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# On the Property of Subalgebras of Evolution Algebras

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**Abstract** In this paper we study subalgebras of complex finite dimensional evolution algebras. We obtain the classification of nilpotent evolution algebras whose any subalgebra is an evolution subalgebra with a basis which can be extended to a natural basis of algebra. Moreover, we formulate three conjectures related to the description of such non-nilpotent algebras.

**Keywords** Evolution algebra · Evolution subalgebra · Complete evolution algebra · Nilpotency

**Mathematics Subject Classification (2010)** 17D92 · 17D99

## 1 Introduction

Nowadays, the algebraic approach is effectively used in the study of the genetics and dynamical systems in population biology. In 20s and 30s of the last century the new object was introduced to mathematics, which was the product of the interactions between Mendelian

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genetics and mathematics. One of the first scientists who gave an algebraic interpretation of the " $\times$ " sign, which indicated sexual reproduction was Serebrowsky [13]. Etherington introduced the formal language of abstract algebra to the study of the genetics [6, 7]. An algebraic approach in genetics consists of the study of various types of genetic algebras (such as algebras of free, "self-reproductive" and bisexual populations, Bernstein algebras and etc.). Until 1980s, the most comprehensive reference in this area was Wörz-Busekros's book [15]. A good survey on algebraic structure of genetic inheritance is the Reed's article [11]. More recent results, such as genetic evolution in genetic algebras, can be found in the Lyubich's book [10].

Recently in the book of J.P. Tian [14] a new type of evolution algebra has been introduced. This algebra describes some evolution laws of the genetics. The study of evolution algebras constitutes a new subject both in algebra and the theory of dynamical systems. In the Tian's book a foundation of the framework of the theory of evolution algebras is established and some applications of evolution algebras in the theory of stochastic processes and genetics are discussed.

Evolution algebras are in general non-associative and do not belong to any of the well-known classes of the non-associative algebras. In fact, nilpotency, right nilpotency and solvability might be interpreted in a biological way as a various types of vanishing ("annihilation") of populations. Although an evolution algebra is an abstract system, it gives an insight for the study of non-Mendelian genetics. For instance, an evolution algebra can be applied to the inheritance of organelle genes. One can predict, in particular, all possible mechanisms to establish the homoplasmy of cell populations.

Recently, Rozikov and Tian [12] studied algebraic structures of evolution algebras associated with Gibbs measures defined on some graphs. In the papers [2, 5, 9] derivations, some properties of chain of evolution algebras and dibaric property of evolution algebras are studied. A connection between certain algebraic properties of evolution algebras (right nilpotency, nilpotency, solvability and etc.) and matrix of the structural constants have been investigated in [1, 3, 4].

It is remarkable that a subalgebra and an ideal of a genetic algebra of population, biologically can be interpreted correspondingly as a subpopulation and a dominant subpopulation with respect to mating.

This paper is devoted to study of subalgebras of finite-dimensional evolution algebras.

In order to achieve our goal we organize the paper as follows. In Section 2, we give some necessary notions and preliminary results about evolution algebras. We consider several types of subalgebras of evolution algebras and present examples of difference of such subalgebras as well. Section 3 is devoted to description of evolution algebras of permutations for which any subalgebra is an evolution subalgebra with a natural basis which can be extended to a natural basis of the algebra (complete evolution algebra). In order to list two-dimensional evolution algebras we identify their subalgebras. In Section 4, we classify the nilpotent complex complete evolution algebras. In Section 5, we formulate three conjectures related to the description of such non-nilpotent algebras.

Through the paper all algebras are assumed to be complex and finite-dimensional.

## 2 Preliminaries

In this section we give necessary definitions and preliminary results to obtain main results of the paper. Let us define the main object of this work - evolution algebra.

**Definition 2.1** [14] Let $E$ be an algebra over a field $F$. If it admits a basis $\{e_1, e_2, \dots\}$ such that

$$e_i \cdot e_j = 0 \quad for\ i \neq j, \qquad e_i \cdot e_i = \sum_k a_{i,k} e_k \quad for\ any\ i,$$

then algebra $E$ is called evolution algebra.

The basis $\{e_1, e_2, \dots\}$ is said to be *natural basis of evolution algebra* $E$. It is remarkable that this type of algebra depends on the natural basis $\{e_1, e_2, \dots\}$.

We denote by $A = (a_{ij})$ the matrix of the structural constants of the evolution algebra $E$.

**Definition 2.2** [14] Let $E$ be an evolution algebra and $E_1$ be a subspace of $E$. If $E_1$ has a natural basis $\{e_i \mid i \in \Lambda_1\}$ which can be extended to a natural basis $\{e_j \mid j \in \Lambda\}$ of $E$, then $E_1$ is called an evolution subalgebra, where $\Lambda_1$ and $\Lambda$ are index sets and $\Lambda_1$ is a subset of $\Lambda$.

In fact, for a linear subspace $E_1$ of evolution algebra $E$ we can consider three concepts of a subalgebra.

(1) $E_1$ is a subalgebra in ordinary sense;
(2) $E_1$ is a subalgebra and there exists a natural basis of $E_1$;
(3) $E_1$ is a subalgebra and there exists a natural basis of $E_1$ which can be extended to the natural basis of $E$.

Note that Definition 2.2 agrees with the third concept of a subalgebra.

Below we present examples which show that concepts 1–3 are different in general.

*Example 2.3* Let $E$ be a three-dimensional evolution algebra with a natural basis $\{e_1, e_2, e_3\}$ and the table of multiplication

$$e_1 \cdot e_1 = e_1 + e_2, \quad e_2 \cdot e_2 = -e_1 - e_2, \quad e_3 \cdot e_3 = e_2 + e_3.$$

It is not difficult to see that $E_1 =< e_1 + e_2, e_2 + e_3 >$ is a subalgebra, but $E_1$ is not an evolution subalgebra (that is, there does not exist a natural basis of $E_1$).

Indeed, if we assume the contrary, i.e., in the subspace $E_1$ there exists a natural basis $\{f_1, f_2\}$, then

$$f_1 = \alpha_1(e_1 + e_2) + \alpha_2(e_2 + e_3), \quad f_2 = \beta_1(e_1 + e_2) + \beta_2(e_2 + e_3),$$

with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$.

From the condition $f_1 \cdot f_2 = 0$ we derive

$$\alpha_1 = \alpha_2 = 0 \quad \text{or} \quad \alpha_2 = \beta_2 = 0 \quad \text{or} \quad \beta_1 = \beta_2 = 0.$$

Consequently, we get a contradiction with the assumption that $\{f_1, f_2\}$ is a natural basis of $E_1$.

