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The algebraic and geometric classification of nilpotent Novikov algebras☆

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ABSTRACT

This paper is devoted to the complete algebraic and geometric classification of 4-dimensional nilpotent Novikov algebras over $\mathbb{C}$.

0. Introduction

The algebraic classification (up to isomorphism) of algebras of dimension n from a certain variety defined by a certain family of polynomial identities is a classic problem in the theory of non-associative algebras. There are many results related to the algebraic classification of small-dimensional algebras in the varieties of Jordan, Lie, Leibniz, Zinbiel and many other algebras [1,9,11,14–17,25,26,31,34,41,42]. Another interesting direction in the classification of algebras is the geometric classification. There are many results related to the geometric classification of Jordan, Lie, Leibniz, Zinbiel and many other algebras [3,6,8,11,13,26–29,35,36,39–41,44]. In the present paper, we give the algebraic and geometric classification of 4-dimensional nilpotent Novikov algebras introduced by Novikov and Balinsky in [5].

The variety of Novikov algebras is defined by the following identities:

$$\begin{aligned}(xy)z &= (xz)y,\\ (xy)z - x(yz) &= (yx)z - y(xz).\end{aligned}$$

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It contains commutative associative algebras as a subvariety. On the other hand, the variety of Novikov algebras is the intersection of the variety of right commutative algebras (defined by the first Novikov identity) and the variety of left symmetric (Pre-Lie) algebras (defined by the second Novikov identity). Also, a Novikov algebra with the commutator multiplication gives a Lie algebra, and Novikov algebras are related to Tortken and Novikov–Poisson algebras [19,47]. The systematic study of Novikov algebras started after the paper of Zelmanov where all finite-dimensional simple Novikov algebras over the complex field were classified [49]. The first nontrivial examples of infinite-dimensional simple Novikov algebras were constructed in [24]. Also, simple Novikov algebras were described in infinite-dimensional case and over fields of positive characteristic [43,46,48]. The algebraic classification of 3-dimensional Novikov algebras was given in [4], and for some classes of 4-dimensional algebras, it was given in [7]; the geometric classification of 3-dimensional Novikov algebras was given in [6]. Many other purely algebraic properties of Novikov algebras were studied in a series by papers of Dzhumadildaev [20–23].

Our method for classifying nilpotent Novikov algebras is based on the calculation of central extensions of nilpotent algebras of smaller dimensions from the same variety. Central extensions play an important role in quantum mechanics: one of the earlier encounters is through Wigner´s theorem, which states that a symmetry of a quantum mechanical system determines an (anti-)unitary transformation of a Hilbert space. Another area of physics where one encounters central extensions is the quantum theory of conserved currents of a Lagrangian. These currents span an algebra which is closely related to the so-called affine Kac–Moody algebras, the universal central extensions of loop algebras. Central extensions are useful in physics because the symmetry group of a quantized system is usually a central extension of the classical symmetry group, and in the same way the corresponding symmetry Lie algebra of the quantum system is, in general, a central extension of the classical symmetry algebra. Kac–Moody algebras have been conjectured to be the symmetry groups of a unified superstring theory. The centrally extended Lie algebras play a dominant role in quantum field theory, particularly in conformal field theory, string theory and in M-theory. In the theory of Lie groups, Lie algebras and their representations, a Lie algebra extension is an enlargement of a given Lie algebra g by another Lie algebra h. Extensions arise in several ways. For example, the trivial extension is obtained as a direct sum of two Lie algebras. Other types are split and central extensions. Extensions arise naturally, for instance, when one constructs a Lie algebra from projective group representations. The algebraic study of central extensions of Lie and non-Lie algebras has been an important topic for years [2,32,33,37,45,50]. First, Skjelbred and Sund used central extensions of Lie algebras to obtain a classification of nilpotent Lie algebras [45]. After that, using the method described by Skjelbred and Sund, all non-Lie central extensions of all 4-dimensional Malcev algebras were described [33], and also all non-associative central extensions of 3-dimensional Jordan algebras [32], all anticommutative central extensions of the 3-dimensional anticommutative algebras [12], and all central extensions of the 2-dimensional algebras [10]. Note that the Skjelbred–Sund method of central extensions is an important tool in the classification of nilpotent algebras (see, for example, [30]), which was used to describe all 4-dimensional nilpotent associative algebras [17], all 4-dimensional nilpotent bicommutative algebras [38], all 5-dimensional nilpotent Jordan algebras [31], all 5-dimensional nilpotent restricted Lie algebras [15], all 6-dimensional nilpotent Lie algebras [14,16], all 6-dimensional nilpotent Malcev algebras [34] and some others.

𝔑𝔢𝔪𝔞𝔯𝔨. Note that, the algebraic classification of all 4-dimension nilpotent Novikov algebras can be obtained as a corollary from [7], but in our opinion the method used in this paper is not clear to understand and many calculations were omitted. Our algebraic classification is obtained by another method and it confirms the result from [7].

1. The algebraic classification of nilpotent Novikov algebras

1.1. Method of classification of nilpotent algebras

Throughout this paper, we use the notations and methods well written in [10,32,33], which we have adapted for the Novikov case with some modifications. Further in this section we give some important definitions.

Let $(\mathbf{A}, \cdot)$ be a Novikov algebra over $\mathbb{C}$ and $\mathbb{V}$ a vector space over $\mathbb{C}$. The $\mathbb{C}$-linear space $\mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ is defined as the set of all bilinear maps $\theta\colon \mathbf{A}\times\mathbf{A} \longrightarrow \mathbb{V}$ such that

$$\begin{aligned} \theta(xy, z) &= \theta(xz, y), \\ \theta(xy, z) - \theta(x, yz) &= \theta(yx, z) - \theta(y, xz). \end{aligned}$$

These elements will be called *cocycles*. For a linear map f from $\mathbf{A}$ to $\mathbb{V}$, if we define $\delta f\colon \mathbf{A}\times\mathbf{A} \longrightarrow \mathbb{V}$ by $\delta f(x, y) = f(xy)$, then $\delta f \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$. We define $\mathrm{B}^2(\mathbf{A}, \mathbb{V}) = \{\theta = \delta f \ : f \in \mathrm{Hom}(\mathbf{A}, \mathbb{V})\}$. We define the *second cohomology space* $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$ as the quotient space $\mathrm{Z}^2(\mathbf{A}, \mathbb{V}) \big/ \mathrm{B}^2(\mathbf{A}, \mathbb{V})$.

Let $\mathrm{Aut}(\mathbf{A})$ be the automorphism group of $\mathbf{A}$ and let $\phi \in \mathrm{Aut}(\mathbf{A})$. For $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ define the action of the group $\mathrm{Aut}(\mathbf{A})$ on $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$ by $\phi\theta(x, y) = \theta(\phi(x), \phi(y))$. It is easy to verify that $\mathrm{B}^2(\mathbf{A}, \mathbb{V})$ is invariant under the action of $\mathrm{Aut}(\mathbf{A})$. So, we have an induced action of $\mathrm{Aut}(\mathbf{A})$ on $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$.

Let $\mathbf{A}$ be a Novikov algebra of dimension m over $\mathbb{C}$ and $\mathbb{V}$ be a $\mathbb{C}$-vector space of dimension k. For $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$, define on the linear space $\mathbf{A}_\theta = \mathbf{A}\oplus\mathbb{V}$ the bilinear product "$[-, -]_{\mathbf{A}_\theta}$" by $\left[x + x', y + y'\right]_{\mathbf{A}_\theta} = xy + \theta(x, y)$ for all $x, y \in \mathbf{A}$, $x', y' \in \mathbb{V}$. The algebra $\mathbf{A}_\theta$ is called a *k-dimensional central extension* of $\mathbf{A}$ by $\mathbb{V}$. One can easily check that $\mathbf{A}_\theta$ is a Novikov algebra if and only if $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$.

