NONTRANSLATION INVARIANT GIBBS MEASURES FOR MODELS WITH UNCOUNTABLE SET OF SPIN VALUES ON A CAYLEY TREE

U. A. ROZIKOV and G. I. BOTIROV

Institute of mathematics, 81, Mirzo Ulug'bek str., 100125 Tashkent, Uzbekistan
(e-mails: rozikovu@yandex.ru, botirovg@yandex.ru)

(*Received May 26, 2017 — Revised September 20, 2017*)

We consider models with nearest-neighbour interactions and with the set $[0, 1]$ of spin values, on a Cayley tree of order $k \geq 1$. It is known that the "splitting Gibbs measures" of the model can be described by solutions of a nonlinear integral equation. Recently, by solving this integral equation some periodic (in particular translation invariant) splitting Gibbs measures were found. In this paper we give three constructions of new sets of nontranslation invariant splitting Gibbs measures. Our constructions are based on known solutions of the integral equation (1.5).

Mathematics Subject Classifications (2010). 82B05, 82B20 (primary); 60K35 (secondary)

Keywords: Cayley tree, configuration, Gibbs measures, uniqueness.

1. Introduction

Let us first give necessary definitions, then explain what is the main problem; secondly we give the history of its solutions and then formulate the part of the problem which we want to solve in this paper.

A *Cayley tree* Γ^k of order $k \geq 1$ is an infinite tree, i.e. a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L the set of edges.

Two vertices x and y are called *nearest neighbours* if there exists an edge $l \in L$ connecting them. We will use the notation $l = \langle x, y\rangle$.

A collection of nearest neighbour pairs $\langle x, x_1\rangle, \langle x_1, x_2\rangle, ..., \langle x_{d-1}, y\rangle$ is called a *path* from x to y. The *distance* $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x to y.

For a fixed $x^0 \in V$, called the *root*, we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \qquad V_n = \bigcup_{m=0}^{n} W_m, \qquad L_n = \{\langle x, y\rangle \in L : x, y \in V_n\}$$

and denote

$$S_k(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \qquad x \in W_n,$$

the set of *direct successors* of x on the Cayley tree of order k.

We consider models where the spin takes values in the set $[0, 1]$, and spins are assigned to the vertices of the tree. For $A \subset V$, a *configuration* σ_A on A is an arbitrary function $\sigma_A : A \to [0, 1]$. Denote by $\Omega_A = [0, 1]^A$ the set of all configurations on A. We denote $\Omega = [0, 1]^V$.

The *Hamiltonian* of the model is

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \xi_{\sigma(x)\sigma(y)}, \tag{1.1}$$

where $J \in R\setminus\{0\}$ and $\xi : (u, v) \in [0, 1]^2 \to \xi_{uv} \in R$ is a given bounded, measurable function.

Let λ be the Lebesgue measure on $[0, 1]$. On the set of all configurations on A the *a priori* measure λ_A is introduced as the $|A|$-fold product of the measure λ, where $|A|$ denotes the cardinality of A.

We consider a standard sigma-algebra $\mathcal{B}$ of subsets of $\Omega = [0, 1]^V$ generated by the measurable cylinder subsets.

A probability measure μ on $(\Omega, \mathcal{B})$ is called a *Gibbs measure* (corresponding to the Hamiltonian H) if it satisfies the DLR equation, namely for any $n = 1, 2, \ldots$ and $\sigma_n \in \Omega_{V_n}$,

$$\mu\left(\left\{\sigma \in \Omega :\ \sigma\big|_{V_n} = \sigma_n\right\}\right) = \int_{\Omega} \mu(\mathrm{d}\omega)\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n),$$

where

$$\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n) = \frac{1}{Z_n\left(\omega\big|_{W_{n+1}}\right)} \exp\left(-\beta H\left(\sigma_n \,||\, \omega\big|_{W_{n+1}}\right)\right),$$

and $\beta = \frac{1}{T}$, $T > 0$ is temperature. Furthermore, $\sigma\big|_{V_n}$ and $\omega\big|_{W_{n+1}}$ denote the restrictions of configurations $\sigma, \omega \in \Omega$ to V_n and W_{n+1}, respectively. Next, $\sigma_n : x \in V_n \mapsto \sigma_n(x)$ is a configuration in V_n and

$$H\left(\sigma_n \,||\, \omega\big|_{W_{n+1}}\right) = -J \sum_{\langle x,y\rangle \in L_n} \xi_{\sigma_n(x)\sigma_n(y)} - J \sum_{\langle x,y\rangle :\, x\in V_n, y\in W_{n+1}} \xi_{\sigma_n(x)\omega(y)}.$$