*Example 2.4* Let $E$ be a three-dimensional evolution algebra with a natural basis $\{e_1, e_2, e_3\}$ and the following table of multiplication

$$e_1 \cdot e_1 = e_1 + e_2 + e_3, \quad e_2 \cdot e_2 = -e_1 - e_2 + e_3, \quad e_3 \cdot e_3 = 0.$$

It is not difficult to see that $E_1 =< e_1 + e_2, e_3 >$ is an evolution algebra with a natural basis $\{e_1 + e_2, e_3\}$, but this basis can not be extended to the natural basis of evolution algebra $E$.

If we assume that there exists a natural basis $\{f_1, f_2\}$ of $E_1$ such that $\{f_1, f_2, f_3\}$ is the natural basis of $E$, then

$$f_1 = \alpha_1(e_1 + e_2) + \alpha_2 e_3, \quad f_2 = \beta_1(e_1 + e_2) + \beta_2 e_3, \quad f_3 = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3.$$

From conditions $f_1 \cdot f_3 = f_2 \cdot f_3 = 0$ we deduce $\alpha_1 = \beta_1 = 0$ or $\gamma_1 = \gamma_2 = 0$. Therefore, we get a contradiction with the assumption that $\{f_1, f_2, f_3\}$ is a basis.

For the sake of convenience, we introduce the following definition.

**Definition 2.5** An evolution algebra $E$ is said to be complete if any subalgebra of $E$ is an evolution subalgebra with a natural basis which can be extended to the natural basis of $E$.

In [14] the conditions for basis transformations that preserve naturalness of the basis are given. The relation between the matrices of structure constants in a new and old natural basis is established in terms of a new defined operation on matrices, as well. Since that relation is not practical for our further purposes, we give the following brief version of isomorphisms.

Let us consider non-singular linear transformation $T$ of a given natural basis $\{e_1, \dots, e_n\}$ with a matrix $(t_{ij})_{1\le i,j\le n}$ in this basis and

$$f_i = \sum_{j=1}^{n} t_{ij} e_j, \ 1 \le i \le n.$$

This transformation is isomorphism if and only if $f_i \cdot f_j = 0$ for all $i \ne j$.

In the following theorem we present a list (up to isomorphism) of 2-dimensional evolution algebras.

**Theorem 2.6** [4] *Any 2-dimensional non-abelian evolution algebra E is isomorphic to one of the following, pairwise non-isomorphic, algebras:*

1. $\dim E^2 = 1$
   - $E_1: \ e_1e_1 = e_1,$
   - $E_2: \ e_1e_1 = e_1, \ \ e_2e_2 = e_1,$
   - $E_3: \ e_1e_1 = e_1 + e_2, \ \ e_2e_2 = -e_1 - e_2,$
   - $E_4: \ e_1e_1 = e_2.$
2. $\dim E^2 = 2$
   - $E_5: \ e_1e_1 = e_1 + a_2e_2, \ \ e_2e_2 = a_3e_1 + e_2, \ \ 1 - a_2a_3 \ne 0,$ *where* $E_5(a_2, a_3) \cong E_5'(a_3, a_2),$
   - $E_6: \ e_1e_1 = e_2, \ \ e_2e_2 = e_1 + a_4e_2,$
     *where for* $a_4 \ne 0, \ E_6(a_4) \cong E_6(a_4') \ \Leftrightarrow \ \frac{a_4'}{a_4} = \cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}$ *for some* $k = 0, 1, 2.$

Consider the following $k$-dimensional evolution algebras

$$ES_k: \quad \begin{cases} e_i \cdot e_i = e_{i+1}, \ 1 \le i \le k-1, \\ e_k \cdot e_k = e_1, \end{cases} \qquad EN_k: \quad \begin{cases} e_i \cdot e_i = e_{i+1}, \ 1 \le i \le k-1, \\ e_k \cdot e_k = 0. \end{cases}$$

In [8] the authors describe a complex evolution algebra $E_{n,\pi}(a_1, a_2, \dots, a_n)$ with a basis $\{e_1, e_2, \dots, e_n\}$ and the table of multiplications as follows:

$$\begin{cases} e_i \cdot e_i = a_i e_{\pi(i)}, & 1 \le i \le n, \\ e_i \cdot e_j = 0, & i \ne j, \end{cases}$$

where $\pi$ is an element of the group of permutations $S_n$. Namely, the following assertion is true.

**Theorem 2.7** *An arbitrary evolution algebra $E_{n,\pi}(a_1, a_2, \dots, a_n)$ is isomorphic to the direct sum of evolution algebras $ES_{p_1}, ES_{p_2}, \dots, ES_{p_s}, EN_{k_1}, EN_{k_2}, \dots, EN_{k_r}$, i.e.,*

$$E_{n,\pi}(a_1, a_2, \dots, a_n) \cong ES_{p_1} \oplus ES_{p_2} \oplus \dots \oplus ES_{p_s} \oplus EN_{k_1} \oplus EN_{k_2} \oplus \dots \oplus EN_{k_r},$$

*where* $\sum_{i=1}^{s} p_i + \sum_{j=1}^{r} k_j = n$

We introduce the following sequences:

$$E^1 = E, \quad E^k = \sum_{i=1}^{k-1} E^i E^{k-i}, \ k \ge 1;$$

$$E^{<1>} = E, \quad E^{<k>} = E^{<k-1>} E, \ k \ge 1.$$

**Definition 2.8** [1] An evolution algebra $E$ is called nilpotent (resp. right nilpotent) if there exists $n \in \mathbb{N}$ (resp. $s \in \mathbb{N}$) such that $E^n = 0$ (resp. $E^{<s>} = 0$) and the minimal such number is called the index of nilpotency (resp. right nilpotency).

In the paper [1], it is proved that the notions of nilpotentcy and right nilpotentcy are equivalent. Moreover, the following theorem is true.

**Theorem 2.9** [1] *Let $E$ be an $n$-dimensional evolution algebra. Then $E$ is nilpotent if and only if the matrix of structure constants $A$ can be transformed by permutation of the natural basis to the following form:*

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & 0 & a_{23} & \dots & a_{2n} \\ 0 & 0 & 0 & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

The next theorem gives the classification of evolution algebras of maximal possible index of nilpotency.

