Algebr Represent Theor (2014) 17:945–969
DOI 10.1007/s10468-013-9426-y

The Classification of Non-Characteristically Nilpotent Filiform Leibniz Algebras

A. K. Khudoyberdiyev · M. Ladra and B. A. Omirov

Received: 12 March 2013 / Accepted: 7 May 2013 / Published online: 23 May 2013

Abstract In this paper we investigate the derivations of filiform Leibniz algebras. Recall that the set of filiform Leibniz algebras of fixed dimension is decomposed into three non-intersected families. We found sufficient conditions under which filiform Leibniz algebras of the first family are characteristically nilpotent. Moreover, for the first family we classify non-characteristically nilpotent algebras by means of Catalan numbers. In addition, for the rest two families of filiform Leibniz algebras we describe non-characteristically nilpotent algebras, i.e., those filiform Leibniz algebras which lie in the complementary set to those characteristically nilpotent.

Keywords Lie algebra · Leibniz algebra · Derivation · Nilpotency · Characteristically nilpotent algebra · Catalan numbers

Mathematics Subject Classifications (2010) 17A32 · 17A36 · 17B30

1 Introduction

In 1955, Jacobson [13] proved that every Lie algebra over a field of characteristic zero admitting a non-singular derivation is nilpotent. The problem whether the inverse

Presented by Alain Verschoren and Peter Littelmann.

A. K. Khudoyberdiyev (✉) · B. A. Omirov
Institute of Mathematics, 29, Do'rmon yo'li srt., 100125, Tashkent, Uzbekistan
e-mail: khabror@mail.ru

B.A. Omirov
e-mail: omirovb@mail.ru

M. Ladra
Department of Algebra, University of Santiago de Compostela,
15782, Santiago de Compostela, Spain
e-mail: manuel.ladra@usc.es

of this statement is correct remained open until an example of an 8-dimensional nilpotent Lie algebra all of whose derivations are nilpotent was constructed in [7]. They called such type of algebras characteristically nilpotent Lie algebras.

If all derivations of an algebra are nilpotent (inner derivations are nilpotent, as well), then by Engel's theorem we conclude that a characteristically nilpotent Lie algebra is nilpotent. The inverse statement is not true, because there exist nilpotent Lie algebras admitting non-nilpotent derivations. Therefore, the subset of characteristically nilpotent Lie algebras is strictly embedded into the set of nilpotent Lie algebras.

The papers [6, 14, 17] and others are devoted to the investigation of characteristically nilpotent Lie algebras. The classification of nilpotent Lie algebras till dimension 8 shows that there are no characteristically nilpotent Lie algebras in dimensions less than 7. Moreover, it is shown that there exist characteristically nilpotent Lie algebras in each dimension from 7 till 13-dimensional. Taking into account that a direct sum of characteristically nilpotent Lie algebras is characteristically nilpotent, then we have the existence of characteristically nilpotent Lie algebras in each finite dimension starting from 7.

It was conjectured for a long time that there are "a few" algebras of this kind, and only in [15], it was proved that every irreducible component of the variety of complex filiform Lie algebras of dimension greater than 7 contains a Zariski open set, consisting of characteristically nilpotent Lie algebras. This implies that there are "many" characteristically nilpotent Lie algebras, and hence they play an important role in the description of the variety of nilpotent Lie algebras.

The notion of Leibniz algebra has been introduced in [18] as a non-antisymmetric generalization of Lie algebras. During the last 20 years the theory of Leibniz algebras has been actively studied and many results of the theory of Lie algebras have been extended to Leibniz algebras (see, e.g. [1, 3, 9]). In particular, an analogue of Jacobson's theorem was proved for Leibniz algebras [16]. Moreover, it is shown that similarly to the case of Lie algebras for Leibniz algebras the inverse of Jacobson's statement does not hold. In [21], analogously as for Lie algebras, the notion of characteristically nilpotent Leibniz algebra was defined and some families of characteristically nilpotent filiform Leibniz algebras were found. Moreover, there was presented a characterization of of characteristically nilpotency of some filiform Leibniz algebras. Due to the existence of an example of a characteristically nilpotent Leibniz algebra which does not satisfy the condition of [21], the characterization is not correct.