Finally,

$$Z_n\left(\omega\big|_{W_{n+1}}\right) = \int_{\Omega_{V_n}} \exp\left(-\beta H\left(\widetilde{\sigma}_n \,||\, \omega\big|_{W_{n+1}}\right)\right)\lambda_{V_n}(d\widetilde{\sigma}_n).$$

The *main problem* for a given Hamiltonian is to describe all its Gibbs measures. See [8] for a general definition of Gibbs measure, motivations why these measures are important and the theory of such measures.

This main problem is not completely solved even for simple Ising or Potts models on a Cayley tree with a finite set of spin values. Mainly this problem is solved for the class of splitting Gibbs measures (SGMs) [11] (Markov chains [8]), which are limiting Gibbs measures constructed by Kolmogorov's extension theorem of the following finite-dimensional distributions: given $n = 1, 2, \ldots,$ consider the probability distribution μ_n on Ω_{V_n} defined by

$$\mu_n(\sigma_n) = Z_n^{-1} \exp\Big(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\Big), \tag{1.2}$$

where $h : x \in V \mapsto h_x = (h_{t,x}, t \in [0,1]) \in R^{[0,1]}$ be mapping of $x \in V \setminus \{x^0\}$. Here Z_n is the corresponding partition function. The probability distributions μ_n are compatible if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$,

$$\int_{\Omega_{W_n}} \mu_n(\sigma_{n-1} \vee \omega_n) \lambda_{W_n}(d(\omega_n)) = \mu_{n-1}(\sigma_{n-1}). \tag{1.3}$$

Here $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of σ_{n-1} and ω_n.

To see that a SGM satisfies the DLR equation, we consider any finite volume D and note that for any finite n which is sufficiently large we have

$$\mu_n\left(\left\{\omega \in \Omega :\ \omega\big|_D = \sigma_D\right\}\right) = \int_\Omega \mu_n(d\varphi) \nu_\varphi^D(\sigma_D), \tag{1.4}$$

which follows from the compatibility property of the finite-volume Gibbs measures.

For the model (1.1) on the Cayley tree, in [10], the problem of describing the SGMs was reduced to the description of the solutions of the following integral equation

$$f(t,x) = \prod_{y \in S_k(x)} \frac{\int_0^1 K(t,u) f(u,y) du}{\int_0^1 K(0,u) f(u,y) du}. \tag{1.5}$$

Here, $K(t,u) = \exp(J\beta\xi_{tu})$ and the unknown function is $f(t,x) > 0$, $x \in V$, $t \in [0,1]$ and $du = \lambda(du)$ is the Lebesgue measure.

If a solution $f(t,x)$ is given then the corresponding SGM μ on Ω is such that, for any n and $\sigma_n \in \Omega_{V_n}$,

$$\mu\Big(\Big\{\sigma\Big|_{V_n} = \sigma_n\Big\}\Big) = Z_n^{-1} \exp\Big(-\beta H(\sigma_n) + \sum_{x \in W_n} \ln f(\sigma(x), x)\Big).$$

A splitting Gibbs measure is called *translation invariant measure* if it corresponds to a solution $f(t,x)$ which does not depend on $x \in V$, i.e. $f(t,x) = f(t)$ for any $x \in V$.

In this paper we only deal with splitting Gibbs measures, therefore we omit the word "splitting" in the following text.

Note that the analysis of solutions to (1.5) is not easy. This difficulty depends on the given function ξ (i.e. on $K(t,u) > 0$).

Let us list known results about solutions of (1.5) and the Gibbs measures corresponding to them:

In [10] for $k = 1$ it was shown that the integral equation has a unique solution. In the case $k \geq 2$ some models (with the set $[0,1]$ of spin values) which have a unique Gibbs measure are constructed.