**Theorem 2.10** [1] *Any n-dimensional complex evolution algebra with maximal index of nilpotency is isomorphic to one of the pairwise non-isomorphic algebras with the following matrix of structural constants*

$$\begin{pmatrix} 0 & 1 & a_{13} & \dots & a_{1,n-1} & 0 \\ 0 & 0 & 1 & \dots & a_{2,n-1} & 0 \\ 0 & 0 & 0 & \dots & a_{3,n-1} & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

*where one of non-zero $a_{ij}$ can be chosen equal to 1.*

The set of all evolution algebras whose matrices of structural constants have the form of Theorem 2.10 is denoted by $ZN^n$.

## 3 Main result

First we investigate which evolution algebras of the list of Theorem 2.6 are complete (or not).

**Proposition 3.1** *Evolution algebras $E_1$ and $E_4$ are complete.*

*Proof* Since $E$ is two-dimensional, then any non-trivial subalgebra of $E$ is one-dimensional. Let $E_1'$ be a one-dimensional subalgebra of $E_1$ and $E_1' =< x >$ with $x = A_1e_1 + A_2e_2$. Consider

$$x \cdot x = (A_1e_1 + A_2e_2) \cdot (A_1e_1 + A_2e_2) = A_1^2 e_1$$

On the other hand,

$$x \cdot x = \alpha x = \alpha(A_1e_1 + A_2e_2).$$

Then $A_1^2 = \alpha A_1$ and $\alpha A_2 = 0$.

- If $\alpha = 0$, then $A_1 = 0$ and $\{e_2\}$ is the basis of $E_1'$. Obviously, this basis is extendable to the natural basis $\{e_1, e_2\}$ of $E_1$.
- If $\alpha \neq 0$, then $A_1 = \alpha$, $A_2 = 0$ and $\{e_1\}$ is the basis of $E_1'$, which is also extendable to the natural basis of $E_1$.

The proof of the proposition regarding the algebra $E_4$ is carried out in a similar way. □

**Proposition 3.2** *Evolution algebras $E_2$, $E_3$, $E_5$ and $E_6$ are not complete.*

*Proof* **1.** Let $E_2'$ be a one-dimensional subalgebra of $E_2$ and $E_2' =< x >$ with $x = A_1e_1 + A_2e_2$. The equality $x \cdot x = \alpha x$ implies $A_1^2 + A_2^2 = \alpha A_1$ and $\alpha A_2 = 0$. We are seeking a subalgebra with a natural basis that can not be extended to a basis of the algebra. Thus, $\alpha = 0$.

We set $A_2 = iA_1$. Then $x = A_1(e_1 + ie_2)$. Let us assume that the basis $\{x\}$ can be extended to the natural basis of $E_2$, that is, there exists $y \in E_2$ such that $\{x, y\}$ is a natural basis of $E_2$. Let $y = B_1e_1 + B_2e_2$, then

$$0 = x \cdot y = A_1(e_1 + ie_2) \cdot (B_1e_1 + B_2e_2) = A_1(B_1 + iB_2)e_1.$$

Hence $B_2 = iB_1$ and we obtain a contradiction with the linear independence of elements $x$ and $y$. Thus, the evolution algebra $E_2$ is not complete.

**2.** Let $E_3' =< x >$ be a one-dimensional subalgebra of $E_3$ with $x = A_1e_1 + A_2e_2$. Putting $A_1 = A_2 = 1$ we conclude that $E_3' =< x >$ is a subalgebra. Let us assume that $x = e_1 + e_2$ can be extended to the natural basis of $E_3$, then there exists $y = B_1e_1 + B_2e_2$, such that $\{x, y\}$ is a natural basis of $E_3$.

From the following equality

$$0 = x \cdot y = (e_1 + e_2) \cdot (B_1e_1 + B_2e_2) = (B_1 - B_2)e_1 + (B_1 - B_2)e_2,$$

we derive $B_2 = B_1$, which is a contradiction with the condition of $\{x, y\}$ being a basis.

Therefore, evolution algebra $E_3$ is not complete.

**3.** The element $x = A_1e_1 + A_2e_2$ forms a basis of a one-dimensional subalgebra of $E_5$. Therefore, $\alpha x = x \cdot x$ for some $\alpha \in \mathbb{C}$. Note, that the condition $dim E_5 = 2$ implies $x \cdot x \neq 0$ (consequently $\alpha \neq 0$). Without loss of generality we can assume that $\alpha = 1$. Then $x = x \cdot x$ deduce

$$\begin{cases} A_1^2 + A_2^2 a_3 = A_1, \\ A_1^2 a_2 + A_2^2 = A_2. \end{cases} \tag{3.1}$$

It is not difficult to check that the system of Eq. 3.1 has a solution $A_1, A_2$ such that $A_1A_2 \neq 0$. Indeed, if $a_2 = a_3 = 0$, then $A_1 = A_2 = 1$ is a solution of the Eq. 3.1.

Let us assume that $(a_2, a_3) \neq (0, 0)$ then, without loss of generality, we can suppose $a_3 \neq 0$. Then from the Eq. 3.1 we have

$$A_2 = \frac{A_1}{a_3}((a_2a_3 - 1)A_1 + 1), \tag{3.2}$$

$$A_1^3 + \frac{2}{a_2a_3 - 1}A_1^2 + \frac{a_3 + 1}{a_2a_3 - 1}A_1 - \frac{1}{(a_2a_3 - 1)^2} = 0. \tag{3.3}$$

Note that the Eq. 3.3 with respect to $A_1$ has three solutions and one of them is not equal to $-\frac{1}{a_2a_3-1}$. Recall that all solutions equal to $-\frac{1}{a_2a_3-1}$ has the following cubic equation

$$A_1^3 + \frac{3}{a_2a_3 - 1}A_1^2 + \frac{3}{(a_2a_3 - 1)^2}A_1 + \frac{1}{(a_2a_3 - 1)^3} = 0.$$

Therefore, the Eq. 3.1 has a solution $A_1, A_2$ with $A_1A_2 \neq 0$. Consequently, there exists a subalgebra $E_5' =< x >$ with $x = A_1e_1 + A_2e_2$, where $A_1A_2 \neq 0$.

The basis of this subalgebra can not be extended to a natural basis of $E_5$. Indeed, if $y = B_1e_1 + B_2e_2$ with the condition that $\{x, y\}$ is the natural basis of $E$, then

$$0 = x \cdot y = (A_1e_1 + A_2e_2) \cdot (B_1e_1 + B_2e_2) = (A_1B_1 + A_2B_2a_3)e_1 + (A_1B_1a_2 + A_2B_2)e_2,$$

which implies

$$A_1B_1 + A_2B_2a_3 = 0,$$
$$A_1B_1a_2 + A_2B_2 = 0.$$

Since $A_1A_2(1 - a_2a_3) \neq 0$, we get $B_1 = B_2 = 0$. It is a contradiction with the condition of $\{x, y\}$ being a basis.