It is known that the class of all filiform Leibniz algebras is split into three non-intersecting families [3, 9], where one of the families contains filiform Lie algebras and the other two families come out from naturally graded non-Lie filiform Leibniz algebras. An isomorphism criterion for these two families of filiform Leibniz algebras have been given in [9].

In this paper, as opposed to [21], we find out a characterization of characteristically nilpotency of filiform Leibniz algebras (see Theorems 3.2, 3.8 and 3.10). In addition, we described, up to isomorphism, the class of filiform Leibniz algebras complementary to characteristically nilpotent filiform Leibniz algebras. Note that filiform Leibniz algebras were classified only up to dimension less than 10 in [8, 20, 23, 24]. Here we classify non-characteristically nilpotent non-Lie filiform Leibniz algebras for any fixed dimension. Recall that non-characteristically nilpotent filiform Lie algebras are described in [11].

The classification of non-characteristically nilpotent Leibniz algebras plays an important role in the structure theory of solvable Leibniz algebras. In the theory of finite dimensional Leibniz algebras it is known the description of solvable Leibniz algebras with a given nilradical based on properties of non-nilpotent derivations of the nilradical. Hence, solvable Leibniz algebras can have only non-characteristically nilpotent nilradical. Therefore, it is very crucial to indicate non-characteristically nilpotent Leibniz algebras. The papers [19, 25, 26] are devoted to classifications of solvable Lie algebras with various types of nilradical. The solvable Leibniz algebras with null-filiform and naturally graded filiform nil-radical are classified in [4, 5].

Catalan numbers are a well-known sequence of numbers and they are involved in a lot of branches of mathematics (combinatorics, graph theory, probability theory and many others). In the present paper we classify some kinds of non-characteristically nilpotent filiform Leibniz algebras in terms of p-th Catalan numbers.

In order to achieve our goal, we have organized the paper as follows: in Section 2 we present necessary definitions and results that will be used in the rest of the paper. In Section 3 we describe characteristically nilpotent filiform non-Lie Leibniz algebras and give the classification of non-characteristically nilpotent filiform non-Lie Leibniz algebras.

Throughout the paper all the spaces and algebras are assumed finite dimensional.

2 Preliminaries

In this section we give necessary definitions and preliminary results.

Definition 2.1 An algebra $(L, [-,-])$ over a field F is called a Leibniz algebra if for any $x, y, z \in L$, the so-called Leibniz identity

$$\big[[x, y], z\big] = \big[[x, z], y\big] + \big[x, [y, z]\big]$$

holds.

For a Leibniz algebra L consider the following central lower series:

$$L^1 = L, \qquad L^{k+1} = [L^k, L^1] \qquad k \geq 1.$$

Since the notions of right nilpotency and nilpotency coincide [2], we can define nilpotency as follows:

Definition 2.2 A Leibniz algebra L is called nilpotent if there exists $s \in \mathbb{N}$ such that $L^s = 0$.

Definition 2.3 A Leibniz algebra L is said to be filiform if $\dim L^i = n - i$, where $n = \dim L$ and $2 \leq i \leq n$.

The following theorem decomposes all $(n+1)$-dimensional filiform Leibniz algebras into three families of algebras.