Call the set $\mathrm{Ann}(\theta)=\{x\in \mathbf{A}:\theta(x,\mathbf{A})+\theta(\mathbf{A},x)=0\}$ the *annihilator* of θ. We recall that the *annihilator* of an algebra $\mathbf{A}$ is defined as the ideal $\mathrm{Ann}(\mathbf{A})=\{x\in\mathbf{A}: x\mathbf{A}+\mathbf{A}x=0\}$. Observe that $\mathrm{Ann}(\mathbf{A}_\theta)=(\mathrm{Ann}(\theta)\cap \mathrm{Ann}(\mathbf{A}))\oplus\mathbb{V}$.

The following result shows that every algebra with a non-zero annihilator is a central extension of a smaller-dimensional algebra.

Lemma 1. *Let* $\mathbf{A}$ *be an n-dimensional Novikov algebra such that* $\dim(\mathrm{Ann}(\mathbf{A}))=m\neq 0$. *Then there exists, up to isomorphism, a unique* $(n-m)$*-dimensional Novikov algebra* $\mathbf{A}'$ *and a bilinear map* $\theta\in \mathrm{Z}^2(\mathbf{A},\mathbb{V})$ *with* $\mathrm{Ann}(\mathbf{A})\cap\mathrm{Ann}(\theta)=0$, *where* $\mathbb{V}$ *is a vector space of dimension m, such that* $\mathbf{A}\cong {\mathbf{A}'}_\theta$ *and* $\mathbf{A}/\mathrm{Ann}(\mathbf{A})\cong\mathbf{A}'$.

Proof. Let $\mathbf{A}'$ be a linear complement of $\mathrm{Ann}(\mathbf{A})$ in $\mathbf{A}$. Define a linear map $P:\mathbf{A}\longrightarrow\mathbf{A}'$ by $P(x+v)=x$ for $x\in\mathbf{A}'$ and $v\in\mathrm{Ann}(\mathbf{A})$, and define a multiplication on $\mathbf{A}'$ by $[x,y]_{\mathbf{A}'}=P(xy)$ for $x,y\in\mathbf{A}'$. For $x,y\in\mathbf{A}$, we have

$$P(xy)=P((x-P(x)+P(x))(y-P(y)+P(y)))=P(P(x)P(y))=[P(x),P(y)]_{\mathbf{A}'}.$$

Since P is a homomorphism $P(\mathbf{A})=\mathbf{A}'$ is a Novikov algebra and $\mathbf{A}/\mathrm{Ann}(\mathbf{A})\cong\mathbf{A}'$, which gives us the uniqueness. Now, define the map $\theta:\mathbf{A}'\times\mathbf{A}'\longrightarrow\mathrm{Ann}(\mathbf{A})$ by $\theta(x,y)=xy-[x,y]_{\mathbf{A}'}$. Thus, $\mathbf{A}'_\theta$ is $\mathbf{A}$ and therefore $\theta\in\mathrm{Z}^2(\mathbf{A},\mathbb{V})$ and $\mathrm{Ann}(\mathbf{A})\cap\mathrm{Ann}(\theta)=0$. □

Definition 2. Let $\mathbf{A}$ be an algebra and I be a subspace of $\mathrm{Ann}(\mathbf{A})$. If $\mathbf{A}=\mathbf{A}_0\oplus I$ then I is called an *annihilator component* of $\mathbf{A}$.

Definition 3. A central extension of an algebra $\mathbf{A}$ without annihilator component is called a *non-split central extension.*

Our task is to find all central extensions of an algebra $\mathbf{A}$ by a space $\mathbb{V}$. In order to solve the isomorphism problem we need to study the action of $\mathrm{Aut}(\mathbf{A})$ on $\mathrm{H}^2(\mathbf{A},\mathbb{V})$. To do that, let us fix a basis $e_1,\ldots,e_s$ of $\mathbb{V}$, and $\theta\in\mathrm{Z}^2(\mathbf{A},\mathbb{V})$. Then θ can be uniquely written as $\theta(x,y)=\sum_{i=1}^{s}\theta_i(x,y)e_i$, where $\theta_i\in\mathrm{Z}^2(\mathbf{A},\mathbb{C})$. Moreover, $\mathrm{Ann}(\theta)=\mathrm{Ann}(\theta_1)\cap\mathrm{Ann}(\theta_2)\cap\cdots\cap\mathrm{Ann}(\theta_s)$. Furthermore, $\theta\in\mathrm{B}^2(\mathbf{A},\mathbb{V})$ if and only if all $\theta_i\in\mathrm{B}^2(\mathbf{A},\mathbb{C})$. It is not difficult to prove (see [33, Lemma 13]) that given a Novikov algebra $\mathbf{A}_\theta$, if we write as above $\theta(x,y)=\sum_{i=1}^{s}\theta_i(x,y)e_i\in\mathrm{Z}^2(\mathbf{A},\mathbb{V})$ and $\mathrm{Ann}(\theta)\cap\mathrm{Ann}(\mathbf{A})=0$, then $\mathbf{A}_\theta$ has an annihilator component if and only if $[\theta_1],[\theta_2],\ldots,[\theta_s]$ are linearly dependent in $\mathrm{H}^2(\mathbf{A},\mathbb{C})$.

Let $\mathbb{V}$ be a finite-dimensional vector space over $\mathbb{C}$. The *Grassmannian* $G_k(\mathbb{V})$ is the set of all k-dimensional linear subspaces of $\mathbb{V}$. Let $G_s\left(\mathrm{H}^2(\mathbf{A},\mathbb{C})\right)$ be the Grassmannian of subspaces of dimension s in $\mathrm{H}^2(\mathbf{A},\mathbb{C})$. There is a natural action of $\mathrm{Aut}(\mathbf{A})$ on $G_s\left(\mathrm{H}^2(\mathbf{A},\mathbb{C})\right)$. Let $\phi\in\mathrm{Aut}(\mathbf{A})$. For $W=\langle[\theta_1],[\theta_2],\ldots,[\theta_s]\rangle\in G_s\left(\mathrm{H}^2(\mathbf{A},\mathbb{C})\right)$ define $\phi W=\langle[\phi\theta_1],[\phi\theta_2],\ldots,[\phi\theta_s]\rangle$. We denote the orbit of $W\in G_s\left(\mathrm{H}^2(\mathbf{A},\mathbb{C})\right)$ under the action of $\mathrm{Aut}(\mathbf{A})$ by $\mathrm{Orb}(W)$. Given

$$W_1=\langle[\theta_1],[\theta_2],\ldots,[\theta_s]\rangle,\ W_2=\langle[\vartheta_1],[\vartheta_2],\ldots,[\vartheta_s]\rangle\in G_s\left(\mathrm{H}^2(\mathbf{A},\mathbb{C})\right),$$

we easily have that if $W_1=W_2$, then $\bigcap_{i=1}^s\mathrm{Ann}(\theta_i)\cap\mathrm{Ann}(\mathbf{A})=\bigcap_{i=1}^s\mathrm{Ann}(\vartheta_i)\cap\mathrm{Ann}(\mathbf{A})$, and therefore we can introduce the set

$$\mathbf{T}_s(\mathbf{A})=\left\{W=\langle[\theta_1],[\theta_2],\ldots,[\theta_s]\rangle\in G_s\left(\mathrm{H}^2(\mathbf{A},\mathbb{C})\right):\bigcap_{i=1}^{s}\mathrm{Ann}(\theta_i)\cap\mathrm{Ann}(\mathbf{A})=0\right\},$$

which is stable under the action of $\mathrm{Aut}(\mathbf{A})$.

Now, let $\mathbb{V}$ be an s-dimensional linear space and let us denote by $\mathbf{E}(\mathbf{A},\mathbb{V})$ the set of all *non-split s-dimensional central extensions* of $\mathbf{A}$ by $\mathbb{V}$. By above, we can write

$$\mathbf{E}(\mathbf{A},\mathbb{V})=\left\{\mathbf{A}_\theta:\theta(x,y)=\sum_{i=1}^{s}\theta_i(x,y)e_i\ \text{ and }\ \langle[\theta_1],[\theta_2],\ldots,[\theta_s]\rangle\in\mathbf{T}_s(\mathbf{A})\right\}.$$

We also have the following result, which can be proved as in [33, Lemma 17].