Therefore, two dimensional evolution algebra $E_5$ is not complete.

**4.** The proof that the algebra $E_6$ is not complete is analogous. □

Next, we present a result on preservation of the completeness of an evolution algebra which is a direct sum of a complete evolution algebra and an abelian algebra.

**Proposition 3.3** *Let $E$ be an $n$-dimensional complete evolution algebra. Then the evolution algebra $E \oplus \mathbb{C}^k$ is also complete.*

*Proof* Let $\{e_1, e_2, \dots, e_n, h_1, h_2, \dots, h_k\}$ be a basis of $E \oplus \mathbb{C}^k$ and $M$ be an $s$-dimensional subalgebra of $E \oplus \mathbb{C}^k$. We set $\{x_1, x_2, \dots, x_s\}$ as a basis of $M$ and $x_i = \sum\limits_{j=1}^{n} \beta_{i,j} e_j + \sum\limits_{j=1}^{k} \gamma_{i,j} h_j$.

Consider

$$x_i \cdot x_j = \sum_{t=1}^{n} \beta_{i,t} \beta_{j,t} \sum_{k=1}^{n} a_{t,k} e_k, \quad 1 \le i, j \le s.$$

Since $x_i \cdot x_j$ belong to $M$, then the elements $\sum\limits_{t=1}^{n} \beta_{i,t}\beta_{j,t} \sum\limits_{k=1}^{n} a_{t,k} e_k$ are expressed by linear combinations of elements $y_i = \sum\limits_{j=1}^{n} \beta_{i,j} e_j,\ 1 \le i \le s$. Consider $N =< y_1, y_2, \dots, y_s >$. It is easy to see that $N$ is a subalgebra of $E$ of dimension $s' \le s$.

For the sake of convenience, by renumeration of indexes, we can assume that basis of $N$ is $\{y_1, y_2, \dots, y_s'\}$.

If $s' = s$, then using conditions of proposition we can find a natural basis $\{y_1, y_2, \dots, y_s, z_1, z_2, \dots, z_{n-s}\}$ of $E$. Thus the following basis $\{x_1, x_2, \dots, x_s, z_1, z_2, \dots, z_{n-s}, h_1, h_2 \dots, h_k\}$ is a natural basis of $E \oplus C^k$.

If $s' < s$, then by elementary transformation of matrices we conclude

$$\begin{pmatrix} \beta_{1,1} & \dots & \beta_{1,n} & \gamma_{1,1} & \dots & \gamma_{1,k} \\ \beta_{2,1} & \dots & \beta_{2,n} & \gamma_{2,1} & \dots & \gamma_{2,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{s,1} & \dots & \beta_{s,n} & \gamma_{s,1} & \dots & \gamma_{s,k} \end{pmatrix} \sim \begin{pmatrix} \beta_{1,1} & \dots & \beta_{1,n} & \gamma_{1,1} & \dots & \gamma_{1,k} \\ \beta_{2,1} & \dots & \beta_{2,n} & \gamma_{2,1} & \dots & \gamma_{2,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{s',1} & \dots & \beta_{s',n} & \gamma_{s',1} & \dots & \gamma_{s',k} \\ 0 & \dots & 0 & \gamma'_{s'+1,1} & \dots & \gamma'_{s'+1,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & \gamma'_{s,1} & \dots & \gamma'_{s,k} \end{pmatrix}.$$

Hence, the following elements

$$x'_i = \begin{cases} \sum\limits_{j=1}^{n} \beta_{i,j} e_j + \sum\limits_{j=1}^{k} \gamma_{i,j} h_j & 1 \le i \le s' \\ \sum\limits_{j=1}^{k} \gamma'_{i,j} h_j & s'+1 \le i \le s \end{cases}$$

form the natural basis of $M$.

Now, we show that this basis is extendable to the natural basis of $E \oplus \mathbb{C}^k$. Due to $N$ being a subalgebra of $E$, we derive the existence of a natural basis $\{y_1, y_2, \dots, y_{s'}, z_1, z_2, \dots, z_{n-s'}\}$ of $E$. It is not difficult to check that the following basis

$$\{x'_1, x'_2, \dots, x'_{s'}, z_1, z_2, \dots, z_{n-s'}, x'_{s'+1}, x'_{s'+2}, \dots, x'_s, h'_1, h'_2 \dots, h_{k+s'-s}\}$$

is the natural basis of $E \oplus \mathbb{C}^k$, where $\{h'_1, h'_2 \dots, h_{k+s'-s}\}$ are the complementary basis elements to $\{x'_{s'+1}, x'_{s'+2}, \dots, x'_s\}$ in $\mathbb{C}^k$. □

Let $E$ be an $n$-dimensional evolution algebra such that $E = E_1 \oplus E_2$, where $E_1$ and $E_2$ are evolution subalgebras of $E$.

**Proposition 3.4** *Let $E$ be a complete evolution algebra. Then the subalgebras $E_1$ and $E_2$ are also complete.*

*Proof* Let $E_1'$ be a subalgebra of $E_1$, then $E_1'$ is a subalgebra of $E$. Therefore there exists a natural basis $\{e_1', e_2', \dots, e_m'\}$ of $E_1'$ which can be extended to the natural basis $\{e_1', e_2', \dots, e_m', x_{m+1}, x_{m+2}, \dots, x_n\}$ of $E$. Since $E = E_1 \oplus E_2$, then $x_j = y_j + z_j$ with $y_j \in E_1, z_j \in E_2, m+1 \le j \le n$. From $e_i' \cdot x_k = 0$ and $x_k \cdot x_t = 0$ we deduce $e_i' \cdot y_k = 0$ and $y_k \cdot y_t = z_k \cdot z_t = 0$. Since $\{e_1', e_2', \dots, e_m', x_{m+1}, x_{m+2}, \dots, x_n\}$ is a basis of $E$, then any element of $E_1$ belongs to $< e_1', e_2', \dots, e_m', y_{m+1}, y_{m+2}, \dots, y_n >$. Choose $y_{j_1}, y_{j_2}, \dots, y_{j_k}$ from the elements $y_{m+1}, y_{m+2}, \dots, y_n$ so that $\{e_1', e_2', \dots, e_m', y_{j_1}, y_{j_2}, \dots, y_{j_k}\}$ is a basis of $E_1$. Thus, $E_1$ is complete. □

The next example shows that the converse assertion of Proposition 3.4 is not true in general.