In [3] several models with nearest-neighbour interactions and with the set $[0, 1]$ of spin values, on a Cayley tree of order $k \geq 2$ are constructed. It is proved that each of the constructed models has at least two translational invariant Gibbs measures, i.e. the equation (1.5) has at least two solutions $f(t, x)$ which are independent of the vertices x of the tree.

In [4] a condition on $K(t, u)$ is found under which the corresponding integral equation (1.5) has a unique solution independent of x (i.e. uniqueness of the translation invariant Gibbs measure).

In [5] for a specifically chosen $K(t, u)$ it is shown that under certain conditions on the parameters of the model there are at least two translation invariant Gibbs measures (i.e. there are phase transitions).

In [9] the authors considered a model on a Cayley tree of order $k \geq 2$, where the function ξ depends on a parameter $\theta \in [0, 1)$. It is show that for $\theta \in [0, \frac{5}{3k}]$ the model has a unique translation invariant Gibbs measure. If $\theta \in (\frac{5}{3k}, 1)$ there is a phase transition, in particular there are three translation invariant Gibbs measures.

Paper [12] deals with a class of Gibbs measures which are periodic and also a Markov chain. It is shown that the period must be either 1 or 2. If $k = 1$ or the interaction is weak enough, the period is 1, i.e. every such Gibbs measure is translation invariant. For $k = 2$, a class of interactions is constructed admitting at least two Gibbs measures with period 2. For k sufficiently large, an interaction is given admitting at least four Gibbs measures with period two.

In [6] the translation invariant Gibbs measures for a function $K(t, u)$ are investigated by properties of positive fixed points of quadratic operators. Under some conditions it is shown that there are two and three positive fixed points.

We note that in the mentioned above papers the existence of a Gibbs measure is proved by directly solving Eq. (1.5) for concrete by chosen $K(t, u)$.

In this paper our *aim* is slightly different: we mainly will construct new solutions of (1.5) by means of its known solutions. To do this we will adapt to our models the construction methods which were used for models with a *finite* set of spin values (see [1, 2, 8, 11]).

2. Nontranslation invariant Gibbs measures

2.1. ART construction

In [1] for the Ising model (with the set $\{-1, 1\}$ of spin values) the authors constructed a class of new Gibbs measures by extending the known Gibbs measures defined on a Cayley tree of order k_0 to a Cayley tree of higher order $k > k_0$. Their construction is called the ART-construction [7].

In this subsection we adapt the ART-construction to models with an uncountable set of spin values.

For a given $H(\sigma)$ of the model (1.1), denote by $\mathcal{G}_k(H)$ the set of *all* splitting Gibbs measures on the Cayley tree of order $k \geq 2$. By $|M|$ we denote the number of elements of a set M.

The main result of this subsection is the following theorem.

THEOREM 1. *Take* $k_0, k \in \{2, 3, \dots\}$, *such that* $k > k_0$. *If* $|\mathcal{G}_{k_0}(H)| \geq 2$ *and* $K(t, u)$ *is such that*

$$\int_0^1 (K(t, u) - K(0, u))du = 0, \qquad \forall t \in [0, 1], \tag{2.1}$$

then for each $\mu \in \mathcal{G}_{k_0}(H)$ *there is a* $\nu = \nu(\mu) \in \mathcal{G}_k(H)$.

Proof: By our assumptions we have that $\mathcal{G}_{k_0}(H)$ contains at least two elements. Condition (2.1) guarantees that $f(t, x) \equiv 1$ is a solution of Eq. (1.5) for *any* $k \geq 2$. Denote by μ_1 the Gibbs measure which corresponds to this solution.

