Contemporary Mathematics
Volume **672**, 2016
http://dx.doi.org/10.1090/conm/672/13463

Description of solvable Leibniz algebras with four-dimensional nilradical

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Abstract. In this work we describe 5-dimensional solvable Leibniz algebras with four-dimensional non-Lie nilradical.

1. Introduction.

According to the structural theory of Lie algebras a finite-dimensional Lie algebra is written as a semidirect sum of its semisimple subalgebra and the solvable radical (Levi's theorem). The semisimple part is a direct sum of simple Lie algebras which are completely classified in fifties of the last century. At the same period the essential progress has been made in the solvable part by Mal'cev reducing the problem of classification of solvable Lie algebras to that of nilpotent Lie algebras. Since then all the classification results have been related to the nilpotent part.

Leibniz algebras, a "noncommutative version" of Lie algebras, were introduced in 1993 by Jean-Louis Loday [**7**]. During the last 20 years the theory of Leibniz algebras has been actively studied and many results on Lie algebras have been extended to Leibniz algebras (see, e.g. [**1**, **2**]). Particularly, in 2011 the analogue of Levi's theorem has been proven by D. Barnes [**2**]. He showed that any finite-dimensional complex Leibniz algebra is decomposed into a semidirect sum of the solvable radical and a semisimple Lie algebra. As above, the semisimple part can be composed by simple Lie algebras and the main issue in the classification problem of finite-dimensional complex Leibniz algebras is to study the solvable part. Therefore the classification of solvable Leibniz algebras is important to construct finite-dimensional Leibniz algebras.

Owing to a result of [**8**], a new approach to study of solvable Lie algebras by using their nilradicals was developed [**9**, **10**], etc. The analogue of Mubarakzjanov's [**8**] results has been applied for Leibniz algebras case in [**4**] which shows the importance of the consideration of their nilradicals in Leibniz algebras case as well. The papers [**3**–**6**] are also devoted to the study of solvable Leibniz algebras by considering their nilradicals. In particular, in the work [**3**] there are complete lists of isomorphism classes of four dimensional complex solvable Leibniz algebras.

The focus of the present work is on classification of five-dimensional Leibniz algebras. Note that the dimension of the nilradical of 5-dimensional solvable Leibniz algebras are equal to three or four. It should be remark that, the description of

Key words and phrases. Leibniz algebra, solvability, nilradical, derivation.

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5-dimensional solvable Leibniz algebras with three-dimensional nilradical has been given in [**11**]. Therefore, we deal with the study of 5-dimensional solvable Leibniz algebras with four-dimensional nilradical.

Throughout the work all the algebras (vector spaces) considered are finite-dimensional and over the field of complex numbers. Also in tables of multiplications of algebras we give nontrivial products only.

2. Preliminaries.

This section is devoted to recalling some basic notions and concepts used through the work.

DEFINITION 2.1. *A vector space with bilinear bracket* $(L, [\cdot,\cdot])$ *is called a Leibniz algebra if for any* $x, y, z \in L$ *the so-called Leibniz identity*

$$\big[x,[y,z]\big] = \big[[x,y],z\big] - \big[[x,z],y\big],$$

holds.

The set $Ann_r(L) = \{x \in L : [y,x] = 0, \ \forall y \in L\}$ is called *the right annihilator of* L. It is observed that for any $x, y \in L$ the elements $[x,x]$ and $[x,y]+[y,x]$ are always in $Ann_r(L)$, and that is $Ann_r(L)$ is a two-sided ideal of L.

For a given Leibniz algebra $(L, [\cdot,\cdot])$ the sequences of two-sided ideals defined recursively as follows:

$$L^1 = L, \ L^{k+1} = [L^k, L], \ k \geq 1, \qquad L^{[1]} = L, \ L^{[s+1]} = [L^{[s]}, L^{[s]}], \ s \geq 1.$$

are said to be *the lower central* and *the derived series of* L, respectively.

DEFINITION 2.2. *A Leibniz algebra* L *is said to be nilpotent (respectively, solvable), if there exists* $n \in \mathbb{N}$ $(m \in \mathbb{N})$ *such that* $L^n = 0$ *(respectively,* $L^{[m]} = 0$*). The minimal number* n *(respectively,* m*) with such property is said to be the index of nilpotency (respectively, solvability) of the algebra* L.

