On Asymptotic Structure of the Critical Galton–Watson Branching Processes with Infinite Variance and Allowing Immigration

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2 authors:

Azam Imomov
Karshi State University

Erkin Tukhtaev
Karshi State University

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Azam A. Imomov and Erkin E. Tukhtaev

Abstract We observe Galton-Watson Branching Processes with possible immigration. The main results of the paper are as follows. In the absence of immigration, an integral form of the generating function of the invariant measure in the its domain of definition is obtained. In the existing literature, only the "local" form of this function in the neighborhood of point 1 was known (see [16]). For the processes with immigration, we establish two theorems. The first one establishes a formula showing the asymptotic form of the generating function of transition probabilities. This generalizes the result of Pakes [10] in the sense that he found a similar formula only at point 1. In Theorem 3, we find the rate of convergence to invariant measures for processes with an infinite variance of the individuals transformation law and an infinite mean of the individuals immigration law.

1 Introduction

Let $\{X_n, n \in \mathbb{N}_0\}$ be the Galton-Watson Branching Process allowing Immigration (GWPI), where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{N} = \{1, 2, \ldots\}$. This is a homogeneous discrete-time Markov chain with state space $\mathcal{S} \subset \mathbb{N}_0$ and whose transition probabilities are

$$p_{ij} = \text{coefficient of } s^j \text{ in } h(s)\big(f(s)\big)^i, \quad s \in [0,1),$$

where $h(s) = \sum_{j\in\mathcal{S}} h_j s^j$ and $f(s) = \sum_{j\in\mathcal{S}} p_j s^j$ are probability generating functions (PGFs). The variable X_n is interpreted as the population size in GWPI at the mo-

Azam A. Imomov
Department of Mathematics, Karshi State University, Karshi city, Uzbekistan,
e-mail: imomov_azam@mail.ru

Erkin E. Tukhtaev
Department of Mathematics, Karshi State University, Karshi city, Uzbekistan,
e-mail: tukhtaev_erkin@mail.ru

ment n. An evolution of the process will occur by the following scheme. An initial state is empty that is $X_0 = 0$ and the process starts owing to immigrants. Each individual at time n produces j progeny with probability p_j independently of each other so that $p_0 > 0$. Simultaneously in the population i immigrants arrive with probability h_i in each moment $n \in \mathbb{N}$. These individuals undergo further transformation obeying the reproduction law $\{p_j\}$ and n-step transition probabilities $p_{ij}^{(n)} := \mathsf{P}\{X_{n+k} = j \,|\, X_k = i\}$ for any $k \in \mathbb{N}$ are given by

$$\mathcal{P}_n^{(i)}(s) := \sum_{j\in\mathcal{S}} p_{ij}^{(n)} s^j = \big(f_n(s)\big)^i \prod_{k=0}^{n-1} h\big(f_k(s)\big) \quad \textit{for any} \;\; i \in \mathcal{S}, \tag{1}$$

where $f_n(s)$ is the n-fold iteration of PGF $f(s)$; see for example [9]. Note that function $f_n(s)$ generates the distribution law of the number of individuals at the time n in the process without immigration; see Section 2 below. Thus the transition probabilities $\left\{p_{ij}^{(n)}\right\}$ are completely defined by the probabilities $\{p_j\}$ and $\{h_j\}$.

Classification of states of the chain $\{X_n\}$ is one of the fundamental problems in theory of GWPI. Direct differentiation of (1) gives

$$\mathsf{E}\big[X_n | X_0 = i\big] = \begin{cases} an + i & , \quad \textit{when} \;\; m = 1, \\ \left(\dfrac{a}{m-1} + i\right) m^n - \dfrac{a}{m-1} \, , & \quad \textit{when} \;\; m \neq 1, \end{cases}$$

where $m := f'(1-) = \sum_{j\in\mathcal{S}} j p_j$ is the mean per-capita offspring number and $a := h'(1-) = \sum_{j\in\mathcal{S}} j h_j$ is the average number of immigration distribution law. The received formula for $\mathsf{E}\big[X_n | X_0 = i\big]$ shows that classification of states of GWPI depends on a value of the parameter m. Process $\{X_n\}$ is classified as sub-critical, critical and supercritical if $m < 1$, $m = 1$ and $m > 1$ accordingly.