Now, for any $\mu \in \mathcal{G}_{k_0}(H) \setminus \{\mu_1\}$, we shall construct a Gibbs measure $\nu = \nu(\mu)$ which is a measure on the Cayley tree of order $k > k_0$. As mentioned in the previous section, to each measure $\mu \in \mathcal{G}_{k_0}(H)$ corresponds a unique function $f(t, x) = f_\mu(t, x)$ which satisfies (1.5) on Γ^{k_0}. Construct a function $g(t, x) \equiv g_\mu(t, x)$ on Γ^k as follows. Let V^k be the set of all vertices of the Cayley tree Γ^k. Since $k_0 < k$ one can consider V^{k_0} as a subset of V^k. Define the following function,

$$g(t, x) = \begin{cases} f_\mu(t, x), & \text{if} \quad x \in V^{k_0}, \\ 1, & \text{if} \quad x \in V^k \setminus V^{k_0}. \end{cases} \tag{2.2}$$

Now we shall check that (2.2) satisfies (1.5) on Γ^k. Let $x \in V^{k_0} \subset V^k$. We have

$$\begin{aligned} g(t, x) &= \prod_{y \in S_k(x)} \frac{\int_0^1 K(t, u) g(u, y) du}{\int_0^1 K(0, u) g(u, y) du} \\ &= \prod_{y \in S_k(x) \cap V^{k_0}} \frac{\int_0^1 K(t, u) f_\mu(u, y) du}{\int_0^1 K(0, u) f_\mu(u, y) du} \prod_{y \in S_k(x) \cap (V^k \setminus V^{k_0})} \frac{\int_0^1 K(t, u) du}{\int_0^1 K(0, u) du} \end{aligned}$$

For the first product we use $S_k(x) \cap V^{k_0} = S_{k_0}(x)$ and for the second product we use the condition (2.1), then we get

$$g(t, x) = \prod_{y \in S_{k_0}(x)} \frac{\int_0^1 K(t, u) f_\mu(u, y) du}{\int_0^1 K(0, u) f_\mu(u, y) du} = f_\mu(t, x).$$

If $x \in V^k \setminus V^{k_0}$ then $S_k(x) \subset V^k \setminus V^{k_0}$. Therefore $g(u, y) = 1$, for any $y \in S_k(x)$ and we have

$$g(t, x) = \prod_{y \in S_k(x)} \frac{\int_0^1 K(t, u) du}{\int_0^1 K(0, u) du} = 1.$$

Thus $g(t, x), x \in V^k$, satisfies the integral equation (1.5) and we denote by $\nu = \nu(\mu)$ the Gibbs measure which corresponds to $g(t, u)$. By the construction one can see that $\nu(\mu_1) \neq \nu(\mu_2)$ if $\mu_1 \neq \mu_2$ and the measure ν is not translation invariant. $\square$

Now let us give some examples where the conditions of Theorem 1 are satisfied.

EXAMPLE 1. Let $k = 2$. In the model (1.1) take

$$\xi_{tu} = \frac{1}{\beta J} \ln \left(1 + \frac{14}{15} \cdot \sqrt[5]{4\left(t - \frac{1}{2}\right)\left(u - \frac{1}{2}\right)}\right), \qquad t, u \in [0, 1].$$

Then, for the kernel $K(t, u)$ of (1.5) we have

$$K(t, u) = 1 + \frac{14}{15} \cdot \sqrt[5]{4\left(t - \frac{1}{2}\right)\left(u - \frac{1}{2}\right)}.$$

In [3] it was shown that this model has at least two Gibbs measures and the condition (2.1) is satisfied, i.e. $f(t, x) \equiv 1$ is a solution to (1.5).

EXAMPLE 2. Consider the case $k = 3$ and

$$K(t, u) = 1 + \frac{1}{2}\sqrt[7]{4\left(t - \frac{1}{2}\right)\left(u - \frac{1}{2}\right)}.$$

This model also satisfies the conditions of Theorem 1 and has at least two Gibbs measures (see [3]).

For other examples satisfying conditions of Theorem 1 see [3], [5], [12].

2.2. The Bleher–Ganikhodjaev construction

Here we will adapt the Bleher–Ganikhodjaev construction of [2] for the model (1.1).

If an arbitrary edge $\langle x^0, x^1\rangle = l \in L$ is deleted from the Cayley tree Γ^k, it splits into two components – two semi-infinite (half) trees Γ_0^k and Γ_1^k. Consider the half tree Γ_0^k, and denote by V^0 the set of its vertices. Namely the root x^0 has k nearest neighbours.

