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TRANSLATION-INVARIANT GIBBS MEASURES OF A MODEL ON CAYLEY TREE

Botirov G. I.¹

RESUME

We consider a model where the spin takes values in the set $[0, 1]^d$, and is assigned to the vertexes of the Cayley tree. We reduce the problem of describing the "splitting Gibbs measures" of the model to the description of the solutions of some non-linear integral equation. For a concrete form of the Kernel of the integral equation we show the uniqueness of solution.

Keywords: Cayley tree, model, Hammerstein's integral operator, Gibbs measures.

INTRODUCTION

It is known that a central problem in the theory of Gibbs measures is to describe infinite-volume (or limiting) Gibbs measures corresponding to a given Hamiltonian (see [1], [7]).

In [3] the Potts model with a *countable* set of spin values on a Cayley tree is considered and it was showed that the set of translation-invariant splitting Gibbs measures of the model contains at most one point, independently on parameters of the Potts model with countable set of spin values on the Cayley tree. This is a crucial difference from the models with a finite set of spin values, since the last ones may have more than one translation-invariant Gibbs measures.

In [2] was continued the investigation from [4] and considered a model with nearest-neighbor interactions and local state space given by the uncountable set $[0, 1]$ on a Cayley tree Γ^k of order $k \geq 2$. The translation-invariant Gibbs measures are studied via a non-linear functional equation and we prove non-uniqueness of translation-invariant Gibbs measures in the right parameter regime for all $k \geq 2$ and not only for $k \in \{2, 3\}$ as in [4]. In [5] models (Hamiltonians) with-nearest-neighbor interactions and with the set $[0, 1]$ of spin values, on a Cayley tree Γ^k of order $k \geq 1$ were studied.

This paper is a continuation and also generalization of our investigations [2, 4, 6, 5]. We reduced the considering problem to the description of the solutions of some nonlinear integral equation. Then for $k = 1$ and a given Hamiltonian we show that the integral equation has a unique solution.

DEFINITIONS

The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L the set of edges. Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them. We will use the notation $l = \langle x, y \rangle$. A collection of nearest neighbor pairs $\langle x, x_1 \rangle, \langle x_1, x_2 \rangle, \dots, \langle x_{d-1}, y \rangle$ is called a *path* from x to y . The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x to y .

For a fixed $x^0 \in V$, called the root, we set

$$W_n := \{x \in V | d(x, x^0) = n\}, \quad V_n := \bigcup_{m=0}^n W_m$$

and denote

$$S(x) := \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n,$$

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the set of direct successors of x .

For $A \subset V$ a configuration σ_A on A is an arbitrary function $\sigma_A : A \rightarrow [0, 1]^d$.

Denote $\Omega_A := [0, 1]^{d|A|}$ the set of all configurations on A . Here $|A|$ denotes the cardinality of A . A configuration σ on V is then defined as a function $x \in V \rightarrow \sigma(x) \in [0, 1]^d$; the set of all configurations is $[0, 1]^d$. The (formal) Hamiltonian of the model is:

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} \xi_{\sigma(x)\sigma(y)}, \quad (1)$$

where $J \in \mathbb{R} \setminus \{0\}$ and $\xi : (u, v) \in [0, 1]^{2d} \rightarrow \xi_{uv} \in \mathbb{R}$ is a given bounded, measurable function.

Let λ be the Lebesgue measure on $[0, 1]^d$. On the set of all configurations on A the a priori measure λ_A is introduced as the $|A|$ fold product of the measure λ . We consider a standard sigma-algebra $\mathbb{B}$ of subsets of $\Omega = [0, 1]^{dV}$ generated by the measurable cylinder subsets. A probability measure μ on $(\Omega, \mathbb{B})$ is called a Gibbs measure (with Hamiltonian H) if it satisfies the DLR equation, namely for any $n = 1, 2, \dots$ and $\sigma_n \in \Omega_{V_n}$:

$$\mu(\{\sigma \in \Omega : \sigma|_{V_n} = \sigma_n\}) = \int_{\Omega} \mu(d\omega) \nu_{\omega|_{W_{n+1}}}^{V_n}(\sigma_n),$$

where $\nu_{\omega|_{W_{n+1}}}^{V_n}$ is the conditional Gibbs density

$$\nu_{\omega|_{W_{n+1}}}^{V_n}(\sigma_n) := \frac{1}{Z_n(\omega|_{W_{n+1}})} \exp\left(-\beta H(\sigma_n | \omega|_{W_{n+1}})\right),$$

and $\beta = \frac{1}{T}$, $T > 0$ is temperature.