Theorem 2.4 (Omirov and Rakhimov [22]) *Any complex $(n+1)$-dimensional filiform Leibniz algebra admits a basis $\{e_0, e_1, \dots, e_n\}$ such that the table of multiplication of the algebra has one of the following forms:*

$$F_1(\alpha_3, \alpha_4, \dots, \alpha_n, \theta): \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0, e_1] = \sum\limits_{k=3}^{n-1} \alpha_k e_k + \theta e_n, & \\ [e_i, e_1] = \sum\limits_{k=i+2}^{n} \alpha_{k+1-i} e_k, & 1 \leq i \leq n-2, \end{cases}$$

$$F_2(\beta_3, \beta_4, \dots, \beta_n, \gamma): \begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_0, e_1] = \sum\limits_{k=3}^{n} \beta_k e_k, & \\ [e_1, e_1] = \gamma e_n, & \\ [e_i, e_1] = \sum\limits_{k=i+2}^{n} \beta_{k+1-i} e_k, & 2 \leq i \leq n-2, \end{cases}$$

$$F_3(\theta_1, \theta_2, \theta_3) = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \leq i \leq n-1, \\ [e_0, e_i] = -e_{i+1}, & 2 \leq i \leq n-1, \\ [e_0, e_0] = \theta_1 e_n, & \\ [e_0, e_1] = -e_2 + \theta_2 e_n, & \\ [e_1, e_1] = \theta_3 e_n, & \\ [e_i, e_j] = -[e_j, e_i] \in \mathrm{lin} < e_{i+j+1}, e_{i+j+2}, \dots, e_n >, & 1 \leq i < j < n-1, \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = \alpha(-1)^i e_n, & 1 \leq i \leq n-1, \end{cases}$$

where $\alpha \in \{0, 1\}$ for odd n and $\alpha = 0$ for even n. Moreover, the structure constants of an algebra from $F_3(\theta_1, \theta_2, \theta_3)$ should satisfy the Leibniz identity.

It is easy to see that algebras of the first and the second families are non-Lie algebras. Note that if $(\theta_1, \theta_2, \theta_3) = (0, 0, 0)$, then an algebra of the third class is a Lie algebra and if $(\theta_1, \theta_2, \theta_3) \neq (0, 0, 0)$, then it is a non-Lie Leibniz algebra.

Further we will use the following lemma.

Lemma 2.5 (Gómez and Omirov [9]) *For any $0 \leq p \leq n-k,\ 3 \leq k \leq n$, the following equality holds:*

$$\sum_{i=k}^{n} a(i) \sum_{j=i+p}^{n} b(i, j) e_j = \sum_{j=k+p}^{n} \sum_{i=k}^{j-p} a(i) b(i, j) e_j.$$

Let us present an isomorphism criterion for the first and second families of non-Lie filiform Leibniz algebras.

Theorem 2.6 (Gómez and Omirov [9])

(a) Two algebras from the families $F_1(\alpha_3, \alpha_4, \dots, \alpha_n, \theta)$ and $F_1^{'}(\alpha_3', \alpha_4', \dots, \alpha_n', \theta')$ are isomorphic if and only if there exist $A, B \in \mathbb{C}$ such that $A(A+B) \neq 0$ and the following conditions hold:

$$\alpha_3' = \frac{(A+B)}{A^2}\alpha_3,$$

$$\begin{aligned}
\alpha_t' = \frac{1}{A^{t-1}} \Bigg((A+B)\alpha_t - \sum_{k=3}^{t-1} \Bigg(& \binom{k-1}{k-2} A^{k-2} B \alpha_{t+2-k} \\
& + \binom{k-1}{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^{t} \alpha_{t+3-i_1} \alpha_{i_1+1-k} \\
& + \binom{k-1}{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^{t} \sum_{i_1=k+3}^{i_2} \alpha_{t+3-i_2} \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1-k} + \cdots \\
& + \binom{k-1}{1} A B^{k-2} \sum_{i_{k-3}=2k-2}^{t} \sum_{i_{k-4}=2k-2}^{i_{k-3}} \cdots \sum_{i_1=2k-2}^{i_2} \alpha_{t+3-i_{k-3}} \\
& \cdot \alpha_{i_{k-3}+3-i_{k-4}} \cdot \cdots \cdot \alpha_{i_2+3-i_1} \alpha_{i_1+5-2k} \\
& + B^{k-1} \sum_{i_{k-2}=2k-1}^{t} \sum_{i_{k-3}=2k-1}^{i_{k-2}} \cdots \sum_{i_1=2k-1}^{i_2} \alpha_{t+3-i_{k-2}} \\
& \cdot \alpha_{i_{k-2}+3-i_{k-3}} \cdot \cdots \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+4-2k} \Bigg) \cdot \alpha_k' \Bigg),
\end{aligned}$$