Denoting $h(t, x) = \ln f(t, x)$ we write Eq. (1.5) as

$$h(t, x) = \sum_{y \in S_k(x)} \ln \frac{\int_0^1 K(t, u)e^{h(u,y)}du}{\int_0^1 K(0, u)e^{h(u,y)}du}. \tag{2.3}$$

On the set $C[0, 1]$ of continuous functions define the following nonlinear operator

$$Af(t) = \ln \frac{\int_0^1 K(t, u)e^{f(u)}du}{\int_0^1 K(0, u)e^{f(u)}du}, \tag{2.4}$$

where $K(t, u) > 0$.

CONDITION 1. Assume that $K(t, u) > 0$ is continuous on $[0, 1]^2$, i.e. $K(\cdot, \cdot) \in C^+[0, 1]^2$, and there is $\alpha \equiv \alpha_K \in [0, 1)$ such that

$$|Af(t) - Ag(t)| \leq \alpha|f(t) - g(t)|, \qquad \forall f, g \in C[0, 1], \forall t \in [0, 1].$$

CONDITION 2. Assume that there are at least two translation invariant solutions, say $h(t,x) \equiv h(t) \in C[0,1]$ and $h(t,x) \equiv \eta(t) \in C[0,1]$, to Eq. (2.3), i.e. they are fixed points for the operator kA,

$$h(t) = kAh(t) = k \ln \frac{\int_0^1 K(t,u) e^{h(u)} du}{\int_0^1 K(0,u) e^{h(u)} du}, \qquad \eta(t) = kA\eta(t). \tag{2.5}$$

REMARK 1. If Condition 1 is satisfied then to satisfy Condition 2 it is necessary that $\frac{1}{k} \le \alpha < 1$.

We use $h(t)$ and $\eta(t)$ to construct an uncountable set of new solutions to (2.3).
Consider an infinite path $\pi = \{x^0 = x_0 < x_1 < \dots\}$ (the notation $x < y$ meaning that paths from the root to y go through x). Associate to this path a collection $h^\pi = \{h^\pi_{t,x} : x \in V^0, t \in [0,1]\}$ given by

$$h^\pi_{t,x} = \begin{cases} h(t), & \text{if } x \prec x_n,\ x \in W_n, \\ \eta(t), & \text{if } x_n \prec x,\ x \in W_n, \\ h_{t,x_n}, & \text{if } x = x_n, \end{cases} \tag{2.6}$$

$n = 1, 2, \dots$ where $x \prec x_n$ (resp. $x_n \prec x$) means that x is on the left (resp. right) from the path π and h_{t,x_n} are specific numbers, some conditions on these numbers will be given below.

THEOREM 2. *If Conditions 1 and 2 are satisfied, then for any infinite path π there exists a unique set of numbers $h^\pi = \{h^\pi_{t,x}\}$ satisfying equations (2.3) and (2.6).*

Proof: On W_n, we define the set

$$h^{(n)}_{t,x} = \begin{cases} h(t), & \text{if } x \prec x_n,\ x \in W_n, \\ \eta(t), & \text{if } x_n \prec x,\ x \in W_n, \\ h^{(n)}_{t,x}, & \text{if } x = x_n, \end{cases} \tag{2.7}$$

where $h^{(n)}_{t,x_n} \in (h^{\min}(t), h^{\max}(t))$, $\forall t \in [0,1]$, is an arbitrary number and $h^{\min}(t)$, $h^{\max}(t)$ are translation invariant solutions of (2.3), i.e.

$$h^\epsilon(t) = kAh^\epsilon(t), \qquad \epsilon = \min, \max. \tag{2.8}$$

We extend the definition of $h^{(n)}_{t,x}$ for all $x \in V_n = \cup_{m=0}^n W_m$ using recursion Eqs. (2.3) and prove that the limit

$$h_{t,x} = \lim_{n\to\infty} h^{(n)}_{t,x} \tag{2.9}$$

exists for every fixed $x \in V^0$ and is independent of the choice of $h^{(n)}_{t,x}$ for $x = x_n$.
If $x \in W_{n-1}$ and $x \prec x_{n-1}$, then for any $y \in S_k(x)$ we have $y \prec x_n$, therefore

$$h^{(n)}_{t,x} = \sum_{y \in S_k(x)} \ln \frac{\int_0^1 K(t,u) e^{h^{(n)}_{u,y}} du}{\int_0^1 K(0,u) e^{h^{(n)}_{u,y}} du} = kAh(t) = h(t).$$

