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Solvable Leibniz algebras with triangular nilradicals

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ABSTRACT

In this paper the description of solvable Lie algebras with triangular nilradicals is extended to Leibniz algebras. It is proven that the matrices of the left and the right operators on the elements of Leibniz algebra have the upper triangular forms. We establish that solvable Leibniz algebra of a maximal possible dimension with a given triangular nilradical is a Lie algebra. Furthermore, solvable Leibniz algebras with triangular nilradicals of the low dimensions are classified.

1. Introduction

Leibniz algebras were introduced at the beginning of the 90s of the past century by J.-L. Loday [3]. They are a generalization of well-known Lie algebras, which admit a remarkable property that an operator of the right multiplication is a derivation.

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From the classical theory of Lie algebras it is well known that the study of finite-dimensional Lie algebras was reduced to the nilpotent ones [11,12]. In the Leibniz algebra case there is an analogue of Levi's theorem [4]. Namely, the decomposition of a Leibniz algebra into a semidirect sum of its solvable radical and a semisimple Lie algebra is obtained. The semisimple part can be described from simple Lie ideals (see [5]) and therefore, the main focus is to study the solvable radical.

The analysis of several works devoted to the study of solvable Lie algebras (for example [1,2,13–15], where solvable Lie algebras with various types of the nilradical were studied, such as naturally graded filiform and quasi-filiform algebras, Abelian, triangular, etc.) shows that we can also apply similar methods to solvable Leibniz algebras with a given nilradical. In fact, any solvable Lie algebra can be represented as an algebraic sum of a nilradical and its complimentary vector space. Mubarakzjanov proposed a method, which claims that the dimension of the complimentary vector space does not exceed the number of nil-independent derivations of the nilradical [12]. Extension of this method to Leibniz algebras is shown in [6]. Usage of this method yields a classification of solvable Leibniz algebras with the given nilradicals in [6–10].

In this article we present the description of solvable Leibniz algebras whose nilradical is a Lie algebra of upper triangular matrices. Since in the work [14] solvable Lie algebras with the triangular nilradical are studied, we reduce our study to non-Lie Leibniz algebras.

Recall that in [14] solvable Lie algebras with the triangular nil-radicals of minimum and maximum possible dimensions were described. Moreover, uniqueness of a Lie algebra of maximal possible dimension with the given triangular nilradical is established.

In order to realize a goal of our study we organize the paper as follows. In Section 2 we give the necessary preliminary results. Section 3 is devoted to the description of finite-dimensional solvable Leibniz algebras with the upper triangular nilradical. We establish that such Leibniz algebras of minimum and maximum possible dimensions are Lie algebras. Finally, in Section 4 we present the complete description of the results of Section 3 in the low dimensions.

Throughout the paper we consider finite-dimensional vector spaces and algebras over the field $\mathbb{C}$. Moreover, in a multiplication table of an algebra omitted products are assumed to be zero and if it is not stated otherwise, we will consider non-nilpotent solvable algebras.

2. Preliminaries

In this section we give the basic concepts and the results used in the studying of Leibniz algebras with the triangular nilradicals.

Definition 2.1. An algebra $(L, [-,-])$ over a field F is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity

$$\big[x,[y,z]\big] = \big[[x,y],z\big] - \big[[x,z],y\big]$$

holds.

Every Lie algebra is a Leibniz algebra, but the bracket in the Leibniz algebra does not possess a skew-symmetric property.

Definition 2.2. For a given Leibniz algebra L the sequences of two-sided ideals defined recursively as follows:

$$L^1 = L, \qquad L^{k+1} = \big[L^k, L\big], \quad k \geq 1, \qquad L^{[1]} = L, \qquad L^{[s+1]} = \big[L^{[s]}, L^{[s]}\big], \quad s \geq 1$$

are called the lower central and the derived series of L, respectively.

Definition 2.3. A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists $n \in \mathbb{N}$ (respectively, $m \in \mathbb{N}$) such that $L^n = 0$ (respectively, $L^{[m]} = 0$).

It is easy to see that a sum of any two nilpotent ideals is nilpotent. Therefore the maximal nilpotent ideal always exists.

Definition 2.4. The maximal nilpotent ideal of a Leibniz algebra is said to be a nilradical of the algebra.

Recall that a linear map $d : L \to L$ of a Leibniz algebra L is called a derivation if for all $x, y \in L$ the following condition holds:

$$d\big([x,y]\big) = \big[d(x),y\big] + \big[x,d(y)\big].$$

For a given element x of a Leibniz algebra L we consider a right multiplication operator $R_x : L \to L$ defined by $R_x(y) = [y,x]$, $\forall y \in L$ and a left multiplication operator $L_x : L \to L$ defined by $L_x(y) = [x,y]$, $\forall y \in L$. It is easy to check that the operator R_x is a derivation. Derivations of this kind are called *inner derivations*.

Linear maps $f_1, \ldots, f_k$ are called *nil-independent*, if

$$\alpha_1 f_1 + \alpha_2 f_2 + \ldots + \alpha_k f_k$$

is not nilpotent for all values α_i, except simultaneously zero.

Let R be a solvable Leibniz algebra with a nilradical N. We denote by Q the complementary vector space of the nilradical N in the algebra R.

Proposition 2.5. *(See [6].) Let R be a solvable Leibniz algebra and N – its nilradical. Then the dimension of the complementary vector space Q is not greater than the maximal number of nil-independent derivations of N.*

Let us consider a finite-dimensional Lie algebra $T(n)$ of the upper-triangular matrices with $n \geq 3$ over the field of complex numbers. The products of the basis elements $\{N_{ij} \mid 1 \leq i < j \leq n\}$ of $T(n)$, where N_{ij} is a matrix with the only non-zero entry at i-th row and j-th column equal to 1, are computed by a rule

$$[N_{ij}, N_{kl}] = \delta_{jk} N_{il} - \delta_{il} N_{kj}. \tag{1}$$

For a natural number f let $G(n, f)$ be a set of solvable Lie algebras of dimension $\frac{1}{2}n(n-1) + f$ with a nilradical $T(n)$. Let $Q = \langle X^1, X^2, \ldots, X^f \rangle$, where Q is the complementary vector space of the nilradical $T(n)$ to an algebra from $G(n, f)$.