$$\begin{aligned}
\theta' = \frac{1}{A^{n-1}} \Bigg(A\theta + B\alpha_n - \sum_{k=3}^{n-1} \Bigg(& \binom{k-1}{k-2} A^{k-2} B \alpha_{n+2-k} \\
& + \binom{k-1}{k-3} A^{k-3} B^2 \sum_{i_1=k+2}^{n} \alpha_{n+3-i_1} \alpha_{i_1+1-k} \\
& + \binom{k-1}{k-4} A^{k-4} B^3 \sum_{i_2=k+3}^{n} \sum_{i_1=k+3}^{i_2} \alpha_{n+3-i_2} \alpha_{i_2+3-i_1} \alpha_{i_1-k} + \cdots \\
& + \binom{k-1}{1} A B^{k-2} \sum_{i_{k-3}=2k-2}^{n} \sum_{i_{k-4}=2k-2}^{i_{k-3}} \cdots \sum_{i_1=2k-2}^{i_2} \alpha_{n+3-i_{k-3}} \\
& \times \alpha_{i_{k-3}+3-i_{k-4}} \dots \alpha_{i_2+3-i_1} \alpha_{i_1+5-2k} \\
& + B^{k-1} \sum_{i_{k-2}=2k-1}^{n} \sum_{i_{k-3}=2k-1}^{i_{k-2}} \cdots \sum_{i_1=2k-1}^{i_2} \alpha_{n+3-i_{k-2}} \\
& \cdot \alpha_{i_{k-2}+3-i_{k-3}} \cdot \cdots \cdot \alpha_{i_2+3-i_1} \cdot \alpha_{i_1+4-2k} \Bigg) \cdot \alpha_k' \Bigg),
\end{aligned}$$

where $4 \leq t \leq n$.

(b) Two algebras from the families $F_2(\beta_3, \beta_4, \ldots, \beta_n, \gamma)$ and $F_2'(\beta_3', \beta_4', \ldots, \beta_n', \gamma')$ are isomorphic if and only if there exist $A, B, D \in \mathbb{C}$ such that $AD \neq 0$ and the following conditions hold:

$$\gamma' = \frac{D^2}{A^n}\gamma,$$

$$\beta_3' = \frac{D}{A^2}\beta_3,$$

$$\begin{aligned}\beta_t' = \frac{1}{A^{t-1}}\Bigg(D\beta_t - \sum_{k=3}^{t-1}\Bigg(&\binom{k-1}{k-2}A^{k-2}B\beta_{t+2-k} + \binom{k-1}{k-3}A^{k-3}B^2\sum_{i_1=k+2}^{t}\beta_{t+3-i_1}\cdot\beta_{i_1+1-k}\\ &+\binom{k-1}{k-4}A^{k-4}B^3\sum_{i_2=k+3}^{t}\sum_{i_1=k+3}^{i_2}\beta_{t+3-i_2}\cdot\beta_{i_2+3-i_1}\cdot\beta_{i_1-k} + \cdots\\ &+\binom{k-1}{1}AB^{k-2}\sum_{i_{k-3}=2k-2}^{t}\sum_{i_{k-4}=2k-2}^{i_{k-3}}\cdots\sum_{i_1=2k-2}^{i_2}\beta_{t+3-i_{k-3}}\beta_{i_{k-3}+3-i_{k-4}}\cdots\beta_{i_2+3-i_1}\beta_{i_1+5-2k}\\ &+\ B^{k-1}\sum_{i_{k-2}=2k-1}^{t}\sum_{i_{k-3}=2k-1}^{i_{k-2}}\cdots\sum_{i_1=2k-1}^{i_2}\beta_{t+3-i_{k-2}}\beta_{i_{k-2}+3-i_{k-3}}\cdots\beta_{i_2+3-i_1}\beta_{i_1+4-2k}\Bigg)\beta_k'\Bigg),\end{aligned}$$