Lemma 4. *Let* $\mathbf{A}_\theta,\mathbf{A}_\vartheta\in\mathbf{E}(\mathbf{A},\mathbb{V})$. *Suppose that* $\theta(x,y)=\sum_{i=1}^{s}\theta_i(x,y)e_i$ *and* $\vartheta(x,y)=\sum_{i=1}^{s}\vartheta_i(x,y)e_i$. *Then the Novikov algebras* $\mathbf{A}_\theta$ *and* $\mathbf{A}_\vartheta$ *are isomorphic if and only if*

$$\mathrm{Orb}\langle[\theta_1],[\theta_2],\ldots,[\theta_s]\rangle=\mathrm{Orb}\langle[\vartheta_1],[\vartheta_2],\ldots,[\vartheta_s]\rangle.$$

This shows that there exists a one-to-one correspondence between the set of $\mathrm{Aut}(\mathbf{A})$-orbits on $\mathbf{T}_s(\mathbf{A})$ and the set of isomorphism classes of $\mathbf{E}(\mathbf{A},\mathbb{V})$. Consequently we have a procedure that allows us, given a Novikov algebra $\mathbf{A}'$ of dimension $n-s$, to construct all non-split central extensions of $\mathbf{A}'$. This procedure is:

Procedure

(1) For a given Novikov algebra $\mathbf{A}'$ of dimension $n-s$, determine $\mathrm{H}^2(\mathbf{A}',\mathbb{C})$, $\mathrm{Ann}(\mathbf{A}')$ and $\mathrm{Aut}(\mathbf{A}')$.
(2) Determine the set of $\mathrm{Aut}(\mathbf{A}')$-orbits on $\mathbf{T}_s(\mathbf{A}')$.
(3) For each orbit, construct the Novikov algebra associated with a representative of it.

1.2. Notations

Let $\mathbf{A}$ be a Novikov algebra with a basis $e_1, e_2, \dots, e_n$. Then by Δ_{ij} we denote the bilinear form $\Delta_{ij}: \mathbf{A}\times\mathbf{A} \longrightarrow \mathbb{C}$ with $\Delta_{ij}(e_l, e_m) = \delta_{il}\delta_{jm}$. Then the set $\left\{\Delta_{ij} : 1 \le i, j \le n\right\}$ is a basis for the space of the bilinear forms on $\mathbf{A}$. Then every $\theta \in \mathrm{Z}^2(\mathbf{A},\mathbb{C})$ can be uniquely written as $\theta = \sum\limits_{1\le i,j\le n} c_{ij}\Delta_{ij}$, where $c_{ij}\in\mathbb{C}$. Let us fix the following notations:

$\mathcal{N}_j^{i*}$	—	jth i-dimensional nilpotent Novikov algebra with identity $xyz = 0$
$\mathcal{N}_j^{i}$	—	jth i-dimensional nilpotent "pure" Novikov algebra (without identity $xyz = 0$)
$\mathfrak{N}_i$	—	i-dimensional algebra with zero product
$(\mathbf{A})_{i,j}$	—	jth i-dimensional central extension of $\mathbf{A}$.

1.3. The algebraic classification of 3-dimensional nilpotent Novikov algebras

There are no nontrivial 1-dimensional nilpotent Novikov algebras. There is only one nontrivial 2-dimensional nilpotent Novikov algebra (it is the non-split central extension of 1-dimensional algebra with zero product):

$$\mathcal{N}_{01}^{2*} \quad : \quad (\mathfrak{N}_1)_{2,1} \quad : \quad e_1e_1 = e_2.$$

Thanks to [10] we have the description of all central extensions of $\mathcal{N}_{01}^{2*}$ and $\mathfrak{N}_2$. Choosing the Novikov algebras from the central extensions of these algebras, we have the classification of all non-split 3-dimensional nilpotent Novikov algebras:

$$\begin{array}{lllllll}
\mathcal{N}_{02}^{3*} & : & (\mathfrak{N}_2)_{3,1} & : & e_1e_1 = e_3, & e_2e_2 = e_3; & \\
\mathcal{N}_{03}^{3*} & : & (\mathfrak{N}_2)_{3,2} & : & e_1e_2 = e_3, & e_2e_1 = -e_3; & \\
\mathcal{N}_{04}^{3*}(\lambda) & : & (\mathfrak{N}_2)_{3,3} & : & e_1e_1 = \lambda e_3, & e_2e_1 = e_3, & e_2e_2 = e_3, \quad \lambda \neq 0; \\
\mathcal{N}_{04}^{3*}(0) & : & (\mathfrak{N}_2)_{3,3} & : & e_1e_2 = e_3, & & \\
\mathcal{N}_{01}^{3} & : & (\mathcal{N}_{01}^{2*})_{3,1} & : & e_1e_1 = e_2, & e_2e_1 = e_3; & \\
\mathcal{N}_{02}^{3}(\lambda) & : & (\mathcal{N}_{01}^{2*})_{3,2} & : & e_1e_1 = e_2, & e_1e_2 = e_3, & e_2e_1 = \lambda e_3, \quad \lambda \in \mathbb{C}.
\end{array}$$

1.4. The algebraic classification of 4-dimensional nilpotent Novikov algebras

Recall that the class defined by the identities $(xy)z = 0$ and $x(yz) = 0$ lies in the intersection of the varieties of algebras defined by polynomial identities of degree 3, such as Leibniz algebras, Zinbiel algebras or associative algebras. On the other side, every algebra defined by the identities $(xy)z = 0$ and $x(yz) = 0$ is a central extension of some suitable algebra with zero product. The list of all non-anticommutative 4-dimensional algebras defined by the identities $(xy)z = 0$ and $x(yz) = 0$ can be found in [18]. Note that there is only one 4-dimensional nilpotent anticommutative algebra with identity $(xy)z = 0$. Obviously every algebra from this list is a 4-dimensional nilpotent "non-pure" Novikov algebra. The aim of the present part of the work is to find all 4-dimensional nilpotent "pure" Novikov algebras which do not belong to the class of algebras defined by the identities $(xy)z = 0$ and $x(yz) = 0$.

Now we are ready to state the main result of this part of the paper. The proof of the present theorem is based on the classification of 3-dimensional nilpotent Novikov algebras and the results of Section 1.5.

Theorem 5. *Let $\mathcal{N}$ be a nonzero 4-dimensional nilpotent "pure" Novikov algebra over $\mathbb{C}$. Then, $\mathcal{N}$ is isomorphic to one of the algebras listed in Table A (see Appendix).*

1.5. 1-Dimensional central extensions of 3-dimensional nilpotent Novikov algebras

1.5.1. The description of second cohomology spaces of 3-dimensional nilpotent Novikov algebras

In the following table we give the description of the second cohomology space of 3-dimensional nilpotent Novikov algebras

$\mathbf{A}$	Z^2 ($\mathbf{A}$)	$\mathrm{B}^2(\mathbf{A})$	$\mathrm{H}^2(\mathbf{A})$
$\mathcal{N}_{01}^{3*}$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{21}, \Delta_{31}, \Delta_{33} \rangle$	$\langle \Delta_{11} \rangle$	$\langle [\Delta_{12}], [\Delta_{13}], [\Delta_{21}], [\Delta_{31}], [\Delta_{33}] \rangle$
$\mathcal{N}_{02}^{3*}$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22} \rangle$	$\langle \Delta_{11} + \Delta_{22} \rangle$	$\langle [\Delta_{12}], [\Delta_{21}], [\Delta_{22}] \rangle$
$\mathcal{N}_{03}^{3*}$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22} \rangle$	$\langle \Delta_{12} - \Delta_{21} \rangle$	$\langle [\Delta_{11}], [\Delta_{21}], [\Delta_{22}] \rangle$
$\mathcal{N}_{04}^{3*}(\lambda)_{\lambda\neq 0}$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22} \rangle$	$\langle \lambda\Delta_{11} + \Delta_{21} + \Delta_{22} \rangle$	$\langle [\Delta_{12}], [\Delta_{21}], [\Delta_{22}] \rangle$
$\mathcal{N}_{04}^{3*}(0)$	$\left\langle \begin{array}{l} \Delta_{11}, \Delta_{12}, \Delta_{13}, \\ \Delta_{21}, \Delta_{22}, \Delta_{23} - \Delta_{32} \end{array} \right\rangle$	$\langle \Delta_{12} \rangle$	$\left\langle \begin{array}{l} [\Delta_{11}], [\Delta_{13}], [\Delta_{21}], \\ [\Delta_{22}], [\Delta_{23}] - [\Delta_{32}] \end{array} \right\rangle$
$\mathcal{N}_{01}^{3}$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} - \Delta_{31} \rangle$	$\langle \Delta_{11}, \Delta_{21} \rangle$	$\langle [\Delta_{12}], [\Delta_{13}] - [\Delta_{31}] \rangle$
$\mathcal{N}_{02}^{3}(\lambda)$	$\left\langle \begin{array}{l} \Delta_{11}, \Delta_{12}, \Delta_{21}, \\ (2-\lambda)\Delta_{13} + \lambda(\Delta_{22} + \Delta_{31}) \end{array} \right\rangle$	$\left\langle \begin{array}{l} \Delta_{11}, \\ \Delta_{12} + \lambda\Delta_{21} \end{array} \right\rangle$	$\left\langle \begin{array}{l} [\Delta_{21}], \\ (2-\lambda)[\Delta_{13}] + \lambda([\Delta_{22}] + [\Delta_{31}]) \end{array} \right\rangle$

