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On Application of Slowly Varying Functions with Remainder in the Theory of Galton-Watson Branching Process

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We investigate an application of slowly varying functions (in sense of Karamata) in the theory of Galton-Watson branching processes. Consider the critical case so that the generating function of the per-capita offspring distribution has the infinite second moment, but its tail is regularly varying with remainder. We improve the Basic Lemma of the theory of critical Galton-Watson branching processes and refine some well-known limit results.

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1. Introduction and preliminaries

A conception of slow variation (or more general – regular variation) was initiated first by Jovan Karamata in [7, 8]. Zolotarev [15] one of the first demonstrated an encouraging perspective of application of the conception of slow variation in probability theory, in particular in the theory of stochastic branching processes. Afterwards Slack [13, 14] and Seneta [9], [10, 12] prove principally new limit theorems for branching processes using slowly varying (SV) functions. Remind that real-valued, positive and measurable function $\ell(x)$ is said to be SV at infinity in sense of Karamata if $\ell(\lambda x)/\ell(x) \to 1$ as $x \to \infty$ for each $\lambda > 0$. A function $\mathsf{V}(x)$ is said to be regularly varying at infinity with index of regular variation $\rho \in \mathbb{R}_+$ if it in the form $\mathsf{V}(x) = x^\rho \ell(x)$, where $\ell(x)$ is SV at infinity. We refer the reader to [1, 3] and [11] for more information.

Let $F(s) = \sum_{j\in\mathbb{N}_0} p_j s^j$ denote an offspring probability generating function (PGF) of Galton-Watson (GW) branching process, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{N} = \{1, 2, \ldots\}$. Supposing that $p_0 > 0$ we consider the case when the mean per-capita offspring number $\sum_{j\in\mathbb{N}} jp_j = 1$, that is the

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process is critical, see [2]. Moreover we assume that PGF $F(s)$ for $0 \leqslant s < 1$ has the following representation:

$$F(s) = s + (1-s)^{1+\nu}\mathcal{L}\left(\frac{1}{1-s}\right), \tag{1}$$

where $0 < \nu \leqslant 1$ and $\mathcal{L}(t)$ is SV at infinity. By the criticality of our process the condition (1) implies that the second moment $F''(1-) = \infty$. This includes the case $F''(1-) < \infty$ when $\nu = 1$ and $\mathcal{L}(t) \to F''(1-)/2$ as $t \to \infty$.

Let Z_n be the population size in n-th generation. Process evolution is characterized by transition probabilities $P_{ij}(n) := \mathbb{P}\{Z_n = j \,|Z_0 = i\}$. In this interpretation $p_j = \mathbb{P}\{Z_1 = j\}$ provided that $\mathbb{P}\{Z_0 = 1\} = 1$. A PGF

$$F_n(s) = \sum_{j \in \mathbb{N}_0} P_{1j}(n) s^j$$

is the n-fold iteration of $F(s)$, see [2]. Further by the symbol $\mathcal{H} = \min\{n : Z_n = 0\}$ we denote a time of extinction of GW process. Write $R_n(s) := 1 - F_n(s)$ and needless to say $Q_n := \mathbb{P}\{\mathcal{H} > n\} = R_n(0)$.

The following theorem is known.

Theorem S [14]. *If the condition* (1) *holds then*

$$\lim_{n\to\infty} \mathbb{P}\left\{Q_n Z_n \leqslant x \;\middle|\; \mathcal{H} > n\right\} = G(x),$$

where $G(x)$ has the Laplace transform

$$\Psi(\theta) = 1 - \left(1 + \theta^{-\nu}\right)^{-1/\nu}.$$

By arguments of Slack [14] one can be shown that if the condition (1) holds then

$$Q_n^{\nu}\mathcal{L}\left(\frac{1}{Q_n}\right) \sim \frac{1}{\nu n} \quad \text{as} \quad n \to \infty. \tag{2}$$

Slack [14] also has shown that

$$\mathcal{U}_n(s) := \frac{F_n(s) - F_n(0)}{F_n(0) - F_{n-1}(0)} \longrightarrow U(s) \quad \text{as} \quad n \to \infty, \tag{3}$$

for $0 \leqslant s < 1$, where $U\left(F(s)\right) = U(s) + 1$ and

$$U(s) = \frac{1 + o(1)}{\nu\,(1-s)^{\nu}\,\mathcal{L}\left(1/(1-s)\right)} \quad \text{as} \quad s \uparrow 1.$$

Combining (1), (2) and (3) we have

$$\mathcal{U}_n(s) \sim U_n(s) := \left[1 - \frac{R_n(s)}{Q_n}\right]\nu n \quad \text{as} \quad n \to \infty.$$

So we have proved the following lemma as a generalization of the assertion (2) for all $s \in [0, 1)$.