where $4 \leq t \leq n-1$*,*

$$\begin{aligned}\beta_n' = \frac{BD\gamma}{A^n} + \frac{1}{A^{n-1}}\Bigg(D\beta_n - \sum_{k=3}^{n-1}\Bigg(&\binom{k-1}{k-2}A^{k-2}B\beta_{n+2-k}\\ &+\binom{k-1}{k-3}A^{k-3}B^2\sum_{i_1=k+2}^{n}\beta_{n+3-i_1}\cdot\beta_{i_1+1-k}\\ &+\binom{k-1}{k-4}A^{k-4}B^3\sum_{i_2=k+3}^{n}\sum_{i_1=k+3}^{i_2}\beta_{n+3-i_2}\cdot\beta_{i_2+3-i_1}\cdot\beta_{i_1-k} + \cdots\\ &+\binom{k-1}{1}AB^{k-2}\sum_{i_{k-3}=2k-2}^{n}\sum_{i_{k-4}=2k-2}^{i_{k-3}}\cdots\sum_{i_1=2k-2}^{i_2}\beta_{n+3-i_{k-3}}\beta_{i_{k-3}+3-i_{k-4}}\cdots\beta_{i_2+3-i_1}\beta_{i_1+5-2k}\\ &+\ B^{k-1}\sum_{i_{k-2}=2k-1}^{n}\sum_{i_{k-3}=2k-1}^{i_{k-2}}\cdots\sum_{i_1=2k-1}^{i_2}\beta_{n+3-i_{k-2}}\beta_{i_{k-2}+3-i_{k-3}}\cdots\beta_{i_2+3-i_1}\beta_{i_1+4-2k}\Bigg)\beta_k'\Bigg),\end{aligned}$$

where $\binom{m}{n}$ are the binomial coefficients.

Derivations of Leibniz algebras are defined as usual:

Definition 2.7 A linear transformation d of a Leibniz algebra L is called a derivation if for any $x, y \in L$

$$d([x, y]) = [d(x), y] + [x, d(y)].$$

A nilpotent Leibniz algebra is called *characteristically nilpotent* if all its derivations are nilpotent. As it was mentioned in Section 1, the class of characteristically nilpotent Leibniz algebras is a subclass of the nilpotent Leibniz algebras.

In [21] the following characterization of characteristically nilpotency is obtained.

Theorem 2.8 (Omirov [21]) *A Leibniz algebra of the family* $F_1(\alpha_3, \alpha_4, \dots, \alpha_n, \theta)$ *is characteristically nilpotent if there exist* i, j $(3 \le i \ne j \le n)$ *such that* $\alpha_i \alpha_j \ne 0$.

Further we shall need the notion of Catalan numbers. The Catalan numbers are defined as follows:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}.$$

The generalized Catalan numbers or p-th Catalan numbers were defined in [12] by the formula

$$C_n^p = \frac{1}{(p-1)n+1}\binom{pn}{n}.$$

Obviously, 2-th Catalan numbers are usual Catalan numbers.