The above described population process was considered first by Heathcote [4] in 1965. Further long-term properties of $\mathcal{S}$ and a problem of existence and uniqueness of invariant measures of GWPI were investigated by Seneta [15], Pakes [11], [12] and by many other authors. Therein some moment conditions for PGF $f(s)$ and $h(s)$ was required to be satisfied. In aforementioned works of Seneta the ergodic properties of $\{X_n\}$ were investigated. He has proved that when $m \leq 1$ the process $\{X_n\}$ has an invariant measure $\{\mu_k, \, k \in \mathcal{S}\}$ which is unique up to multiplicative constant. Pakes [12] have shown that in supercritical case $\mathcal{S}$ is transient. In the critical case $\mathcal{S}$ can be transient, null-recurrent or ergodic. In this case, if in addition to assume that $2b := f''(1-) < \infty$, properties of $\mathcal{S}$ depend on value of parameter $\lambda = a\big/b$: if $\lambda > 1$ or $\lambda < 1$, then $\mathcal{S}$ is transient or null-recurrent accordingly. In the case when $\lambda = 1$, Pakes [11] studied necessary and sufficient conditions for a null-recurrence property. Limiting distribution law for critical process $\{X_n\}$ was found first by Seneta [14]. He has proved that the normalized process $X_n\big/(bn)$ has limiting Gamma distribution with density function $\Gamma^{-1}(\lambda) x^{\lambda-1} e^{-x}$ provided that $0 < \lambda < \infty$, where $x > 0$ and $\Gamma(*)$ is Euler's Gamma function. This result has been established also by Pakes [11] without reference to Seneta. Afterwards Pakes [9], [10], has obtained principally new results for all cases $m < \infty$ and $b = \infty$.

Throughout the paper we keep on the critical case only and $b = \infty$. Our reasoning will bound up with elements of slow variation theory in sense of Karamata; see [13]. Remind that real-valued, positive and measurable function $L(x)$ is said to be slowly varying (SV) at infinity if $L(\lambda x)/L(x) \to 1$ as $x \to \infty$ for each $\lambda > 0$. We refer the reader to [1], [2] and [13] for more information.

In the second Section we study invariant measures of the simple Galton-Watson (GW) Process $\{Z_n\}$. In Theorem 1 appears an integral form of PGF $U(s)$ of an invariant measure of the process $\{Z_n\}$ and an asymptotic form of derivative $U'(s)$ neighborhood of the point 1. This theorem expands Slack's result [16] in the sense that he found only locally representation of the function $U(s)$ in a neighborhood of the point 1.

In the third Section we investigate invariant properties of GWPI. We observe an asymptotic expansion of $\mathcal{P}_n(s) := \mathcal{P}_n^{(0)}(s)$ supposing that $h'(1-) = \sum_{j\in\mathcal{S}} jh_j = \infty$ but PGF $h(s)$ regularly varies; see representation $[h_\delta]$ below.

2 Invariant measures of GW Process

Let $\{Z_n, n \in \mathbb{N}_0\}$ be the simple GW Branching Process without immigration given by offspring PGF $f(s)$. Discussing this case we will assume that the offspring PGF $f(s)$ has the following representation:

$$f(s) = s + (1-s)^{1+\nu}\mathcal{L}\left(\frac{1}{1-s}\right), \qquad [f_\nu]$$

where $0 < \nu < 1$ and $\mathcal{L}(x)$ is SV at infinity. By the criticality of the process the condition $[f_\nu]$ implies that $b = \infty$. This includes the case $b < \infty$ when $\nu = 1$ and $\mathcal{L}(t) \to b$ as $t \to \infty$.