where $\mathcal{N}_{01}^{3*} = \mathcal{N}_{01}^{2*} \oplus \mathbb{C}e_3$.

Remark 6. Since $\mathrm{Z}^2 = \langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{22} \rangle$ of the algebras $\mathcal{N}_{02}^{3*}, \mathcal{N}_{03}^{3*}, \mathcal{N}_{04}^{3*}(\lambda)_{\lambda\neq 0}$ and center of these algebras is $\{e_3\}$, then 1-dimensional central extensions of these algebras give us four dimensional algebras with two dimensional center. Note that four dimensional algebras with two dimensional center are isomorphic to the algebras 2-dimensional central extensions of 2-dimensional nilpotent Novikov algebras. Thanks to [10] we have the description of all non-split 2-dimensional central extensions of 2-dimensional nilpotent Novikov algebras:

$$\mathcal{N}_{03}^{4} \quad : \quad (\mathcal{N}_{01}^{2*})_{4,1} \quad : \quad e_1e_1 = e_2, \quad e_1e_2 = e_4, \quad e_2e_1 = e_3.$$

1.5.2. Central extensions of $\mathcal{N}_{01}^{3*}$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \nabla_2 = [\Delta_{13}], \nabla_3 = [\Delta_{21}], \nabla_4 = [\Delta_{31}], \nabla_5 = [\Delta_{33}].$$

The automorphism group of $\mathcal{N}_{01}^{3*}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ u & x^2 & w \\ z & 0 & y \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ \alpha_3 & 0 & 0 \\ \alpha_4 & 0 & \alpha_5 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & x^3\alpha_1 & wx\alpha_1 + y(x\alpha_2 + z\alpha_5) \\ x^3\alpha_3 & 0 & 0 \\ x(w\alpha_3 + y\alpha_4) + yz\alpha_5 & 0 & y^2\alpha_5 \end{pmatrix},$$

the action of $\mathrm{Aut}(\mathcal{N}_{01}^{3*})$ on subspace $\left\langle \sum_{i=1}^{5} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{5} \alpha_i^* \nabla_i \right\rangle$, where

$$\begin{aligned} \alpha_1^* &= x^3\alpha_1; \\ \alpha_2^* &= wx\alpha_1 + yx\alpha_2 + yz\alpha_5; \\ \alpha_3^* &= x^3\alpha_3; \\ \alpha_4^* &= wx\alpha_3 + yx\alpha_4 + yz\alpha_5; \\ \alpha_5^* &= y^2\alpha_5. \end{aligned}$$

It is easy to see that the elements $\alpha_1\nabla_1 + \alpha_3\nabla_3$ and $\alpha_2\nabla_2 + \alpha_4\nabla_4 + \alpha_5\nabla_5$ give algebras with 2-dimensional annihilator, which were described before. Since we are interested only in new algebras, we have the following cases:

(1) $\alpha_1 \neq 0, \alpha_3 \neq 0, \alpha_5 \neq 0$, then:

  (a) if $\alpha_1 \neq \alpha_3$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_1}}, y = \frac{1}{\sqrt{\alpha_5}}, w = \frac{y(\alpha_2-\alpha_4)}{\alpha_3-\alpha_1}, z = \frac{x(\alpha_1\alpha_4-\alpha_2\alpha_3)}{\alpha_5(\alpha_3-\alpha_1)}$, we have the representative $\langle \nabla_1 + \alpha\nabla_3 + \nabla_5 \rangle_{\alpha\neq 0;1}$.

  (b) if $\alpha_1 = \alpha_3$, $\alpha_2 \neq \alpha_4$, then choosing $x = \frac{(\alpha_2-\alpha_4)^2}{\alpha_1\alpha_5}, y = \frac{(\alpha_2-\alpha_4)^3}{\alpha_1\alpha_5^2}, w = -\frac{yx\alpha_4+yz\alpha_5}{x\alpha_1}$, we have the representative $\langle \nabla_1 + \nabla_2 + \nabla_3 + \nabla_5 \rangle$.

  (c) if $\alpha_1 = \alpha_3$, $\alpha_2 = \alpha_4$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_1}}, y = \frac{1}{\sqrt{\alpha_5}}, w = -\frac{yx\alpha_4+yz\alpha_5}{x\alpha_1}$, we have the representative $\langle \nabla_1 + \nabla_3 + \nabla_5 \rangle$.

(2) $\alpha_1 \neq 0, \alpha_3 \neq 0, \alpha_5 = 0$, then:

  (a) if $\alpha_1\alpha_4 \neq \alpha_2\alpha_3$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_1}}, w = -\frac{y\alpha_4}{\alpha_3}, y = \frac{x^2\alpha_1}{\alpha_2\alpha_3-\alpha_1\alpha_4}$, we have the representative $\langle \nabla_1 + \nabla_2 + \alpha\nabla_3 \rangle_{\alpha\neq 0}$.

  (b) if $\alpha_1\alpha_4 = \alpha_2\alpha_3$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_1}}, w = -\frac{y\alpha_4}{\alpha_3}$, we have the representative $\langle \nabla_1 + \alpha\nabla_3 \rangle_{\alpha\neq 0}$.

(3) $\alpha_1 \neq 0, \alpha_3 = 0, \alpha_5 \neq 0$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_1}}, w = -\frac{yx\alpha_2 + yz\alpha_5}{x\alpha_1}, z = -\frac{x\alpha_4}{\alpha_5}, y = \frac{1}{\sqrt{\alpha_5}}$, we have the representative $\langle \nabla_1 + \nabla_5 \rangle$.
(4) $\alpha_1 = 0, \alpha_3 \neq 0, \alpha_5 \neq 0$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_3}}, z = -\frac{x\alpha_2}{\alpha_5}, w = -\frac{yx\alpha_4 + yz\alpha_5}{x\alpha_3}, y = \frac{1}{\sqrt{\alpha_5}}$, we have the representative $\langle \nabla_3 + \nabla_5 \rangle$.
(5) $\alpha_1 = 0, \alpha_3 \neq 0, \alpha_5 = 0$, then:
   (a) if $\alpha_2 \neq 0$, then choosing $y = \frac{x^2\alpha_3}{\alpha_2}, x = \frac{1}{\sqrt[3]{\alpha_3}}, w = -\frac{y\alpha_4}{\alpha_3}$, we have the representative $\langle \nabla_2 + \nabla_3 \rangle$.
   (b) if $\alpha_2 = 0$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_3}}, w = -\frac{y\alpha_4}{\alpha_3}$, we have the representative $\langle \nabla_3 \rangle$.
(6) $\alpha_1 \neq 0, \alpha_3 = 0, \alpha_5 = 0$, then:
   (a) if $\alpha_4 \neq 0$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_1}}, w = -\frac{y\alpha_2}{\alpha_1}, y = \frac{x^2\alpha_1}{\alpha_4}$, we have the representative $\langle \nabla_1 + \nabla_4 \rangle$.
   (b) if $\alpha_4 = 0$, then choosing $x = \frac{1}{\sqrt[3]{\alpha_1}}, w = -\frac{y\alpha_2}{\alpha_1}$, we have the representative $\langle \nabla_1 \rangle$.