H. W. Gould developed a generalization of the n-th Catalan numbers, also called Rothe numbers or Rothe/Hagen coefficients of the first type (see [10]), as follows:

$$A_n(x, z) = \frac{x}{x+zn}\binom{x+zn}{n},$$

together with their convolution formula

$$\sum_{k=0}^{n} A_k(x, z) A_{n-k}(y, z) = A_n(x+y, z). \tag{2.1}$$

Note that $A_n(1, p)$ is the p-th Catalan number C_n^p.

From the convolution formula 2.1, it is not difficult to obtain the following formula:

$$\sum_{k=1}^{n} C_k^p C_{n-k}^p = \frac{2n}{(p-1)n+p+1} C_{n+1}^p. \tag{2.2}$$

3 The Main Results

Since filiform characteristically nilpotent Lie algebras are already in detail studied in [14, 15], we shall consider only non-Lie Leibniz algebras.

In this section we describe characteristically nilpotent filiform non-Lie Leibniz algebras and give the classification of non-characteristically nilpotent filiform non-Lie Leibniz algebras.

3.1 Characteristically Nilpotent Filiform Leibniz Algebras of the Family $F_1(\alpha_3, \alpha_4, \dots, \alpha_n, \theta)$

Let L be a filiform Leibniz algebra from the family $F_1(\alpha_3, \alpha_4, \dots, \alpha_n, \theta)$. The following proposition describes the derivations of such algebras.

Proposition 3.1 *The derivations of the filiform Leibniz algebras from the family $F_1(\alpha_3, \alpha_4, \ldots, \alpha_n, \theta)$ have the following form:*

$$\begin{pmatrix}
a_0 & a_1 & a_2 & a_3 & \ldots & a_{n-2} & a_{n-1} & a_n \\
0 & a_0+a_1 & a_2 & a_3 & \ldots & a_{n-2} & b_{n-1} & b_n \\
0 & 0 & 2a_0+a_1 & a_2+a_1\alpha_3 & \ldots & a_{n-3}+a_1\alpha_{n-2} & a_{n-2}+a_1\alpha_{n-1} & a_{n-1}+a_1\alpha_n \\
0 & 0 & 0 & 3a_0+a_1 & \ldots & a_{n-4}+2a_1\alpha_{n-2} & a_{n-3}+2a_1\alpha_{n-1} & a_{n-2}+2a_1\alpha_{n-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & (n-2)a_0+a_1 & a_2+(n-3)a_1\alpha_3 & a_3+(n-3)a_1\alpha_4 \\
0 & 0 & 0 & 0 & \ldots & 0 & (n-1)a_0+a_1 & a_2+(n-2)a_1\alpha_3 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & na_0+a_1
\end{pmatrix}$$

where

$$a_0(\theta - \alpha_n) = 0, \quad a_1(\alpha_n - \theta) = a_{n-1} - b_{n-1}, \quad \alpha_3(a_1 - a_0) = 0,$$

$$\alpha_k(a_1 - (k-2)a_0) = \frac{k}{2} a_1 \sum_{j=4}^{k} \alpha_{j-1}\alpha_{k-j+3}, \quad 4 \le k \le n. \tag{3.1}$$

Proof Let L be a filiform Leibniz algebra from the family $F_1(\alpha_3, \alpha_4, \ldots, \alpha_n, \theta)$ and let $d : L \to L$ be a derivation of L.

Put

$$d(e_0) = \sum_{k=0}^{n} a_k e_k, \quad d(e_1) = \sum_{k=0}^{n} b_k e_k.$$

By the property of the derivation, we have

$$\begin{aligned}
d(e_2) &= d([e_0, e_0]) = [d(e_0), e_0] + [e_0, d(e_0)] = \left[\sum_{k=0}^{n} a_k e_k, e_0\right] + \left[e_0, \sum_{k=0}^{n} a_k e_k\right] \\
&= (a_0 + a_1)e_2 + \sum_{k=3}^{n} a_{k-1} e_k + a_0 e_2 + a_1 \left(\sum_{k=3}^{n-1} \alpha_k e_k + \theta e_n\right) \\
&= (2a_0 + a_1)e_2 + \sum_{k=3}^{n-1} (a_{k-1} + a_1\alpha_k)e_k + (a_{n-1} + a_1\theta)e_n.
\end{aligned}$$