Let $L_n = \{\langle x, y \rangle \in L : x, y \in V_n\}$ and Ω_{V_n} is the set of configurations in V_n (and Ω_{W_n} that in W_n). Furthermore, $\sigma|_{V_n}$ and $\omega|_{W_n}$ denote the restrictions of configurations $\sigma, \omega \in \Omega$ to V_n and W_{n+1} , respectively. Next, $\sigma_n : x \in V_n \mapsto \sigma_n(x)$ is a configuration in V_n and $H(\sigma_n | \omega|_{W_{n+1}})$ is defined as the sum $H(\sigma_n) + U(\sigma_n, \omega|_{W_{n+1}})$ where

$$H(\sigma_n) = -J \sum_{\langle x, y \rangle \in L_n} \xi_{\sigma_n(x)\sigma_n(y)},$$

$$U(\sigma_n, \omega|_{W_{n+1}}) = -J \sum_{\substack{\langle x, y \rangle \in L \\ x \in V_n, y \in W_{n+1}}} \xi_{\sigma_n(x)\omega(y)}.$$

Finally, $Z_n(\omega|_{W_{n+1}})$ stands for the partition function in V_n , with the boundary condition $\omega|_{W_{n+1}}$:

$$Z_n(\omega|_{W_{n+1}}) = \int_{\Omega_{V_n}} \exp\left(-\beta H(\tilde{\sigma}_n | \omega|_{W_{n+1}})\right) \lambda_{V_n}(d\tilde{\sigma}_n).$$

Write $x < y$ if the path from x^0 to y goes through x . Call vertex y a direct successor of x if $y > x$ and x, y are nearest neighbors. Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k+1$.

Let $h : x \in V \mapsto h_x = (h_{t,x}, t \in [0, 1]^d) \in \mathbb{R}^{[0,1]^d}$ be a mapping of $x \in V \setminus \{x^0\}$. Given $n = 1, 2, \dots$, consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) := Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x), x}\right). \quad (2)$$

Here, as before, $\sigma_n : x \in V_n \mapsto \sigma(x)$ and Z_n is the corresponding partition function:

$$Z_n := \int_{\Omega_{V_n}} \exp \left(-\beta H(\tilde{\sigma}_n) + \sum_{x \in W_n} h_{\tilde{\sigma}(x), x} \right) \lambda_{V_n}(d\tilde{\sigma}_n). \quad (3)$$

The probability distributions $\mu^{(n)}$ are called compatible if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\int_{\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) \lambda_{W_n}(d(\omega_n)) = \mu^{(n-1)}(\sigma_{n-1}). \quad (4)$$

Here $\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}$ is the concatenation of σ_{n-1} and ω_n . In this case, because of the Kolmogorov extension theorem, there exists a unique measure μ on Ω_V such that, for any n and $\sigma_n \in \Omega_{V_n}$, $\mu \left(\left\{ \sigma \Big|_{V_n} = \sigma_n \right\} \right) = \mu^{(n)}(\sigma_n)$ [7].

AN INTEGRAL EQUATION AND TRANSLATIONAL-INVARIANT SOLUTION

Theorem 1. The probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \dots$ in (2) are compatible if for any $x \in V \setminus \{x^0\}$ the following equation holds:

$$f((t_1, \dots, t_d), x) = \prod_{y \in S(x)} \frac{\int_{[0,1]^d} \exp(J\beta\xi_{(t_1, \dots, t_d)(u_1, u_2, \dots, u_d)}) f((u_1, \dots, u_d), y) du_1 \dots du_d}{\int_{[0,1]^d} \exp(J\beta\xi_{(0, \dots, 0)(u_1, \dots, u_d)}) f((u_1, \dots, u_d), y) du_1 \dots du_d}. \quad (5)$$

where $f((t_1, t_2, \dots, t_d), x) = \exp(h_{(t_1, t_2, \dots, t_d), x} - h_{(0, 0, \dots, 0), x})$, $(t_1, t_2, \dots, t_d) \in [0, 1]^d$ and $du_1 du_2 \dots du_d = \lambda(du_1 du_2 \dots du_d)$ is the Lebesgue measure.