Denote

$$\left[N_{ij}, X^\alpha\right] = \sum_{1 \leq q-p < n} a^\alpha_{ij,pq} N_{pq}, \qquad \left[X^\alpha, N_{ij}\right] = \sum_{1 \leq q-p < n} b^\alpha_{ij,pq} N_{pq},$$
$$\left[X^\alpha, X^\beta\right] = \sum_{1 \leq q-p < n} \sigma^{\alpha\beta}_{pq} N_{pq}, \tag{2}$$

where $1 \leq \alpha, \beta \leq f$ and $a^\alpha_{ij,pq}, b^\alpha_{ij,pq}, \sigma^{\alpha\beta}_{pq} \in \mathbb{C}$, $p < q \leq n$.

Let N be a vector column $(N_{12}N_{23} \ldots N_{(n-1)n}N_{13}N_{24} \ldots N_{(n-2)n} \ldots N_{1n})^T$ then we have

$$R_{X^\alpha}(N) = A^\alpha N, \qquad L_{X^\alpha}(N) = B^\alpha N,$$

where $A^\alpha = (a^\alpha_{ij,pq})$ and $B^\alpha = (b^\alpha_{ij,pq})$, $1 \leq i < j \leq n$, $1 \leq p < q \leq n$ are $\frac{1}{2}n(n-1) \times \frac{1}{2}n(n-1)$ complex matrices.

The following lemma provides some information about the structure matrices above.

Lemma 2.6. *(See [14].) The structure matrices $A^\alpha = (a^\alpha_{ij,pq})$, $1 \leq i < j \leq n$, $1 \leq p < q \leq n$ have the following properties:*

(*i*) *They are upper triangular;*
(*ii*) *The only off-diagonal matrix elements that do not vanish identically and cannot be annulled by a redefinition of the elements X^α are:*

$$a^\alpha_{12,2n}, \quad a^\alpha_{i(i+1),1n} \quad (2 \leq i \leq n-2), \quad a^\alpha_{(n-1)n,1(n-1)};$$

(*iii*) *The diagonal elements $a^\alpha_{i(i+1),i(i+1)}$, $1 \leq i \leq n-1$ are arbitrary. The other diagonal elements satisfy*

$$a^\alpha_{ik,ik} = \sum_{p=i}^{k-1} a^\alpha_{p(p+1),p(p+1)}, \quad k > i+1.$$

Lemma 2.7. *(See [14].) The maximal number of non-nilpotent elements is*

$$f_{max} = n - 1.$$

3. Main result

We denote by $L(n, f)$ a set of all non-nilpotent solvable Leibniz algebras with a nilradical $T(n)$ and a complementary vector space $\langle X^1, X^2, \dots, X^f\rangle$.

For the brevity, further we shall not write the products $[N_{ij}, N_{kl}]$ which are easily obtained from (1).

Using the notations (2) we have

$$R_{X^\alpha}(N) = A^\alpha N, \qquad L_{X^\alpha}(N) = B^\alpha N,$$

where $A^\alpha = (a^\alpha_{ij,pq})$, $B^\alpha = (b^\alpha_{ij,pq})$, $1 \le i < j \le n$, $1 \le p < q \le n$.

Since the proof of the assertions concerning the elements of the matrix A^α in Lemma 2.6 uses only the property of derivation, one can check that it obviously extends to our case of the Leibniz algebras. For the matrix B^α however, we have the next result.

Lemma 3.1. *The following relations hold:*

$$b^\alpha_{ij,pq} = -a^\alpha_{ij,pq}, \quad i+1<j,\ (p,q) \ne (1,n)$$

Proof. From Lemma 2.6 we conclude

$$\begin{aligned}
[N_{12}, X^\alpha] &= a^\alpha_{12,12} N_{12} + a^\alpha_{12,2n} N_{2n}, \\
[N_{i(i+1)}, X^\alpha] &= a^\alpha_{i(i+1),i(i+1)} N_{i(i+1)} + a^\alpha_{i(i+1),1n} N_{1n}, \quad 2 \le i \le n-2, \\
[N_{(n-1)n}, X^\alpha] &= a^\alpha_{(n-1)n,(n-1)n} N_{(n-1)n} + a^\alpha_{(n-1)n,1(n-1)} N_{1(n-1)}, \\
[N_{ij}, X^\alpha] &= \sum_{p=i}^{j-1} a^\alpha_{p(p+1),p(p+1)} N_{ij}, \quad i+1<j.
\end{aligned}$$

It is easy to see that $[X^\alpha, N_{12}] + [N_{12}, X^\alpha]$ belongs to the right annihilator of the algebra of $L(n, f)$. From the chain of equalities

$$\begin{aligned}
0 = \big[N_{12}, [X^\alpha, N_{12}] + [N_{12}, X^\alpha]\big] &= \left[N_{12}, \sum_{i=3}^{n-1} b^\alpha_{12,2i} N_{2i} + \big(a^\alpha_{12,2n} + b^\alpha_{12,2n}\big) N_{2n}\right] \\
&= \sum_{i=3}^{n-1} b^\alpha_{12,2i} N_{1i} + \big(a^\alpha_{12,2n} + b^\alpha_{12,2n}\big) N_{1n},
\end{aligned}$$

we deduce $b^\alpha_{12,2j} = 0$, $3 \le j \le n-1$ and $b^\alpha_{12,2n} = -a^\alpha_{12,2n}$.

Similarly, from

$$0 = \left[N_{1i}, \left[X^\alpha, N_{12}\right] + \left[N_{12}, X^\alpha\right]\right] = \left[N_{1i}, \sum_{j=i+1}^{n} b_{ij}^\alpha N_{ij}\right] = \sum_{j=i+1}^{n} b_{ij}^\alpha N_{1j}, \quad i > 2,$$

we derive $b_{12,ij}^\alpha = 0$, $2 < i < j \leq n$.