It is easy to see that the sum of any two nilpotent ideals is nilpotent. Therefore the maximal nilpotent ideal always exists.

DEFINITION 2.3. *The maximal nilpotent ideal of a Leibniz algebra is said to be a nilradical of the algebra.*

DEFINITION 2.4. *A linear map* $d \colon L \to L$ *of a Leibniz algebra* $(L, [\cdot,\cdot])$ *is said to be a derivation if for all* $x, y \in L$*, the following condition holds:*

$$d([x,y]) = [d(x), y] + [x, d(y)].$$

The set of all derivations of L *is denoted by* $Der(L)$.

For a given element x of a Leibniz algebra L, the right multiplication operator $R_x \colon L \to L$, defined by $R_x(y) = [y,x], y \in L$ is a derivation. In fact, a Leibniz algebra is characterized by this property of the right multiplication operators. As in Lie case this kind derivations are said to be *inner derivations*. Let the set of all inner derivations of a Leibniz algebra L denote by $R(L)$, i.e. $R(L) = \{R_x \mid x \in L\}$. The set $R(L)$ inherits the Lie algebra structure from $Der(L)$:

$$[R_x, R_y] = R_x \circ R_y - R_y \circ R_x = R_{[y,x]}.$$

Let L be a 5-dimensional solvable Leibniz algebra. Then it can be written in the form $L = N + Q$, where N is the nilradical and Q is the complementary subspace.

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Similar to the case of Lie algebras, for the solvable Leibniz algebra L we have the inequality $\dim N \geq \dfrac{\dim L}{2}$. Therefore, we get $\dim N \geq 3$. Since the description of 5-dimensional solvable Leibniz algebras with three-dimensional nilradical has been given in [**11**], we consider case of $dimN = 4$.

If index of nilpotency of the nilradical N equals to 5 or 4, then N is null-filiform or filiform algebra, respectively. Since solvable Leibniz algebras with null-filiform or filiform nilradicals were classified in the papers [**4**–**6**], we consider case of nilindex of nilradical is equal to 3.

Below we present the list of all the four-dimensional non-Lie nilpotent Leibniz algebras with the index of nilpotency is equal to 3 from [**1**].

$$\begin{array}{llllll}
\lambda_1: & [e_1,e_2]=e_3, & [e_2,e_1]=e_4, & [e_2,e_2]=-e_3; & & \\
\lambda_2(\alpha): & [e_1,e_1]=e_3, & [e_1,e_2]=e_4, & [e_2,e_1]=-\alpha e_3, & [e_2,e_2]=-e_4; & \\
\lambda_3: & [e_1,e_1]=e_4, & [e_1,e_2]=e_4, & [e_2,e_1]=-e_4, & [e_3,e_3]=e_4; & \\
\lambda_4: & [e_1,e_1]=e_4, & [e_1,e_2]=e_3, & [e_2,e_1]=-e_3, & [e_2,e_2]=-2e_3+e_4; & \\
\mu_1(\alpha): & [e_1,e_1]=e_4, & [e_1,e_2]=\alpha e_4, & [e_2,e_1]=-\alpha e_4, & [e_2,e_2]=e_4, & [e_3,e_3]=e_4; \\
\mu_2: & [e_1,e_2]=e_4, & [e_1,e_3]=e_4, & [e_2,e_1]=-e_4, & [e_2,e_2]=e_4, & [e_3,e_1]=e_4; \\
\mu_3: & [e_1,e_2]=e_3, & [e_2,e_1]=e_4; & & & \\
\mu_4: & [e_1,e_1]=e_4, & [e_1,e_2]=e_3, & [e_2,e_1]=-e_3; & & \\
\mu_5: & [e_1,e_1]=e_3, & [e_1,e_2]=e_4; & & & \\
\mu_6(\alpha): & [e_1,e_2]=e_4, & [e_2,e_1]=\frac{1+\alpha}{1-\alpha}e_4, & [e_2,e_2]=e_3; & \alpha\neq 1 & \\
\mu_7: & [e_1,e_2]=e_4, & [e_2,e_1]=-e_4, & [e_3,e_3]=e_4. & &
\end{array}$$

In the following proposition we describe the derivations of the four-dimensional nilpotent non-Lie Leibniz algebras.

