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Local and 2-local derivations of solvable Leibniz algebras

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We show that any local derivation on the solvable Leibniz algebras with model or abelian nilradicals, whose dimension of complementary space is maximal is a derivation. We show that solvable Leibniz algebras with abelian nilradicals, which have 1 dimension complementary space, admit local derivations which are not derivations. Moreover, similar problem concerning 2-local derivations of such algebras is investigated and an example of solvable Leibniz algebra is given such that any 2-local derivation on it is a derivation, but which admits local derivations which are not derivations.

Keywords: Leibniz algebras; solvable algebras; nilpotent algebras; derivation; inner derivation; local derivation; 2-local derivation.

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1. Introduction

In recent years, non-associative analogues of classical constructions have become of interest in connection with their applications in many branches of mathematics and physics. The notions of local and 2-local derivations are also popular for some non-associative algebras such as the Lie and Leibniz algebras.

The notions of local derivations were introduced in 1990 by Kadison [10] and Larson and Sourour [11]. Later in 1997, Šemrl introduced the notions of 2-local derivations and 2-local automorphisms on algebras [14]. The main problems concerning these notions are to find conditions under which all local (2-local) derivations become (global) derivations and to present examples of algebras with local (2-local) derivations that are not derivations.

Investigation of local derivations on Lie algebras was initiated in papers in [5, 9]. Ayupov and Kudaybergenov have proved that every local derivation on semi-simple Lie algebras is a derivation and gave examples of nilpotent finite-dimensional Lie algebras with local derivations which are not derivations. In [2], local derivations of solvable Lie algebras are investigated and it is shown that in the class of solvable Lie algebras, there exist algebras which admit local derivations which are not ordinary derivation and also algebras for which every local derivation is a derivation. Moreover, it is proved that every local derivation on a finite-dimensional solvable Lie algebra with model nilradical and maximal dimension of complementary space is a derivation. In [6], local derivations and automorphism of complex finite-dimensional simple Leibniz algebras are investigated. They proved that in all local derivations on the finite-dimensional complex, simple Leibniz algebras are automatically derivations and it is shown that filiform Leibniz algebras admit local derivations which are not derivations.

Several papers have been devoted to the similar notions and corresponding problems for 2-local derivations and automorphisms of finite-dimensional Lie and Leibniz algebras [3, 4, 6, 7, 9]. Namely, in [7], it is proved that every 2-local derivation on the semi-simple Lie algebras is a derivation and that each finite-dimensional nilpotent Lie algebra, with dimension larger than two admits 2-local derivation which is not a derivation. Concerning 2-local automorphism, Chen and Wang in [9] proved that if $\mathcal{L}$, is a simple Lie algebra of type A_l, D_l or $E_k, (k = 6, 7, 8)$ over an algebraically closed field of characteristic zero, then every 2-local automorphism of $\mathcal{L}$, is an automorphism. Finally, in [3], Ayupov and Kudaybergenov generalized this result of [9] and proved that every 2-local automorphism of the finite-dimensional semi-simple Lie algebras over an algebraically closed field of characteristic zero is an automorphism. Moreover, they also showed that every nilpotent Lie algebra with finite-dimensional larger than two admits 2-local automorphisms, which are not automorphisms.

In [8, 15] the authors investigated 2-local derivations on infinite-dimensional Lie algebras over a field of characteristic zero. They proved that all 2-local derivations on the Witt algebra as well as on the positive Witt algebra are (global) derivations, and gave an example of infinite-dimensional Lie algebra with a 2-local derivation which is not a derivation.

In this paper, we study local and 2-local derivations of solvable Leibniz algebras. We show that any local derivation on solvable Leibniz algebras with model or abelian nilradicals, whose dimension of complementary space is maximal is a derivation, but solvable Leibniz algebras with abelian nilradical, who have 1 dimension complementary space, admit local derivations which are not derivations. Moreover, similar problems concerning 2-local derivations of such algebras are investigated.

2. Preliminaries

In this section, we give some necessary definitions and preliminary results.

Definition 2.1 ([12]). A vector space with a bilinear bracket $(\mathcal{L}, [\cdot,\cdot])$ is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity

$$\big[x, [y, z]\big] = \big[[x, y], z\big] - \big[[x, z], y\big],$$

holds.

