O'ZBEKISTON MATEMATIKA JURNALI

УЗБЕКСКИЙ МАТЕМАТИЧЕСКИЙ ЖУРНАЛ

УзМЖ

Содержание

Ground states for Potts model with competing interactions on cayley tree
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Maqolada Keli daraxtida aniqlangan to'rtta spin qiymatli o'zaro ta'sir radiusi ikkiga teng bo'lgan Potts modeli qaralgan. Bu model uchun davriy asosiy holatlar qurilgan va Peyerlis sharti tekshirilgan.

Рассматривается модель Поттса с четермя значениями спина и с радиусом взаимодействия два на дереве Кэли. Дается периодические основные состояния этой модели. Также проверяется условие Пайерлса.

1. Introduction

The state of a physical system having the lowest possible potential energy. For example, an electron in the lowest energy orbital in a hydrogen atom is in a ground state. The ground state of a physical system tends to be stable unless energy is applied to it from the outside; states that are not the ground state have a tendency to revert to the ground state, giving off energy in the process.

The ground states for models on the cubic lattice Z^d were studied in many works (see e.g. [5,6,8,9]). The Ising model, with two values of spin 1 was considered and became actively researched in the 1990̆s and afterwards (see for example [1ЦЗ,10]).

In [11] an Ising model on a Cayley tree with competing interactions is considered. The goal of this paper is to study of (periodic and non periodic) ground states and to verify the Peierls condition for the Potts model with competing interactions.

For the Ising model with competing interactions, in [12] the set of all weakly periodic ground states corresponding to normal divisors of indices 2 and 4 of the group representation of the Cayley tree is described.

In the Pirogov-Sinai theory configurations can be described by contours which satisfy Peierls condition. This theory provides tools for a very detailed knowledge of structure of Gibbs measures in a region in relevant parameters space [13].

Pirogov and Sinai developed a theory of phase transitions in systems satisfying Peierls condition. W. Holsztynski and J. Slawny give a criterium for the Peierls condition to hold and apply it to several systems. In particular they proved that ferromagnetic system satisfies the Peierls condition iff its (internal) symmetry group is finite. And using an algebraic argument they show that in two dimensions the symmetry groups of reduced translation invariant systems is finite [7].

The Potts model is a generalization of the Ising model in which each lattice site contains an entity (a spin) that can be in one of q states. Potts models are

useful for describing the absorption of molecules on crystalline surfaces and the behavior of foams, for example, and exhibit a discontinuous or continuous phase transition depending on the value of q.

The paper is organized as follows. In Section 2 we give definitions of Cayley tree, configuration space and the Potts model with competing interactions. In Section 3 we construct periodic ground states. In Section 4 we check Peierls condition for the model.

2. Definitions and statement of problem

2.1. Cayley tree.

The Cayley tree Γ^k of order $k \geq 1$ is the infinite tree (i.e., a cycle-free graph) each of whose vertices has exactly $k + 1$ outgoing edges. Let $\Gamma^k = (V, L, i)$, where V is the set of vertices of Γ^k, and L is the set of edges, and i is the incidence function, which takes each edge $l \in L$ to its endpoints $x, y \in V$. If $i(l) = \{x, y\}$, then the vertices x and y are called nearest neighbors, and we write $< x, y >$. The distance $d(x, y), x, y \in V$, on the Cayley tree is defined by the formula

$$d(x, y) = \min\{d : \exists x = x_0, \ x_1, \ldots, x_{d-1}, \ x_d = y \in V$$
$$\text{such that } x = < x_0, x_1 >, \ldots, < x_{d-1}, x_d >\}.$$

There exists a one-to-one correspondence between the set V of vertices of the Cayley tree of order $k \geq 1$ and the group G_k that is the free product of $k + 1$ cyclic groups of order 2 with generators $a_1, a_2, \ldots, a_{k+1}$ [8].

2.2 Configuration space.

Let $\Phi = \{1, 2, \ldots, q\}$, $q \geq 2$. A configurations on V is defined as a function $x \in V \to \sigma(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^V$. Let $A \subset V$. By Ω_A we denote the space of configurations defined on A.

2.3 The Potts model.

R.B. Potts defined the former model in 1952, at the suggestion of C. Domb. He actually defined two models. The first is known as the "Z_n model and supposes that at each site of a lattice there is a two-dimensional unit vector which can point in one of N equally spaced directions. Two adjacent vectors interact with interaction energy proportional to their scalar product.

The second model is the one that will be discussed here, and referred to simply as "the Potts model". This can be formulated on any graph $\wp$, i.e. on any set of sites, and edges joining pairs of sites.