Proposition 2.5. *The matrix forms of the derivations of* $\lambda_i, i=\overline{1,4}$ *and* $\mu_k, k=\overline{1,7}$ *are represented as follows:*

$$D(\lambda_1)=\begin{pmatrix} a_1 & 0 & a_2 & a_3\\ 0 & a_1 & a_4 & a_5\\ 0 & 0 & 2a_1 & 0\\ 0 & 0 & 0 & 2a_1\end{pmatrix},\quad D(\lambda_2(\alpha))=\begin{pmatrix} a_1 & 0 & a_2 & a_3\\ 0 & a_1 & a_4 & a_5\\ 0 & 0 & 2a_1 & 0\\ 0 & 0 & 0 & 2a_1\end{pmatrix},$$

$$D(\lambda_3)=\begin{pmatrix} a_1 & a_2 & 0 & a_3\\ 0 & a_1 & 0 & a_4\\ 0 & 0 & a_1 & a_5\\ 0 & 0 & 0 & 2a_1\end{pmatrix},\quad D(\lambda_4)=\begin{pmatrix} a_1 & 0 & a_2 & a_3\\ 0 & a_1 & a_4 & a_5\\ 0 & 0 & 2a_1 & 0\\ 0 & 0 & 0 & 2a_1\end{pmatrix},$$

$$D(\mu_1(\alpha))=\begin{pmatrix} a_1 & a_2 & 0 & a_3\\ -a_2 & a_1 & 0 & a_4\\ 0 & 0 & a_1 & a_5\\ 0 & 0 & 0 & 2a_1\end{pmatrix},\quad D(\mu_1(0)=\begin{pmatrix} a_1 & a_2 & a_3 & a_4\\ -a_2 & a_1 & a_5 & a_6\\ -a_3 & -a_5 & a_1 & a_7\\ 0 & 0 & 0 & 2a_1\end{pmatrix},$$

$$D(\mu_2)=\begin{pmatrix} a_1 & -a_2 & 0 & a_3\\ 0 & a_1 & a_2 & a_4\\ 0 & 0 & a_1 & a_5\\ 0 & 0 & 0 & 2a_1\end{pmatrix},\quad D(\mu_3)=\begin{pmatrix} a_1 & 0 & a_3 & a_4\\ 0 & a_2 & a_5 & a_6\\ 0 & 0 & a_1+a_2 & 0\\ 0 & 0 & 0 & a_1+a_2\end{pmatrix},$$

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$$D(\mu_4) = \begin{pmatrix} a_1 & a_3 & a_4 & a_5 \\ 0 & a_2 & a_6 & a_7 \\ 0 & 0 & a_1+a_2 & 0 \\ 0 & 0 & 0 & 2a_1 \end{pmatrix}, \quad D(\mu_5) = \begin{pmatrix} a_1 & a_3 & a_4 & a_5 \\ 0 & a_2 & a_6 & a_7 \\ 0 & 0 & 2a_1 & a_3 \\ 0 & 0 & 0 & a_1+a_2 \end{pmatrix},$$

$$D(\mu_6(\alpha)) = \begin{pmatrix} a_1 & a_3 & a_4 & a_5 \\ 0 & a_2 & a_6 & a_7 \\ 0 & 0 & 2a_1 & \frac{2}{1-\alpha}a_3 \\ 0 & 0 & 0 & a_1+a_2 \end{pmatrix}, \quad D(\mu_7) = \begin{pmatrix} a_1 & a_3 & 0 & a_4 \\ a_5 & a_2 & 0 & a_6 \\ 0 & 0 & a_1+a_2 & a_7 \\ 0 & 0 & 0 & \frac{a_1+a_2}{2} \end{pmatrix}.$$

In the following proposition we give the information of the five-dimensional solvable Leibniz algebra with nilradical λ_1, $\lambda_2(\alpha)$, λ_3, λ_4.

PROPOSITION 2.6. *There is no a five-dimensional solvable Leibniz algebra with four-dimensional nilradical* λ_1, $\lambda_2(\alpha)$, λ_3, λ_4.

