\ [y_j, x] = 0, & j \in \{m_1, m_1 + m_2, \dots, m_1 + m_2 + \dots + m_s\}. \end{cases} \tag{1}$$

for some $x \in L_0 \setminus L_0^2$.

According to the Theorem 7 from [1] we have the description of single-generated Leibniz superalgebras, which have nilindex $n + m + 1$. If the number of generators is greater than two, then the superalgebra has nilindex less than $n + m$. Therefore, we should consider the case of two-generated superalgebras.

Since the superalgebra $L = L_0 \oplus L_1$ has two generators, then we have the following possible cases: both generators lie in L_0; one generator lies in L_0 and the another one lies in L_0; both generators lie in L_1.

Since $m \neq 0$ we omit the case where both generators lie in even part.

Let us consider the second possible case.

Lemma 1. Let one generator lie in L_0 and another one lie in L_1. Then x_1 and y_1 can be chosen as generators of L. Moreover, in the equality (1) instead of the element x we can take x_1.

Proof. The proof is similar to the proof of Lemma 3.1 in [5]. □

Due to Lemma 1 further we shall suppose that $\{x_1, y_1\}$ are generators of the Leibniz superalgebra L. Therefore,

$$L^2 = \{x_2, x_3, \dots, x_n, y_2, y_3, \dots, y_m\}.$$

Let us introduce the notations:

$$[x_i, y_1] = \sum_{j=2}^{m} \alpha_{i,j} y_j, \ 1 \leq i \leq n, \quad [y_i, y_1] = \sum_{j=2}^{n} \beta_{i,j} x_j, \ 1 \leq i \leq m. \tag{2}$$

Firstly we consider the case where $dim(L^3)_0 = n-1$, then $dim(L^3)_1 = m-2$. i.e.

$$L^3 = \{x_2, x_3, \dots, x_n, y_3, \dots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \dots, y_m\},$$

where $(B_1, B_2) \neq (0,0)$.

We summarize our main result in considered cases in the following

Theorem 1. Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with the characteristic sequence equal to $(n_1, \dots, n_k | m_1, \dots, m_s)$, where $n_1 \leq n-2, \ m_1 \leq m-1$ and let $dim(L^3)_0 = n-1$ with $y_2 \notin L^3$. Then L has the nilindex less than $n+m$.

Proof. Let us assume the contrary, i.e. the nilindex of the superalgebra L is equal to $n+m$. Then from the conditions of the Theorem for the powers of the superalgebra L we obtain

$$L^h = \{x_2, x_3, \dots, x_n, y_h, y_{h+1}, \dots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \dots, y_m\}, \ h \geq 3,$$

$$L^{h+1} = \{x_3, \dots, x_n, y_h, y_{h+1}, \dots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \dots, y_m\},$$

$$L^{h+2} = \{x_3, \dots, x_n, y_{h+1}, \dots, y_{m_1}, B_1 y_2 + B_2 y_{m_1+1}, y_{m_1+2}, \dots, y_m\}.$$

Since $y_h \notin L^{h+2}$, it follows that

$$\alpha_{2,h} \neq 0, \quad \alpha_{i,h} = 0 \quad \text{for} \quad i > 2.$$

Consider the product

$$[[y_{h-1}, y_1], y_1] = \frac{1}{2}[y_{h-1}, [y_1, y_1]] = \frac{1}{2}[y_{h-1}, \sum_{i=2}^{n} \beta_{1,i} x_i].$$

The element y_{h-1} belongs to L^{h-1} and the elements $x_2, x_3, \dots, x_n$ lie in L^3. Hence $\frac{1}{2}[y_{h-1}, \sum\limits_{i=2}^{n} \beta_{1,i} x_i] \in L^{h+2}$. Since $y_h \notin L^{h+2}$, we obtain that $[[y_{h-1}, y_1], y_1] = \sum\limits_{j \geq h+1} (*) y_j$.

On the other hand,

$$[[y_{h-1}, y_1], y_1] = [x_2, y_1] = \alpha_{2,h} y_h + \sum_{j=h+1}^{m} \alpha_{2,j} y_j.$$

Comparing the coefficients at the basic elements we obtain $\alpha_{2,h} = 0$, which is a contradiction with the assumption that the superalgebra L has the nilindex equal to $n+m$ and therefore the assertion of the theorem is proved. $\square$

If $y_2 \in L^3$ then $B_2 = 0$ and the following theorem is true.

Theorem 2. Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with the characteristic sequence equal to $(n_1, \dots, n_k | m_1, \dots, m_s)$, where $n_1 \le n-2,\ m_1 \le m-1$ and let $dim(L^3)_0 = n-1$ with $y_2 \in L^3$. Then L has the nilindex less than $n+m$.