We consider the Potts model with competing interactions, where the spin takes values in the set $\Phi = \{0, 1, 3, \ldots, q\}$, on the Cayley tree which is defined by the following Hamiltonian

$$H(\sigma) = J_1 \sum_{\substack{<x,y>, \\ x,y \in V}} \delta_{\sigma(x)\sigma(y)} + J_2 \sum_{\substack{x,y \in V, \\ d(x,y)=2}} \delta_{\sigma(x)\sigma(y)} \tag{1}$$

where $J_1, J_2 \in R$ are coupling constants and σ a configuration on V [4].

In this paper we consider the case $q = 4$.

3. Ground states

Ground states can be defined in two ways, yielding, roughly, zero-temperature version of Gibbs states and equilibrium states, respectively. The first definition is that any local perturbation of the state increases its energy; the second, applicable to periodic configurations, is that specific energy is minimal.

For a pair of configurations σ and φ coinciding almost everywhere, i.e., everywhere except at a finite number of points, we consider the relative Hamiltonian $H(\sigma, \varphi)$, of the difference between the energies of the configurations σ and φ, i.e.,

$$H(\sigma, \varphi) = J_1 \sum_{\substack{<x,y> \\ x,y \in V}} (\delta_{\sigma(x)\sigma(y)} - \delta_{\varphi(x)\varphi(y)}) + J_2 \sum_{\substack{x,y \in V: \\ d(x,y)=2}} (\delta_{\sigma(x)\sigma(y)} - \delta_{\varphi(x)\varphi(y)}) \tag{2}$$

where $J = (J_1, J_2) \in R^2$ is an arbitrary fixed parameter.

We suppose that M is the set of unit balls with vertices in V. The restriction of a configuration σ on a ball $b \in M$ is called a bounded configuration σ_b. We define the energy of the configuration σ_b on the ball b as

$$U(\sigma_b) \equiv U(\sigma_b, J) = \frac{1}{2} J_1 \sum_{\substack{<x,y>, \\ x,y \in b}} \delta_{\sigma(x)\sigma(y)} + J_2 \sum_{\substack{x,y \in b: \\ d(x,y)=2}} \delta_{\sigma(x)\sigma(y)} \tag{3}$$

where $J = (J_1, J_2) \in R$.

Using a combinatorial calculations one can prove the following

Lemma 3.1. 1) Let σ_b be a configuration with $\sigma_b(c_b) = i$, (where c_b is the center of the ball b), and $|\{x : \sigma_b(x) = 1\}| = m$, $|\{x : \sigma(x) = 2\}| = n$, $|\{x : \sigma(x) = 3\}| = l$, $|\{x : \sigma(x) = 4\}| = r$. Here $|A|$ denotes the number of elements of A. Then $U(\sigma_b)$ has the following form

$$U(\sigma_b) \equiv U_{i,k}(m,n,l,r,J_1,J_2) = \frac{1}{2}(\delta_{1i}m + \delta_{2i}n + \delta_{3i}l + \delta_{3i}l)J_1 + (C_m^2 + C_n^2 + C_l^2 + C_r^2)J_2 \tag{4}$$

where $m, n, l, r \in N \cup \{0\}$, $m + n + l + r = k + 1$ and $J = (J_1, J_2) \in R^2$.

2) For any configuration σ_b we have

$$U(\sigma_b) \in \{U_{i,k}(m,n,l,r,J_1,J_2): \ m,n,l,r, \in N \cup \{0\}, \text{ with} \\ m+n+l+r = k+1 \text{ and } J = (J_1, J_2) \in R^2\}.$$

The following lemma can be easily proved.

Lemma 3.2. Relative Hamiltonian (2) has the form

$$H(\sigma, \varphi) = \sum_{b \in M} (U(\sigma_b) - U(\varphi_b)).$$

Denote

$$F_p^{(i)} \equiv F_p^{(i)}(\sigma_b) = \{j \in N_k : \sigma_b(c_b) = i, \ \sigma_b(a_j) = p\}, \ p = 1, 2, 3, 4;$$
$$\Omega_{m,n,l}^{(i)} = \{\sigma_b : \ \sigma_b(c_b) = i, \ |F_1^{(i)}| = m, |F_2^{(i)}| = n, |F_3^{(i)}| = l\}.$$

Let S_4 be the group of permutations on $\{1, 2, 3, 4\}$.
$C_{m,n,l}^{(i)} = \bigcup_{\pi \in S_4} \pi(\Omega_{m,n,l}^{(i)})$, where for $\pi = (\pi(1), \pi(2), \pi(3), \pi(4)) \in S_4$ we put
$\pi(\Omega_{m,n,l}^{(i)}) = \{\pi\sigma : \sigma \in \Omega_{m,n,l}^{(i)}\}$ with $(\pi\sigma)(x) = \pi(\sigma(x))$.