*Example 3.5* Let $E$ be a 4-dimensional evolution algebra which is a direct sum of two-dimensional evolution algebras $E_1$ and $E_2$, where

$$E_1 : e_1 \cdot e_1 = e_2; \qquad E_2 : e_3 \cdot e_3 = e_4.$$

Clearly, $E_1$ and $E_2$ are complete evolution algebras, but $E$ is not complete. Indeed, the subalgebra $L =< e_1 + e_3, e_2 + e_4 >$ is not an evolution subalgebra.

In the following proposition we identify complete evolution algebras among the algebras of the type $E_{n,\pi}(a_1, a_2, \dots, a_n)$.

**Proposition 3.6** *Let $E$ be an $n$-dimensional complete evolution algebra of the type $E_{n,\pi}(a_1, a_2, \dots, a_n)$. Then $E$ is isomorphic to one of the following non-isomorphic algebras:*

$$ES_1 \oplus \mathbb{C}^{n-1}, \qquad EN_s \oplus \mathbb{C}^{n-s}, \qquad ES_1 \oplus EN_s \oplus \mathbb{C}^{n-s-1}.$$

*Proof* Let $E$ be an algebra of the type $E_{n,\pi}(a_1, a_2, \dots, a_n)$, then by Theorem 2.7 we have

$$E \cong ES_{p_1} \oplus ES_{p_2} \oplus \dots \oplus ES_{p_s} \oplus EN_{k_1} \oplus EN_{k_2} \oplus \dots \oplus EN_{k_r}.$$

From Proposition 3.4 we obtain that algebras $ES_{p_i}$ and $EN_{k_i}$ are complete.

If there exists $p_j \ge 2$ with $1 \le j \le s$ then, we have

$$ES_{p_j} : \quad \begin{cases} e_i \cdot e_i = e_{i+1}, & 1 \le i \le p_j - 1, \\ e_{p_j} \cdot e_{p_j} = e_1, \end{cases}$$

This algebra is not complete, because the one-dimensional subalgebra $< x >$ with $x = e_1 + e_2 + \dots + e_{p_j}$ is not an evolution subalgebra. Thus, $p_j = 1$ for any $j \in \{1, \dots, s\}$.

If there exist $i$ and $j$ such that $p_i = p_j = 1$, then from Example 3.5 we conclude that $E$ is not complete. Therefore, we can assume $p_1 = 1$ and $p_j = 0$ for $2 \le j \le s$.

Let us suppose that there exist $i$ and $j$ such that $k_i \ge 2, k_j \ge 2$. Without loss of generality we can assume $i = 1, \ j = 2$ and $k_1 \ge k_2$. We denote by $\{e_1, e_2, \dots, e_{k_1}\}$ and $\{f_1, f_2, \dots, f_{k_2}\}$ the basis of $EN_{k_1}$ and $EN_{k_2}$, respectively. Then $M =< x_1, x_2, \dots, x_{k_2} >$ with $x_i = e_{k_1-k_2+i} + f_i, \ 1 \le i \le k_2$ form a subalgebra of $E$ with the following products

$$x_i \cdot x_i = x_{i+1}, \quad 1 \le i \le k_2 - 1, \qquad x_{k_2} \cdot x_{k_2} = 0.$$

It is not difficult to check that $M$ is not an evolution subalgebra. Thus, we get a contradiction with the assumption that there exist $i$ and $j$ such that $k_i \geq 2, k_j \geq 2$. Therefore, we can assume $k_j = 1$ for $2 \leq j \leq r$.

Since $EN_1$ is a one-dimensional algebra with trivial multiplication, then by Proposition 3.3 it is enough to consider the case $s = r = 1$, that is, we reduce the study to $ES_p \oplus EN_k$ with $p \in \{0, 1\}$.

- In the case of $p = 1$ and $k = 1$ we obtain the algebra $ES_1 \oplus \mathbb{C}^{n-1}$;
- In the case of $p = 1$ and $k \geq 2$ we obtain the algebra $ES_1 \oplus EN_k \oplus \mathbb{C}^{n-k-1}$;
- In the case of $p = 0$, we obtain the algebra $EN_k \oplus \mathbb{C}^{n-k}$.

It is not difficult to check that all obtained algebras $ES_1 \oplus \mathbb{C}^{n-1}$, $EN_s \oplus \mathbb{C}^{n-s}$, $ES_1 \oplus EN_s \oplus \mathbb{C}^{n-s-1}$ are complete. □

## 4 Nilpotent Case

Let $E$ be an $n$-dimensional non-abelian evolution algebra with a natural basis $\{e_1, e_2, \ldots, e_n\}$. By transformation of the basic elements we get the following table of multiplication

$$e_i^2 \neq 0, \quad 1 \leq i \leq k, \qquad e_i^2 = 0, \quad k+1 \leq i \leq n, \quad k \leq n. \tag{4.1}$$

We consider the notation given in Theorem 2.9.

**Proposition 4.1** *Let $rank(A) < k$. Then $E$ is not complete.*

*Proof* We prove the statement of the proposition by the contrary. Let us assume that $rank(A) = s < k$, then there exist indexes $i_1, i_2, \ldots, i_s$ such that the elements $e_{i_1}^2, e_{i_2}^2, \ldots, e_{i_s}^2$ are linearly independent. For the sake of convenience assume that $e_1^2, e_2^2, \ldots, e_s^2$ are linearly independent.

Consider the non-trivial linear combination

$$\alpha_1 e_1^2 + \alpha_2 e_2^2 + \cdots + \alpha_s e_s^2 + \alpha_{s+1} e_{s+1}^2 = 0.$$

Since $\alpha_{s+1} \neq 0$ (otherwise we obtain trivial linear combination) we get

$$e_{s+1}^2 = -\frac{\alpha_1}{\alpha_{s+1}} e_1^2 - \frac{\alpha_2}{\alpha_{s+1}} e_2^2 - \cdots - \frac{\alpha_s}{\alpha_{s+1}} e_s^2.$$

Due to existence $\alpha_i \neq 0$ for some $1 \leq i \leq s$, without loss of generality, we can assume $\alpha_1 \neq 0$.

For the element $x = \sqrt{\alpha_1} e_1 + \sqrt{\alpha_2} e_2 + \cdots + \sqrt{\alpha_s} e_s + \sqrt{\alpha_{s+1}} e_{s+1}$ we have $x \cdot x = 0$. Hence, $< x >$ is a one-dimensional subalgebra. Consequently, there exists a natural basis $\{x, y_2, y_3, \ldots, y_n\}$ of $E$.