PROOF. Let us assume the contrary and L be a 5-dimensional Leibniz algebra with nilradical λ_1. We choose a basis $\{e_1, e_2, e_3, e_4, x\}$ of L, where $\{e_1, e_2, e_3, e_4\}$ – a basis of λ_1. Restriction of the right multiplication operator R_x to λ_1 is non-nilpotent derivation of λ_1. Then using Proposition 2.5 we get

$$\begin{aligned} &[e_1, e_2] = e_3, && [e_1, x] = a_1e_1 + a_2e_3 + a_3e_4, && [e_3, x] = 2a_1e_3, \\ &[e_2, e_1] = e_4, && [e_2, x] = a_1e_2 + a_4e_3 + a_5e_4, && [e_4, x] = 2a_1e_4. \\ &[e_2, e_2] = -e_3, \end{aligned}$$

Put

$$[x, e_1] = \sum_{i=1}^{4} \alpha_i e_i, \quad [x, e_2] = \sum_{i=1}^{4} \beta_i e_i, \quad [x, x] = \sum_{i=1}^{4} \delta_i e_i.$$

Applying the Leibniz identity to the triple e_1, x, e_2 as follow

$$[e_1, [x, e_2]] = [e_1, \sum_{i=1}^{4} \beta_i e_i] = \beta_2 e_3,$$

$$[e_1, [x, e_2]] = [[e_1, x], e_2] - [[e_1, e_2], x] = a_1e_3 - 2a_1e_3 = -a_1e_3,$$

we obtain $\beta_2 = -a_1$.

Similarly applying the Leibniz identity

$$0 = [x, [e_1, e_2]] = [[x, e_1], e_2] - [[x, e_2], e_1] = (\alpha_1 - \alpha_2)e_3 - \beta_2 e_4,$$

we get $\beta_2 = 0$, therefore $a_1 = 0$. However, it contradicts to the non-nilpotent of the R_x. The cases $\lambda_2(\alpha)$, λ_3 and λ_4 are proved similarly.

□

THEOREM 2.7. *Let L be 5-dimensional solvable Leibniz algebra with 4-dimensional non-Lie nilradical N. If nilindex of N is equal to 3, then L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$I_1(\alpha): \begin{cases} [e_1, e_1] = e_4, & [e_1, x] = e_1 - \alpha e_2, & [x, e_1] = -e_1 + \alpha e_2, \\ [e_1, e_2] = \alpha e_4, & [e_2, x] = \alpha e_1 + e_2, & [x, e_2] = -\alpha e_1 - e_2, \\ [e_2, e_1] = -\alpha e_4, & [e_3, x] = e_3, & [x, e_3] = -e_3, \\ [e_2, e_2] = e_4, & [e_4, x] = 2e_4, & [e_3, e_3] = e_4. \end{cases}$$

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$$I_2: \begin{cases} [e_1,e_1]=e_4, & [e_1,x]=e_1-ie_2, & [x,e_1]=-e_1+ie_2, \\ [e_1,e_2]=ie_4, & [e_2,x]=ie_1+e_2, & [x,e_2]=-ie_1-e_2+e_4, \\ [e_2,e_1]=-ie_4, & [e_3,x]=e_3, & [x,e_3]=-e_3, \\ [e_2,e_2]=e_4, & [e_4,x]=2e_4, & [e_3,e_3]=e_4. \end{cases}$$

$$II_1: \begin{cases} [e_1,e_2]=e_4, & [e_1,x]=e_1-e_2, & [x,e_1]=-e_1+e_2, \\ [e_1,e_3]=e_4, & [e_2,x]=e_2+e_3, & [x,e_2]=-e_2-e_3, \\ [e_2,e_1]=-e_4, & [e_3,x]=e_3, & [x,e_3]=-e_3, \\ [e_2,e_2]=e_4, & [e_4,x]=2e_4, & [e_3,e_1]=e_4. \end{cases}$$

$$III_1(\alpha): \begin{cases} & [e_1,x]=e_1, & [x,e_1]=-e_1, \\ [e_1,e_2]=e_3, & [e_2,x]=\alpha e_2, & [x,e_2]=-\alpha e_2, \\ [e_2,e_1]=e_4, & [e_3,x]=(1+\alpha)e_3, & [x,e_3]=-e_3+\alpha e_4, \\ & [e_4,x]=(1+\alpha)e_4, & [x,e_4]=e_3-\alpha e_4. \end{cases}$$

$$III_2: \begin{cases} [e_1,e_2]=e_3, & [e_1,x]=e_1+e_3, & [x,e_1]=-e_1+e_4, \\ [e_2,e_1]=e_4, & [e_3,x]=e_3, & [x,e_3]=-e_3, \\ & [e_4,x]=e_4, & [x,e_4]=e_3. \end{cases}$$