From the equality

$$0 = \left[N_{i(i+1)}, \left[X^\alpha, N_{12}\right] + \left[N_{12}, X^\alpha\right]\right], \quad i \geq 2,$$

we get

$$b_{12,12}^\alpha = -a_{12,12}^\alpha, \qquad b_{12,1i}^\alpha = 0, \quad 3 \leq i \leq n-1.$$

Therefore, we obtain

$$\left[X^\alpha, N_{12}\right] = -a_{12,12}^\alpha N_{12} - a_{12,2n}^\alpha N_{2n} + b_{12,1n}^\alpha N_{1n}.$$

Applying analogous argumentations as we used above for the products with $k \geq 2$,

$$\begin{gathered}
\left[N_{1k}, \left[X^\alpha, N_{i(i+1)}\right] + \left[N_{i(i+1)}, X^\alpha\right]\right], \\
\left[N_{i(i+1)}, \left[X^\alpha, N_{i(i+1)}\right] + \left[N_{i(i+1)}, X^\alpha\right]\right], \quad 2 \leq i \leq n-2, \\
\left[N_{1k}, \left[X^\alpha, N_{(n-1)n}\right] + \left[N_{(n-1)n}, X^\alpha\right]\right], \quad \left[N_{i(i+1)}, \left[X^\alpha, N_{(n-1)n}\right] + \left[N_{(n-1)n}, X^\alpha\right]\right], \\
\left[N_{1k}, \left[X^\alpha, N_{ij}\right] + \left[N_{ij}, X^\alpha\right]\right], \quad \left[N_{i(i+1)}, \left[X^\alpha, N_{ij}\right] + \left[N_{ij}, X^\alpha\right]\right], \quad 1 < j-i < n-1, \\
\left[N_{1k}, \left[X^\alpha, N_{1n}\right] + \left[N_{1n}, X^\alpha\right]\right], \quad \left[N_{i(i+1)}, \left[X^\alpha, N_{1n}\right] + \left[N_{1n}, X^\alpha\right]\right],
\end{gathered}$$

we obtain

$$\begin{aligned}
\left[X^\alpha, N_{i(i+1)}\right] &= -a_{i(i+1),i(i+1)}^\alpha N_{i(i+1)} + b_{i(i+1),1n}^\alpha N_{1n}, \quad 2 \leq i \leq n-2, \\
\left[X^\alpha, N_{(n-1)n}\right] &= -a_{(n-1)n,(n-1)n}^\alpha N_{(n-1)n} - a_{(n-1)n,1(n-1)}^\alpha N_{1(n-1)} + b_{(n-1)n,1n}^\alpha N_{1n}, \\
\left[X^\alpha, N_{ij}\right] &= -\sum_{p=i}^{j-1} a_{p(p+1),p(p+1)}^\alpha N_{ij} + b_{ij,1n}^\alpha N_{1n}, \quad 1 < j-i < n-1, \\
\left[X^\alpha, N_{1n}\right] &= b_{1n,1n}^\alpha N_{1n}.
\end{aligned}$$

From the chain of equalities

$$\begin{aligned}
\left[X^\alpha, N_{1n}\right] &= \left[X^\alpha, [N_{12}, N_{2n}]\right] = \left[\left[X^\alpha, N_{12}\right], N_{2n}\right] - \left[\left[X^\alpha, N_{2n}\right], N_{12}\right] \\
&= \left[-a_{12,12}^\alpha N_{12} - a_{12,2n}^\alpha N_{2n} + b_{12,1n}^\alpha N_{1n}, N_{2n}\right]
\end{aligned}$$

$$- \left[-\sum_{p=2}^{n-1} a^{\alpha}_{p(p+1),p(p+1)} N_{2n} + b^{\alpha}_{2n,1n} N_{1n}, N_{12} \right]$$

$$= -a^{\alpha}_{12,12} N_{1n} - \sum_{p=2}^{n-1} a^{\alpha}_{p(p+1),p(p+1)} N_{1n} = -\sum_{p=1}^{n-1} a^{\alpha}_{p(p+1),p(p+1)} N_{1n},$$

we get $[X^{\alpha}, N_{1n}] = -\sum_{p=1}^{n-1} a^{\alpha}_{p(p+1),p(p+1)} N_{1n}$.

By induction on j we will prove

$$\left[X^{\alpha}, N_{i(i+j)}\right] = -\sum_{p=i}^{i+j-1} a^{\alpha}_{p(p+1),p(p+1)} N_{i(i+j)}, \quad j - i \geq 2. \tag{3}$$

By the induction hypothesis, the following equalities hold

$$\begin{aligned}\left[X^{\alpha}, N_{i(i+2)}\right] &= \left[X^{\alpha}, [N_{i(i+1)}, N_{(i+1)(i+2)}]\right] \\ &= \left[\left[X^{\alpha}, N_{i(i+1)}\right], N_{(i+1)(i+2)}\right] - \left[\left[X^{\alpha}, N_{(i+1)(i+2)}\right], N_{i(i+1)}\right] \\ &= -\sum_{p=i}^{i+1} a^{\alpha}_{p(p+1),p(p+1)} N_{i(i+2)}, \quad 1 \leq i \leq n-2.\end{aligned}$$

Let us suppose that (3) holds for j and we will show it for $j + 1$.

For $i + j + 1 \leq n - 1$ we have

$$\begin{aligned}\left[X^{\alpha}, N_{i(i+j+1)}\right] &= \left[X^{\alpha}, [N_{i(i+j)}, N_{(i+j)(i+j+1)}]\right] \\ &= \left[\left[X^{\alpha}, N_{i(i+j)}\right], N_{(i+j)(i+j+1)}\right] - \left[\left[X^{\alpha}, N_{(i+j)(i+j+1)}\right], N_{i(i+j)}\right] \\ &= \left[-\sum_{p=i}^{i+j-1} a^{\alpha}_{p(p+1),p(p+1)} N_{i(i+j)}, N_{(i+j)(i+j+1)} \right] \\ &\quad - \left[-a^{\alpha}_{(i+j)(i+j+1),(i+j)(i+j+1)} N_{(i+j)(i+j+1)} \right. \\ &\quad \left. + b^{\alpha}_{(i+j)(i+j+1),1n} N_{1n}, N_{i(i+j)} \right] \\ &= -\sum_{p=i}^{i+j} a^{\alpha}_{p(p+1),p(p+1)} N_{i(i+j+1)}.\end{aligned}$$

