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# Anisotropic Ising Model with Countable Set of Spin Values on Cayley Tree

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*In this paper we investigate of an infinite system of functional equations for the Ising model with competing interactions and countable spin values* $0, 1, \ldots$ *and non zero filed on a Cayley tree of order two. We derived an infinite system of functional equations for the Ising model that is we describe conditions on* $h_x$ *guaranteeing compatibility of distributions* $\mu^{(n)}(\sigma_n)$.

*Keywords: Cayley tree, Ising model, Gibbs measures, functional equations, compatibility of distributions measures.*
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## Introduction

It is well known the Ising model is the simplest and most famous example of lattice model, moreover its behavior is wonderfully rich. In [1] some physical motivations why the Ising model on a Cayley tree is interesting are given. In [2] and in [3] the existence of a phase transition for the Ising modle on the Cayley tree for $k \geqslant 2$ is established.

In [4] the Potts model with countable set $\Phi$ of spin values on $Z^d$ was considered and it was proved that with respect to Poisson distribution on $\Phi$ the set of limiting Gibbs measure is not empty. In [5] the Potts model with a nearest neighbor interaction and countable set of spin values on a Cayley tree.

If the interactions of an atom with its nearest neighbors is independent of direction, the model is called isotropic; otherwise, when the interaction energy depends on the direction of the neighbor, such as its horizontal versus vertical neighbors, the model is called anisotropic.

In [6, 7] considered models with nearest-neighbor interactions and with the set $[0, 1]$ of spin values, on a Cayley tree of order $k \geqslant 1$.

In this paper we investigate countable spin Ising model on a Cayley tree. These countable spin models, which we rigorously define shortly, generalize the classical Ising models in that the spin variables $\sigma$ are not restricted to the two $\pm 1$ values but instead may assume any natural number values.

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# 1. Definition

The Cayley tree (Bethe lattice) $\Gamma^k$ of order $k \geqslant 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where $V$ is the set of vertices and $L$ the set of edges. Two vertices $x$ and $y$ are called *nearest neighbors* if there exists an edge $l \in L$ connecting them and we denote $l =< x, y >$. A collection of nearest neighbor pairs $< x, x_1 >, < x_1, x_2 >, \ldots, < x_{d-1}, y >$ is called a *path* from $x$ to $y$. The distance $d(x,y)$ on the Cayley tree is the number of edges of the shortest path from $x$ and $y$.

For a fixed $x^0 \in V$, called the root, we set

$$W_n = \{x \in V | d(x, x^0) = n\}, \qquad V_n = \bigcup_{m=1}^{n} W_m$$

and denote

$$S(x) = \{y \in W_{n+1} : d(x,y) = 1\}, \quad x \in W_n,$$

the set of direct successors of $x$.

# 2. Hamiltonian and measure

The Ising model was invented by the physicist Wilhelm Lenz (1920), who gave it as a problem to his student Ernst Ising. The one-dimensional Ising model has no phase transition and was solved by Ising (1925) himself.

We consider the Ising model with competing interactions and countable set of spin values on the Cayley tree of order two.

The vertices $x$ and $y$ are called *second neighbor* which is denoted by $\rangle x, y\langle$, if there exists a vertex $z \in V$ such that $x, z$ and $y, z$ are nearest-neighbors. We will consider only second neighbors $\rangle x, y\langle$, for which there exist $n$ such that $x, y \in W_n$.

In this paper we consider model where the spin takes values in the set of all non negative integer numbers $\Phi := 0, 1, ...$, and is assigned to the of the tree. A configuration $\sigma$ on $V$ is then defined as a function $x \in V \mapsto \sigma(x) \in \Phi$; the set of all configurations is $\Phi^V$.

We consider the Ising model with competing interactions on the Cayley tree which is defined by the following Hamiltonian

$$H(\sigma) = -J \sum_{\substack{<x,y> \\ x,y \in V}} \sigma(x)\sigma(y) - J_1 \sum_{\substack{>x,y< \\ x,y \in V}} \sigma(x)\sigma(y) \qquad (2.1)$$

where $J, J_1 \in R$ are constants.

Write $x < y$ if the path from $x^0$ to $y$ goes through $x$. Call vertex $y$ a direct successor of $x$ if $y > x$ and $x, y$ are nearest neighbors. Denote by $S(x)$ the set of direct successors of $x$. Observe that any vertex $x \neq x^0$ has $k$ direct successors and $x^0$ has $k+1$.