Consider PGF $f_n(s) := \mathsf{E}\left[s^{Z_n} | Z_0 = 1\right]$ and write $R_n(s) := 1 - f_n(s)$. Evidently $Q_n := R_n(0)$ is the survival probability of the process. By arguments of Slack [16] one can be shown that if the condition $[f_\nu]$ holds then

$$Q_n^\nu \cdot \mathcal{L}\left(\frac{1}{Q_n}\right) \sim \frac{1}{\nu n} \quad as \;\; n \to \infty. \qquad (2)$$

Slack [16] also has shown that

$$\mathcal{U}_n(s) := \frac{f_n(s) - f_n(0)}{f_n(0) - f_{n-1}(0)} \longrightarrow U(s) \qquad (3)$$

for $s \in [0,1)$, where the limit function $U(s)$ satisfies the Abel equation

$$U(f(s)) = U(s) + 1, \qquad (4)$$

so that $U(s)$ is PGF of invariant measure for the GW process $\{Z_n\}$. Combining $[f_\nu]$, (2) and (3) and considering properties of the process $\{Z_n\}$ we have

$$\mathcal{U}_n(s) \sim U_n(s) := \left[1 - \frac{R_n(s)}{Q_n}\right] \nu n \quad as \;\; n \to \infty.$$

So we proved the following lemma.

Lemma 1. *If the condition* $[f_\nu]$ *holds then*

$$R_n(s) = \frac{\mathcal{N}(n)}{(\nu n)^{1/\nu}} \cdot \left[1 - \frac{U_n(s)}{\nu n}\right], \tag{5}$$

where the function $\mathcal{N}(x)$ *is SV at infinity and*

$$\mathcal{N}(n) \cdot \mathcal{L}^{1/\nu}\left(\frac{(\nu n)^{1/\nu}}{\mathcal{N}(n)}\right) \longrightarrow 1 \quad as \;\; n \to \infty, \tag{6}$$

and the function $U_n(s)$ *satisfies the following properties:*

- $U_n(s) \longrightarrow U(s)$ *as* $n \to \infty$ *so that the equation* (4) *holds;*
- $\lim_{s\uparrow 1} U_n(s) = \nu n$ *for each fixed* $n \in \mathbb{N}$*;*
- $U_n(0) = 0$ *for each fixed* $n \in \mathbb{N}$.

Apparently, this lemma is generalization of (2) and is established with a simpler proof than as shown in [5].

Further writing

$$\Lambda(y) := \frac{f(1-y) - (1-y)}{y} = y^\nu \mathcal{L}\left(\frac{1}{y}\right),$$

we establish the following important assertion.

Lemma 2. *If the condition* $[f_\nu]$ *holds then*

- *the following relation is true:*

$$\frac{\partial R_n(s)}{\partial s} = -\psi_n(s) \frac{R_n(s)\Lambda\big(R_n(s)\big)}{(1-s)\Lambda(1-s)},$$

  where $\psi_n(s)$ *is continuous increasing on* $s \in [0,1]$*, for all* $n \in \mathbb{N}$ *and*

$$\frac{f'(s)}{f'\big(f_n(s)\big)} < \psi_n(s) < 1;$$

- *the following asymptotic relation is true:*

$$\frac{\partial R_n(s)}{\partial s} \sim -\psi(s) \frac{R_n(s)\Lambda\big(R_n(s)\big)}{(1-s)\Lambda(1-s)} \quad as \;\; n \to \infty,$$

where $\psi(s)$ is continuous increasing on $s \in [0,1]$, so that

$$f'(s) \le \psi(s) \le 1;$$

- *the following locally asymptotic relation is true:*

$$\frac{\partial R_n(s)}{\partial s} = \frac{R_n(s)\Lambda\big(R_n(s)\big)}{(1-s)\Lambda(1-s)}\big(1+\phi(1-s)\big) \quad \text{as} \;\; s \uparrow 1 \;\; \text{and} \;\; n \to \infty,$$

where $\phi(y) = -(1+\nu)\Lambda(y)\big(1+o(1)\big)$ as $y \downarrow 0$.

The statements of the last lemma will play an important role in the proof of Theorem 1 below.