Now we have all new 4-dimensional nilpotent Novikov algebras constructed from $\mathcal{N}_{01}^{3*}$:

$$\mathcal{N}_{04}^4(\alpha),\ \mathcal{N}_{05}^4,\ \mathcal{N}_{06}^4(\alpha)_{\alpha\neq 0},\ \mathcal{N}_{07}^4,\ \mathcal{N}_{08}^4,\ \mathcal{N}_{09}^4.$$

The multiplication tables of these algebras can be found in Appendix.

1.5.3. Central extensions of $\mathcal{N}_{04}^{3*}(0)$

Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \nabla_2 = [\Delta_{13}], \nabla_3 = [\Delta_{21}], \nabla_4 = [\Delta_{22}], \nabla_5 = [\Delta_{23}] - [\Delta_{32}].$$

The automorphism group of $\mathcal{N}_{04}^{3*}(0)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ z & t & xy \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & \alpha_2 \\ \alpha_3 & \alpha_4 & \alpha_5 \\ 0 & -\alpha_5 & 0 \end{pmatrix} \phi = \begin{pmatrix} x(x\alpha_1 + z\alpha_2) & \alpha^* & x^2y\alpha_2 \\ y(x\alpha_3 + z\alpha_5) & y^2\alpha_4 & xy^2\alpha_5 \\ 0 & -xy^2\alpha_5 & 0 \end{pmatrix},$$

the action of $\mathrm{Aut}(\mathcal{N}_{04}^{3*}(0))$ on the subspace $\langle \sum_{i=1}^5 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^5 \alpha_i^* \nabla_i \rangle$, where

$$\begin{aligned} \alpha_1^* &= x(x\alpha_1 + z\alpha_2); \\ \alpha_2^* &= x^2y\alpha_2; \\ \alpha_3^* &= y(x\alpha_3 + z\alpha_5); \\ \alpha_4^* &= y^2\alpha_4; \\ \alpha_5^* &= xy^2\alpha_5. \end{aligned}$$

It is easy to see that the elements $\alpha_1\nabla_1 + \alpha_3\nabla_3 + \alpha_4\nabla_4$ give algebras which are central extensions of 2-dimensional algebras. We have the following new cases:

(1) $\alpha_4 = \alpha_5 = 0, \alpha_2 \neq 0$, then:
   (a) if $\alpha_3 = 0$, then choosing $z = -\frac{x\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_2 \rangle$.
   (b) if $\alpha_3 \neq 0$, then choosing $z = -\frac{x\alpha_1}{\alpha_2}$ and $x = \frac{\alpha_3}{\alpha_2}$, we have the representative $\langle \nabla_2 + \nabla_3 \rangle$.
(2) $\alpha_4 = \alpha_2 = 0, \alpha_5 \neq 0$, then:
   (a) if $\alpha_1 = 0$, then choosing $z = -\frac{x\alpha_3}{\alpha_5}$, we have the representative $\langle \nabla_5 \rangle$.
   (b) if $\alpha_1 \neq 0$, then choosing $z = -\frac{x\alpha_3}{\alpha_5}$ and $x = \frac{y^2\alpha_5}{\alpha_1}$, we have the representative $\langle \nabla_1 + \nabla_5 \rangle$.
(3) $\alpha_4 = 0, \alpha_5 \neq 0, \alpha_2 \neq 0$, then:
   (a) if $\alpha_1\alpha_5 - \alpha_2\alpha_3 = 0$, then choosing $z = -\frac{x\alpha_3}{\alpha_5}$ and $x = \frac{y\alpha_5}{\alpha_2}$, we have the representative $\langle \nabla_2 + \nabla_5 \rangle$.
   (b) if $\alpha_1\alpha_5 - \alpha_2\alpha_3 \neq 0$, then choosing $z = -\frac{x\alpha_3}{\alpha_5}, y = \frac{\alpha_1\alpha_5 - \alpha_2\alpha_3}{\alpha_2\alpha_5}$ and $x = \frac{\alpha_1\alpha_5 - \alpha_2\alpha_3}{\alpha_2^2}$, we have the representative $\langle \nabla_1 + \nabla_2 + \nabla_5 \rangle$.

(4) $\alpha_4 \neq 0, \alpha_5 = 0, \alpha_2 \neq 0$, then:

(a) if $\alpha_3 = 0$, then choosing $z = -\frac{x\alpha_1}{\alpha_2}$ and $y = \frac{x^2\alpha_2}{\alpha_4}$, we have the representative $\langle \nabla_2 + \nabla_4 \rangle$.

(b) if $\alpha_3 \neq 0$, then choosing $z = -\frac{x\alpha_1}{\alpha_2}$, $y = \frac{\alpha_3^2}{\alpha_2\alpha_4}$ and $x = \frac{\alpha_3}{\alpha_2}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_4 \rangle$.

(5) $\alpha_4 \neq 0, \alpha_5 \neq 0, \alpha_2 = 0$, then:

(a) if $\alpha_1 = 0$, then choosing $z = -\frac{x\alpha_3}{\alpha_5}$ and $x = \frac{\alpha_4}{\alpha_5}$, we have the representative $\langle \nabla_4 + \nabla_5 \rangle$.

(b) if $\alpha_1 \neq 0$, then choosing $z = -\frac{x\alpha_3}{\alpha_5}$, $y = \sqrt{\frac{\alpha_4\alpha_1}{\alpha_5^2}}$ and $x = \frac{\alpha_4}{\alpha_5}$, we have the representative $\langle \nabla_1 + \nabla_4 + \nabla_5 \rangle$.

(6) $\alpha_4 \neq 0, \alpha_5 \neq 0, \alpha_2 \neq 0$, then choosing $z = -\frac{x\alpha_3}{\alpha_5}$, $y = \frac{\alpha_2\alpha_4}{\alpha_5^2}$, $x = \frac{\alpha_4}{\alpha_5}$ and $\alpha = \frac{\alpha_5(\alpha_1\alpha_5 - \alpha_2\alpha_3)}{\alpha_2^2\alpha_4}$, we have the representative $\langle \alpha\nabla_1 + \nabla_2 + \nabla_4 + \nabla_5 \rangle$.

Now we have all new 4-dimensional nilpotent Novikov algebras constructed from $\mathcal{N}_{04}^{3*}$:

$\mathcal{N}_{10}^4, \ \ldots, \mathcal{N}_{20}^4(\alpha)$.

The multiplication tables of these algebras can be found in Appendix.

1.5.4. Central extensions of $\mathcal{N}_{01}^3$

Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \nabla_2 = [\Delta_{13}] - [\Delta_{31}].$$

The automorphism group of $\mathcal{N}_{01}^3$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ y & x^2 & 0 \\ z & xy & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & 0 \\ -\alpha_2 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & x^3\alpha_1 + x^2y\alpha_2 & x^4\alpha_2 \\ \alpha^{**} & 0 & 0 \\ -x^4\alpha_2 & 0 & 0 \end{pmatrix},$$

the action of $\mathrm{Aut}(\mathcal{N}_{01}^3)$ on the subspace $\left\langle \sum_{i=1}^{2} \alpha_i \nabla_i \right\rangle$ is given by $\left\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \right\rangle$, where

$$\begin{aligned} \alpha_1^* &= x^3\alpha_1 + x^2y\alpha_2 \\ \alpha_2^* &= x^4\alpha_2. \end{aligned}$$

Since 2-dimensional central extensions of two dimensional algebras were already considered, we have $\alpha_2 \neq 0$. Choosing $y = -\frac{\alpha_1 y}{\alpha_2}$, $x = \frac{1}{\sqrt[4]{\alpha_2}}$, we have the representative $\langle \nabla_2 \rangle$.

Now we have only one new 4-dimensional nilpotent Novikov algebra constructed from $\mathcal{N}_{01}^3$:

$\mathcal{N}_{21}^4$.

The multiplication table of this algebra can be found in Appendix.