Here, we adopt the right Leibniz identity; since the bracket is not skew-symmetric, there exists the version corresponding to the left Leibniz identity,

$$\big[[x, y], z\big] = \big[x, [y, z]\big] - \big[y, [x, z]\big].$$

Let $\mathcal{L}$ be a Leibniz algebra. For a Leibniz algebra $\mathcal{L}$, consider the following lower central and derived sequences:

$$\mathcal{L}^1 = \mathcal{L}, \quad \mathcal{L}^{k+1} = [\mathcal{L}^k, \mathcal{L}^1], \quad k \geq 1,$$
$$\mathcal{L}^{[1]} = \mathcal{L}, \quad \mathcal{L}^{[s+1]} = [\mathcal{L}^{[s]}, \mathcal{L}^{[s]}], \quad s \geq 1.$$

Definition 2.2. A Leibniz algebra $\mathcal{L}$ is called nilpotent (respectively, solvable), if there exists $p \in \mathbb{N}$ $(q \in \mathbb{N})$ such that $\mathcal{L}^p = 0$ (respectively, $\mathcal{L}^{[q]} = 0$).The minimal number p (respectively, q) with such property is said to be the index of nilpotency (respectively, of solvability) of the algebra $\mathcal{L}$.

Note that any Leibniz algebra $\mathcal{L}$ contains a unique maximal solvable (respectively, nilpotent) ideal, called the radical (respectively, nilradical) of the algebra.

A derivation on a Leibniz algebra $\mathcal{L}$ is a linear map $D : \mathcal{L} \to \mathcal{L}$ which satisfies the Leibniz rule:

$$D([x, y]) = [D(x), y] + [x, D(y)] \quad \text{for any } \ x, y \in \mathcal{L}. \tag{2.1}$$

The set of all derivations of a Leibniz algebra $\mathcal{L}$ is a Lie algebra with respect to the usual matrix commutator and it is denoted by $\mathrm{Der}(\mathcal{L})$.

For any element $x \in \mathcal{L}$, the operator of right multiplication $ad_x : \mathcal{L} \to \mathcal{L}$, defined as $ad_x(z) = [z, x]$ is a derivation, and derivations of this form are called inner derivation. The set of all inner derivations of $\mathcal{L}$, denoted by $ad(\mathcal{L})$, is an ideal in $\mathrm{Der}(\mathcal{L})$.

For a finite-dimensional nilpotent Leibniz algebra N and for the matrix of the linear operator ad_x denote by $C(x)$ the descending sequence of its Jordan blocks' dimensions. Consider the lexicographical order on the set $C(N) = \{C(x) \mid x \in N\}$.

Definition 2.3. The sequence

$$\left(\max_{x \in N \setminus N^2} C(x) \right)$$

is said to be the characteristic sequence of the nilpotent Leibniz algebra N.

Definition 2.4. A linear operator Δ is called a local derivation, if for any $x \in \mathcal{L}$, there exists a derivation $D_x : \mathcal{L} \to \mathcal{L}$ (depending on x) such that $\Delta(x) = D_x(x)$. The set of all local derivations on $\mathcal{L}$ we denote by $\mathrm{LocDer}(\mathcal{L})$.

Definition 2.5. A map $\nabla : \mathcal{L} \to \mathcal{L}$ (not necessary linear) is called a 2-local derivation, if for any $x, y \in \mathcal{L}$, there exists a derivation $D_{x,y} \in \mathrm{Der}(\mathcal{L})$ such that

$$\nabla(x) = D_{x,y}(x), \quad \nabla(y) = D_{x,y}(y).$$

2.1. *Solvable Leibniz algebras with abelian nilradical*

Let $\mathbf{a}_n$ be the n-dimensional abelian algebra and let R be a solvable Leibniz algebra with nilradical $\mathbf{a}_n$. Take a basis $\{f_1, f_2, \dots, f_n, x_1, x_2, \dots x_k\}$ of R, such that $\{f_1, f_2, \dots, f_n\}$ is a basis of $\mathbf{a}_n$. In [1], such solvable algebras in case of $k = n$ are classified and it is proved that any $2n$-dimensional solvable Leibniz algebra with nilradical $\mathbf{a}_n$ is isomorphic and the direct sum of two-dimensional algebras, i.e., isomorphic to the algebra

$$\mathcal{L}_t : [f_j, x_j] = f_j, \quad [x_j, f_j] = \alpha_j f_j, \ \ 1 \leq j \leq n,$$

where $\alpha_j \in \{-1, 0\}$ and t is the number of zero parameters α_j.

Moreover, in the following theorem, the classification of $(n+1)$-dimensional solvable Leibniz algebras with n-dimensional abelian nilradical is given.

Theorem 2.6 ([1]). *Let R be a $(n+1)$-dimensional solvable Leibniz algebra with n-dimensional abelian nilradical. If R has a basis $\{f_1, f_2, \dots, f_n, x\}$ such that the operator $ad_x|_{\boldsymbol{a}_n}$ has Jordan block form, then it is isomorphic to one of the following two non-isomorphic algebras*:

$$R_1 : \begin{cases} [f_i, x] = f_i + f_{i+1}, & 1 \leq i \leq n-1, \\ [f_n, x] = f_n, \end{cases}$$

$$R_2 : \begin{cases} [f_i, x] = f_i + f_{i+1}, & 1 \leq i \leq n-1, \\ [f_n, x] = f_n, \\ [x, f_i] = -f_i - f_{i+1}, & 1 \leq i \leq n-1, \\ [x, f_n] = -f_n. \end{cases}$$

In the following propositions, we present the general form of derivations of the algebras $\mathcal{L}_t$, R_1 and R_2.