Consider now the function

$$\mathcal{M}_n(s) = 1 - \frac{\Lambda\big(R_n(s)\big)}{\Lambda\big(Q_n\big)}. \tag{7}$$

It follows from (5) and from SV properties, that

$$\begin{aligned}\mathcal{M}_n(s) &= 1 - \left(\frac{R_n(s)}{Q_n}\right)^{\nu} \frac{\mathcal{L}\big(1/R_n(s)\big)}{\mathcal{L}\big(1/Q_n\big)} \\ &\sim 1 - \left(1 - \frac{U_n(s)}{\nu n}\right)^{\nu} = \frac{U_n(s)}{n}\big(1+\kappa_n(s)\big) \quad \text{as} \;\; n \to \infty,\end{aligned}$$

where $\kappa_n(s) = \mathcal{O}\big(1/n\big)$ uniformly in $s \in [0,1)$.

Thus we have the following

Lemma 3. *If the condition $[f_\nu]$ holds then*

$$n \cdot \mathcal{M}_n(s) \longrightarrow U(s) \quad \text{as} \;\; n \to \infty, \tag{8}$$

where $U(s)$ is PGF of invariant measure of GW process.

The following statement gives an asymptotical representation for $\Lambda\big(R_n(s)\big)$ and can be substituted for Lemma 1.

Lemma 4. *If the condition $[f_\nu]$ holds then*

$$\Lambda\big(R_n(s)\big) = \frac{\Lambda\,(1-s)}{\Lambda\,(1-s)\,\nu n + 1}\big(1+o(1)\big) \quad \text{as} \;\; n \to \infty. \tag{9}$$

Remark 1. The asymptotic relation (9) seems in appearance, to be an analogue of classical form of the Basic Lemma of the theory of critical GW Branching Processes with finite variance, in which $b = f''(1-)/2$ instead of ν and $\Lambda(x) \equiv x$; see, for instance, [7, Lemma 1 (ii)].

Remark 2. Along with all applications, the second statement of Lemma 2 in combining with the formula (9), provide an opportunity to find an asymptotic representation of the transition probability $P_{11}(n) := \mathsf{P}\left\{Z_n = 1 \,\middle|\, Z_0 = 1\right\}$ as $n \to \infty$, since $f_n'(0) = P_{11}(n)$. In fact, we obtain

$$P_{11}(n) \sim \frac{\psi(0)}{p_0} \cdot \frac{\mathcal{N}(n)}{(\nu n)^{1+1/\nu}} \quad as \;\; n \to \infty,$$

where $p_1 < \psi(0) < 1$ and $\mathcal{N}(\cdot)$ is SV defined in (6).

Now using (5)–(9) and considering Lemma 2, we track down an explicit form of PGF $U(s)$ and the asymptote of its derivative.

Theorem 1. *If the condition* $[f_\nu]$ *holds then*

- *the PGF* $U(s)$ *is form of*

$$U(s) = \int\limits_0^s \frac{\psi(u)}{(1-u)\Lambda(1-u)}\, du,$$

where $\psi(s)$ *is continuous increasing on* $s \in [0,1]$*, so that*

$$f'(s) \le \psi(s) \le 1;$$

- *the function* $U'(s)$ *has the following locally asymptotic form:*

$$U'(s) = \frac{1}{(1-s)\Lambda(1-s)}\big(1 + \phi(1-s)\big) \quad as \;\; s \uparrow 1,$$

where $\phi(y) = -(1+\nu)\Lambda(y)\big(1 + o(1)\big)$ *as* $y \downarrow 0$.

3 Invariant measures of GWPI

In this Section we consider GWPI. First of all we recall the following theorem which was proved by Pakes [10].