Let us introduce the following denotations

$$y_i = \sum_{j=1}^{n} \beta_{i,j} e_j, \quad 2 \leq i \leq n.$$

Consider

$$0 = x \cdot y_i = \left( \sum_{j=1}^{s+1} \sqrt{\alpha_j} e_j \right) \cdot \left( \sum_{j=1}^{n} \beta_{i,j} e_j \right) = \sum_{j=1}^{s+1} \sqrt{\alpha_j} \beta_{i,j} e_j^2$$

$$= \sum_{j=1}^{s} \sqrt{\alpha_j} \beta_{i,j} e_j^2 - \sqrt{\alpha_{s+1}} \beta_{i,s+1} \sum_{j=1}^{s} \frac{\alpha_j}{\alpha_{s+1}} e_j^2 = \sum_{j=1}^{s} \left( \sqrt{\alpha_j} \beta_{i,j} - \sqrt{\alpha_{s+1}} \beta_{i,s+1} \frac{\alpha_j}{\alpha_{s+1}} \right) e_j^2.$$

Thus,

$$\sqrt{\alpha_j} \beta_{i,j} - \sqrt{\alpha_{s+1}} \beta_{i,s+1} \frac{\alpha_j}{\alpha_{s+1}} = 0, \quad 2 \le i \le n, \ 1 \le j \le s. \tag{4.2}$$

For $j = 1$ in the restrictions (4.2) we obtain

$$\beta_{i,s+1} = \sqrt{\frac{\alpha_{s+1}}{\alpha_1}} \beta_{i,1}, \quad 2 \le i \le n.$$

We have that $\{x, y_2, y_3, \dots, y_n\}$ and $\{e_1, e_2, \dots, e_n\}$ are two bases of $E$. Then the matrix of change of basis has the following form:

$$B = \begin{pmatrix} \sqrt{\alpha_1} & \dots & \sqrt{\alpha_s} & \sqrt{\alpha_{s+1}} & 0 & \dots & 0 \\ \beta_{2,1} & \dots & \beta_{2,s} & \sqrt{\frac{\alpha_{s+1}}{\alpha_1}} \beta_{2,1} & \beta_{2,s+2} & \dots & \beta_{2,n} \\ \beta_{3,1} & \dots & \beta_{3,s} & \sqrt{\frac{\alpha_{s+1}}{\alpha_1}} \beta_{3,1} & \beta_{3,s+2} & \dots & \beta_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{n,1} & \dots & \beta_{n,s} & \sqrt{\frac{\alpha_{s+1}}{\alpha_1}} \beta_{n,1} & \beta_{n,s+2} & \dots & \beta_{n,n} \end{pmatrix}.$$

Since $det(B) = 0$ we get a contradiction. Thus, the algebra $E$ is not complete. □

In the following theorem we describe the nilpotent complete evolution algebras.

**Theorem 4.2** *An arbitrary nilpotent complete evolution algebra is isomorphic to*

$$\widetilde{E} \oplus \mathbb{C}^{n-k},$$

*where* $\widetilde{E} \in ZN^k$.

*Proof* Let $E$ be a nilpotent complete evolution algebra with the table of multiplication (4.1). Then the matrix $A$ has the form:

$$A = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \dots & a_{1,k+1} & \dots & a_{1,n} \\ 0 & 0 & a_{2,3} & \dots & a_{2,k+1} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a_{k,k+1} & \dots & a_{k,n} \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

Putting $e'_{k+1} = \sum\limits_{j=k+1}^{n} a_{k,j} e_j$ we can assume $e_k^2 = e_{k+1}$, that is, we can always suppose $a_{k,k+1} = 1$ and $a_{k,j} = 0$ for $k + 2 \le j \le n$.

Let $a_{1,2}a_{2,3}\dots a_{k-1,k} = 0$. Then we denote by $t$ the greatest number such that $a_{t,t+1} = 0$, i.e., $a_{i,i+1} \neq 0$ for $t+1 \leq i \leq k+1$. If $a_{i,i+1} = 0$ for all $1 \leq i \leq k-1$, then we put $t = k$.

Consider the subalgebra $E_1 =< e_t + e_{t+1}, e_{t+2}, \dots, e_n >$. Then there exists a natural basis $\{y_1, y_2, \dots, y_t, e_t + e_{t+1}, e_{t+2}, \dots, e_n\}$ of $E$.

We set $y_i = \sum\limits_{j=1}^{n} \beta_{i,j} e_j$ with $1 \leq i \leq t$. Then

$$0 = (e_t + e_{t+1}) \cdot y_i = \beta_{i,t} e_t^2 + \beta_{i,t+1} e_{t+1}^2.$$

Due to Proposition 4.1 we conclude that $rank(A) = k$. It implies that $e_t^2$ and $e_{t+1}^2$ are linearly independent. Therefore, $\beta_{i,t} = \beta_{i,t+1} = 0,\ 1 \leq i \leq t$. We have two bases in $E : \{x, y_2, y_3, \dots, y_n\}$ and $\{e_1, e_2, \dots, e_n\}$. Then, the matrix of changes of basis has the following form:

$$B = \begin{pmatrix} \beta_{1,1} & \dots & \beta_{1,t-1} & 0 & 0 & \beta_{1,t+2} & \dots & \beta_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_{t,1} & \dots & \beta_{t,t-1} & 0 & 0 & \beta_{t,t+2} & \dots & \beta_{t,n} \\ 0 & \dots & 0 & 1 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Hence, $det(B) = 0$ and we get a contradiction. Therefore, $a_{1,2}a_{2,3}\dots a_{k-1,k} \neq 0$.

Taking the following change of basis:

$$\begin{cases} e'_1 = a_{1,2}^{-1/2} a_{2,3}^{-1/4} \dots a_{k-1,k}^{-1/2^{k-1}} e_1, \\ e'_2 = a_{2,3}^{-1/2} a_{3,4}^{-1/4} \dots a_{k-1,k}^{-1/2^{k-2}} e_2, \\ \dots\dots\dots\dots\dots\dots \\ e'_{k-1} = a_{k-1,k}^{-1/2} e_{k-1}, & k \leq i \leq n, \\ e'_i = e_i, \end{cases}$$

we can assume $a_{1,2} = a_{2,3} = \dots = a_{k-1,k} = 1$.