1.5.5. Central extensions of $\mathcal{N}_{02}^3(\lambda)$

Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \nabla_2 = (2-\lambda)[\Delta_{13}] + \lambda([\Delta_{22}] + [\Delta_{31}]).$$

The automorphism group of $\mathcal{N}_{02}^3(\lambda)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ y & x^2 & 0 \\ z & xy(1+\lambda) & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & (2-\lambda)\alpha_2 \\ \alpha_1 & \lambda\alpha_2 & 0 \\ \lambda\alpha_2 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & (2+2\lambda-\lambda^2)x^2y\alpha_2 & (2-\lambda)x^4\alpha_2 \\ x^3\alpha_1 + \lambda(2+\lambda)x^2y\alpha_2 & \lambda x^4\alpha_2 & 0 \\ \lambda x^4\alpha_2 & 0 & 0 \end{pmatrix},$$

the action of $\mathrm{Aut}(\mathcal{N}^3_{02}(\lambda))$ on the subspace $\langle \sum_{i=1}^{2} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{2} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{rcl} \alpha_1^* & = & x^3\alpha_1 + \lambda^2(\lambda-1)x^2y\alpha_2, \\ \alpha_2^* & = & x^4\alpha_2. \end{array}$$

Since 2-dimensional central extensions of two dimensional algebras were already considered, we have $\alpha_2 \neq 0$. We have the following cases:

(1) if $\lambda \neq 0, 1$, then choosing $x = \frac{1}{\sqrt[4]{\alpha_2}}$, $y = -\frac{x\alpha_1}{\lambda^2(\lambda-1)\alpha_2}$, we have the representative $\langle \nabla_2 \rangle$.
(2) if $\lambda = 0$ or $\lambda = 1$, and $\alpha_1 = 0$, then choosing $x = \frac{1}{\sqrt[4]{\alpha_2}}$, we have the representative $\langle \nabla_2 \rangle$.
(3) if $\lambda = 0$ or $\lambda = 1$, and $\alpha_1 \neq 0$, then choosing $x = \frac{\alpha_1}{\alpha_2}$, we have the representative $\langle \nabla_1 + \nabla_2 \rangle$.

Now we have all new 4-dimensional algebras constructed from $\mathcal{N}^3_{02}(\lambda)$:

$$\mathcal{N}^4_{22}(\lambda),\ \mathcal{N}^4_{23},\ \mathcal{N}^4_{24}.$$

The multiplication tables of these algebras can be found in Appendix.

2. The geometric classification of nilpotent Novikov algebras

2.1. Definitions and notation

Given an n-dimensional vector space $\mathbb{V}$, the set $\mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V}) \cong \mathbb{V}^* \otimes \mathbb{V}^* \otimes \mathbb{V}$ is a vector space of dimension n^3. This space has the structure of the affine variety $\mathbb{C}^{n^3}$. Indeed, let us fix a basis $e_1, \ldots, e_n$ of $\mathbb{V}$. Then any $\mu \in \mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ is determined by n^3 structure constants $c_{ij}^k \in \mathbb{C}$ such that $\mu(e_i \otimes e_j) = \sum_{k=1}^{n} c_{ij}^k e_k$. A subset of $\mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ is *Zariski-closed* if it can be defined by a set of polynomial equations in the variables c_{ij}^k $(1 \le i, j, k \le n)$.

Let T be a set of polynomial identities. The set of algebra structures on $\mathbb{V}$ satisfying polynomial identities from T forms a Zariski-closed subset of the variety $\mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$. We denote this subset by $\mathbb{L}(T)$. The general linear group $GL(\mathbb{V})$ acts on $\mathbb{L}(T)$ by conjugations:

$$(g * \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y)$$

for $x, y \in \mathbb{V}$, $\mu \in \mathbb{L}(T) \subset \mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ and $g \in GL(\mathbb{V})$. Thus, $\mathbb{L}(T)$ is decomposed into $GL(\mathbb{V})$-orbits that correspond to the isomorphism classes of algebras. Let $O(\mu)$ denote the orbit of $\mu \in \mathbb{L}(T)$ under the action of $GL(\mathbb{V})$ and $\overline{O(\mu)}$ denote the Zariski closure of $O(\mu)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two n-dimensional algebras satisfying the identities from T, and let $\mu, \lambda \in \mathbb{L}(T)$ represent $\mathcal{A}$ and $\mathcal{B}$, respectively. We say that $\mathcal{A}$ degenerates to $\mathcal{B}$ and write $\mathcal{A} \to \mathcal{B}$ if $\lambda \in \overline{O(\mu)}$. Note that in this case we have $\overline{O(\lambda)} \subset \overline{O(\mu)}$. Hence, the definition of a degeneration does not depend on the choice of μ and λ. If $\mathcal{A} \not\cong \mathcal{B}$, then the assertion $\mathcal{A} \to \mathcal{B}$ is called a *proper degeneration*. We write $\mathcal{A} \not\to \mathcal{B}$ if $\lambda \notin \overline{O(\mu)}$.

Let $\mathcal{A}$ be represented by $\mu \in \mathbb{L}(T)$. Then $\mathcal{A}$ is *rigid* in $\mathbb{L}(T)$ if $O(\mu)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is called irreducible if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called an *irreducible component*. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. The algebra $\mathcal{A}$ is rigid in $\mathbb{L}(T)$ if and only if $\overline{O(\mu)}$ is an irreducible component of $\mathbb{L}(T)$.

Given the spaces U and W, we write simply $U > W$ instead of $\dim U > \dim W$.

2.2. Method of the description of degenerations of algebras

In the present work we use the methods applied to Lie algebras in [8,28,29,44]. First of all, if $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \not\cong \mathcal{B}$, then $\mathfrak{Der}(\mathcal{A}) < \mathfrak{Der}(\mathcal{B})$, where $\mathfrak{Der}(\mathcal{A})$ is the Lie algebra of derivations of $\mathcal{A}$. We compute the dimensions of algebras of derivations and check the assertion $\mathcal{A} \to \mathcal{B}$ only for such $\mathcal{A}$ and $\mathcal{B}$ that $\mathfrak{Der}(\mathcal{A}) < \mathfrak{Der}(\mathcal{B})$.

To prove degenerations, we construct families of matrices parametrized by t. Namely, let $\mathcal{A}$ and $\mathcal{B}$ be two algebras represented by the structures μ and λ from $\mathbb{L}(T)$ respectively. Let $e_1, \ldots, e_n$ be a basis of $\mathbb{V}$ and c_{ij}^k $(1 \le i, j, k \le n)$ be the structure constants of λ in this basis. If there exist $a_i^j(t) \in \mathbb{C}$ $(1 \le i, j \le n,\ t \in \mathbb{C}^*)$ such that $E_i^t = \sum_{j=1}^{n} a_i^j(t) e_j$ $(1 \le i \le n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{C}^*$, and the structure constants of μ in the basis $E_1^t, \ldots, E_n^t$ are such rational functions $c_{ij}^k(t) \in \mathbb{C}[t]$ that $c_{ij}^k(0) = c_{ij}^k$, then $\mathcal{A} \to \mathcal{B}$. In this case $E_1^t, \ldots, E_n^t$ is called a *parametrized basis* for $\mathcal{A} \to \mathcal{B}$.

Since the variety of 4-dimensional nilpotent Novikov algebras contains infinitely many non-isomorphic algebras, we have to do some additional work. Let $\mathcal{A}(*) := \{\mathcal{A}(\alpha)\}_{\alpha \in I}$ be a series of algebras, and let $\mathcal{B}$ be another algebra. Suppose that for $\alpha \in I$, $\mathcal{A}(\alpha)$ is represented by the structure $\mu(\alpha) \in \mathbb{L}(T)$ and $B \in \mathbb{L}(T)$ is represented by the structure λ. Then we say that $\mathcal{A}(*) \to \mathcal{B}$ if $\lambda \in \overline{\{O(\mu(\alpha))\}_{\alpha \in I}}$, and $\mathcal{A}(*) \not\to \mathcal{B}$ if $\lambda \notin \overline{\{O(\mu(\alpha))\}_{\alpha \in I}}$.