Theorem P1 [10]. *If* $m = 1$ *then*

$$p_{00}^{(n)} \sim K \exp\left\{ \int\limits_1^{e^n} \frac{\ln h(1 - \varphi(y))}{y}\, dy \right\} \quad as \;\; n \to \infty,$$

where $\varphi(y)$ *is decreasing SV-function. If*

$$\sum_{m=0}^{\infty} \Big[\big(1 - h(f_m(0))\big)\big(1 - f'(f_m(0))\big) \Big] < \infty,$$

then

$$p_{00}^{(n)} \sim K_1 \exp\left\{ \int_0^{f_n(0)} \frac{\ln h(y)}{f(y)-y} dy \right\} \quad as \ \ n \to \infty.$$

Herein K and K_1 are some constants.

Since this point we everywhere will consider the case that immigration PGF $h(s)$ has the following form:

$$1 - h(s) = (1-s)^{\delta} \ell\left(\frac{1}{1-s}\right), \qquad [h_\delta]$$

where $0 < \delta < 1$ and $\ell(x)$ is SV at infinity. The assumption $[h_\delta]$ implies that an average number of immigration distribution law is infinite i.e. $\sum_{j \in \mathcal{S}} j h_j = \infty$, but $\sum_{j \in \mathcal{S}} j^{\delta} h_j < \infty$.

Our results appear provided that conditions $[f_\nu]$ and $[h_\delta]$ hold and $\delta > \nu$. As it has been shown in [10] that in this case $\mathcal{S}$ is ergodic. Namely we improve statements of Theorem P1. Herewith we put forward an additional requirement concerning $\mathcal{L}(x)$ and $\ell(x)$. So since $\mathcal{L}(x)$ is SV we can write

$$\frac{\mathcal{L}(\lambda x)}{\mathcal{L}(x)} = 1 + \alpha(x) \qquad [\mathcal{L}_\alpha]$$

for each $\lambda > 0$, where $\alpha(x) \to 0$ as $x \to \infty$. Henceforth we suppose that some positive function $g(x)$ is given so that $g(x) \to 0$ and $\alpha(x) = o\big(g(x)\big)$ as $x \to \infty$. In this case $\mathcal{L}(x)$ is called SV with remainder $\alpha(x)$; see [2, p. 185, condition SR3]. Wherever we exploit the condition $[\mathcal{L}_\alpha]$ we will suppose that

$$\alpha(x) = o\left(\frac{\mathcal{L}(x)}{x^{\nu}}\right) \quad as \ \ x \to \infty. \tag{10}$$

And also by perforce we suppose the condition

$$\frac{\ell(\lambda x)}{\ell(x)} = 1 + \beta(x) \qquad [\ell_\beta]$$

for each $\lambda > 0$, where

$$\beta(x) = o\left(\frac{\ell(x)}{x^{\delta}}\right) \quad as \ \ x \to \infty.$$

Since $f_n(s) \uparrow 1$ for all $s \in [0,1)$ in virtue of (1) it sufficiently to observe the case $i = 0$ as $n \to \infty$. Denote

$$\mathcal{P}_n(s) := \mathcal{P}_n^{(0)}(s).$$

The following theorem is a generalization of Theorem P1.

Theorem 2. *Let conditions $[f_\nu]$, $[h_\delta]$ hold. If $\delta > \nu$ then*

$$\mathcal{P}_n(s) \sim K(s)\exp\left\{-\int_s^{f_n(s)} \frac{1-h(y)}{f(y)-y}\Big[1+\delta(1-y)\Big]dy\right\}$$

as $n\to\infty$, where $K(s)$ is a bounded function for $s\in[0,1)$ and $\delta(x)\to 0$ as $x\downarrow 0$. If in addition, the conditions $[\mathcal{L}_\alpha]$ and (10) *are satisfied then*

$$\delta(x) = \mathcal{O}\big(\Lambda\,(x)\big) \quad as \;\; x\downarrow 0.$$

The next result directly follows from Theorem 2 setting $x=0$ there.

Corollary 1. *Let conditions $[f_\nu]$, $[h_\delta]$ hold. If $\delta>\nu$ then*

$$p_{00}^{(n)} \sim A\exp\left\{-\frac{1}{\delta-\nu}\cdot \mathsf{L}\left(\frac{(\nu n)^{1/\nu}}{\mathcal{N}(n)}\right)\right\} \quad as \;\; n\to\infty,$$

where A is a positive constant, $\mathsf{L}:=\ell/\mathcal{L}$ and $\mathcal{N}(x)$ is SV defined in (6)*.*

Further we need the following result which is an improved analog of the Basic Lemma of the theory of critical GW processes.