Proof. Necessity. Suppose that (4) holds; we will prove (5). Substituting (1) into (4), obtain that for any configurations $\sigma_{n-1} : x \in V_{n-1} \rightarrow \sigma_{n-1}(x) \in [0, 1]^d$:

$$\begin{aligned} \frac{Z_{n-1}}{Z_n} \int_{\Omega_{W_n}} \exp \left(\sum_{x \in W_{n-1}} \sum_{y \in S(x)} (J\beta\xi_{\sigma_{n-1}(x)\omega_n(y)} + h_{\omega_n(y), y}) \right) \lambda_{W_n}(d\omega_n) = \\ = \exp \left(\sum_{x \in W_{n-1}} h_{\sigma_{n-1}(x), x} \right), \end{aligned} \quad (6)$$

where $\omega_n : x \in W_n \rightarrow \omega_n(x)$.

From (6) we get:

$$\begin{aligned} \frac{Z_{n-1}}{Z_n} \int_{\Omega_{W_n}} \prod_{x \in W_{n-1}} \prod_{y \in S(x)} \exp(J\beta\xi_{\sigma_{n-1}(x)\omega_n(y)} + h_{\omega_n(y), y}) = \\ = \prod_{x \in W_{n-1}} \exp(h_{\sigma_{n-1}(x), x}). \end{aligned} \quad (7)$$

Consequently, for any $(t_1, t_2, \dots, t_d) \in [0, 1]^d$ we have

$$\begin{aligned} \prod_{y \in S(x)} \frac{\int_{[0,1]^d} \exp(J\beta\xi_{(t_1, \dots, t_d)(u_1, u_2, \dots, u_d)} + h_{(u_1, \dots, u_d), y}) du_1 \dots du_d}{\int_{[0,1]^d} \exp(J\beta\xi_{(0, \dots, 0)(u_1, \dots, u_d)} + h_{(u_1, \dots, u_d), y}) du_1 \dots du_d} = \\ = \exp(h_{(t_1, t_2, \dots, t_d), x} - h_{(0, 0, \dots, 0), x}), \end{aligned} \quad (8)$$

which implies (5).

Sufficiency. Suppose that (5) holds. It is equivalent to the representation

$$\prod_{y \in S(x) [0,1]^d} \int \exp(J\beta \xi_{(t_1, \dots, t_d)}(u_1, u_2, \dots, u_d) + h_{(u_1, \dots, u_d), y}) du_1 \dots du_d = a(x) \exp(h_{(t_1, t_2, \dots, t_d), x}), \quad (9)$$

for some function $a(x) > 0$, $x \in V$. We have LHS of (1) is equal to the following

$$\frac{1}{Z_n} \exp(-\beta H(\sigma_{n-1})) \lambda_{V_{n-1}}(d(\sigma_n)) \times \prod_{x \in W_{n-1}} \prod_{y \in S(x) [0,1]^d} \int \exp(J\beta \xi_{\sigma_{n-1}(x)} + h_{(u_1, \dots, u_d), y}) du_1 \dots du_d. \quad (10)$$

Substituting (9) into (10) and denoting $A_n(x) := \prod_{x \in W_{n-1}} a(x)$, we get RHS of (10) is equal to the following

$$\frac{A_{n-1}}{Z_n} \exp(-\beta H(\sigma_{n-1})) \lambda_{V_{n-1}}(d(\sigma_n)) \times \prod_{x \in W_{n-1}} h_{\sigma_{n-1}(x), x}. \quad (11)$$

Since $\mu^{(n)}$, $n \geq 1$ is a probability, we should have

$$\int_{\Omega_{V_{n-1}}} \lambda_{V_{n-1}}(d\sigma_{n-1}) \int_{\Omega_{W_n}} \lambda_{W_n}(d\sigma_n) \mu^{(n)}(\sigma_{n-1}, \omega_n) = 1. \quad (12)$$

Hence from (11) we get (4).

We are going to solve equation (5) in the class of translation-invariant functions $f((t_1, t_2, \dots, t_d), x) = f(t_1, t_2, \dots, t_d)$, for all $x \in V$. For such functional equation (5) can be written as

$$f(t) = \left(\frac{\int_{[0,1]^d} K(t_1, t_2, \dots, t_d; u_1, u_2, \dots, u_d) f((u_1, u_2, \dots, u_d), y) du_1 \dots du_d}{\int_{[0,1]^d} K(0, 0, \dots, 0; u_1, u_2, \dots, u_d) f((u_1, u_2, \dots, u_d), y) du_1 \dots du_d} \right)^k, \quad (13)$$

where $K(t, u) = \exp(J\beta \xi_{tu}) > 0$, $f(t) > 0$, $t, u \in [0, 1]^d$.