The following chain of equalities completes the proof of the equality (3)

$$\begin{aligned}\left[X^{\alpha}, N_{in}\right] &= \left[X^{\alpha}, [N_{i(n-1)}, N_{(n-1)n}]\right] \\ &= \left[X^{\alpha}, N_{i(n-1)}\right], N_{(n-1)n}] - \left[X^{\alpha}, N_{(n-1)n}\right], N_{i(n-1)}] \\ &= \left[-\sum_{p=i}^{n-2} a^{\alpha}_{p(p+1),p(p+1)} N_{i(n-1)}, N_{(n-1)n} \right]\end{aligned}$$

$$
\begin{aligned}
&- \big[-a^{\alpha}_{(n-1)n,(n-1)n} N_{(n-1)n} - a^{\alpha}_{(n-1)n,1(n-1)} N_{1(n-1)} \\
&+ b^{\alpha}_{(n-1)n,1n} N_{1n}, N_{i(n-1)}\big] \\
&= -\sum_{p=i}^{n-1} a^{\alpha}_{p(p+1),p(p+1)} N_{in}.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
\big[X^{\alpha}, N_{12}\big] &= -a^{\alpha}_{12,12} N_{12} - a^{\alpha}_{12,2n} N_{2n} + b^{\alpha}_{12,1n} N_{1n}. \\
\big[X^{\alpha}, N_{i(i+1)}\big] &= -a^{\alpha}_{i(i+1),i(i+1)} N_{i(i+1)} + b^{\alpha}_{i(i+1),1n} N_{1n}, \quad 2 \le i \le n-2, \\
\big[X^{\alpha}, N_{(n-1)n}\big] &= -a^{\alpha}_{(n-1)n,(n-1)n} N_{(n-1)n} - a^{\alpha}_{(n-1)n,1(n-1)} N_{1(n-1)} + b^{\alpha}_{(n-1)n,1n} N_{1n}, \\
\big[X^{\alpha}, N_{ij}\big] &= -\sum_{p=i}^{j-1} a^{\alpha}_{p(p+1),p(p+1)} N_{ij}, \quad j > i+1.
\end{aligned}
$$

A comparison of the above products with the notations in (2) completes the proof of the lemma. □

Lemma 3.2. *For $1 \le \alpha, \beta \le n$ we have $[X^{\alpha}, X^{\beta}] = \sigma^{\alpha\beta} N_{1n}$ for some $\sigma^{\alpha\beta} \in \mathbb{C}$.*

Proof. Consider

$$
\begin{aligned}
\big[N_{12}, \big[X^{\alpha}, X^{\beta}\big]\big] &= \big[\big[N_{12}, X^{\alpha}\big], X^{\beta}\big] - \big[\big[N_{12}, X^{\beta}\big], X^{\alpha}\big] \\
&= \big[a^{\alpha}_{12,12} N_{12} + a^{\alpha}_{12,2n} N_{2n}, X^{\beta}\big] - \big[a^{\beta}_{12,12} N_{12} + a^{\beta}_{12,2n} N_{2n}, X^{\alpha}\big] \\
&= a^{\alpha}_{12,12}\big(a^{\beta}_{12,12} N_{12} + a^{\beta}_{12,2n} N_{2n}\big) + a^{\alpha}_{12,2n}\left(\sum_{p=2}^{n-1} a^{\beta}_{p(p+1),p(p+1)} N_{2n}\right) \\
&\quad - a^{\beta}_{12,12}\big(a^{\alpha}_{12,12} N_{12} + a^{\alpha}_{12,2n} N_{2n}\big) - a^{\beta}_{12,2n}\left(\sum_{p=2}^{n-1} a^{\alpha}_{p(p+1),p(p+1)} N_{2n}\right) \\
&= \Bigg(a^{\alpha}_{12,12} a^{\beta}_{12,2n} - a^{\beta}_{12,12} a^{\alpha}_{12,2n} - \sum_{p=2}^{n-1} a^{\alpha}_{p(p+1),p(p+1)} a^{\beta}_{12,2n} \\
&\quad + \sum_{p=2}^{n-1} a^{\beta}_{p(p+1),p(p+1)} a^{\alpha}_{12,2n}\Bigg) N_{2n}.
\end{aligned}
$$

On the other hand,

$$
\big[N_{12}, \big[X^{\alpha}, X^{\beta}\big]\big] = \left[N_{12}, \sum_{1 \le q-p < n} \sigma^{\alpha\beta}_{pq} N_{pq}\right] = \sum_{i=3}^{n} \sigma^{\alpha\beta}_{2i} N_{1i}.
$$

Comparing coefficients at the basis elements we derive

$$\sigma_{2i}^{\alpha\beta} = 0, \quad 3 \le i \le n.$$

For $2 \le i \le n-2$ we consider the chain of equalities

$$\begin{aligned}
\big[N_{i(i+1)}, \big[X^\alpha, X^\beta\big]\big] &= \big[\big[N_{i(i+1)}, X^\alpha\big], X^\beta\big] - \big[\big[N_{i(i+1)}, X^\beta\big], X^\alpha\big] \\
&= a_{i(i+1),i(i+1)}^{\alpha}\big(a_{i(i+1),i(i+1)}^{\beta} N_{i(i+1)} + a_{i(i+1),1n}^{\beta} N_{1n}\big) \\
&\quad + a_{i(i+1),1n}^{\alpha} \sum_{p=1}^{n-1} a_{p(p+1),p(p+1)}^{\beta} N_{1n} \\
&\quad - a_{i(i+1),i(i+1)}^{\beta}\big(a_{i(i+1),i(i+1)}^{\alpha} N_{i(i+1)} + a_{i(i+1),1n}^{\alpha} N_{1n}\big) \\
&\quad - a_{i(i+1),1n}^{\beta} \sum_{p=1}^{n-1} a_{p(p+1),p(p+1)}^{\alpha} N_{1n} \\
&= \Bigg(a_{i(i+1),i(i+1)}^{\alpha} a_{i(i+1),1n}^{\beta} + a_{i(i+1),1n}^{\alpha} \sum_{p=1}^{n-1} a_{p(p+1),p(p+1)}^{\beta} \\
&\quad - a_{i(i+1),i(i+1)}^{\beta} a_{i(i+1),1n}^{\alpha} - a_{i(i+1),1n}^{\beta} \sum_{p=1}^{n-1} a_{p(p+1),p(p+1)}^{\alpha}\Bigg) N_{1n}.
\end{aligned}$$