Proposition 2.7 ([1]). *Any derivation D of the algebra $\mathcal{L}_t$ has the following form*:

$$D(f_j) = a_j f_j, \quad D(x_j) = \alpha_j b_j f_j, \ \ 1 \leq j \leq n.$$

Proposition 2.8. *Any derivation D of the algebras R_1 and R_2 has the following form*:

$$\mathrm{Der}(R_1) : D(f_i) = \sum_{j=i+1}^{n} \alpha_{j-i+1} f_j + \alpha_1 f_i.$$

$$\mathrm{Der}(R_2) : \begin{cases} D(f_i) = \displaystyle\sum_{j=i+1}^{n} \alpha_{j-i+1} f_j + \alpha_1 f_i, & 1 \leq i \leq n, \\ D(x) = \displaystyle\sum_{j=1}^{n} \beta_j f_j. \end{cases}$$

2.2. *Solvable Leibniz algebras with model nilradical*

Let N be a nilpotent Leibniz algebra with the characteristic sequence $(m_1, \ldots, m_s)$, and with the table of multiplication

$$N_{m_1,\ldots,m_s} : [e_i^t, e_1^1] = e_{i+1}^t, \quad 1 \leq t \leq s, \ 1 \leq i \leq m_t - 1.$$

The algebra $N_{m_1,\ldots,m_s}$ usually is said to be model Leibniz algebra.

Theorem 2.9 ([13]). *An solvable Leibniz algebra R with nilradical $N_{m_1,\ldots,m_s}$, such that* $\mathrm{Dim}R - \mathrm{Dim}N_{m_1,\ldots,m_s} = s$, *is isomorphic to the algebra*:

$$R(N_{m_1,\ldots,m_s}, s) : \begin{cases} [e_i^t, e_1^1] = e_{i+1}^t, & 1 \leq t \leq s, \ 1 \leq i \leq m_t - 1, \\ [e_i^1, x_1] = i e_i^1, & 1 \leq i \leq m_1, \\ [e_i^t, x_1] = (i-1) e_i^t, & 2 \leq t \leq s, \ 2 \leq i \leq m_t, \\ [e_i^t, x_t] = e_i^t, & 2 \leq t \leq s, \ 1 \leq i \leq m_t, \\ [x_1, e_1^1] = -e_1^1, & \end{cases}$$

where $\{x_1, \ldots, x_s\}$ is a basis of the complementary vector space.

Proposition 2.10 ([13]). *Any derivation D of the algebra* $\mathrm{Der}(R(N_{m_1,\ldots,m_s}, s))$ *has the following form*:

$$\begin{aligned} &D(e_i^1) = i\alpha_1 e_i^1 + \alpha_2 e_{i+1}^1, && 1 \leq i \leq m_1 - 1, \\ &D(e_{m_1}^1) = m_1 \alpha_1 e_{m_1}^1, && \\ &D(e_i^t) = ((i-1)\alpha_1 + \beta_t) e_i^t + \alpha_2 e_{i+1}^t, && 2 \leq t \leq s, \ 1 \leq i \leq m_t - 1, \\ &D(e_{m_t}^t) = ((m_t - 1)\alpha_1 + \beta_t) e_{m_t}^t, && 2 \leq t \leq s, \\ &D(x_1) = -\alpha_2 e_1^1. && \end{aligned}$$

Remark 2.11. Any derivation on the solvable Leibniz algebra $R(N_{m_1,\ldots,m_s}, s)$ is an inner derivation.

3. Local Derivation of Solvable Leibniz Algebras

3.1. *Local derivation of solvable Leibniz algebra* $R(N_{m_1,\ldots,m_s}, s)$

Now, we shall give the main result concerning local derivations of solvable Leibniz algebra $R(N_{m_1,\ldots,m_s}, s)$.

Theorem 3.1. *Any local derivation on the solvable Leibniz algebra $R(N_{m_1,\dots,m_s}, s)$ is a derivation.*

Proof. Let Δ be a local derivation on $R(N_{m_1,\dots,m_s}, s)$, then we have

$$\Delta(x_i) = \sum_{j=1}^{s} a_{i,j} x_j + \sum_{p=1}^{s}\sum_{j=1}^{m_p} b_{i,j}^{p} e_j^p, \quad \Delta(e_i^t) = \sum_{j=1}^{s} c_{i,j}^{t} x_j + \sum_{p=1}^{s}\sum_{j=1}^{m_p} d_{i,j}^{t,p} e_j^p.$$