Moreover, the basis transformation

$$e''_j = e'_j + \sum_{i=k+2}^{n} a_{j-1,i} e'_i + \sum_{i=k+2}^{n} \left( \sum_{t=j}^{k-1} a_{t,i} \left( \sum_{p=1}^{t-j+1} (-1)^p \prod_{h=1}^{p} a_{j-2+h,t+1-p+h} \right) \right) e'_i,$$
$$2 \leq j \leq k,$$

implies that the algebra $E$ belongs to the family of algebras $ZN^{k+1} \oplus \mathbb{C}^{n-k-1}$. Taking into account the result of Proposition 3.3 it is enough to prove that any evolution algebra of the set $ZN^n$ is complete.

Indeed, if a subalgebra $M$ of $\widetilde{E}$ (where $\widetilde{E} \in ZN^k$) contains an element $e_j + \sum\limits_{s=j+1}^{k} \beta_s e_s$, then $\{e_j, e_{j+1}, \dots, e_k\} \subseteq M$. Hence, algebra $\widetilde{E}$ has only subalgebras of the form $E_i =<$

$e_i, e_{i+1}, \dots, e_k >$. It is not difficult to see that the subalgebras $E_i$ are evolution subalgebras of $\widetilde{E}$. □

## 5 Conjectures

In this section we formulate two related conjectures. A positive answer to the first conjecture implies a positive answer for the second one. In fact, the correctness of the second conjecture completes the description of complete evolution algebras.

**Conjecture 5.1** *Let $A = (a_{i,j})_{1\le i,j\le n}$ be a complex invertible matrix. Then the following system of equations*

$$\begin{pmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_n^2 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{2,1} & \dots & a_{n,1} \\ a_{1,2} & a_{2,2} & \dots & a_{n,2} \\ \vdots & \vdots & \dots & \vdots \\ a_{1,n} & a_{2,n} & \dots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \tag{5.1}$$

*has a solution $(x_1, x_2, \dots, x_n)$ such that $x_i \neq 0$ for all $i$.*

- If $n = 1$, then this conjecture is obviously true.
- If $n = 2$, then we consider subcases:

1. Subcase $(a_{1,2}, a_{2,1}) = (0, 0)$. Then $a_{1,1}a_{2,2} \neq 0$ and we have a solution $x_1 = a_{1,1}, x_2 = a_{2,2}$.
2. Subcase $(a_{1,2}, a_{2,1}) \neq (0, 0)$. Then, without loss of generality, we can assume $a_{1,2} \neq 0$. Putting $x_2 = \frac{1}{a_{1,2}}(x_1^2 - a_{1,1}x_1)$, we get

$$x_1(x_1^3 - 2a_{1,1}x_1^2 + (a_{1,1}^2 - a_{1,2}a_{2,2})x_1 + a_{1,2}(a_{1,2}a_{2,1} - a_{2,2}a_{1,1})) = 0. \tag{5.2}$$

Since $a_{1,2}(a_{1,2}a_{2,1} - a_{2,2}a_{1,1}) \neq 0$, the Eq. 5.2 has three non-trivial solution. Moreover,

$$x_1^3 - 2a_{1,1}x_1^2 + (a_{1,1}^2 - a_{1,2}a_{2,2})x_1 + a_{1,2}(a_{1,2}a_{2,1} - a_{2,2}a_{1,1}) \neq (x - a_{1,1})^3.$$

From this inequality we deduce that Eq. 5.2 has a solution $x_1$ different from 0 and $a_{1,1}$. Hence, $x_2 = \frac{1}{a_{1,2}}(x_1^2 - a_{1,1}x_1) \neq 0$.

Thus, Conjecture 5.1 for the case $n = 2$ is correct, as well.

Now we present two consequences of Conjecture 5.1 about the description of complete evolution algebras.

**Conjecture 5.2** *Let $E$ be an $n$-dimensional ($n \geq 2$) evolution algebra with the natural basis $\{e_1, e_2, \dots, e_n\}$ and invertible matrix $A$. Then $E$ is not complete.*

Indeed, if we consider $x \cdot x = x$ with $x = \sum_{i=1}^{n} x_i e_i$, then comparing the coefficients at the basic elements $e_i$, we obtain the system of Eq. 5.1. Due to $det A \neq 0$ and according to Conjecture 5.1 we get the existence of a solution $(x_1, x_2, \dots, x_n)$ such that $x_i \neq 0$ for all $i$. Therefore, $E_1 =< x >$ is a subalgebra of $E$. However, this subalgebra is not an evolution subalgebra and the assumption of Conjecture 5.2 is correct.

**Conjecture 5.3** *Let $E$ be an $n$-dimensional non-nilpotent complete evolution algebra. Then $E$ is isomorphic to one of the following, pairwise non-isomorphic, algebras:*

$$ES_1 \oplus \mathbb{C}^{n-1}, \qquad ES_1 \oplus \widetilde{E} \oplus \mathbb{C}^{n-s-1},$$

*where $\widetilde{E} \in ZN^s$ is a nilpotent evolution algebra with maximal index of nilpotency.*

**Explanation of Conjecture 5.3**

Let $E$ be an $n$-dimensional non-nilpotent complete evolution algebra with the table of multiplication (4.1).

Note that the table of multiplication (4.1) for $k = 1$ gives the algebra $ES_1 \oplus \mathbb{C}^{n-1}$. Therefore, further we assume $k \geq 2$.

Let us introduce the denotations $x_{s,t} = (a_{s,1}, a_{s,2}, \dots, a_{s,t})$ for $1 \leq s \leq t$ and $1 \leq t \leq k$.

Note that there are no $s'$ and $s''$ such that $x_{s',k} = x_{s'',k} = (0, 0, \dots, 0)$. In fact, if there exist $s'$ and $s''$, then the subalgebra $E_1 =< e_{s'} + e_{s''}, e_{k+1}, \dots, e_n >$ is not an evolution subalgebra.

It is not difficult to see that the non-zero vectors $x_{s_1,k}, x_{s_2,k}, \dots, x_{s_t,k}$ are linearly independent. Otherwise there exists a non-trivial linear combination

$$\alpha_1 x_{s_1,k} + \alpha_2 x_{s_2,k} + \cdots + \alpha_t x_{s_t,k} = 0,$$

and the subalgebra $E_1 =< \sqrt{\alpha_1} e_{s_1} + \sqrt{\alpha_2} e_{s_2} + \cdots + \sqrt{\alpha_t} e_{s_t}, e_{k+1}, e_{k+2}, \dots, e_n >$ is not an evolution subalgebra.

**Iteration 1** Let us assume that all vectors $x_{s,k}$ are non-zero (there are $k$-pieces), then the determinant of the main minor of order $k$ is non-zero.