In addition, the following identity holds

$$\begin{aligned}
\big[N_{i(i+1)}, \big[X^\alpha, X^\beta\big]\big] &= \Bigg[N_{i(i+1)}, \sum_{k=1}^{i-1} \sigma_{ki}^{\alpha\beta} N_{ki} + \sum_{j=i+2}^{n} \sigma_{(i+1)j}^{\alpha\beta} N_{(i+1)j}\Bigg] \\
&= -\sum_{k=1}^{i-1} \sigma_{ki}^{\alpha\beta} N_{k(i+1)} + \sum_{j=i+2}^{n} \sigma_{(i+1)j}^{\alpha\beta} N_{ij}.
\end{aligned}$$

Therefore,

$$\sigma_{ki}^{\alpha\beta} = \sigma_{js}^{\alpha\beta} = 0, \quad 1 \le k \le i-1,\ 2 \le i \le n-2,\ 3 \le j \le n-1,\ j+1 \le s \le n$$

and

$$\big[X^\alpha, X^\beta\big] = \sigma_{1(n-1)}^{\alpha\beta} N_{1(n-1)} + \sigma_{1n}^{\alpha\beta} N_{1n}.$$

Similar arguments for the products

$$\big[N_{(n-1)n}, \big[X^\alpha, X^\beta\big]\big]$$

yield $\sigma^{\alpha\beta}_{1(n-1)} = 0$, which completes the proof of the lemma. For convenience let us omit the lower indexes of $\sigma^{\alpha\beta}_{1n}$. □

From the Leibniz identity

$$[X^\alpha, [N_{i(i+1)}, X^\alpha]] = [[X^\alpha, N_{i(i+1)}], X^\alpha] - [[X^\alpha, X^\alpha], N_{i(i+1)}]$$

for $1 \le i \le n-1$ we obtain restrictions:

$$a^\alpha_{i(i+1),i(i+1)}(a^\alpha_{i(i+1),1n} + b^\alpha_{i(i+1),1n}) = 0, \quad 2 \le i \le n-2,$$
$$a^\alpha_{12,12} b^\alpha_{12,1n} = a^\alpha_{(n-1)n,(n-1)n} b^\alpha_{(n-1)n,1n} = 0.$$

Let us list again the obtained products between the basis elements. For $1 \le \alpha \le f$ we have

$$\begin{cases} [N_{12}, X^\alpha] = a^\alpha_{12,12} N_{12} + a^\alpha_{12,2n} N_{2n}, \\ [N_{i(i+1)}, X^\alpha] = a^\alpha_{i(i+1),i(i+1)} N_{i(i+1)} + a^\alpha_{i(i+1),1n} N_{1n}, \quad 2 \le i \le n-2, \\ [N_{(n-1)n}, X^\alpha] = a^\alpha_{(n-1)n,(n-1)n} N_{(n-1)n} + a^\alpha_{(n-1)n,1(n-1)} N_{1(n-1)}, \\ [N_{ij}, X^\alpha] = \sum_{p=i}^{j-1} a^\alpha_{p(p+1),p(p+1)} N_{ij}, \quad j > i+1, \\ [X^\alpha, N_{12}] = -a^\alpha_{12,12} N_{12} - a^\alpha_{12,2n} N_{2n} + b^\alpha_{12,1n} N_{1n}, \\ [X^\alpha, N_{i(i+1)}] = -a^\alpha_{i(i+1),i(i+1)} N_{i(i+1)} + b^\alpha_{i(i+1),1n} N_{1n}, \quad 2 \le i \le n-2, \\ [X^\alpha, N_{(n-1)n}] = -a^\alpha_{(n-1)n,(n-1)n} N_{(n-1)n} - a^\alpha_{(n-1)n,1(n-1)} N_{1(n-1)} + b^\alpha_{(n-1)n,1n} N_{1n}, \\ [X^\alpha, N_{ij}] = -\sum_{p=i}^{j-1} a^\alpha_{p(p+1),p(p+1)} N_{ij}, \quad j > i+1, \\ [X^\alpha, X^\beta] = \sigma^{\alpha\beta} N_{1n}, \end{cases}$$

with restrictions on parameters:

$$a^\alpha_{i(i+1),i(i+1)}(a^\alpha_{i(i+1),1n} + b^\alpha_{i(i+1),1n}) = 0, \quad 2 \le i \le n-2,$$
$$a^\alpha_{12,12} b^\alpha_{12,1n} = a^\alpha_{(n-1)n,(n-1)n} b^\alpha_{(n-1)n,1n} = 0.$$

Note that for solvable non-Lie Leibniz algebras of the set $L(n, f)$ the following equality holds

$$[X^\gamma, N_{1n}] = [N_{1n}, X^\gamma] = 0, \quad 1 \le \gamma \le f. \tag{4}$$

Indeed, if we assume the contrary, then taking into account that $[X^\gamma, N_{1n}] = -[N_{1n}, X^\gamma]$ we can assume $[X^\gamma, N_{1n}] \ne 0$ for some $\gamma \in \{1, \dots, f\}$.