Let $\mathcal{A}(*)$, $\mathcal{B}$, $\mu(\alpha)$ $(\alpha \in I)$ and λ be as above. To prove $\mathcal{A}(*) \to \mathcal{B}$ it is enough to construct a family of pairs $(f(t), g(t))$ parametrized by $t \in \mathbb{C}^*$, where $f(t) \in I$ and $g(t) \in GL(\mathbb{V})$. Namely, let $e_1, \ldots, e_n$ be a basis of $\mathbb{V}$ and c_{ij}^k $(1 \le i, j, k \le n)$

be the structure constants of λ in this basis. If we construct $a_i^j : \mathbb{C}^* \to \mathbb{C}$ $(1 \le i, j \le n)$ and $f : \mathbb{C}^* \to I$ such that $E_i^t = \sum_{j=1}^n a_i^j(t) e_j$ $(1 \le i \le n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{C}^*$, and the structure constants of $\mu_{f(t)}$ in the basis $E_1^t, \ldots, E_n^t$ are such rational functions $c_{ij}^k(t) \in \mathbb{C}[t]$ that $c_{ij}^k(0) = c_{ij}^k$, then $\mathcal{A}(*) \to \mathcal{B}$. In this case $E_1^t, \ldots, E_n^t$ and $f(t)$ are called a parametrized basis and a *parametrized index* for $\mathcal{A}(*) \to \mathcal{B}$, respectively.

We now explain how to prove $\mathcal{A}(*) \not\to \mathcal{B}$. Note that if $\mathfrak{Der}\ \mathcal{A}(\alpha) > \mathfrak{Der}\ \mathcal{B}$ for all $\alpha \in I$ then $\mathcal{A}(*) \not\to \mathcal{B}$. One can use also the following generalization of Lemma from [29], whose proof is the same as the proof of Lemma.

Lemma 7. *Let $\mathfrak{B}$ be a Borel subgroup of $GL(\mathbb{V})$ and $\mathcal{R} \subset \mathbb{L}(T)$ be a $\mathfrak{B}$-stable closed subset. If $\mathcal{A}(*) \to \mathcal{B}$ and for any $\alpha \in I$ the algebra $\mathcal{A}(\alpha)$ can be represented by a structure $\mu(\alpha) \in \mathcal{R}$, then there is $\lambda \in \mathcal{R}$ representing $\mathcal{B}$.*

2.3. The geometric classification of 4-dimensional nilpotent Novikov algebras

The main result of the present section is the following theorem.

Theorem 8. *The variety of 4-dimensional nilpotent Novikov algebras has two irreducible components defined by infinite families of algebras $\mathcal{N}_{20}^4(\alpha)$ and $\mathcal{N}_{22}^4(\lambda)$.*

Proof. Recall that the full description of the degeneration system of 4-dimensional non-pure Novikov algebras was given in [39]. Using the cited result, we can see that the variety of 4-dimensional non-pure Novikov algebras has two irreducible components given by the following families of algebras:

$$\begin{array}{llllll} \mathfrak{N}_2(\alpha) & e_1e_1 = e_3, & e_1e_2 = e_4, & e_2e_1 = -\alpha e_3, & e_2e_2 = -e_4 & \\ \mathfrak{N}_3(\alpha) & e_1e_1 = e_4, & e_1e_2 = \alpha e_4, & e_2e_1 = -\alpha e_4, & e_2e_2 = e_4, & e_3e_3 = e_4. \end{array}$$

Now we can prove that the variety of 4-dimensional nilpotent Novikov algebras has two irreducible components. One can easily compute that

$$\mathfrak{Der}\ \mathcal{N}_{20}^4(\alpha) = 3, \ \mathfrak{Der}\ \mathcal{N}_{22}^4(\lambda)_{\lambda \neq 0,1} = 3.$$

Since the dimensions of derivations of these algebras are the smallest possible in this variety, the families of algebras $\mathcal{N}_{20}^4(\alpha)$ and $\mathcal{N}_{22}^4(\lambda)$ give two irreducible components. The list of all necessary degenerations is given in Table B (see Appendix). □

Table A
The list of 4-dimensional nilpotent "pure" Novikov algebras.

$\mathcal{N}_{01}^4$	:	$e_1e_1 = e_2$	$e_2e_1 = e_3$				
$\mathcal{N}_{02}^4(\lambda)$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_2e_1 = \lambda e_3$			
$\mathcal{N}_{03}^4$	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_3$			
$\mathcal{N}_{04}^4(\alpha)$	:	$e_1e_1 = e_2,$	$e_1e_2 = e_4,$	$e_2e_1 = \alpha e_4,$	$e_3e_3 = e_4,$		
$\mathcal{N}_{05}^4$	:	$e_1e_1 = e_2,$	$e_1e_2 = e_4,$	$e_1e_3 = e_4,$	$e_2e_1 = e_4,$	$e_3e_3 = e_4$	
$\mathcal{N}_{06}^4(\alpha)_{\alpha \neq 0}$	:	$e_1e_1 = e_2,$	$e_1e_2 = e_4,$	$e_1e_3 = e_4,$	$e_2e_1 = \alpha e_4$		
$\mathcal{N}_{07}^4$	:	$e_1e_1 = e_2,$	$e_2e_1 = e_4,$	$e_3e_3 = e_4$			
$\mathcal{N}_{08}^4$	:	$e_1e_1 = e_2,$	$e_1e_3 = e_4,$	$e_2e_1 = e_4$			
$\mathcal{N}_{09}^4$	:	$e_1e_1 = e_2,$	$e_1e_2 = e_4,$	$e_3e_1 = e_4$			
$\mathcal{N}_{10}^4$	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$				
$\mathcal{N}_{11}^4$	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4,$			
$\mathcal{N}_{12}^4$	:	$e_1e_2 = e_3$	$e_2e_3 = e_4$	$e_3e_2 = -e_4,$			
$\mathcal{N}_{13}^4$	:	$e_1e_2 = e_3$	$e_1e_1 = e_4$	$e_2e_3 = e_4$	$e_3e_2 = -e_4,$		
$\mathcal{N}_{14}^4$	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_3 = e_4$	$e_3e_2 = -e_4,$		
$\mathcal{N}_{15}^4$	:	$e_1e_2 = e_3$	$e_1e_1 = e_4$	$e_1e_3 = e_4$	$e_2e_3 = e_4$	$e_3e_2 = -e_4,$	
$\mathcal{N}_{16}^4$	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_4,$			
$\mathcal{N}_{17}^4$	:	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_2e_2 = e_4,$		
$\mathcal{N}_{18}^4$	:	$e_1e_2 = e_3$	$e_2e_2 = e_4$	$e_2e_3 = e_4$	$e_3e_2 = -e_4,$		
$\mathcal{N}_{19}^4$	:	$e_1e_2 = e_3$	$e_1e_1 = e_4$	$e_2e_2 = e_4$	$e_2e_3 = e_4$	$e_3e_2 = -e_4,$	
$\mathcal{N}_{20}^4(\alpha)$	:	$e_1e_2 = e_3$	$e_1e_1 = \alpha e_4$	$e_1e_3 = e_4$	$e_2e_2 = e_4$	$e_2e_3 = e_4$	$e_3e_2 = -e_4,$
$\mathcal{N}_{21}^4$	:	$e_1e_1 = e_2,$	$e_2e_1 = e_3,$	$e_1e_3 = e_4,$	$e_3e_1 = -e_4$		
$\mathcal{N}_{22}^4(\lambda)$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = (2-\lambda)e_4$	$e_2e_1 = \lambda e_3$	$e_2e_2 = \lambda e_4$	$e_3e_1 = \lambda e_4$
$\mathcal{N}_{23}^4$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = 2e_4$	$e_2e_1 = e_4$		
$\mathcal{N}_{24}^4$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_4$	$e_3e_1 = e_4,$

Table B
Degenerations of 4-dimensional nilpotent Novikov algebras.