Similarly, for $x \in W_{n-1}$ and $x_{n-1} \prec x$, we get $h_{t,x}^{(n)} = \eta(t)$. Consequently, for any $x \in W_m$, $m \leq n$ we have

$$h_{t,x}^{(n)} = \begin{cases} h(t), & \text{if } x \prec x_m,\ x \in W_m, \\ \eta(t), & \text{if } x_m \prec x,\ x \in W_m. \end{cases} \tag{2.10}$$

This implies that the limit (2.9) exists for $x \in W_m$, $x \neq x_m$ and

$$h_{t,x} = \begin{cases} h(t), & \text{if } x \prec x_m,\ x \in W_m, \\ \eta(t), & \text{if } x_m \prec x,\ x \in W_m. \end{cases}$$

Therefore, we only need to establish that the limit (2.9) exists for $x = x_m$. For $1 \leq l \leq n$ we have

$$h_{t,x_{l-1}}^{(n)} = \sum_{y \in S_k(x_{l-1})} \ln \frac{\int_0^1 K(t,u) e^{h_{u,y}^{(n)}} du}{\int_0^1 K(0,u) e^{h_{u,y}^{(n)}} du} = \sum_{y \in S_k(x_{l-1})} A h_{t,y}^{(n)}. \tag{2.11}$$

Consider two sets $\{\bar{h}_{t,x}^{(n)}, x \in V_n\}$ and $\{\tilde{h}_{t,x}^{(n)}, x \in V_n\}$ which correspond to two values $\bar{h}_{t,x}^{(n)}$ and $\tilde{h}_{t,x}^{(n)}$ for $x = x_n$, in (2.7), then since $\tilde{h}_{t,y}^{(n)} = \bar{h}_{t,y}^{(n)}$, $\forall t \in [0,1]$ and for any $y \neq x_l$, $y \in W_l$, from (2.11) we get

$$\tilde{h}_{t,x_{l-1}}^{(n)} - \bar{h}_{t,x_{l-1}}^{(n)} = A\tilde{h}_{t,x_l}^{(n)} - A\bar{h}_{t,x_l}^{(n)}. \tag{2.12}$$

Consequently, by Condition 1 we get

$$\left| \tilde{h}_{t,x_{l-1}}^{(n)} - \bar{h}_{t,x_{l-1}}^{(n)} \right| \leq \alpha \left| \tilde{h}_{t,x_l}^{(n)} - \bar{h}_{t,x_l}^{(n)} \right|. \tag{2.13}$$

Iterating this inequality we obtain

$$\left| \tilde{h}_{t,x_m}^{(n)} - \bar{h}_{t,x_m}^{(n)} \right| \leq \alpha^{n-m} \left| \tilde{h}_{t,x_n}^{(n)} - \bar{h}_{t,x_n}^{(n)} \right|. \tag{2.14}$$

For arbitrary $N, M > n$, we now consider the sets $\{h_{t,x}^{(N)}, x \in V_N\}$ and $\{h_{t,x}^{(M)}, x \in V_M\}$, $t \in [0,1]$, determined by initial conditions of the form (2.7) for $x \in W_N$ and $x \in W_M$, respectively, and by recursion equations (2.3). We set $\bar{h}_{t,x_n}^{(n)} = h_{t,x_n}^{(N)}$, $\tilde{h}_{t,x_n}^{(n)} = h_{t,x_n}^{(M)}$. Then inequalities (2.14) imply

$$\left| h_{t,x_m}^{(N)} - h_{t,x_m}^{(M)} \right| \leq \alpha^{n-m} \left| h_{t,x_n}^{(N)} - h_{t,x_n}^{(M)} \right| \leq 2h_0^{\max} \alpha^{n-m}.$$