Simplifying the following products using the Leibniz identity

$$\left[X^\gamma, \left[N_{12}, X^\alpha\right] + \left[X^\alpha, N_{12}\right]\right], \quad \left[X^\gamma, \left[N_{i(i+1)}, X^\alpha\right] + \left[X^\alpha, N_{i(i+1)}\right]\right],$$
$$\left[X^\gamma, \left[N_{(n-1)n}, X^\alpha\right] + \left[X^\alpha, N_{(n-1)n}\right]\right], \quad \left[X^\gamma, \left[X^\alpha, X^\beta\right] + \left[X^\beta, X^\alpha\right]\right],$$
$$\left[X^\gamma, \left[X^\alpha, X^\alpha\right]\right],$$

we obtain

$$b^\alpha_{12,1n} = b^\alpha_{(n-1)n,1n} = \sigma^{\alpha\alpha} = 0, \qquad b^\alpha_{i(i+1),1n} = -a^\alpha_{i(i+1),1n}, \qquad \sigma^{\alpha\beta} = -\sigma^{\beta\alpha}.$$

Thus we get a Lie algebra, which is a contradiction.

Corollary 3.3. *For a Leibniz algebra of the set $L(n,1)$ the matrices of the left and right operators $A = (a_{ij,pq})$, $B = (b_{ij,pq})$ have the following properties:*

(1) The maximum number of the off-diagonal elements of the matrix A is $n-1$;
(2) The maximum number of the off-diagonal elements of the matrix B is $n+1$.

Theorem 3.4. *Any Leibniz algebra from $L(n,n-1)$ is a Lie algebra.*

Proof. Making a suitable change of a basis we can assume that the operator R_{X^1} acts as follows

$$\begin{aligned}
\left[N_{12}, X^1\right] &= N_{12} + a^1_{12,2n} N_{2n}, \\
\left[N_{i(i+1)}, X^1\right] &= a^1_{i(i+1),1n} N_{1n}, \quad 2 \le i \le n-2, \\
\left[N_{(n-1)n}, X^1\right] &= a^1_{(n-1)n,1(n-1)} N_{1(n-1)}, \\
\left[N_{1j}, X^1\right] &= N_{1j}, \quad j > 2.
\end{aligned}$$

Since $[N_{1n}, X^1] = N_{1n}$, then from Eq. (4) it follows that the algebra is a Lie algebra. □

So we present a description of solvable Leibniz algebras with the nilradical $T(n)$. Moreover, in the case of the maximal possible dimension we show that this algebra is a Lie algebra.

4. Illustration for low dimensions

In this section we give the classification of Leibniz algebras with nilradicals $T(3)$ and $T(4)$.

Note that the Lie algebra $T(3)$ is nothing but the Heisenberg algebra $H(1)$. Solvable Leibniz algebras with the Heisenberg nilradical were described in [10].

Therefore we consider the case $n = 4$. We know that the complimentary vector space to the nilradical $T(4)$ has a dimension less than 4. In case when the dimension of the complementary space is equal to 3 we obtain a Lie algebra (see Theorem 3.4), which falls into the classification already obtained in [14]. So we will consider the dimension of the complimentary vector space to be equal to 1 and 2.

Note that the commutators of the elements in the nilradical $T(4)$ have the form

$$\begin{cases} [N_{12}, N_{23}] = -[N_{23}, N_{12}] = N_{13} \\ [N_{12}, N_{24}] = -[N_{24}, N_{12}] = N_{14} \\ [N_{13}, N_{34}] = -[N_{34}, N_{13}] = N_{14} \\ [N_{23}, N_{34}] = -[N_{34}, N_{23}] = N_{24}. \end{cases} \tag{5}$$

Through this section in the table of multiplications of considered algebras these commutators will be skipped.

4.1. The Leibniz algebras $L(4, 1)$

From the previous section we have that an algebra from $L(4, 1)$ admits a basis $\{N_{12}, N_{23}, N_{34}, N_{13}, N_{24}, N_{14}, X\}$ in which the table of multiplication has the following form:

$$\begin{cases} [N_{12}, X] = a_{12,12}N_{12} + a_{12,24}N_{24}, \\ [X, N_{12}] = -a_{12,12}N_{12} - a_{12,24}N_{24} + b_{12,14}N_{14}, \\ [N_{23}, X] = a_{23,23}N_{23} + a_{23,14}N_{14}, \\ [X, N_{23}] = -a_{23,23}N_{23} + b_{23,14}N_{14}, \\ [N_{34}, X] = -(a_{12,12} + a_{23,23})N_{34} + a_{34,13}N_{13}, \\ [X, N_{34}] = (a_{12,12} + a_{23,23})N_{34} - a_{34,13}N_{13} + b_{34,14}N_{14}, \\ [N_{13}, X] = -[X, N_{13}] = (a_{12,12} + a_{23,23})N_{13}, \\ [N_{24}, X] = -[X, N_{24}] = -a_{12,12}N_{24}, \\ [X, X] = \sigma_{14}N_{14}, \end{cases} \tag{6}$$

where

$$a_{12,12}b_{12,14} = a_{23,23}(a_{23,14} + b_{23,14}) = (a_{12,12} + a_{23,23})b_{34,14} = 0.$$

Since $L(4, 1)$ is a non-nilpotent Leibniz algebra we have $(a_{12,12}, a_{23,23}) \neq (0, 0)$.

Case 1. Let $a_{12,12} = 0$. Then $a_{23,23} \neq 0$, $b_{23,14} = -a_{23,14}$ and $b_{34,14} = 0$.

Taking the change of basis as follows:

$$X' = \frac{1}{a_{23,23}}X, \qquad N'_{23} = N_{23} + \frac{a_{23,14}}{a_{23,23}}N_{14}, \qquad N'_{34} = N_{34} - \frac{a_{34,13}}{2a_{23,23}}N_{13}$$

the multiplication (6) transforms into

$$\begin{aligned}
&[N_{12}, X] = a_{12,24}N_{24}, && [X, N_{12}] = -a_{12,24}N_{24} + b_{12,14}N_{14},\\
&[N_{23}, X] = -[X, N_{23}] = N_{23}, && [N_{34}, X] = -[X, N_{34}] = -N_{34},\\
&[N_{13}, X] = -[X, N_{13}] = N_{13}, && [X, X] = \sigma_{14}N_{14},
\end{aligned}$$

where $(b_{12,14}, \sigma_{14}) \neq (0, 0)$.

Case 2. Let $a_{12,12} \neq 0$, then $b_{12,14} = 0$. Taking the change of basis $X' = \frac{1}{a_{12,12}}X$, we can assume $a_{12,12} = 1$.