Lemma 5 ([6]). *Let conditions $[f_\nu]$, $[\mathcal{L}_\alpha]$ and* (10) *hold. Then*

$$\frac{1}{\Lambda\big(R_n(s)\big)} - \frac{1}{\Lambda\,(1-s)} = \nu n + \frac{1+\nu}{2}\cdot \ln\big(1+\nu n\Lambda\,(1-s)\big) + \rho_n(s),$$

where $\rho_n(s)=o\,(\ln n)+\sigma_n(s)$ and, $\sigma_n(s)$ is bounded uniformly in $s\in[0,1)$ and converges to the limit $\sigma(s)$ as $n\to\infty$, which is a bounded function for $s\in[0,1)$.

We make sure that at the conditions of second part of Theorem 2 PGF $\mathcal{P}_n(s)$ converges to a limit $\pi(s)$ which we denote by the power series representation

$$\pi(s) = \sum_{j\in\mathcal{S}} \pi_j s^j.$$

Now using Lemma 5, we can establish a speed rate of this convergence in the following theorem.

Theorem 3. *Let conditions $[f_\nu]$, $[h_\delta]$ hold and $\delta>\nu$. Then $\mathcal{P}_n(s)$ converges to $\pi(s)$ which generates the invariant measures $\{\pi_j\}$ for GWPI. The convergence is uniform over compact subsets of the open unit disc. If in addition, the conditions $[\mathcal{L}_\alpha]$,* (10) *and $[\ell_\beta]$ are fulfilled then*

$$\mathcal{P}_n(s) = \pi(s)\left(1+\Delta_n(s)\mathcal{N}_\delta\left(\frac{1}{R_n(s)}\right)\right),$$

where $\mathcal{N}_\delta(x)=\mathcal{N}^\delta(x)\ell(x)$, the function $\mathcal{N}(x)$ is defined in (6)*, herein*

$$\Delta_n(s) = \frac{1}{\delta - \nu} \frac{1}{\big(\nu_n(s)\big)^{\delta/\nu - 1}} - \frac{1+\nu}{2\nu} \frac{\ln\big[\nu_n(s)\big]}{\big(\nu_n(s)\big)^{\delta/\nu}} \big(1 + o(1)\big)$$

as $n \to \infty$ *and* $\nu_n(s) = \nu n + \Lambda^{-1}(1-s)$.

The following result is direct consequence of Theorem 3.

Corollary 2. *If conditions of Theorem 3 hold then*

$$p_{00}^{(n)} = \pi_0 \cdot \Big(1 + \Delta_n \mathcal{N}_\delta(n)\Big),$$

where $\mathcal{N}_\delta(n)$ *is SV at infinity and*

$$\Delta_n = \frac{1}{\delta - \nu} \frac{1}{(\nu n)^{\delta/\nu - 1}} - \frac{1+\nu}{2\nu} \frac{\ln n}{(\nu n)^{\delta/\nu}} \big(1 + o(1)\big) \quad as \;\; n \to \infty.$$

Remark 3. The analogous result as in Theorem 2 has been proved in [8] for $\delta = 1$ and $f'''(1-) < \infty$.

Conclusion

In this report, we consider and study the model of the evolution of the population size of homogeneous individuals, called the branching process allowing immigration. The main goal of the work is to study the asymptotic properties of the process trajectory in the interpretation of transition probabilities in the minimal moment conditions.

In the monograph [3, pp. 29–31], part of the treatment of gene fixation was interpreted in terms of invariant (stationary) measures. Hope that the properties of the invariant measures of the simple GW Branching Process established in Theorem 1, and the asymptotic formulas for transition probabilities for the GWPI (Theorem 2 and Theorem 3), showing approximations to invariant measures, can be useful in theoretical aspects of applied problems similar to those described in [3].

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