Let D be a derivation on $R(N_{m_1,\dots,m_s}, s)$, then by Proposition 2.10, we obtain

$$\begin{aligned}
&D(e_i^1) = i\alpha_i e_i^1 + \beta_i e_{i+1}^1, && 1 \le i \le m_1 - 1,\\
&D(e_{m_1}^1) = m_1 \alpha_{m_1} e_{m_1}^1, &&\\
&D(e_1^t) = \sigma_t e_1^t + \theta_t e_2^t, && 2 \le t \le s,\\
&D(e_i^t) = ((i-1)\lambda_{i,t} + \mu_{i,t}) e_i^t + \delta_{i,t} e_{i+1}^t, && 2 \le t \le s,\ \ 2 \le i \le m_t - 1,\\
&D(e_{m_t}^t) = ((m_t - 1)\xi_{i,t} + \eta_{i,t}) e_{m_t}^t, && 2 \le t \le s,\\
&D(x_1) = -\gamma e_1^1. &&
\end{aligned}$$

Considering the equalities

$$\begin{aligned}
\Delta(x_j) &= D_{x_j}(x_j), \quad 1 \le j \le s,\\
\Delta(e_i^t) &= D_{e_i^t}(e_i^t), \quad 1 \le t \le s,\ \ 1 \le i \le m_t,
\end{aligned}$$

we have

$$\begin{cases}
\displaystyle\sum_{j=1}^{s} c_{i,j}^{1} x_j + \sum_{p=1}^{s}\sum_{j=1}^{m_p} d_{i,j}^{1,p} e_j^p = i\alpha_i e_i^1 + \beta_i e_{i+1}^1, & 1 \le i \le m_1 - 1,\\
\displaystyle\sum_{j=1}^{s} c_{m_1,j}^{1} x_j + \sum_{p=1}^{s}\sum_{j=1}^{m_p} d_{m_1,j}^{1,p} e_j^p = m_1 \alpha_{m_1} e_{m_1}^1, &\\
\displaystyle\sum_{j=1}^{s} c_{1,j}^{t} x_j + \sum_{p=1}^{s}\sum_{j=1}^{m_p} d_{1,j}^{t,p} e_j^p = \sigma_t e_1^t + \theta_t e_2^t, & 2 \le t \le s,\\
\displaystyle\sum_{j=1}^{s} c_{i,j}^{t} x_j + \sum_{p=1}^{s}\sum_{j=1}^{m_p} d_{i,j}^{t,p} e_j^p &\\
\qquad = ((i-1)\lambda_{i,t} + \mu_{i,t}) e_i^t + \delta_{i,t} e_{i+1}^t, & 2 \le t \le s,\ \ 2 \le i \le m_t - 1,\\
\displaystyle\sum_{j=1}^{s} c_{m_p,j}^{t} x_j + \sum_{p=1}^{s}\sum_{j=1}^{m_p} d_{m_p,j}^{t,p} e_j^p &\\
\qquad = ((m_t - 1)\xi_{m_p,t} + \eta_{m_p,t}) e_{m_t}^t, & 2 \le t \le s,\\
\displaystyle\sum_{j=1}^{s} a_{1,j} x_j + \sum_{p=1}^{s}\sum_{j=1}^{m_p} b_{1,j}^{p} e_j^p = -\gamma e_1^1, &\\
\displaystyle\sum_{j=1}^{s} a_{1,j} x_j + \sum_{p=1}^{s}\sum_{j=1}^{m_p} b_{1,j}^{p} e_j^p = 0, & 2 \le i \le n.
\end{cases}$$

From the previous restrictions, we get that

$$\begin{aligned}
&\Delta(e_i^1) = d_{i,i}^{1,1} e_i^1 + d_{i,i+1}^{1,1} e_{i+1}^1, && 1 \le i \le m_1 - 1,\\
&\Delta(e_{m_1}^1) = d_{m_1,m_1}^{1,1} e_{m_1}^1, &&\\
&\Delta(e_1^t) = d_{1,1}^{t,t} e_1^t + d_{1,2}^{t,t} e_2^t, && 2 \le t \le s,\\
&\Delta(e_i^t) = d_{i,i}^{t,t} e_i^t + d_{i,i+1}^{t,t} e_{i+1}^t, && 2 \le t \le s,\ \ 2 \le i \le m_t - 1,\\
&\Delta(e_{m_t}^t) = d_{m_t,m_t}^{t,t} e_{m_t}^t, && 2 \le t \le s,\\
&\Delta(x_1) = b_{1,1}^1 e_1^1. &&
\end{aligned}$$

Consider

$$\Delta(e_1^1 + e_1^t) = d_{1,1}^{1,1} e_1^1 + d_{1,2}^{1,1} e_2^1 + d_{1,1}^{t,t} e_1^t + d_{1,2}^{t,t} e_2^t.$$