Proof. We shall prove the assertion of the theorem by the contrary, i.e. we assume that nilindex of the superalgebra L equal to $n+m$. The condition $y_2 \in L^3$ implies

$$L^3 = \{x_2, x_3, \dots, x_n, y_2, \dots, y_{m_1}, y_{m_1+2}, \dots, y_m\}.$$

Then $\alpha_{1,m_1+1} \neq 0$ and $\alpha_{i,m_1+1} = 0$ for $i \ge 2$. The element y_2 is generated from products $[x_i, y_1]$, $i \ge 2$ which implies $y_2 \in L^4$. Since $[y_{m_1+1}, y_1] = [[x_1, y_1], y_1] = \frac{1}{2}[x_1, [y_1, y_1]] = \frac{1}{2}[x_1, \sum\limits_{i\ge 2}(*)x_i]$ and x_2 is a generator of the Leibniz algebra L_0 then x_2 can not generated from the product $[y_{m_1+1}, y_1]$. By asterisks $(*)$ we denote the appropriate coefficients at the basic elements of superalgebra. Thereby x_2 also belongs to L^4.

Consider the equality

$$[[x_1, y_1], x_1] = [x_1, [y_1, x_1]] + [[x_1, x_1], y_1] = [x_1, y_2] - [\sum_{i\ge 3}(*)x_i, y_1].$$

From this it follows that the product $[[x_1, y_1], x_1]$ belongs to L^5 (and therefore belongs to L^4).

On the other hand,

$$[[x_1, y_1], x_1] = [\sum_{j=2}^{m} \alpha_{1,j} y_j, x_1] = \alpha_{1,2} y_3 + \dots + \alpha_{1,m_1-1} y_{m_1} + \alpha_{1,m_1+1} y_{m_1+2} + \dots + \alpha_{1,m-1} y_m.$$

Since $\alpha_{1,m_1+1} \neq 0$, we obtain that $y_{m_1+2} \in L^4$. Thus, we have $L^4 = \{x_2, x_3, \dots, x_n, y_2, \dots, y_{m_1}, y_{m_1+2}, \dots, y_m\}$, that is $L^4 = L^3$. It is a contradiction to nilpotency of the superalgebra L.

Thus, we get a contradiction with the assumption that the superalgebra L has the nilindex equal to $n+m$ and therefore the assertion of the theorem is proved. $\square$

From Theorems 1 and 2 we obtain that a Leibniz superalgebra L with condition $dim(L^3)_0 = n-1$ has the nilindex less than $n+m$.

The investigation of Leibniz superalgebras with property $dim(L^3)_0 = n-2$ shows that the restriction on to nilindex depends on the structure of the Leibniz algebra L_0. Below we present some necessary remarks on nilpotent Leibniz algebras.

Let $A = \{z_1, z_2, \dots, z_n\}$ be an n-dimensional nilpotent Leibniz algebra with the nilindex l $(l < n)$.

Proposition 1. [5] Let $gr(A)$ be a naturally graded non-Lie Leibniz algebra. Then $dimA^3 \le n-4$.

The results on nilindex of the superalgebra under the condition $dim(L^3)_0 = n-2$ are established in the following two propositions.

Proposition 2. Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with the characteristic sequence $(n_1, \dots, n_k | m_1, \dots m_s)$, where $n_1 \le n-2,\ m_1 \le m-1$, $dim(L^3)_0 = n-2$ and $dimL_0^3 \le n-4$. Then L has the nilindex less than $n+m$.

Proof. The proof is similar to the proof of Proposition 3.1. in [5]. □

From Proposition 2 we conclude that the Leibniz superalgebra $L = L_0 \oplus L_1$ with the characteristic sequence $(n_1, \dots, n_k | m_1, \dots, m_s)$, where $n_1 \leq n-2,\ m_1 \leq m-1$ and nilindex $n+m$ can appear only if $dim L_0^3 \geq n-3$. Taking into account the condition $n_1 \leq n-2$ and properties of naturally graded subspaces $gr(L_0)_1,\ gr(L_0)_2$ we get $dim L_0^3 = n-3$. Then from Proposition 1 the naturally graded Leibniz algebra $gr(L_0)$ is a Lie algebra, i.e. the following multiplication rules hold

$$\begin{cases} [x_1, x_1] = \gamma_{1,4}x_4 + \gamma_{1,5}x_5 + \dots + \gamma_{1,n}x_n, \\ [x_2, x_1] = x_3, \\ [x_1, x_2] = -x_3 + \gamma_{2,4}x_4 + \gamma_{2,5}x_5 + \dots + \gamma_{2,n}x_n, \\ [x_2, x_2] = \gamma_{3,4}x_4 + \gamma_{3,5}x_5 + \dots + \gamma_{3,n}x_n. \end{cases} \tag{3}$$

Proposition 3. Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with the characteristic sequence $(n_1, \dots, n_k | m_1, \dots, m_s)$, where $n_1 \leq n-2,\ m_1 \leq m-1$, $dim(L^3)_0 = n-2$ and $dim L_0^3 = n-3$. Then L has the nilindex less than $n+m$.

Proof. The proof is similar to the proof of Proposition 3.2. in [5]. □

Let us consider the case where both generators lie in the odd part. Then the following Theorem is true.