Subcase 2.1. Let $a_{23,23} = 0$. Then $b_{34,14} = 0$.

Applying the change of basis

$$N'_{12} = N_{12} + \frac{a_{12,24}}{2}N_{24}, \qquad N'_{34} = N_{34} - \frac{a_{34,13}}{2}N_{13}$$

the products (6) simplify to the following:

$$\begin{aligned}
&[N_{12}, X] = -[X, N_{12}] = N_{12}, && [N_{34}, X] = -[X, N_{34}] = -N_{34},\\
&[N_{13}, X] = -[X, N_{13}] = N_{13}, && [N_{24}, X] = -[X, N_{24}] = -N_{24},\\
&[N_{23}, X] = a_{23,14}N_{14}, && [X, N_{23}] = b_{23,14}N_{14},\\
&[X, X] = \sigma_{14}N_{14},
\end{aligned}$$

where $(a_{23,14} + b_{23,14}, \sigma_{14}) \neq (0, 0)$.

Subcase 2.2. Let $a_{23,23} \neq 0$. Then $b_{23,14} = -a_{23,14}$.

Subcase 2.2.1. Let $a_{23,23} = -1$. Then substituting

$$N'_{23} = N_{23} - a_{23,14}N_{14}, \qquad N'_{12} = N_{12} + \frac{a_{12,24}}{2}N_{24}$$

we derive an algebra with the following multiplication table:

$$\begin{aligned}
&[N_{12}, X] = -[X, N_{12}] = N_{12}, && [N_{23}, X] = [X, N_{23}] = -N_{23},\\
&[N_{34}, X] = a_{34,13}N_{13}, && [X, N_{34}] = -a_{34,13}N_{13} + b_{34,14}N_{14},\\
&[N_{24}, X] = -[X, N_{24}] = -N_{24}, && [X, X] = \sigma_{14}N_{14}
\end{aligned}$$

where $(b_{12,14}, \sigma_{14}) \neq (0, 0)$.

Note that by permuting the indices of the basis elements of the above algebra one obtains an algebra from Case 1.

Subcase 2.2.2. Let $a_{23,23} \neq -1$. Then $b_{34,14} = 0$.

Setting

$$N'_{12} = N_{12} + \frac{a_{12,24}}{2} N_{24}, \qquad N'_{23} = N_{23} + \frac{a_{23,14}}{a_{23,23}} N_{14},$$
$$N'_{34} = \sigma_{14}\left(N_{34} - \frac{a_{34,13}}{2(1+a_{23,23})} N_{13}\right), \qquad N'_{24} = \sigma_{14} N_{24}, \qquad N'_{14} = \sigma_{14} N_{14}$$

we get an algebra with the following table of multiplication:

$$\begin{aligned} &[N_{12}, X] = -[X, N_{12}] = N_{12}, && [N_{23}, X] = -[X, N_{23}] = a_{23,23} N_{23}, \\ &[N_{34}, X] = -[X, N_{34}] = -(1 + a_{23,23}) N_{34}, && [N_{13}, X] = -[X, N_{13}] = (1 + a_{23,23}) N_{13}, \\ &[N_{24}, X] = -[X, N_{24}] = -N_{24}, && [X, X] = N_{14}, \end{aligned}$$

where $(1 + a_{23,23}) a_{23,23} \neq 0$.

Non-isomorphisms of the obtained algebras can be easily established by considering the dimensions of derived series of the algebras.

Thus, the following theorem is proved.

Theorem 4.1. *An arbitrary non-Lie Leibniz algebra of the set $L(4,1)$ is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$L_1: \quad \begin{aligned} &[N_{12}, X] = a_{12,24} N_{24}, && [X, N_{12}] = -a_{12,24} N_{24} + b_{12,14} N_{14}, \\ &[N_{23}, X] = -[X, N_{23}] = N_{23}, && [N_{34}, X] = -[X, N_{34}] = -N_{34}, \\ &[N_{13}, X] = -[X, N_{13}] = N_{13}, && [X, X] = \sigma_{14} N_{14}, \end{aligned}$$

where $(b_{12,14}, \sigma_{14}) \neq (0,0)$*;*

$$L_2: \quad \begin{aligned} &[N_{12}, X] = -[X, N_{12}] = N_{12}, && [N_{34}, X] = -[X, N_{34}] = -N_{34} \\ &[N_{13}, X] = -[X, N_{13}] = N_{13}, && [N_{24}, X] = -[X, N_{24}] = -N_{24}, \\ &[N_{23}, X] = a_{23,14} N_{14}, && [X, N_{23}] = b_{23,14} N_{14}, \\ &[X, X] = \sigma_{14} N_{14}, && \end{aligned}$$

where $(a_{23,14} + b_{23,14}, \sigma_{14}) \neq (0,0)$*;*

$$L_3: \quad \begin{aligned} &[N_{12}, X] = -[X, N_{12}] = N_{12}, \\ &[N_{23}, X] = -[X, N_{23}] = a_{23,23} N_{23}, \\ &[N_{34}, X] = -[X, N_{34}] = -(1 + a_{23,23}) N_{34}, \\ &[N_{13}, X] = -[X, N_{13}] = (1 + a_{23,23}) N_{13}, \\ &[N_{24}, X] = -[X, N_{24}] = -N_{24}, \\ &[X, X] = N_{14}, \end{aligned}$$

where $(1 + a_{23,23}) a_{23,23} \neq 0$.