Lemma 1.1. *If the condition* (1) *holds then*

$$R_n(s) = \frac{\mathcal{N}(n)}{(\nu n)^{1/\nu}} \cdot \left[1 - \frac{U_n(s)}{\nu n}\right],$$

where the function $\mathcal{N}(x)$ is SV at infinity and

$$\mathcal{N}(n) \cdot \mathcal{L}^{1/\nu}\left(\frac{(\nu n)^{1/\nu}}{\mathcal{N}(n)}\right) \longrightarrow 1 \quad as \quad n \to \infty,$$

and the function $U_n(s)$ enjoys following properties:

(i) $U_n(s) = U(s)\,(1 + o(1))$ *as* $n \to \infty$,

(ii) $\lim_{s\uparrow 1} U_n(s) = \nu n$ *for each fixed* $n \in \mathbb{N}$,

(iii) $U_n(0) = 0$ *for each fixed* $n \in \mathbb{N}$.

We obtain the Lemma 1.1 by more simple proof rather than as shown in [6]. This lemma is called the Basic Lemma of the theory of critical GW branching process. The following lemma established in [6] is differential analogue of Lemma 1.1

Lemma 1.2. *If the condition* (1) *holds then*

$$\frac{\partial R_n(s)}{\partial s} = -\left(\frac{R_n(s)}{1-s}\right)^{1+\nu} \frac{\mathcal{L}\left(1/R_n(s)\right)}{\mathcal{L}\left(1/(1-s)\right)}.$$

Since $\mathcal{L}(x)$ is SV-function we can write

$$\frac{\mathcal{L}\left(\lambda x\right)}{\mathcal{L}(x)} = 1 + \alpha(x) \tag{4}$$

for each $\lambda > 0$, where $\alpha(x) \to 0$ as $x \to \infty$. Henceforth we suppose that some positive function $g(x)$ is given so that $g(x) \to 0$ and $\alpha(x) = o\big(g(x)\big)$ as $x \to \infty$. In this case $\mathcal{L}(x)$ is called SV with remainder, see [3, p. 185, condition SR3].

We devote this paper to improvement of the Lemma 1.1 provided that the condition (4) holds with given $\alpha(x)$. Subsequently of this we will improve the Lemma 1.2 and define a speed rate in some well-known limit theorems from the theory of critical GW branching process.

2. Improvement of the Basic Lemma and results

Everywhere in this section we suppose the condition (4) holds. Write

$$\Lambda(y) := \frac{F(1-y) - (1-y)}{y} = y^{\nu}\mathcal{L}\left(\frac{1}{y}\right).$$

Note that the function $y\Lambda(y)$ is positive and tends to zero and has a monotone derivative for $y \in (0,1]$ so that $y\Lambda'(y)/\Lambda(y) \to \nu$ as $y \downarrow 0$, see [3, p. 401]. Hence we can write

$$\frac{y\Lambda'(y)}{\Lambda(y)} = \nu + \delta(y),$$

where $\delta(y)$ is continuous and $\delta(y) \to 0$ as $y \downarrow 0$. Integrating this equality we obtain

$$\Lambda(y) = p_0 y^{\nu} \exp\int_1^y \frac{\delta(u)}{u}du.$$

We have considered that $\Lambda(1) = \mathcal{L}(1) = p_0$ in last step. Therefore we have

$$\mathcal{L}\left(\frac{1}{y}\right) = p_0 \exp\int_1^y \frac{\delta(u)}{u}du.$$

Changing variables as $u = 1/t$ in integrand gives

$$\mathcal{L}\left(x\right) = p_0 \exp\int_1^x \frac{\varepsilon(t)}{t}dt, \tag{5}$$

where $\varepsilon(t)$ is continuous and $\varepsilon(t) \to 0$ as $t \to \infty$.