This estimate implies that the sequence $h_{t,x_m}^{(n)}$ satisfies the Cauchy criterion as $n \to \infty$ for a fixed m and a fixed $t \in [0,1]$; therefore, the limit (2.9) exists and is independent of the choice of $h_{t,x_n}^{(n)}$ in (2.7). Because, by construction, the sets $\{h_{t,x}^{(n)}\}$ satisfy equation (2.3) before taking the limit, so does $\{h_{t,x}\}$. The uniqueness of $\{h_{t,x}\}$ obviously follows from the estimate (2.14). □

A real number $r = r(\pi)$, $0 \leq r \leq 1$, can be assigned to the infinite path π (see [2]) and by Theorem 2 the set $h^{\pi(r)}$ (where $\pi(r)$ is the path corresponding to the number $r \in [0, 1]$) satisfying (2.3) is uniquely defined. By the construction of $h^{\pi(r)}$ it is obvious that they are distinct for different $r \in [0, 1]$. We denote by ν_r the Gibbs measure corresponding to $h^{\pi(r)}$, $r \in [0, 1]$. One thus obtains uncountably many Gibbs measures, i.e. we proved the following result.

THEOREM 3. *If Conditions 1 and 2 are satisfied then for any $r \in [0, 1]$ there exists a nontranslation invariant Gibbs measure ν_r and $\nu_r \neq \nu_l$ if $r \neq l$.*

2.3. The Zachary construction

In this subsection we adopt Zachary's construction ([14], [8, p.251]), which was done for the Ising model, for our model (1.1) on the Cayley tree.

CONDITION 3. *Assume $K(t, u) > 0$ such that the operator A,* (2.4)*, is invertible.*

From (2.3) we get that

$$h^{\min}(t) \leq h(t, x) \leq h^{\max}(t), \qquad \forall x \in V, \tag{2.15}$$

where

$$h^{\min}(t) = k \ln \frac{\min_{u\in[0,1]} K(t, u)}{\max_{u\in[0,1]} K(0, u)}, \qquad h^{\max}(t) = k \ln \frac{\max_{u\in[0,1]} K(t, u)}{\min_{u\in[0,1]} K(0, u)}.$$

Under Conditions 2 and 3 we shall construct a continuum of distinct functions $h^{\zeta}_{t,x}$, which satisfy the functional equation (2.3), where $\zeta(t)$ is such that

$$h^{\min}(t) < \zeta(t) < h^{\max}(t), \qquad \forall t \in [0, 1]. \tag{2.16}$$

Take any $\zeta(t)$ with condition (2.16). Define the sequence $\zeta_n(t)$, $n \geq 0$ recursively by $\zeta_0(t) = \zeta(t)$,

$$\zeta_n(t) = kA\zeta_{n+1}(t), \qquad n \geq 0. \tag{2.17}$$

Since the operator A is invertible the definition of $\zeta_n(t)$ given by (2.17) is unambiguous.

Define the function $h^{\zeta}_{t,x}$ by $h^{\zeta}_{t,x} = \zeta_n(t)$ for all $x \in W_n$. Now we check that this function satisfies Eq. (2.3): for any $x \in V$ there is $n \geq 0$ such that $x \in W_n$, consequently, $S_k(x) \subset W_{n+1}$ and we have

$$h^{\zeta}_{t,x} = \sum_{y\in S_k(x)} Ah^{\zeta}_{t,y} = \sum_{y\in S_k(x)} A\zeta_{n+1}(t) = kA\zeta_{n+1}(t) = \zeta_n(t),$$

i.e. the function $h^{\zeta}_{t,x}$ satisfies (2.3) for any t and ζ.

By the construction, distinct functions ζ define distinct functions $h^{\zeta} = \{h^{\zeta}_{t,x}, x \in V, t \in [0, 1]\}$. Denote by μ^{ζ} the Gibbs measure which corresponds to the function h^{ζ}.

Thus we have proved the following

THEOREM 4. *If Conditions* 2 *and* 3 *are satisfied, then for any ζ satisfying* (2.16) *there exists a Gibbs measure μ^{ζ} such that $\mu^{\zeta} \neq \mu^{\eta}$ if $\zeta \neq \eta$.*

2.4. Discussions

The first our construction (Theorem 1) is an adaptation of the ART-construction. In particular, it follows from Theorem 1 that if for the model (1.1) (satisfying the conditions of Theorem 1) there is more than one Gibbs measure on a Cayley tree of order k_0 then it has more than one Gibbs measure for any $k \geq k_0$.