4.2. The Leibniz algebras $L(4,2)$

The classification of Leibniz algebras belonging to $L(4,2)$ is summarized in the following

Theorem 4.2. *An arbitrary non-Lie Leibniz algebra of the set $L(4,2)$ admits a basis $\{N_{12}, N_{23}, N_{34}, N_{13}, N_{24}, N_{14}, X^1, X^2\}$ in which the table of multiplication has the following form:*

$$[N_{12}, X^1] = -[X^1, N_{12}] = N_{12}, \qquad [N_{34}, X^1] = -[X^1, N_{34}] = -N_{34},$$
$$[N_{13}, X^1] = -[X^1, N_{13}] = N_{13}, \qquad [N_{24}, X^1] = -[X^1, N_{24}] = -N_{24},$$
$$[N_{23}, X^2] = -[X^2, N_{23}] = N_{23}, \qquad [N_{34}, X^2] = -[X^2, N_{34}] = -N_{34},$$
$$[N_{13}, X^2] = -[X^2, N_{13}] = N_{13}, \qquad [X^1, X^1] = \sigma^{11} N_{14},$$
$$[X^2, X^2] = \sigma^{22} N_{14}, \qquad [X^1, X^2] = \sigma^{12} N_{14}, \qquad [X^2, X^1] = \sigma^{21} N_{14}.$$

Proof. From Lemmas 3.1 and 3.2 we have

$$\begin{aligned}
[N_{12}, X^1] &= a^1_{12,12} N_{12} + a^1_{12,24} N_{24}, \\
[X^1, N_{12}] &= -a^1_{12,12} N_{12} - a^1_{12,24} N_{24} + b^1_{12,14} N_{14}, \\
[N_{23}, X^1] &= a^1_{23,23} N_{23} + a^1_{23,14} N_{14}, \\
[X^1, N_{23}] &= -a^1_{23,23} N_{23} + b^1_{23,14} N_{14}, \\
[N_{34}, X^1] &= -(a^1_{12,12} + a^1_{23,23}) N_{34} + a^1_{34,13} N_{13}, \\
[X^1, N_{34}] &= (a^1_{12,12} + a^1_{23,23}) N_{34} - a^1_{34,13} N_{13} + b^1_{34,14} N_{14}, \\
[N_{13}, X^1] &= -[X^1, N_{13}] = (a^1_{12,12} + a^1_{23,23}) N_{13}, \\
[N_{24}, X^1] &= -[X^1, N_{24}] = -a^1_{12,12} N_{24}, \\
[N_{12}, X^2] &= a^2_{12,12} N_{12} + a^2_{12,24} N_{24}, \\
[X^2, N_{12}] &= -a^2_{12,12} N_{12} - a^2_{12,24} N_{24} + b^2_{12,14} N_{14}, \\
[N_{23}, X^2] &= a^2_{23,23} N_{23} + a^2_{23,14} N_{14}, \\
[X^2, N_{23}] &= -a^2_{23,23} N_{23} + b^2_{23,14} N_{14}, \\
[N_{34}, X^2] &= -(a^2_{12,12} + a^2_{23,23}) N_{34} + a^2_{34,13} N_{13}, \\
[X^2, N_{34}] &= (a^2_{12,12} + a^2_{23,23}) N_{34} - a^2_{34,13} N_{13} + b^2_{34,14} N_{14}, \\
[N_{13}, X^2] &= -[X^2, N_{13}] = (a^2_{12,12} + a^2_{23,23}) N_{13}, \\
[N_{24}, X^2] &= -[X^2, N_{24}] = -a^2_{12,12} N_{24}
\end{aligned}$$

with the restrictions

$$a^1_{12,12}b^1_{12,14} = a^1_{23,23}\big(a^1_{23,14} + b^1_{23,14}\big) = \big(a^1_{12,12} + a^1_{23,23}\big)b^1_{34,14} = 0,$$
$$a^2_{12,12}b^2_{12,14} = a^2_{23,23}\big(a^2_{23,14} + b^2_{23,14}\big) = \big(a^2_{12,12} + a^2_{23,23}\big)b^2_{34,14} = 0.$$

Taking the change of basis

$$X^{1'} = \frac{a^2_{23,23}}{a^1_{12,12}a^2_{23,23} - a^2_{12,12}a^1_{23,23}}X^1 - \frac{a^1_{23,23}}{a^1_{12,12}a^2_{23,23} - a^2_{12,12}a^1_{23,23}}X^2,$$
$$X^{2'} = -\frac{a^2_{12,12}}{a^1_{12,12}a^2_{23,23} - a^2_{12,12}a^1_{23,23}}X^1 + \frac{a^1_{12,12}}{a^1_{12,12}a^2_{23,23} - a^2_{12,12}a^1_{23,23}}X^2,$$

we deduce

$$\big[N_{12}, X^1\big] = -\big[X^1, N_{12}\big] = N_{12} + a^1_{12,24}N_{24}, \qquad \big[N_{23}, X^1\big] = a^1_{23,14}N_{14},$$
$$\big[X^1, N_{23}\big] = b^1_{23,14}N_{14}, \qquad \big[N_{34}, X^1\big] = -\big[X^1, N_{34}\big] = -N_{34} + a^1_{34,13}N_{13},$$
$$\big[N_{13}, X^1\big] = -\big[X^1, N_{13}\big] = N_{13}, \qquad \big[N_{24}, X^1\big] = -\big[X^1, N_{24}\big] = -N_{24},$$
$$\big[N_{12}, X^2\big] = a^2_{12,24}N_{24}, \qquad \big[X^2, N_{12}\big] = -a^2_{12,24}N_{24} + b^2_{12,14}N_{14},$$
$$\big[N_{23}, X^2\big] = -\big[X^2, N_{23}\big] = N_{23} + a^2_{23,14}N_{14},$$
$$\big[N_{34}, X^2\big] = -\big[X^2, N_{34}\big] = -N_{34} + a^2_{34,13}N_{13},$$
$$\big[N_{13}, X^2\big] = -\big[X^2, N_{13}\big] = N_{13}.$$

Applying the Leibniz identity for the following triples of elements:

$$\big(N_{12}, X^1, X^2\big), \quad \big(N_{23}, X^1, X^2\big), \quad \big(N_{34}, X^1, X^2\big), \quad \big(X^1, N_{23}, X^2\big), \quad \big(X^2, N_{12}, X^1\big)$$

we get

$$a^2_{12,24} = a^1_{23,14} = a^1_{34,13} = a^2_{34,13} = b^1_{23,14} = b^2_{12,14} = 0.$$

Finally, taking the basis transformation:

$$N'_{12} = N_{12} + \frac{a^1_{12,24}}{2}N_{24}, \qquad N'_{23} = N_{23} + a^2_{23,14}N_{14}$$

we obtain the multiplication table listed in the assertion of the theorem. □

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