On the other hand,

$$\begin{aligned}
\Delta(e_1^1 + e_1^t) &= D_{e_1^1+e_1^t}(e_1^1 + e_1^t) = D_{e_1^1+e_1^t}(e_1^1) + D_{e_1^1+e_1^t}(e_1^t)\\
&= \alpha_{e_1^1+e_1^t} e_1^1 + \beta_{e_1^1+e_1^t} e_2^1 + \eta_{e_1^1+e_1^t} e_1^t + \beta_{e_1^1+e_1^t} e_2^t.
\end{aligned}$$

Comparing the coefficients at the basis elements e_2^1 and e_2^t, we get $\beta_{e_1^1+e_1^t} = d_{1,2}^{1,1}$, $\beta_{e_1^1+e_1^t} = d_{1,2}^{t,t}$, which implies

$$d_{1,2}^{t,t} = d_{1,2}^{1,1}, \quad 2 \le t \le s.$$

Similarly, considering $\Delta(e_1^1 + e_i^1)$ for $3 \le i \le m_1 - 1$, we have

$$\begin{aligned}
\Delta(e_1^1 + e_i^1) &= d_{1,1}^{1,1} e_1^1 + d_{1,2}^{1,1} e_2^1 + d_{i,i}^{1,1} e_i^1 + d_{i,i+1}^{1,1} e_{i+1}^1\\
&= D_{e_1^1+e_i^1}(e_1^1 + e_i^1) = D_{e_1^1+e_i^1}(e_1^1) + D_{e_1^1+e_i^1}(e_i^1)\\
&= \alpha_{e_1^1+e_i^1} e_1^1 + \beta_{e_1^1+e_i^1} e_2^1 + i\alpha_{e_1^1+e_i^1} e_i^1 + \beta_{e_1^1+e_i^1} e_{i+1}^1,
\end{aligned}$$

which implies

$$d_{i,i}^{1,1} = i d_{1,1}^{1,1}, \quad d_{i,i+1}^{1,1} = d_{1,2}^{1,1}, \quad 3 \le i \le m_1 - 1.$$

From the equalities,

$$\begin{aligned}
\Delta(e_1^1 + e_2^1) &= d_{1,1}^{1,1} e_1^1 + d_{1,2}^{1,1} e_2^1 + d_{2,2}^{1,1} e_2^1 + d_{2,3}^{1,1} e_3^1\\
&= D_{e_1^1+e_2^1}(e_1^1 + e_2^1) = D_{e_1^1+e_2^1}(e_1^1) + D_{e_1^1+e_2^1}(e_2^1)\\
&= \alpha_{e_1^1+e_2^1} e_1^1 + \beta_{e_1^1+e_2^1} e_2^1 + 2\alpha_{e_1^1+e_2^1} e_2^1 + \beta_{e_1^1+e_2^1} e_3^1,
\end{aligned}$$

and

$$\begin{aligned}
&\Delta(e_1^1 + e_{m_1}^1) = d_{1,1}^{1,1} e_1^1 + d_{1,2}^{1,1} e_2^1 + d_{m_1,m_1}^{1,1} e_{m_1}^1\\
&\Delta(e_1^1 + e_{m_1}^1) = D_{e_1^1+e_{m_1}^1}(e_1^1 + e_{m_1}^1) = D_{e_1^1+e_{m_1}^1}(e_1^1) + D_{e_1^1+e_{m_1}^1}(e_{m_1}^1) =\\
&\qquad = \alpha_{e_1^1+e_{m_1}^1} e_1^1 + \beta_{e_1^1+e_{m_1}^1} e_2^1 + m_1\alpha_{e_1^1+e_{m_1}^1} e_{m_1}^1,
\end{aligned}$$

we get that

$$d_{2,2}^{1,1} = 2d_{1,1}^{1,1}, \quad d_{m_1,m_1}^{1,1} = m_1 d_{1,1}^{1,1}.$$

Now for $2 \leq t \leq s$, $2 \leq i \leq m_t - 1$, we consider

$$\Delta(e_i^t + e_1^t + e_1^1) = d_{i,i}^{t,t} e_i^t + d_{i,i+1}^{t,t} e_{i+1}^t + d_{1,1}^{t,t} e_1^t + d_{1,2}^{t,t} e_2^t + d_{1,1}^{1,1} e_1^1 + d_{1,2}^{1,1} e_2^1.$$

On the other hand,

$$\begin{aligned}\Delta(e_i^t + e_1^t + e_1^1) = D_{e_i^t+e_1^t+e_1^1}(e_i^t + e_1^t + e_1^1) &= ((i-1)\alpha_{e_i^t+e_1^t+e_1^1} + \eta_{e_i^t+e_1^t+e_1^1,t}) e_i^t \\ &+ \beta_{e_i^t+e_1^t+e_1^1} e_{i+1}^t + \eta_{e_i^t+e_1^t+e_1^1,t} e_1^t + \beta_{e_i^t+e_1^t+e_1^1} e_2^t \\ &+ \alpha_{e_i^t+e_1^t+e_1^1} e_1^1 + \beta_{e_i^t+e_1^t+e_1^1} e_2^1.\end{aligned}$$