We define a $\hat{G}_k$ – periodic configuration as a configuration $\sigma(x)$ that is invariant under a finite-index subgroup $\hat{G}_k \in G_k$, i.e., $\sigma(yx) = \sigma(x)$ for any $x \in G_k$ and $y \in \hat{G}_k$. For a given periodic configuration, the index of the subgroup is called the configuration period. A configuration which is invariant under all translations is said to be translationally invariant.

Theorem 3.3. *For any class $C_{m,n,l}^{(i)}$ and for any bounded configuration $\sigma_b \in C_{m,n,l}^{(i)}$ there exists a periodic configuration φ with period non exceeding 4 such that $\varphi_{b'} \in C_{m,n,l}^{(i)}$ for any $b' \in M$ and $\varphi_b = \sigma_b$.*

Proof: For arbitrary given class $C_{m,n,l}^{(i)}$ and $\sigma_b \in C_{m,n,l}^{(i)}$ we shall construct configuration φ as follows. Without loss of generality we can take b as the ball with the center $e \in G_k$ (here e is the identity if G_k), i.e. $b = \{e, a_1, \cdots, a_{k+1}\}$. Assume $\sigma_b(e) = i$ where $i = 1, 2, 3, 4$.

Consider

$$H_0^{(i)} = \{x \in G_k : \sum_{j \in N_k \backslash (F_1^{(i)} \cup F_2^{(i)})} \omega_j(x) - \text{even}, \sum_{j \in F_2^{(i)} \cup F_3^{(i)}} \omega_j(x) - \text{even}\},$$

where $\omega_j(x)$ is the number of a_j in $x \in G_k$. Since $H_0^{(i)}$ is the subgroup of index 4 in G_k, the quotient group has form $G_k/H_0^{(i)} = \{H_0^{(i)}, H_1^{(i)}, H_2^{(i)}, H_3^{(i)}\}$ with the co sets

$$H_1^{(i)} = \{x \in G_k : \sum_{j \in N_k \backslash (F_1^{(i)} \cup F_2^{(i)})} \omega_j(x) - \text{even}, \sum_{j \in F_2^{(i)} \cup F_3^{(i)}} \omega_j(x) - \text{odd}\};$$

$$H_2^{(i)} = \{x \in G_k : \sum_{j \in N_k \backslash (F_1^{(i)} \cup F_2^{(i)})} \omega_j(x) - \text{odd}, \sum_{j \in F_2^{(i)} \cup F_3^{(i)}} \omega_j(x) - \text{even}\};$$

$$H_3^{(i)} = \{x \in G_k : \sum_{j \in N_k \setminus (F_1^{(i)} \cup F_2^{(i)})} \omega_j(x) - odd, \sum_{j \in F_2^{(i)} \cup F_3^{(i)}} \omega_j(x) - odd\}.$$

We continue the bounded configuration $\sigma_b \in C_{m,n,l}^{(i)}$ to whole lattice Γ^k (which we denote by φ) by

$$\varphi(x) = \begin{cases} 1, & \text{if } x \in H_0^{(i)}; \\ 2, & \text{if } x \in H_1^{(i)}; \\ 3, & \text{if } x \in H_2^{(i)}; \\ 4, & \text{if } x \in H_3^{(i)}. \end{cases}$$

So we obtain a periodic configuration φ with period 4 (=index of the subgroup); then by the construction $\varphi_b = \sigma_b$. Now we shall prove that all restrictions $\varphi_{b'}$, $b' \in M$ of the configuration φ belong $C_{m,n,l}^{(i)}$. Let $q_j(x) = |S_1(x) \cap H_j|$, $j = 0, 1, 2, 3$; where $S_1(x) = \{y \in G_k : < x, y >\}$, the set of all nearest neighbors of $x \in G_k$.

Denote $Q(x) = (q_0(x), q_1(x), q_2(x), q_3(x))$. We note [15] that for every $x \in G_k$ there is a permutation π_x of the coordinate of the vector $Q(e)$ (where e as before is the identity of G_k) such that $\pi_x Q(e) = Q(x)$.

Moreover, it is easy to see that

$$Q(x) = \begin{cases} Q(e), & \text{if } x \in H_0^{(0)}; \\ (q_1(e), q_0(e), q_3(e), q_2(e)), & \text{if } x \in H_1^{(i)}; \\ (q_2(e), q_3(e), q_0(e), q_1(e)), & \text{if } x \in H_2^{(i)}; \\ (q_3(e), q_2(e), q_1(e), q_0(e)), & \text{if } x \in H_3^{(i)}. \end{cases}$$