Theorem 3 gives uncountable set of Gibbs measures. Taking any two of these measures (i.e. corresponding to two values of $t \in [0,1]$) one can use the argument of Subsection 2.2. to extend the set of Gibbs measures. Zachary's construction is also a way to give an uncountable set of Gibbs measures.

It is known that the set of all Gibbs measures of the model (1.1) is a nonempty, convex and compact subset in the set of all probability measures on $(\Omega, \mathcal{B})$ (see [8, Chapter 7]). Therefore it is interesting to know the extreme elements (Gibbs measures) of the set of all Gibbs measures. Checking extremality of a given Gibbs measure is a difficult problem. Our constructions of measures in Theorems 1–4 are based on known Gibbs measures. If the known measures are extreme then the measures mentioned in Theorems 1–4 also are extreme. In general, the problem of extremality of measures (mentioned in Theorems 1-4) remains open. Since our analysis is related to nonlinear integral equations, it seems difficult to give examples satisfying Conditions 1–3.

Acknowledgements

We thank all (three) referees for their helpful suggestions.

REFERENCES

[1] H. Akin, U. A. Rozikov and S. Temir: A new set of limiting Gibbs measures for the Ising model on a Cayley tree, *J. Stat. Phys.* **142**(2) (2011), 314–321.

[2] P. M. Bleher and N. N. Ganikhodjaev: On pure phases of the Ising model on the Bethe lattice, *Theor. Probab. Appl.* **35** (1990), 216–227.

[3] Yu. Kh. Eshkobilov, F. H. Haydarov and U. A. Rozikov: Non-uniqueness of Gibbs measures for models with uncountable set of spin values on a Cayley tree, *J. Stat. Phys.* **147**(4) (2012), 779–794.

[4] Yu. Kh. Eshkobilov, F. H. Haydarov and U. A. Rozikov: Uniqueness of Gibbs measure for models with uncountable set of spin values on a Cayley tree, *Math. Phys. Anal. Geom.* **16**(1) (2013), 1–17.

[5] Yu. Kh. Eshkobilov, U. A. Rozikov and G. I. Botirov: Phase transitions for a model with uncountable set of spin values on a Cayley tree, *Lobachevskii J. Math.* **34**(3) (2013), 256–263.

[6] Yu. Kh. Eshkabilov, Sh. D. Nodirov and F. H. Haydarov: Positive fixed points of quadratic operators and Gibbs measures, *Positivity* **20**(4) (2016), 929–943.

[7] D. Gandolfo, M. M. Rakhmatullaev, U. A. Rozikov and J. Ruiz: On free energies of the Ising model on the Cayley tree, *J. Stat. Phys.* **150**(6) (2013), 1201–1217.

[8] H. O. Georgii: *Gibbs Measures and Phase Transitions*, 2nd ed. De Gruyter Studies in Mathematics, vol. 9. Walter de Gruyter, Berlin 2011.

[9] B. Jahnel, C. Kuelske and G. I. Botirov: Phase transition and critical values of a nearest-neighbor system with uncountable local state space on Cayley trees. *Math. Phys. Anal. Geom.* **17**(3-4) (2014), 323–331.

[10] U. A. Rozikov and Yu. Kh. Eshkabilov: On models with uncountable set of spin values on a Cayley tree: Integral equations, *Math. Phys. Anal. Geom.* **13** (2010), 275–286.

[11] U. A. Rozikov: *Gibbs Measures on Cayley trees*, World Sci. Publ., Singapore 2013, 404 pp.

[12] U. A. Rozikov and F. H. Haydarov: Periodic Gibbs measures for models with uncountable set of spin values on a Cayley tree, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **18**(1) (2015), 1550006.
[13] Ya. G. Sinai: *Theory of phase transitions*. Rigorous results. "Nauka", Moscow 1980.
[14] S. Zachary: Countable state space Markov random fields and Markov chains on trees, *Ann. Prob.* **11** (1983), 894–903.