Let $h : x \in V \mapsto h_x = (h_{t,x}, t \in [0,1]) \in \mathbb{R}^{[0,1]}$ be a mapping of $x \in V \setminus \{x^0\}$. Given $n = 1, 2, \ldots$, consider the probability distribution $\mu^{(n)}$ on $\Omega_{V_n}$ defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\right). \qquad (2.2)$$

Here, as before, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and $Z_n$ is the corresponding partition function:

$$Z_n = \int_{\Omega_{V_n}} \exp\left(-\beta H(\widetilde{\sigma}_n) + \sum_{x \in W_n} h_{\widetilde{\sigma}(x),x}\right) \lambda_{V_n}(d\widetilde{\sigma}_n). \tag{2.3}$$

The probability distributions $\mu^{(n)}$ are called compatible if for any $n \geqslant 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\int_{\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n)\lambda_{W_n}(d(\omega_n)) = \mu^{(n-1)}(\sigma_{n-1}). \tag{2.4}$$

Here $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of $\sigma_{n-1}$ and $\omega_n$. In this case, because of the Kolmogorov extension theorem, there exists a unique measure $\mu$ on $\Omega_V$ such that, for any $n$ and $\sigma_n \in \Omega_{V_n}$, $\mu\left(\left\{\sigma\Big|_{V_n} = \sigma_n\right\}\right) = \mu^{(n)}(\sigma_n)$. Such a measure is called a splitting Gibbs measure corresponding to Hamiltonian (2.1) and function $x \mapsto h_x$, $x \neq x^0$.

# 3. An infinite system of functional equations

The following theorem describes conditions on $h_x$ guaranteeing compatibility of distributions $\mu^{(n)}(\sigma_n)$.

**Theorem 3.1.** *Probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$, in (2.2), on Cayley tree of order two, are compatible iff for any $x \in V \setminus \{x^0\}$ the following equation holds:*

$$h^*_{i,x} = F_i(h^*_y, h^*_z, \beta, J), \quad i = 1, 2, \ldots, \tag{3.1}$$

*where $S(x) = \{y, z\}$, $h^*_x = (h_{1,x} - h_{0,x}, h_{2,x} - h_{0,x}, \ldots)$ and*

$$F_i(h^*_y, h^*_z, \beta, J) = \ln \frac{1 + \sum\limits_{\substack{p,q=0\\p+q\neq 0}}^{\infty} \exp\{i\beta J(p+q) + J_1\beta pq + h^*_{p,y} + h^*_{q,z} + \ln\frac{\nu(p)}{\nu(0)} + \ln\frac{\nu(q)}{\nu(0)}\}}{1 + \sum\limits_{\substack{p,q=0\\p+q\neq 0}}^{\infty} \exp\{J_1\beta pq + h^*_{p,y} + h^*_{q,z} + \ln\frac{\nu(p)}{\nu(0)} + \ln\frac{\nu(q)}{\nu(0)}\}}.$$

*Proof. Necessity.* Suppose that (2.4) holds; we will prove (3.1). Substituting (2.2) in (2.4), obtain that for any configurations $\sigma_{n-1} : x \in V_{n-1} \mapsto \sigma_{n-1}(x) \in \Phi$:

$$\frac{1}{Z_n} \sum_{\sigma^{(n)} \in \Phi^{W_n}} \exp\{-\beta H_n(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\} \times \prod_{y \in W_n} \nu(\sigma(y)) =$$
$$= \frac{1}{Z_{n-1}} \exp\{-\beta H_{n-1}(\sigma_{n-1}) + \sum_{x \in W_{n-1}} h_{\sigma_{n-1}(x),x}\}.$$

$$\frac{Z_{n-1}}{Z_n} \sum_{\sigma^{(n)} \in \Phi^{W_n}} \exp\{-\beta H_{n-1}(\sigma_{n-1}) + J\beta \sum_{\substack{x \in W_{n-1}\\y,z \in S(x)}} \sigma(x)\left(\sigma(y) + \sigma(z)\right) + J_1\beta \sum_{\substack{x \in W_{n-1}\\y,z \in S(x)}} \sigma(y)\sigma(z) +$$
$$+ \sum_{x \in W_n} h_{\sigma(x),x}\} \times \prod_{y \in W_n} \nu(\sigma(y)) = \exp\{-\beta H_{n-1}(\sigma_{n-1}) + \sum_{x \in W_{n-1}} h_{\sigma_{n-1}(x),x}\}$$