$$IV_1: \begin{cases} [e_1,e_1]=e_4, & [e_1,x]=e_1+e_2, & [e_2,x]=e_2, \\ [e_1,e_2]=e_3, & [x,e_1]=-e_1-e_2, & [x,e_2]=-e_2, \\ [e_2,e_1]=-e_3, & [e_3,x]=2e_3, & [x,e_3]=-2e_3, \quad [e_4,x]=2e_4. \end{cases}$$

$$IV_2(\alpha): \begin{cases} [e_1,e_1]=e_4, & [e_1,x]=e_1, & [x,e_1]=-e_1, \quad [e_4,x]=2e_4. \\ [e_1,e_2]=e_3, & [e_2,x]=\alpha e_2, & [e_3,x]=(1+\alpha)e_3, \\ [e_2,e_1]=-e_3, & [x,e_2]=-\alpha e_2, & [x,e_3]=-(1+\alpha)e_3, \end{cases}$$

$$IV_3: \begin{cases} [e_1,e_1]=e_4, & [e_2,x]=e_2, & [x,e_1]=-e_1, \\ [e_1,e_2]=e_3, & [e_3,x]=e_3, & [x,e_3]=-e_3. \\ [e_2,e_1]=-e_3, & & \end{cases}$$

$$IV_4(\alpha): \begin{cases} [e_1,e_1]=e_4, & [e_2,x]=e_2, & [x,e_1]=\alpha e_4, \\ [e_1,e_2]=e_3, & [e_3,x]=e_3, & [x,e_2]=-e_2, \\ [e_2,e_1]=-e_3, & & [x,e_3]=-e_3. \end{cases}$$

$$IV_5(\alpha): \begin{cases} [e_1,e_1]=e_4, & [e_2,x]=e_2, & [x,e_1]=\alpha e_4, \\ [e_1,e_2]=e_3, & [e_3,x]=e_3, & [x,e_2]=-e_2, \\ [e_2,e_1]=-e_3, & [x,x]=e_4, & [x,e_3]=-e_3. \end{cases}$$

$$IV_6(\alpha,\delta): \begin{cases} [e_1,e_1]=e_4, & [e_2,x]=e_2+e_3, & [x,e_1]=\alpha e_4, \\ [e_1,e_2]=e_3, & [e_3,x]=e_3, & [x,e_2]=-e_2-e_3, \\ [e_2,e_1]=-e_3, & [x,x]=\delta e_4, & [x,e_3]=-e_3. \end{cases}$$

$$V_1: \begin{cases} [e_1,e_1]=e_3, & [e_2,x]=e_2, & [x,e_2]=-e_2. \\ [e_1,e_2]=e_4, & [e_4,x]=e_4, & \end{cases}$$

$$V_2: \begin{cases} [e_1,e_1]=e_3, & [e_2,x]=e_2, & [x,e_2]=-e_2, \\ [e_1,e_2]=e_4, & [e_4,x]=e_4, & [x,x]=e_3. \end{cases}$$

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$$V_3(\alpha):\left\{\begin{array}{llll}[e_1,e_1]=e_3, & [e_2,x]=e_2, & [x,e_1]=-e_3, & \\ [e_1,e_2]=e_4, & [e_4,x]=e_4, & [x,e_2]=-e_2, & [x,x]=\alpha e_3.\end{array}\right.$$

$$VI_1:\left\{\begin{array}{lll}[e_1,e_2]=e_4, & [e_1,x]=e_1, & [x,e_1]=-e_1,\\ [e_2,e_1]=e_4, & [e_2,x]=e_2, & [x,e_2]=-e_2,\\ [e_2,e_2]=e_4, & [e_3,x]=2e_3, & [e_4,x]=2e_4.\end{array}\right.$$

$$VII_1:\left\{\begin{array}{lll}[e_1,e_2]=e_4, & [e_1,x]=e_1, & [x,e_1]=-e_1,\\ [e_2,e_1]=-e_4, & [e_2,x]=-e_2, & [x,e_2]=e_2.\\ [e_3,e_3]=e_4, & [e_3,x]=e_4, & \end{array}\right.$$