It follows from (5) and (4) that

$$\frac{\mathcal{L}(\lambda x)}{\mathcal{L}(x)} = \exp\int_x^{\lambda x} \frac{\varepsilon(t)}{t} dt = 1 + \alpha(x) \quad \text{as} \quad x \to \infty$$

for each $\lambda > 0$. Hereof

$$\int_x^{\lambda x} \frac{\varepsilon(t)}{t} dt = \ln\left[1 + \alpha(x)\right] = \alpha(x) + \mathcal{O}\left(\alpha^2(x)\right) \quad \text{as} \quad x \to \infty.$$

Using the mean value theorem in the left-hand side of this equality we can be convinced that

$$\varepsilon(x) = \mathcal{O}\big(\alpha(x)\big) \quad \text{as} \quad x \to \infty. \tag{6}$$

Further we will consider a case when

$$\alpha(x) = o\left(\frac{\mathcal{L}(x)}{x^{\nu}}\right) \quad \text{as} \quad x \to \infty. \tag{7}$$

Denote

$$\phi(y) := 1 - F(1-y) = y - y\Lambda(y).$$

In pursuance of reasoning from [14] we obtain the following asymptotic relation:

$$\frac{1}{\Lambda\big(\phi(y)\big)} - \frac{1}{\Lambda(y)} = \nu + \delta(y), \tag{8}$$

where $\delta(y)$ is continuous function so that $\delta(y) \to 0$ as $y \downarrow 0$, see also [3, p. 401].

Further discussions allow us to estimate the tail-part $\delta(y)$ in (8). At first we will prove the following lemma.

Lemma 2.1. *Let conditions* (1), (4) *and* (7) *hold. Then*

$$\mathcal{L}\left(\frac{1}{\phi(y)}\right) = \mathcal{L}\left(\frac{1}{y}\right)\left(1 + o\left(\Lambda(y)\right)\right) \quad \textit{as} \quad y \downarrow 0. \tag{9}$$

Proof. Since the function $\mathcal{L}(x) = x^{\nu}\Lambda(1/x)$ is differentiable, by virtue of the mean value theorem we have

$$\mathcal{L}\left(\frac{x}{1-\Lambda}\right) - \mathcal{L}(x) = \mathcal{L}'\left(\frac{1-\theta\Lambda}{1-\Lambda}x\right)\cdot\frac{x\Lambda}{1-\Lambda}, \tag{10}$$

where $\Lambda := \Lambda(1/x)$ and $0 < \theta < 1$. From integral representation (5) and considering (6) it follows that

$$\mathcal{L}'(u) = \mathcal{L}(u)\frac{\varepsilon(u)}{u} = o\left(\frac{\mathcal{L}^2(u)}{u^{1+\nu}}\right) \quad \textit{as} \quad u \to \infty. \tag{11}$$

Denote $u = (1-\theta\Lambda)x/(1-\Lambda)$. Since $\Lambda(1/x) \to 0$ then $u \sim x$ and $\mathcal{L}(u) \sim \mathcal{L}(x)$ as $x \to \infty$. Therefore using (11) in the equality (10) and after some elementary transformations the assertion (9) readily follows. The lemma is proved. □

Lemma 2.2. *Let conditions* (1), (4) *and* (7) *hold. Then*

$$\frac{1}{\Lambda\big(\phi(y)\big)} - \frac{1}{\Lambda(y)} = \nu + \frac{\nu(\nu+1)}{2}\Lambda(y) + \gamma(y), \tag{12}$$

where $\gamma(y) = o\big(\Lambda(y)\big)$ *as* $y \downarrow 0$.