After some abbreviations we get

$$\frac{Z_{n-1}}{Z_n}\prod_{x\in W_{n-1}}\sum_{\sigma_x^{(n)}=\{\sigma(y),\sigma(z)\}}\exp\left\{J\beta\sigma(x)\left(\sigma(y)+\sigma(z)\right)+J_1\beta\sigma(y)\sigma(z)+h_{\sigma(y),y}+h_{\sigma(z),z}+\right.$$
$$\left.+\ln\nu(\sigma(y))+\ln\nu(\sigma(z))\right\}=\prod_{x\in W_{n-1}}\exp\{h_{\sigma_{n-1}(x),x}\}.$$

Consequently, for any $i\in\Phi$,

$$\frac{e^{h_{0,y}+h_{0,z}+2\ln\nu(0)}+\sum\limits_{\substack{p,q=0\\p+q\neq0}}^{\infty}e^{J\beta i(p+q)+J_1\beta pq+h_{p,y}+h_{q,z}+\ln\nu(p)+\ln\nu(q)}}{e^{h_{0,y}+h_{0,z}+2\ln\nu(0)}+\sum\limits_{\substack{p,q=0\\p+q\neq0}}^{\infty}e^{J_1\beta pq+h_{p,y}+h_{q,z}+\ln\nu(p)+\ln\nu(q)}}=e^{h_{i,x}-h_{0,x}},$$

so that

$$h^*_{i,x}=\ln\frac{1+\sum\limits_{\substack{p,q=0\\p+q\neq0}}^{\infty}e^{J\beta i(p+q)+J_1\beta pq+h^*_{p,y}+h^*_{q,z}}}{1+\sum\limits_{\substack{p,q=0\\p+q\neq0}}^{\infty}e^{J_1\beta pq+h^*_{p,y}+h^*_{q,z}}},$$

where

$$h^*_{i,x}=h_{i,x}-h_{0,x}+\ln\frac{\nu(i)}{\nu(0)}.$$

*Sufficiency.* Let (3.1) is satisfied we will prove (2.4). We have

$$\sum_{p,q=0}^{\infty}\exp\{J\beta i(p+q)+J_1\beta pq+h_{p,y}+h_{q,z}+\ln\nu(p)+\ln\nu(q)\}=a(x)\exp\{h_{i,x}\},\tag{3.2}$$

here $i=0,1,\ldots$ .

Then

$$\text{LHS of}(2.4)=\frac{1}{\mathrm{Z_n}}\exp\{-\beta\mathrm{H_{n-1}}(\sigma_{\mathrm{n-1}})\prod_{\mathrm{x}\in\mathrm{W_{n-1}}}\nu(\sigma(\mathrm{x}))\times$$
$$\times\sum_{\substack{x\in W_{n-1}\\y,z\in S(x)}}\exp\{J\beta\sigma(x)(\sigma(y)+\sigma(z))+J_1\beta\sigma(y)\sigma(z)+h_{\sigma(y),y}+h_{\sigma(z),z}\}\,.\tag{3.3}$$

Substituting (3.2) into (3.3) and denoting $A_n=\prod\limits_{x\in W_{n-1}}a(x)$, we get

$$\text{RHS of}(3.3)=\frac{\mathrm{A_{n_1}}}{\mathrm{Z_n}}\exp(-\beta\mathrm{H_{n-1}}(\sigma_{\mathrm{n-1}}))\prod_{\mathrm{x}\in\mathrm{W_{n-1}}}\mathrm{h}_{\sigma_{\mathrm{n-1}}(\mathrm{x}),\mathrm{x}}\,.\tag{3.4}$$

Since $\mu^{(n)},\ n\geqslant1$ is a probability, we should have

$$\sum_{\sigma_{n-1}}\sum_{\sigma}^{(n)}\mu^{(n)}(\sigma_{n-1},\sigma^{(n-1)})=1.$$

Hence from (3.4) we get $Z_{n-1}A_{n-1}=Z_n$, and (2.4) holds.

**Remark**. From Theorem 3.1. it follows that for any $h=\{h_x,x\in V\}$ satisfying the functional equation (3.1) there exists a unique Gibbs measure $\mu$ and vice versa. However, the analysis of solutions to (3.1) is not easy.

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## Анизотропная модель Изинга со счетным множеством значений спина на дереве Кэли


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*В данной работе мы исследуем бесконечную систему функциональных уравнений для модели Изинга с конкурирующими взаимодействиями, счетными значениями спина* $0, 1, \ldots$ *и ненулевыми данными на дереве Кэли второго порядка. Мы нашли бесконечную систему функциональных уравнений для модели Изинга, в который мы описываем условия на* $h_x$*, гарантирующие совместимость распределений* $\mu^{(n)}(\sigma_n)$.



*Ключевые слова: дерево Кэли, модель Изинга, гиббсовские меры, функциональные уравнения, совместимость распределений мер.*