$\mathcal{N}_{14}^4$	$\to$	$\mathcal{N}_{01}^4$	$E_1^t = t^{-1}(e_1 - e_2), E_2^t = -t^{-2}e_3, E_3^t = -t^{-3}e_4, E_4^t = -e_2$
$\mathcal{N}_{22}^4(\lambda)$	$\to$	$\mathcal{N}_{02}^4(\lambda)$	$E_1^t = e_1, E_2^t = e_2, E_3^t = e_3, E_4^t = t^{-1}e_4$
$\mathcal{N}_{23}^4$	$\to$	$\mathcal{N}_{03}^4$	$E_1^t = te_1, E_2^t = t^2e_2, E_3^t = t^3e_4, E_4^t = t^3e_3$
$\mathcal{N}_{20}^4(-\frac{\beta}{(\beta+1)^2})$	$\to$	$\mathcal{N}_{04}^4(\beta)$	$E_1^t = t^2(-\beta e_1 + \frac{\beta^2}{\beta+1}e_2 - \frac{\beta^2}{\beta+1}e_3 + \frac{\beta^3}{\beta+1}e_4)$ $E_2^t = t^4(-\frac{\beta^3}{\beta+1}e_3 + \frac{2\beta^4}{(\beta+1)^2}e_4), E_3^t = t^3\frac{\beta^2}{\beta+1}e_2, E_4^t = t^6\frac{\beta^4}{(\beta+1)^2}e_4$
$\mathcal{N}_{20}^4(-\frac{1}{4} - \frac{1}{2}\sqrt[3]{\frac{t}{4}})$	$\to$	$\mathcal{N}_{05}^4$	$E_1^t = -\sqrt[3]{4t^2}e_1 + \sqrt[3]{\frac{t^2}{2}}e_2 - \sqrt[3]{\frac{t^2}{2}}e_3, E_2^t = -\sqrt[3]{2t^4}e_3 + (\sqrt[3]{2t^4} - \sqrt[3]{\frac{t^5}{2}})e_4, E_3^t = te_2, E_4^t = t^2e_4$
$\mathcal{N}_{04}^4(\alpha \neq 1)$	$\to$	$\mathcal{N}_{06}^4(\alpha)$	$E_1^t = t(e_1 - \frac{\alpha}{(\alpha-1)^2}e_2 + \frac{\alpha}{\alpha-1}e_3 + \frac{\alpha^2}{(\alpha-1)^4}e_4),$ $E_2^t = t^2(e_2 - \frac{\alpha}{(\alpha-1)^2}e_4), E_3^t = t^2(e_3 - \frac{1}{\alpha-1}e_2 + \frac{\alpha}{(\alpha-1)^3}e_4), E_4^t = t^3e_4$
$\mathcal{N}_{20}^4(0)$	$\to$	$\mathcal{N}_{07}^4$	$E_1^t = t^2(-e_1 + e_2 - e_3 + e_4), E_2^t = t^4(-e_3 + 2e_4), E_3^t = -t^3e_2, E_4^t = t^6e_4$
$\mathcal{N}_{23}^4$	$\to$	$\mathcal{N}_{08}^4$	$E_1^t = te_1, E_2^t = t^2e_2, E_3^t = \frac{1}{2}t^2e_3, E_4^t = t^3e_4$
$\mathcal{N}_{22}^4(2)$	$\to$	$\mathcal{N}_{09}^4$	$E_1^t = t(e_1 - e_2), E_2^t = t^2(e_2 - 3e_3 + 2e_4), E_3^t = t^2(e_3 - 4e_4), E_4^t = 2t^3e_4$
$\mathcal{N}_{11}^4$	$\to$	$\mathcal{N}_{10}^4$	$E_1^t = t^{-1}e_1, E_2^t = t^{-2}e_2, E_3^t = t^{-3}e_3, E_4^t = t^{-4}e_4$
$\mathcal{N}_{20}^4(t-1)$	$\to$	$\mathcal{N}_{11}^4$	$E_1^t = t(e_1 + e_3 - e_4), E_2^t = te_2, E_3^t = t(e_3 - e_4), E_4^t = te_4$
$\mathcal{N}_{13}^4$	$\to$	$\mathcal{N}_{12}^4$	$E_1^t = te_1, E_2^t = e_2, E_3^t = te_3, E_4^t = te_4$
$\mathcal{N}_{15}^4$	$\to$	$\mathcal{N}_{13}^4$	$E_1^t = t^2e_1, E_2^t = te_2, E_3^t = t^3e_3, E_4^t = t^4e_4$
$\mathcal{N}_{15}^4$	$\to$	$\mathcal{N}_{14}^4$	$E_1^t = t^{-1}e_1, E_2^t = t^{-1}e_2, E_3^t = t^{-2}e_3, E_4^t = t^{-3}e_4$
$\mathcal{N}_{20}^4(\frac{1}{t})$	$\to$	$\mathcal{N}_{15}^4$	$E_1^t = t^{-1}e_1, E_2^t = t^{-1}e_2, E_3^t = t^{-2}e_3, E_4^t = t^{-3}e_4$
$\mathcal{N}_{17}^4$	$\to$	$\mathcal{N}_{16}^4$	$E_1^t = t^{-1}e_1, E_2^t = t^{-2}e_2, E_3^t = t^{-3}e_3, E_4^t = t^{-4}e_4$
$\mathcal{N}_{20}^4(t^3 - t)$	$\to$	$\mathcal{N}_{17}^4$	$E_1^t = te_1 + t^2e_3 - t^3e_4, E_2^t = t^2e_2, E_3^t = t^3e_3 - t^4e_4, E_4^t = t^4e_4$
$\mathcal{N}_{19}^4$	$\to$	$\mathcal{N}_{18}^4$	$E_1^t = e_1, E_2^t = t^{-1}e_2, E_3^t = t^{-1}e_3, E_4^t = t^{-2}e_4$
$\mathcal{N}_{20}^4(\frac{1}{t^2})$	$\to$	$\mathcal{N}_{19}^4$	$E_1^t = e_1, E_2^t = t^{-1}e_2, E_3^t = t^{-1}e_3, E_4^t = t^{-2}e_4$
$\mathcal{N}_{22}^4(\frac{1}{t})$	$\to$	$\mathcal{N}_{21}^4$	$E_1^t = e_1, E_2^t = e_2, E_3^t = \frac{1}{t}e_3, E_4^t = -\frac{1}{t^2}e_4$
$\mathcal{N}_{22}^4(t)$	$\to$	$\mathcal{N}_{23}^4$	$E_1^t = e_1 + \frac{1}{t^2(t-1)}e_2, E_2^t = e_2 + \frac{t+1}{t^2(t-1)}e_3 + \frac{1}{t^3(t-1)^2}e_4, E_3^t = e_3 + \frac{2+2t-t^2}{t^2(t-1)}e_4, E_4^t = e_4$
$\mathcal{N}_{22}^4(t+1)$	$\to$	$\mathcal{N}_{24}^4$	$E_1^t = e_1 + \frac{1}{t(t+1)^2}e_2, E_2^t = e_2 + \frac{t+2}{t(t+1)^2}e_3 + \frac{1}{t^3(t+1)^2}e_4, E_3^t = e_3 + \frac{3-t^2}{t(t+1)^2}e_4, E_4^t = e_4$
$\mathcal{N}_{17}^4$	$\to$	$\mathfrak{N}_2(\alpha)_{\alpha\neq 0,1}$	$E_1^t = -\frac{\sqrt{1-\alpha}}{\alpha}te_1 + \frac{\sqrt{1-\alpha}}{\alpha}te_2, E_2^t = \sqrt{1-\alpha}te_1 + \frac{t}{\sqrt{1-\alpha}}e_3, E_3^t = \frac{\alpha-1}{\alpha^2}t^2e_3, E_4^t = -t^2e_4$
$\mathcal{N}_{04}^4(\alpha \neq -1)$	$\to$	$\mathfrak{N}_3(i\frac{\alpha-1}{\alpha+1})$	$E_1^t = t\sqrt[3]{\frac{2}{\alpha+1}}e_1, E_2^t = it\left(\sqrt[3]{\frac{2}{\alpha+1}}e_1 - \sqrt[3]{\frac{2}{\alpha+1}}^2e_2 + e_4\right), E_3^t = te_3, E_4^t = t^2e_4$

Appendix

See Tables A and B.

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