Proof. Write

$$K(y) := \frac{1}{\Lambda\big(\phi(y)\big)} - \frac{1}{\Lambda(y)} = \frac{\mathcal{L}\left(\frac{1}{y}\right) - \left(1-\Lambda(y)\right)^{\nu}\mathcal{L}\left(\frac{1}{\phi(y)}\right)}{\Lambda(y)\cdot\left(1-\Lambda(y)\right)^{\nu}\mathcal{L}\left(\frac{1}{\phi(y)}\right)}.$$

Taking into consideration (9) the last relation becomes

$$K(y) = \frac{1-\big(1-\Lambda(y)\big)^{\nu}}{\Lambda(y)\big(1-\Lambda(y)\big)^{\nu}}\big(1+o\big(\Lambda(y)\big)\big) \quad \text{as} \quad y \downarrow 0. \tag{13}$$

By the Taylor expansion the head part of (13)

$$\frac{1-\big(1-\Lambda(y)\big)^{\nu}}{\Lambda(y)\big(1-\Lambda(y)\big)^{\nu}} = \nu + \frac{\nu(\nu+1)}{2}\Lambda(y) + \mathcal{O}\big(\Lambda^2(y)\big) \quad \text{as} \quad y \downarrow 0.$$

From here and (13) the formula (12) now easily follows. The lemma is proved. □

The following assertion is improved analogy of the Basic Lemma.

Lemma 2.3. *Let conditions* (1), (4) *and* (7) *hold. Then*

$$\frac{1}{\Lambda\big(R_n(s)\big)} - \frac{1}{\Lambda\,(1-s)} = \nu n + \frac{1+\nu}{2}\cdot \ln\left(1+\nu n\Lambda(1-s)\right) + \rho_n(s), \tag{14}$$

where $\rho_n(s) = o\big(\ln n\big) + \sigma_n(s)$ *and,* $\sigma_n(s)$ *is bounded uniformly for* $s \in [0,1)$ *and converges to a limit* $\sigma(s)$ *as* $n \to \infty$ *which is a bounded function of* $s \in [0,1)$.

Proof. Note that $R_{k+1}(s) = \phi\big(R_k(s)\big)$. It is known that $R_k(s) \to 0$ as $k \to \infty$ uniformly for $s \in [0,1)$, see [2, p. 6]. Therefore putting $y = R_k(s)$ it follows from (12) that

$$\frac{1}{\Lambda\big(R_{k+1}(s)\big)} - \frac{1}{\Lambda\big(R_k(s)\big)} = \nu + \frac{\nu(\nu+1)}{2}\Lambda\big(R_k(s)\big) + \gamma\big(R_k(s)\big),$$

where $\gamma(y) = o\big(\Lambda(y)\big)$ as $y \downarrow 0$. Summing both sides of last equality on k from 1 to n we obtain

$$\frac{1}{\Lambda\big(R_n(s)\big)} - \frac{1}{\Lambda\big(R_0(s)\big)} = \nu n + \frac{\nu(\nu+1)}{2}\sum_{k=0}^{n-1}\Lambda\big(R_k(s)\big) + \sum_{k=0}^{n-1}\gamma_k\big(s\big), \tag{15}$$

where $\gamma_k\big(s\big) = o\big(\Lambda\big(R_k(s)\big)\big)$. Since $\Lambda\big(R_n(s)\big) \to 0$ uniformly for $s \in [0,1)$ each of the last two sums on the right-hand side of (15) is $o(n)$ as $n \to \infty$. So that considering $R_0(s) = 1-s$, we have

$$\frac{1}{\Lambda\big(R_n(s)\big)} - \frac{1}{\Lambda\big(1-s\big)} \sim \nu n \quad \textit{as} \quad n \to \infty.$$

Thus and so $\nu n\Lambda\big(R_n(s)\big) \to 1$ uniformly for $s \in [0,1)$ as $n \to \infty$. Hence

$$\sum_{k=0}^{n-1}\Lambda\big(R_k(s)\big) = \mathcal{O}\big(\ln n\big)$$

and $\sum\limits_{k=0}^{n-1}\gamma_k\big(s\big) = o\big(\ln n\big)$ as $n \to \infty$. Thus we obtain

$$\frac{1}{\Lambda\big(R_n(s)\big)} - \frac{1}{\Lambda(1-s)} = \nu n + \mathcal{O}\big(\ln n\big) \quad \text{as} \quad n \to \infty. \tag{16}$$