$$VII_2:\left\{\begin{array}{lll}[e_1,e_2]=e_4, & [e_1,x]=e_1, & [x,e_1]=-e_1,\\ [e_2,e_1]=-e_4, & [e_2,x]=-e_2, & [x,e_2]=e_2,\\ [e_3,e_3]=e_4, & [e_3,x]=e_4, & [x,e_3]=e_4.\end{array}\right.$$

$$VII_3(\alpha):\left\{\begin{array}{llll}[e_1,e_2]=e_4, & [e_1,x]=e_1, & [x,e_1]=-e_1, & \\ [e_2,e_1]=-e_4, & [e_2,x]=-e_2, & [x,e_2]=e_2, & \\ [e_3,e_3]=e_4, & [e_3,x]=e_4, & [x,e_3]=\alpha e_4, & [x,x]=e_4.\end{array}\right.$$

PROOF. Due to Proposition 2.6, we consider case of nilradical N is isomorphic to one of the algebras $\mu_1(\alpha)$, μ_2, μ_3, μ_4, μ_5, $\mu_6(\alpha)$, μ_7.

Let L be 5-dimensional solvable Leibniz algebra, whose nilradical is isomorphic to $\mu_1(\alpha)$, $\alpha\neq 0$. Then there exists a basis $\{e_1,e_2,e_3,e_4,x\}$ such that L is represented by the table of multiplications as follows:

$$\begin{array}{lll}[e_1,e_1]=e_4, & [e_1,x]=a_1e_1+a_2e_2+a_3e_4, & \\ [e_1,e_2]=\alpha e_4, & [e_2,x]=-a_2e_1+a_1e_2+a_4e_4, & \\ [e_2,e_1]=-\alpha e_4, & [e_3,x]=a_1e_3+a_5e_4, & \\ [e_2,e_2]=e_4, & [e_4,x]=2a_1e_4, & [e_3,e_3]=e_4.\end{array}\qquad(1)$$

We get

$$[x,e_1]=\sum_{i=1}^{4}\alpha_ie_i,\quad [x,e_2]=\sum_{i=1}^{4}\beta_ie_i,\quad [x,e_3]=\sum_{i=1}^{4}\gamma_ie_i,\quad [x,x]=\sum_{i=1}^{4}\delta_ie_i.$$

From the table of multiplications (1), it is easy to see that $Ann_r(L)=\{e_4\}$. Then we obtain the following relations for the structure constants

$$\delta_i=0,\ i=\overline{1,3};\quad \alpha_1=-a_1,\ \alpha_2=-a_2,\ \alpha_3=0;$$

$$\beta_1=a_2,\ \beta_2=-a_1,\ \beta_3=0;\quad \gamma_1=0,\ \gamma_2=0,\ \gamma_3=-a_1.$$

Applying the Leibniz identity to the triple $\{x,e_1,e_2\}$ as follow

$$0=[x,[e_1,e_2]]=[[x,e_1],e_2]-[[x,e_2],e_1]=-2(\alpha a_1+a_2)e_4,$$

we get $a_2=-\alpha a_1$ and $\alpha_2=-\alpha\alpha_1$.

Similarly, applying the Leibniz identity to the triples $\{x,x,e_3\}$; $\{x,x,e_1\}$ and $\{x,x,e_2\}$ we obtain the following system for the structure constants

$$\gamma_4=a_5,\ (1+\alpha^2)\alpha_4=-2\alpha a_4-a_3(\alpha^2-1),\ (1+\alpha^2)\beta_4=2\alpha a_3-a_4(\alpha^2-1)\quad(2)$$

Now we are going to consider the possible cases of the parameter α.

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Let $1+\alpha^2 \neq 0$, then from the relation (2), we have

$$\alpha_4 = -\frac{2\alpha a_4 + a_3(\alpha^2-1)}{1+\alpha^2}, \quad \beta_4 = \frac{2\alpha a_3 - a_4(\alpha^2-1)}{1+\alpha^2}.$$