Comparing the coefficients at the basis elements e_i^t, e_1^t and e_1^1, we get

$$\alpha_{e_i^t+e_1^t+e_1^1} = d_{1,1}^{1,1}, \quad (i-1)\alpha_{e_i^t+e_1^t+e_1^1} + \eta_{e_i^t+e_1^t+e_1^1,t} = d_{i,i}^{t,t}, \ \ \eta_{e_i^t+e_1^t+e_1^1,t} = d_{1,1}^{t,t},$$

which implies

$$d_{i,i}^{t,t} = (i-1)d_{1,1}^{1,1} + d_{1,1}^{t,t}, \quad 2 \leq t \leq s, \ \ 2 \leq i \leq m_t - 1.$$

Similarly, from

$$\begin{aligned}\Delta(e_{m_t}^t + e_1^t + e_1^1) &= d_{m_t,m_t}^{t,t} e_{m_t}^t + d_{1,1}^{t,t} e_1^t + d_{1,2}^{t,t} e_2^t + d_{1,1}^{1,1} e_1^1 + d_{1,2}^{1,1} e_2^1 \\ &= D_{e_{m_t}^t+e_1^t+e_1^1}(e_{m_t}^t + e_1^t + e_1^1) \\ &= ((m_t-1)\alpha_{e_{m_t}^t+e_1^t+e_1^1} + \eta_{e_{m_t}^t+e_1^t+e_1^1,t}) e_{m_t}^t + \\ &\quad + \eta_{e_{m_t}^t+e_1^t+e_1^1,t} e_1^t + \beta_{e_{m_t}^t+e_1^t+e_1^1} e_2^t + \alpha_{e_{m_t}^t+e_1^t+e_1^1} e_1^1 \\ &\quad + \beta_{e_{m_t} e_i^t+e_1^t+e_1^1} e_2^1,\end{aligned}$$

we get that

$$d_{m_t,m_t}^{t,t} = (m_t - 1)d_{1,1}^{1,1} + d_{1,1}^{t,t}, \quad 2 \leq t \leq s.$$

Now, we consider

$$\Delta(x_1 + e_2^1) = b_{1,1}^1 e_1^1 + d_{2,2}^{1,1} e_2^1 + d_{2,3}^{1,1} e_3^1.$$

On the other hand,

$$\Delta(x_1 + e_2^1) = D_{x_1+e_2^1}(x_1 + e_2^1) = -\beta_{x_1+e_2^1} e_1^1 + 2\alpha_{x_1+e_2^1} e_2^1 + \beta_{x_1+e_2^1} e_3^1.$$

Comparing the coefficients at the basis elements e_3^1 and e_1^1, we get $\beta_{x_1+e_2^1} = d_{2,3}^{1,1}$, $-\beta_{x_1+e_2^1} = b_{1,1}^1$, which implies

$$b_{1,1}^1 = -d_{2,3}^{1,1} = -d_{1,2}^{1,1}.$$

Thus, we obtain that the local derivation Δ has the following form:

$$\begin{aligned}
&\Delta(e_i^1) = id_{1,1}^{1,1}e_i^1 + d_{1,2}^{1,1}e_{i+1}^1, && 1 \le i \le m_1 - 1,\\
&\Delta(e_{m_1}^1) = m_1 d_{1,1}^{1,1}e_{m_1}^1, && \\
&\Delta(e_i^t) = ((i-1)d_{1,1}^{1,1} + d_{1,1}^{t,t})e_i^t + d_{1,2}^{1,1}e_{i+1}^t, && 2 \le t \le s,\ \ 1 \le i \le m_t - 1,\\
&\Delta(e_{m_t}^t) = ((m_t - 1)d_{1,1}^{1,1} + d_{1,1}^{t,t})e_{m_t}^t, && 2 \le t \le s,\\
&\Delta(x_1) = -d_{1,2}^{1,1}e_1^1. &&
\end{aligned}$$

Proposition 2.10 implies that Δ is a derivation. Hence, every local derivation on $R(N_{m_1,\dots,m_s}, s)$ is a derivation. □

3.2. *Local derivation of solvable Leibniz algebras with abelian nilradical*

Now, we shall give the main result concerning local derivations on solvable Leibniz algebras with abelian nilradicals.