Theorem 3. Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra from $Leib_{n,m}$ with the characteristic sequence equal to $(n_1, \dots, n_k | m_1, \dots, m_s)$, where $n_1 \leq n-2,\ m_1 \leq m-1$ and let both generators lie in L_1. Then L has the nilindex less than $n+m$.

Proof. Let us assume the contrary, i.e. the nilindex of the superalgebra L is equal to $n+m$. Since both generators of the superalgebra L lie in L_1, they are linear combinations of the elements $\{y_1, y_{m_1+1}, \dots, y_{m_1+\dots+m_{s-1}+1}\}$. Without loss of generality we may assume that y_1 and y_{m_1+1} are generators.

Applying similar method as in the proof of the Theorem 3.3 [3] we obtain that $n = m-1$ or $n = m-2$ and

$$L^3 = \{x_2, \dots, x_n, y_2, y_3, \dots, y_{m_1}, y_{m_1+2}, \dots, y_m\}.$$

Let $B_1y_2 + B_2y_{m_1+2}$ the element which do not belong to L^4, then any element of the form $B_1'y_2 + B_2'y_{m_1+2}$ belongs to L^4, where $B_1B_2' - B_2B_1' \neq 0$.

Hence, from the notations

$$[x_1, y_1] = \alpha_{1,2}(B_1y_2 + B_2y_{m_1+2}) + \alpha_{1,m_1+2}(B_1'y_2 + B_2'y_{m_1+2}) + \sum_{j=3, j\neq m_1+2}^{m} \alpha_{1,j}y_j.$$

$$[x_1, y_{m_1+1}] = \delta_{1,2}(B_1y_2 + B_2y_{m_1+2}) + \delta_{1,m_1+2}(B_1'y_2 + B_2'y_{m_1+2}) + \sum_{j=3, j\neq m_1+2}^{m} \delta_{1,j}y_j,$$

we have $(\alpha_{1,2}, \delta_{1,2}) \neq (0,0)$.

Similarly, from the notations

$$[B_1y_2 + B_2y_{m_1+2}, y_1] = \beta_{2,2}x_2 + \beta_{2,3}x_3 + \dots + \beta_{2,n}x_n,$$

$$[B_1y_2 + B_2y_{m_1+2}, y_{m_1+1}] = \gamma_{2,2}x_2 + \gamma_{2,3}x_3 + \dots + \gamma_{2,n}x_n,$$

we obtain the condition $(\beta_{2,2}, \gamma_{2,2}) \neq (0,0)$.

Consider the product

$$[x_1,[y_1,y_1]] = 2[[x_1,y_1],y_1] = 2\alpha_{1,2}[B_1y_2+B_2y_{m_1+2},y_1] + 2\alpha_{1,m_1+2}[B_1'y_2+B_2'y_{m_1+2},y_1]+$$

$$+2\sum_{j=3,j\neq m_1+2}^{m}\delta_{1,j}[y_j,y_1] = 2\alpha_{1,2}\beta_{2,2}x_2 + \sum_{i\geq 3}(*)x_i.$$

On the other hand,

$$[x_1,[y_1,y_1]] = [x_1,\beta_{1,1}x_1+\beta_{1,2}x_2+\cdots+\beta_{1,n}x_n] = \sum_{i\geq 3}(*)x_i.$$

Comparing the coefficients at the basic elements in these equations we obtain $\alpha_{1,2}\beta_{2,2} = 0$.

Similarly, considering the product $[x_1,[y_{m_1+1},y_{m_1+1}]]$, we obtain $\delta_{1,2}\gamma_{2,2}=0$.

From these equations and the conditions $(\beta_{2,2},\gamma_{2,2})\neq(0,0)$, $(\alpha_{1,2},\delta_{1,2})\neq(0,0)$ we easily obtain that the solutions are $\alpha_{1,2}\gamma_{2,2}\neq 0, \beta_{2,2}=\delta_{1,2}=0$ or $\beta_{2,2}\delta_{1,2}\neq 0, \alpha_{1,2}=\gamma_{2,2}=0$.

Consider the following product

$$[[x_1,y_1],y_{m_1+1}] = [x_1,[y_1,y_{m_1+1}]] - [[x_1,y_{m_1+1}],y_1] = -\delta_{1,2}\beta_{2,2}x_2 + \sum_{i\geq 3}(*)x_i.$$

On the other hand,

$$[[x_1,y_1],y_{m_1+1}] = \alpha_{1,2}\gamma_{2,2}x_2 + \sum_{i\geq 3}(*)x_i.$$

Comparing the coefficients of the basic elements in these equations we obtain irregular equation $\alpha_{1,2}\gamma_{2,2} = -\beta_{2,2}\delta_{1,2}$. It is a contradiction with assumption that the nilindex of the superalgebra is equal to the $n+m$. The theorem is proved. □

Thus, the results of the Theorems 1–3 and Propositions 2, 3 show that the Leibniz superalgebras with the characteristic sequence $(n_1,\dots,n_k|m_1,\dots,m_s)$ ($n_1\leq n-2$ and $m_1\leq m-1$) has the nilindex less than $n+m$.

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