Thus, we obtain the following table of multiplications:

$$\begin{array}{lll}
[e_1,e_1]=e_4, & [e_1,x]=a_1e_1-\alpha a_1e_2+a_3e_4, & [x,e_1]=-a_1e_1+\alpha a_1e_2+\alpha_4e_4,\\
[e_1,e_2]=\alpha e_4, & [e_2,x]=\alpha a_1e_1+a_1e_2+a_4e_4, & [x,e_2]=-\alpha a_1e_1-a_1e_2+\beta_4e_4,\\
[e_2,e_1]=-\alpha e_4, & [e_3,x]=a_1e_3+a_5e_4, & [x,e_3]=-a_1e_3+a_5e_4,\\
[e_2,e_2]=e_4, & [e_4,x]=2a_1e_4, & [x,x]=\delta_4e_4.\\
[e_3,e_3]=e_4, & &
\end{array}$$

Taking the change of basis

$$x'=\frac{1}{a_1}x-\frac{\delta_4}{2a_1^2}e_4,\ e_1'=e_1+\frac{\alpha a_4-a_3}{a_1(\alpha^2+1)}e_4,\ e_2'=e_2-\frac{\alpha a_3+a_4}{a_1(\alpha^2+1)}e_4,\ e_3'=e_3-\frac{a_5}{a_1}e_4$$

we can assume that $a_1=1,\ a_3=a_4=a_5=\alpha_4=\beta_4=0$ and we get $I_1(\alpha)$ for $\alpha\neq 0$ and $\alpha\neq \pm i$.

In the case of $1+\alpha^2=0$, by similar argumentation we obtain the algebra I_2.

Let N is isomorphic $\mu_1(0)$. Similar to the previous cases we take a basis $\{e_1,e_2,e_3,e_4,x\}$ of L that the table of multiplication in this basis has the following form:

$$\begin{array}{lll}
[e_1,e_1]=e_4, & [e_1,x]=a_1e_1+a_2e_2+a_3e_3+a_4e_4, & \\
[e_2,e_2]=e_4, & [e_2,x]=-a_2e_1+a_1e_2+a_5e_3+a_6e_4, & \\
[e_3,e_3]=e_4, & [e_3,x]=-a_3e_1-a_5e_2+a_1e_3+a_7e_4, & [e_4,x]=2a_1e_4.
\end{array} \tag{3}$$

It is easy to see that $Ann_r(L)=\{e_4\}$. Then

$$\delta_i=0,\ i=\overline{1,3};\quad \alpha_1=-a_1,\ \alpha_2=-a_2, \alpha_3=-a_3;$$

$$\beta_1=a_2,\ \beta_2=-a_1,\ \beta_3=-a_5;\quad \gamma_1=a_3,\ \gamma_2=a_5,\ \gamma_3=-a_1.$$

Applying the Leibniz identities to the elements of the form $\{x,e_1,e_2\}$, $\{x,e_1,e_3\}$, $\{x,e_2,e_3\}$, $\{x,x,e_1\}$, $\{x,x,e_2\}$ and $\{x,x,e_3\}$ we get

$$a_2=0,\quad a_3=0,\quad a_5=0,\quad \alpha_4=a_4,\quad \beta_4=a_6,\quad \gamma_4=a_7.$$

Thus, we obtain the following table of multiplications:

$$\begin{array}{lll}
[e_1,e_1]=e_4, & [e_1,x]=a_1e_1+a_4e_4, & [x,e_1]=-a_1e_1+a_4e_4,\\
[e_2,e_2]=e_4, & [e_2,x]=a_1e_2+a_6e_4, & [x,e_2]=-a_1e_2+a_6e_4,\\
[e_3,e_3]=e_4, & [e_3,x]=a_1e_3+a_7e_4, & [x,e_3]=-a_1e_3+a_7e_4,\\
 & [e_4,x]=2a_1e_4, & [x,x]=\delta_4e_4.
\end{array}$$

Since $a_1\neq 0$, then taking the following change of basis:

$$x'=\frac{1}{a_1}x-\frac{\delta_4}{2a_1^2}e_4,\ e_1'=e_1-\frac{a_4}{a_1}e_4,\ e_2'=e_2+\frac{a_6}{a_1}e_4,\ e_3'=e_3-\frac{a_7}{a_1}e_4$$

we get $I_1(0)$.

The algebras II_1-VII_3 are obtained by using similar tools to prove I_1 in the cases μ_2, μ_3, μ_4, μ_5, $\mu_6(\alpha)$ and μ_7.

$\square$

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Conclusion. The present work we conclude that there are 8 parametric families and 10 concrete non isomorphic solvable Leibniz algebra structures with four-dimensional non-Lie nilradicals on 5-dimensional complex vector space.

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