Theorem 3.2. *Any local derivation on the algebra $\mathcal{L}_t$ is a derivation.*

Proof. For any local derivation Δ on the algebra $\mathcal{L}_t$, we put the derivation D, such that:

$$D(f_j) = a_j f_j, \quad D(x_j) = \alpha_j b_j f_j, \ \ 1 \le j \le n.$$

Then, we get

$$\Delta(f_j) = D_{f_j}(f_j) = a_j f_j, \quad \Delta(x_j) = D_{x_j}(x_j) = \alpha_j b_j f_j.$$

Hence, Δ is a derivation. □

In the following theorem, we show that $(n+1)$-dimensional solvable Leibniz algebras with n-dimensional abelian nilradical have a local derivation, which is not a derivation.

Theorem 3.3. $(n+1)$*-dimensional solvable Leibniz algebras* R_1 *and* R_2 (*see Theorem* 2.6), *admit a local derivation, which is not a derivation.*

Proof. Let us consider the linear operator Δ on R_1 and R_2, such that

$$\Delta\left(\sum_{i=1}^{n} \xi_i e_i + \xi_{n+1} x\right) = 2\xi_1 e_{n-1} + \xi_2 e_n.$$

By Proposition 2.8, it is not difficult to see that Δ is not a derivation. We show that Δ is a local derivation on R_1 and R_2.

Consider the derivations D_1 and D_2 on the algebras R_1 and R_2, defined as

$$D_1\left(\sum_{i=1}^{n}\xi_i e_i + \xi_{n+1}x\right) = \xi_1 e_{n-1} + \xi_2 e_n,$$

$$D_2\left(\sum_{i=1}^{n}\xi_i e_i + \xi_{n+1}x\right) = \xi_1 e_n.$$

Now, for any $\xi = \sum_{i=1}^{n}\xi_i e_i + \xi_{n+1}x$, we shall find a derivation D, such that $\Delta(\xi) = D(\xi)$.

If $\xi_1 = 0$, then

$$\Delta(\xi) = 0 = D_2(\xi).$$

If $\xi_1 \neq 0$, then setting $D = 2D_1 + tD_2$, where $t = -\frac{\xi_2}{\xi_1}$, we obtain that

$$\Delta(\xi) = 2\xi_1 e_{n-1} + \xi_2 e_n = 2\xi_1 e_{n-1} + (2\xi_2 + t\xi_1)e_n = 2(\xi_1 e_{n-1} + \xi_2 e_n) + t\xi_1 e_n = \\ = 2D_1(\xi) + tD_2(\xi) = D(\xi).$$

Hence, Δ is a local derivation. □

4. 2-Local Derivation of Solvable Leibniz Algebras

4.1. 2-*local derivation of solvable Leibniz algebra* $R(N_{m_1,\dots,m_s}, s)$

Now, we shall give the main result concerning the 2-local derivations of solvable Leibniz algebra $R(N_{m_1,\dots,m_s}, s)$.

Consider an element $q = \sum_{t=1}^{s}\sum_{i=1}^{m_t} e_i^t$ of $R(N_{m_1,\dots,m_s}, s)$.

Lemma 4.1. *Let* ∇ *be a* 2-*local derivation of* $R(N_{m_1,\dots,m_s}, s)$, *such that* $\nabla(q) = 0$. *Then* $\nabla \equiv 0$.

Proof. Take an element $a_{q,\xi} \in R(N_{m_1,\dots,m_s}, s)$, such that

$$\nabla(q) = [q, a_{q,\xi}], \quad \nabla(\xi) = [\xi, a_{q,\xi}].$$

Then

$$0 = \nabla(q) = [q, a_{q,\xi}] = \left[\sum_{t=1}^{s}\sum_{i=1}^{m_t} e_i^t, \sum_{i=1}^{n} c_i x_i + \sum_{t=1}^{s}\sum_{i=1}^{m_t} d_i^t e_i^t\right]$$

$$= \sum_{i=1}^{m_1} c_1 i e_1^1 + \sum_{t=2}^{s}\sum_{i=1}^{m_t} c_1(i-1)e_i^t + \sum_{t=2}^{s}\sum_{i=1}^{m_t} c_t(i-1)e_i^t + \sum_{t=1}^{s}\sum_{i=1}^{m_t-1} d_1^1 e_{i+1}^t,$$

which implies, $c_i = d_1^1 = 0$ for all $1 \leq i \leq n$.