Next, using (16), we have

$$u_k(s) := \Lambda\big(R_k(s)\big) - \frac{1}{\nu k + \Lambda^{-1}(1-s)} = \frac{\mathcal{O}\big(\ln k\big)}{\big(\nu k + \Lambda^{-1}(1-s) + \mathcal{O}\big(\ln k\big)\big)\big(\nu k + \Lambda^{-1}(1-s)\big)} =$$
$$= \frac{\mathcal{O}\big(\ln k\big)}{\big(\nu k + \Lambda^{-1}(1-s)\big)^2 + \mathcal{O}\big(\big(k + \Lambda^{-1}(1-s)\big)\ln k\big)}.$$

Since $0 \leqslant s < 1$, the right-hand side of last equality is $\mathcal{O}\left(\ln k/k^2\right)$. Hence it follows that $\sum_{k\in\mathbb{N}_0} |u_k(s)| < \infty$ for all $s \in [0,1)$. Returning to (15) we see that the sum in second term in (15) is $\sum_{k=0}^{n-1} \frac{\Lambda(1-s)}{\Lambda(1-s)\nu k + 1} + \sum_{k=0}^{n-1} u_k(s)$. In turn by standard arguments [4, p. 544] we see that the expression

$$\sum_{k=0}^{n-1} \frac{\Lambda(1-s)}{\Lambda(1-s)\nu k + 1} - \frac{\ln\left(1 + \nu n \Lambda(1-s)\right)}{\nu}$$

is bounded uniformly for $s \in [0,1)$ and approaches a limit as $n \to \infty$ which is a bounded function of $s \in [0,1)$. Thus since the bound on the $u_k(s)$ is uniform for $s \in [0,1)$, the expression

$$\frac{1}{\Lambda\big(R_n(s)\big)} - \frac{1}{\Lambda\left(1-s\right)} - \nu n - \frac{1+\nu}{2} \cdot \ln\left(1 + \nu n \Lambda(1-s)\right) - \sum_{k=0}^{n-1} \gamma_k\big(s\big)$$

converges to a bounded limit as $n \to \infty$ uniformly for $s \in [0,1)$. Finally, since the second sum in (15) $\sum_{k=0}^{n-1} \gamma_k\big(s\big) = o\big(\ln n\big)$ the formula (14) is fair. The lemma is proved. □

Remark 1. *The assertion* (14) *was proved in [5, pp. 20–21] provided that* $F'''(1-)$ *is finite.*

Now using Lemma 2.3 we can improve the assertion (2). In fact putting $s = 0$ and, after elementary arguments we obtain the following results.

Theorem 2.1. *Let conditions* (1), (4) *and* (7) *hold. Then*

$$\mathbb{P}\big\{\mathcal{H} > n\big\} = \frac{\mathcal{N}(n)}{\big(\nu n\big)^{1/\nu}} \left(1 - \frac{1+\nu}{2\nu^2}\frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right)\right),$$

as $n \to \infty$*, where* $\mathcal{N}(n)$ *is SV-function and defined in Lemma 1.1*

By the same way we obtain the following local limit theorem which is improvement of the analogous result from the paper [6].

Theorem 2.2. *If conditions* (1), (4) *and* (7) *hold, then*

$$\big(\nu n\big)^{1+1/\nu} \cdot P_{11}(n) = \frac{\mathcal{N}_\nu(n)}{p_0} \left(1 - \frac{(1+\nu)^2}{2\nu^2}\frac{\ln n}{n} + o\left(\frac{\ln n}{n}\right)\right),$$

where $\mathcal{N}_\nu(n)\mathcal{N}^{-1}(n) \to 1$ *as* $n \to \infty$.

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О применении медленно меняющихся функций с остатком в теории ветвящихся процессов Гальтона-Ватсона

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В работе мы исследуем применение медленно меняющихся функций (в смысле Карамата) в теории ветвящихся процессов Гальтона-Ватсона. Рассмотрим критический случай такой, что производящая функция распределения прямого потомка одной частицы имеет бесконечный второй момент, но его хвост регулярно меняется с остатком. Мы уточняем основную лемму теории критических ветвящихся процессов Гальтона-Ватсона и улучшаем некоторые известные асимптотические результаты.

Ключевые слова: ветвящийся процесс Гальтона-Ватсона, медленно меняющиеся функции, производящие функции.