We shall find positive continuous solutions to (13), i.e., such that

$$f \in C^+[0, 1]^d = \{f \in C[0, 1]^d : f(x) > 0\}.$$

We define the linear operator $W : C[0, 1]^d \rightarrow C[0, 1]^d$ by

$$(Wf)(t) = \int_{[0,1]^d} K(t_1, \dots, t_d, u_1, \dots, u_d) f(u_1, \dots, u_d) du_1 \dots du_d$$

and defined the linear functional $\omega : C[0, 1]^d \rightarrow \mathbb{R}$ by

$$\omega(f) \equiv (Wf)(0) = \int_{[0,1]^d} K(0, \dots, 0, u_1, \dots, u_d) f(u_1, \dots, u_d) du_1 \dots du_d.$$

Then equation (13) can be written as

$$f(t) = (A_k f)(t) = \left(\frac{(Wf)(t)}{(Wf)(0)} \right)^k, \quad f \in C^+[0, 1]^d, \quad k \geq 1.$$

Proposition 1. If $K(t_1, t_2, \dots, t_d; u_1, u_2, \dots, u_d) = \alpha(t_1, t_2, \dots, t_d) + \alpha(u_1, u_2, \dots, u_d)$ where α is a given positive function, then

$$f(t) = \prod_{i=1}^d \frac{\alpha(t_i) + \sqrt{\int_{[0,1]^d} \alpha^2(u_1, \dots, u_d) du_1 \dots du_d}}{\alpha(0) + \sqrt{\int_{[0,1]^d} \alpha^2(u_1, \dots, u_d) du_1 \dots du_d}} \quad (14)$$

is the unique solution of the equation $A_1 f = f$.

Proof. Let $K(t_1, t_2, \dots, t_d; u_1, u_2, \dots, u_d) = \alpha(t_1, t_2, \dots, t_d) + \alpha(u_1, u_2, \dots, u_d)$. From (13) when $k = 1$ we get

$$f(t) = \frac{\int_{[0,1]^d} (\alpha(t_1, t_2, \dots, t_d) + \alpha(u_1, u_2, \dots, u_d)) f(u_1, \dots, u_d) du_1 \dots du_d}{\int_{[0,1]^d} (\alpha(0, 0, \dots, 0) + \alpha(u_1, u_2, \dots, u_d)) f(u_1, \dots, u_d) du_1 \dots du_d} \quad (15)$$

Hence and denoting

$$\begin{aligned} C_1 &= \int_{[0,1]^d} f(u_1, \dots, u_d) du_1 \dots du_d, \\ C_2 &= \int_{[0,1]^d} \alpha(u_1, \dots, u_d) f(u_1, \dots, u_d) du_1 \dots du_d \end{aligned} \quad (16)$$

we get

$$f(t) = \frac{\alpha(t_1, t_2, \dots, t_d) C_1 + C_2}{\alpha(0, 0, \dots, 0) C_1 + C_2}. \quad (17)$$

Substituting (17) into (16) and we get

$$\begin{aligned} C_1 &= \int_{[0,1]^d} \frac{\alpha(u_1, u_2, \dots, u_d) C_1 + C_2}{\alpha(0, 0, \dots, 0) C_1 + C_2} du_1 \dots du_d, \\ C_2 &= \int_{[0,1]^d} \alpha(u_1, \dots, u_d) \frac{\alpha(u_1, u_2, \dots, u_d) C_1 + C_2}{\alpha(0, 0, \dots, 0) C_1 + C_2} du_1 \dots du_d \end{aligned} \quad (18)$$

From solution of (18) we get (14).

Let $k \geq 2$ in the model (1) and

$$H(\sigma) = -\frac{1}{\beta} J \sum_{\langle x, y \rangle \in E} \ln(\alpha(t_1, t_2, \dots, t_d) + \alpha(u_1, u_2, \dots, u_d)). \quad (19)$$

Theorem 2. For model (19) on $[0, 1]^d$, $\forall J \in \mathbb{R}$ and for any $\beta > 0$ on the Cayley tree Γ^1 (i.e. $k = 1$) there exists a unique splitting Gibbs measure.

Proof. It follows from Theorem 1 and Proposition 2.

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