Thus for any $b' \in M$ we have (i) if $c_{b'} \in H_0^{(i)}$ (where as before $c_{b'}$ is the center of b') then $\varphi_{b'} = \sigma_b$ up to rotation; (ii) if $c_{b'} \in H_1^{(i)}$ then $\varphi_{b'} = \pi_1(\sigma_b)$ where $\pi_1 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \end{pmatrix}$, then $\varphi_{b'} = \sigma_b$ up to rotation; (iii) if $c_{b'} \in H_2^{(i)}$ then $\varphi_{b'} = \pi_2(\sigma_b)$ where $\pi_2 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{pmatrix}$, then $\varphi_{b'} = \sigma_b$ up to rotation; (iv) if $c_{b'} \in H_3^{(i)}$ then $\varphi_{b'} = \pi_3(\sigma_b)$ where $\pi_3 = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix}$, then $\varphi_{b'} = \sigma_b$ up to rotation. Since the energy of a configuration σ_b and $\pi(\sigma_b)$ for arbitrary $\pi \in S_4$ is equal we get $\varphi_{b'} \in C_{m,n,l}^{(i)}$ for any $b' \in M$. Theorem is proved.

Definition 3.4. A configuration σ is called a *ground state* of the relative Hamiltonian H if

$$U(\sigma_b) = \min\{U_{i,k}(m, n, l, J_1, J_2) : i = 1, 2, 3; m, n, l \in N \cup \{0\}$$
$$\text{with } 0 \le m + n + l \le k + 1\} \text{ for any } b \in M.$$

Denote by $A_{i,k}(m, n, l)$ the set of points (J_1, J_2) such that

$$A_{i,k}(m, n, l) = \{(J_1, J_2) : U_{i,k}(m, n, l, J_1, J_2) \le U_{j,k}(m', n', l', J_1, J_2),$$
$$\text{for any } m', n' l' \in N \cup \{0\}, j = 1, 2, 3, 4\}.$$

Let $GS(H)$ be the set of all ground states of the relative Hamiltonian H (see (2)).

Theorem 3.5. (i) If $J = (0,0)$ then $GS(H) = \Omega$.

(ii) If $J \in A_{i,k}(m,n,l)$ then $GS(H) = \{\pi(\sigma_{m,n,l}^{(i)}) : \pi \in S_4\}$, where $\sigma_{m,n,l}^{(i)} \in \Omega$ such that $(\sigma_{m,n,l}^{(i)})_b \in \Omega_{m,n,l}^{(i)}$ for any $b \in M$, $0 \leq m+n+l \leq k+1$, $i = 1,2,3,4$.

Proof: Case (i) is trivial. (ii) for a given configuration σ_b which makes $U(\sigma_b)$ minimal, by Theorem 3.3 one can construct the periodic ground states $\sigma_{m,n,l}^{(i)}$ with period non exceeding four. For each case the exact number of such ground states coincides with number of the configurations σ_b which make $U(\sigma_b)$ minimal. Theorem is proved.

4. The Peierls condition

Denote by $\mathbf{U}$ the collection of all possible values of $U(\sigma_b)$ for any configuration σ_b, $b \in M$.

Put $U^{\min} = \min\{U : U \in \mathbf{U}\}$ and

$$\lambda_0 = \min\{\mathbf{U}\backslash\{U \in \mathbf{U} : U = U^{\min}\}\} - U^{\min}.$$

Definition 4.1. Let GS be the set of all ground states of relative Hamiltonian H. A ball $b \in M$ is said to be an *improper* ball of the configuration σ if $\sigma_b \neq \varphi_b$ for any $\varphi \in GS$. The union of improper balls of a configuration σ is called the *boundary* of the configuration and denoted by $\partial(\sigma)$.

Definition 4.2. The relative Hamiltonian H with the set of ground states GS satisfies the Peierls condition if for any $\varphi \in GS$ and any configuration σ coinciding almost everywhere with φ (i.e. $|\{x \in V : \sigma(x) \neq \varphi(x)\}| < \infty$)

$$H(\sigma, \varphi) \geq \lambda|\partial(\sigma)|$$

where λ is a positive constant which does not depend on σ, and $|\partial(\sigma)|$ is the number of balls in $\partial(\sigma)$.

Theorem 4.3 If $J \neq (0,0)$ then the Peierls condition is satisfied.

Proof: Note that $|\mathbf{U}| = 1$ if and only if $J = (0,0)$, consequently $\lambda_0 > 0$ if $J \neq (0,0)$.

Suppose σ coincides almost everywhere with a ground state $\varphi \in GS(H)$ then we have $U(\sigma_b) - U(\varphi_b) \geq \lambda_0$ for any $b \in \partial(\sigma)$ since φ is a ground state. Thus

$$H(\sigma, \varphi) = \sum_{b \in M} (U(\sigma_b) - U(\varphi_b)) = \sum_{b \in \partial(\sigma)} (U(\sigma_b) - U(\varphi_b)) \geq \lambda_0|\partial(\sigma)|.$$

Therefore, the Peierls condition is satisfied for $\lambda = \lambda_0$. The theorem is proved.

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