Thus,

$$a_{q,\xi} = \sum_{t=2}^{s} d_1^t e_1^t + \sum_{i=2}^{m_1} d_i^1 e_i^1 + \sum_{t=2}^{s}\sum_{i=2}^{m_t} d_i^t e_i^t.$$

Consequently, for any element $\xi \in R(N_{m_1,\dots,m_s}, s)$, we have

$$\nabla(\xi) = [\xi, a_{q,\xi}] = \left[\sum_{i=1}^{n} \lambda_i x_i + \sum_{t=1}^{s}\sum_{i=1}^{m_t} \mu_i^t e_i^t, \sum_{t=2}^{s} d_1^t e_1^t + \sum_{i=2}^{m_1} d_i^1 e_i^1 + \sum_{t=2}^{s}\sum_{i=2}^{m_t} d_i^t e_i^t\right] = 0.$$

□

Theorem 4.2. *Any 2-local derivation of the solvable Leibniz algebra $R(N_{m_1,\dots,m_s}, s)$ is a derivation.*

Proof. Let ∇ be a 2-local derivation of $R(N_{m_1,\dots,m_s}, s)$. Take a derivation $D_{\xi,q}$ such that

$$\nabla(q) = D_{\xi,q}(q).$$

Set $\nabla_1 = \nabla - D_{\xi,q}$. Then ∇_1 is a 2-local derivation, such that $\nabla_1(q) = 0$.

By Lemma 4.1, we get that $\nabla_1 \equiv 0$. Hence, $\nabla = D_{\xi,q}$, i.e., ∇ is a derivation.

□

4.2. *2-local derivation of solvable Leibniz algebras with abelian nilradical*

Now, we shall give the result concerning of 2-local derivations of solvable Leibniz algebras with abelian nilradical.

Theorem 4.3. *The algebra $\mathcal{L}_t$ admits a 2-local derivation which is not a derivation.*

Proof. Let us define a homogeneous non additive function f on $\mathbb{C}^2$ as follows:

$$f(z_1, z_2) = \begin{cases} \dfrac{z_1^2}{z_2}, & \text{if } z_2 \neq 0, \\ 0, & \text{if } z_2 = 0, \end{cases}$$

where $(z_1, z_2) \in \mathbb{C}^2$.

Define the operator ∇ on $\mathcal{L}_t$, such that

$$\nabla(\xi) = f(\xi_1, \xi_{n+1}) f_1, \tag{4.1}$$

for any element $\xi = \sum_{i=1}^{n} \xi_i f_i + \sum_{i=1}^{n} \xi_{n+i} x_i$.

The operator ∇ is not a derivation, since it is not linear.

Let us show that, ∇ is a 2-local derivation. For this purpose, define a derivation D on $\mathcal{L}_t$ by

$$D(\xi) = (a\xi_1 + b\xi_{n+1}) f_1.$$

For each pair of elements ξ and η, we choose a and b, such that $\nabla(\xi) = D(\xi)$ and $\nabla(\eta) = D(\eta)$. Let us rewrite the above equalities as system of linear equations with respect to unknowns a, b as follows:

$$\begin{cases} \xi_1 a + \xi_{n+1} b = f(\xi_1, \xi_{n+1}), \\ \eta_1 a + \eta_{n+1} b = f(\eta_1, \eta_{n+1}). \end{cases} \tag{4.2}$$

Case 1. $\xi_1\eta_{n+1} - \xi_{n+1}\eta_1 = 0$. In this case, since the right-hand side of the system (4.2) is homogeneous, it has infinitely many solutions.

Case 2. $\xi_1\eta_{n+1} - \xi_{n+1}\eta_1 \neq 0$. In this case, the system (4.2) has a unique solution. □

Proposition 4.4. *Any 2-local derivation of the algebra R_1 is a derivation.*

Proof. Let ∇ be a 2-local derivation on R_1, such that $\nabla(f_1) = 0$. Then for any element $\xi = \sum_{i=1}^{n} \xi_i f_i + \xi_{n+1}x \in R_1$, there exists a derivation $D_{f_1,\xi}(\xi)$, such that

$$\nabla(f_1) = D_{f_1,\xi}(f_1), \quad \nabla(\xi) = D_{f_1,\xi}(\xi).$$

Hence,

$$0 = \nabla(f_1) = D_{f_1,\xi}(f_1) = \sum_{i=1}^{n} \alpha_i f_i,$$

which implies, $\alpha_i = 0$ for $1 \leq i \leq n$.

Consequently, from the description of the derivation R_1, we conclude that $D_{f_1,\xi} = 0$. Thus, we obtain that if $\nabla(f_1) = 0$, then $\nabla \equiv 0$.

Let now ∇ be an arbitrary 2-local derivation of R_1. Take a derivation $D_{f_1,\xi}$, such that

$$\nabla(f_1) = D_{f_1,\xi}(f_1) \quad \text{and} \quad \nabla(\xi) = D_{f_1,\xi}(\xi).$$

Set $\nabla_1 = \nabla - D_{f_1,\xi}$. Then ∇_1 is a 2-local derivation, such that $\nabla_1(f_1) = 0$. Hence, $\nabla_1(\xi) = 0$ for all $\xi \in R_1$, which implies $\nabla = D_{f_1,\xi}$. Therefore, ∇ is a derivation. □

Theorem 4.5. *Solvable Leibniz algebra R_2 admits a 2-local derivation which is not a derivation.*

Proof. The proof is similar to the proof of Theorem 4.3. □

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