Then taking the change $e'_i = e_i + \sum\limits_{j=k+1}^{n} \beta_{i,j} e_j, \ 1 \leq i \leq k$, where $\beta_{i,j}$ are found from the following equation

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,k} \\ a_{2,1} & a_{2,2} & \dots & a_{2,k} \\ \vdots & \vdots & \dots & \vdots \\ a_{k,1} & a_{k,2} & \dots & a_{k,k} \end{pmatrix} \begin{pmatrix} \beta_{1,k+1} & \beta_{1,k+2} & \dots & \beta_{1,n} \\ \beta_{2,k+1} & \beta_{2,k+2} & \dots & \beta_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ \beta_{k,k+1} & \beta_{k,k+2} & \dots & \beta_{k,n} \end{pmatrix} = \begin{pmatrix} a_{1,k+1} & a_{1,k+2} & \dots & a_{1,n} \\ a_{2,k+1} & a_{2,k+2} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k,k+1} & a_{k,k+2} & \dots & a_{k,n} \end{pmatrix}, \tag{5.3}$$

and we obtain that the evolution algebra $E$ is isomorphic to the algebra $E' \oplus \mathbb{C}^{n-k}$. The basis $\{e'_1, e'_2, \dots, e'_k\}$ is the natural basis of the evolution algebra $E'$. Due to Proposition 3.3 the evolution algebra $E'$ should be complete, but according to Conjecture 5.2 the algebra $E'$ is not complete. Thus, in this case we get a contradiction.

Let us suppose that there exists some $s_0$ such that $x_{s_0,k} = (0, 0, \dots, 0)$. Without loss of generality, we can assume $s_0 = k$. Then we obtain the multiplication

$$e_i \cdot e_i = \sum_{i=1}^{n} a_{i,j} e_i, \quad 1 \leq i \leq k-1, \quad e_k \cdot e_k = \sum_{i=k+1}^{n} a_{i,j} e_i, \quad e_i \cdot e_i = 0, \quad k+1 \leq i \leq n.$$

Applying a change of basis similar to Eq. 5.3 we can assume $a_{i,j} = 0$ for $1 \leq i \leq k-1, \ k+1 \leq j \leq n$. In addition, choosing $e'_{k+1} = \sum\limits_{i=k+1}^{n} a_{i,j} e_i$, we derive $e_k \cdot e_k = e_{k+1}$.

**Iteration 2** Now we consider the vectors $x_{s,k-1} = (a_{s,1}, a_{s,2}, \dots, a_{s,k-1})$, for $1 \leq s \leq k-1$.

Reduce our study to the case when all vectors $x_{s,k-1}$ are non-zero. Then the main minor of order $k-1$ is non-zero and the equality $x \cdot x = x$ with $x = \sum_{i=1}^{k+1} x_i e_i$ implies the following system of equations

$$\begin{pmatrix} a_{1,1} & a_{2,1} & \dots & a_{k-1,1} & 0 & 0 \\ a_{1,2} & a_{2,2} & \dots & a_{k-1,2} & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ a_{1,k} & a_{2,k} & \dots & a_{k-1,k} & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_k^2 \\ x_{k+1}^2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ x_{k+1} \end{pmatrix}.$$

From Conjecture 5.1 we have the existence of solution $x_i \neq 0$ of the system of equation

$$\begin{pmatrix} a_{1,1} & a_{2,1} & \dots & a_{k-1,1} \\ a_{1,2} & a_{2,2} & \dots & a_{k-1,2} \\ \vdots & \vdots & \dots & \vdots \\ a_{1,k-1} & a_{2,k-1} & \dots & a_{k-1,k-1} \end{pmatrix} \begin{pmatrix} x_1^2 \\ x_2^2 \\ \vdots \\ x_{k-1}^2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{k-1} \end{pmatrix}$$

and $x_k = \sum_{s=1}^{k-1} a_{s,k} x_s^2,\ x_{k+1} = x_k^2$. Therefore, the element $x$ is not extendable to the natural basis of the evolution algebra $E$. We get a contradiction with the assumption that all vectors $x_{s,k-1}$ are non-zero.

Continuing the iterations for all vectors $x_{s,k-2},\ x_{s,k-3}, \dots x_{s,2}$, we conclude that there exists $s_t$ such that $x_{s_t,t} = (0, 0, \dots, 0)$ for all $t$. By shifting basis elements we can assume that $s_t = t$ and we obtain that the evolution algebra $E$ is isomorphic to the following algebra:

$$\begin{aligned} &e_1 \cdot e_1 = \sum_{j=1}^{k} a_{i,j} e_i,\ e_i \cdot e_i = \sum_{j=i+1}^{k} a_{i,j} e_i,\ 2 \leq i \leq k-1, \\ &e_k \cdot e_k = e_{k+1}, \qquad e_i \cdot e_i = 0, \qquad\qquad k+1 \leq i \leq n. \end{aligned}$$

For the element $x = \sum_{i=1}^{k+1} x_i e_i$ the equality $x \cdot x = x$ implies the system of equations as follows

$$\begin{cases} a_{1,1} x_1^2 = x_1, \\ a_{1,2} x_1^2 = x_2, \\ a_{1,3} x_1^2 + a_{2,3} x_2^2 = x_3, \\ \dots\dots\dots\dots\dots\dots \\ a_{1,k} x_1^2 + a_{2,k} x_2^2 + \dots + a_{k-1,k} x_{k-1}^2 = x_k, \\ x_k^2 = x_{k+1}. \end{cases}$$

Taking into account that the algebra $E$ is non-nilpotent, we have $a_{1,1} \neq 0$ and $x_1 = \frac{1}{a_{1,1}}$.

If $(a_{1,2}, a_{1,3}, \dots, a_{1,k}) \neq (0, 0, \dots, 0)$, then there exists a solution $(x_1, \dots x_{k+1})$ such that $x_i \neq 0$ for some $2 \leq i \leq k+1$. Similarly as above we conclude that the evolution algebra $E$ is not complete.

Thus, we get $(a_{1,2}, a_{1,3}, \dots, a_{1,k}) = (0, 0, \dots, 0)$. Hence an $n$-dimensional non-nilpotent complete evolution algebra $E$ is isomorphic to one of the following, pairwise non-isomorphic, algebras:

$$ES_1 \oplus C^{n-1}, \qquad ES_1 \oplus \widetilde{E} \oplus C^{n-s-1}, \qquad \widetilde{E} \in ZN^s.$$

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