By induction we derive

$$d(e_i) = (ia_0 + a_1)e_i + \sum_{k=i+1}^{n} (a_{k-i+1} + (i-1)a_1\alpha_{k-i+2})e_k, \quad 3 \le i \le n.$$

Indeed, if the induction hypothesis is true for i, then for $i+1$ it implies, from the following chain of equalities:

$$\begin{aligned}
d(e_{i+1}) &= d([e_i, e_0]) = [d(e_i), e_0] + [e_i, d(e_0)] \\
&= \left[(ia_0 + a_1)e_i + \sum_{k=i+1}^{n} (a_{k-i+1} + (i-1)a_1\alpha_{k-i+2})e_k, e_0\right] + \left[e_i, \sum_{k=0}^{n} a_k e_k\right] \\
&= (ia_0 + a_1)e_{i+1} + \sum_{k=i+2}^{n} (a_{k-i} + (i-1)a_1\alpha_{k-i+1})e_k + a_0 e_{i+1} + a_1 \sum_{k=i+2}^{n} \alpha_{k-i+1} e_k \\
&= ((i+1)a_0 + a_1)\, e_{i+1} + \sum_{k=i+2}^{n} (a_{k-i} + ia_1\alpha_{k-i+1})e_k.
\end{aligned}$$

Consider the property of derivation:

$$\begin{aligned}
d([e_1, e_0]) &= [d(e_1), e_0] + [e_1, d(e_0)] = \left[\sum_{k=0}^{n} b_k e_k, e_0\right] + \left[e_1, \sum_{k=0}^{n} a_k e_k\right] \\
&= (b_0 + b_1)e_2 + \sum_{k=3}^{n} b_{k-1} e_k + a_0 e_2 + a_1 \sum_{k=3}^{n} \alpha_k e_k \\
&= (a_0 + b_0 + b_1)e_2 + \sum_{k=3}^{n} (b_{k-1} + a_1\alpha_k)e_k.
\end{aligned}$$

On the other hand

$$d([e_1, e_0]) = d(e_2) = (2a_0 + a_1)e_2 + \sum_{k=3}^{n-1} (a_{k-1} + a_1\alpha_k)e_k + (a_{n-1} + a_1\theta)e_n.$$

Comparing the coefficients at the basic elements we have

$$b_0 + b_1 = a_0 + a_1, \quad b_i = a_i, \quad 2 \le i \le n-2, \quad a_1(\alpha_n - \theta) = a_{n-1} - b_{n-1}.$$

Using Lemma 2.5 we obtain

$$\begin{aligned}
d([e_0, e_1]) &= [d(e_0), e_1] + [e_0, d(e_1)] = \left[\sum_{k=0}^{n} a_k e_k, e_1\right] + \left[e_0, \sum_{k=0}^{n} b_k e_k\right] \\
&= a_0\left(\sum_{k=3}^{n-1} \alpha_k e_k + \theta e_n\right) + \sum_{k=1}^{n-2} a_k \sum_{j=k+2}^{n} \alpha_{j-k+1} e_j + b_0 e_2 + b_1\left(\sum_{k=3}^{n-1} \alpha_k e_k + \theta e_n\right)
\end{aligned}$$

$$
\begin{aligned}
&= b_0 e_2 + (a_0 + b_1)\alpha_3 e_3 + \sum_{k=4}^{n-1}(a_0 + b_1)\alpha_k e_k + (a_0 + b_1)\theta e_n + a_1\alpha_3 e_3 \\
&\quad + a_1 \sum_{j=4}^{n} \alpha_j e_j + \sum_{k=4}^{n} a_{k-2} \sum_{j=k}^{n} \alpha_{j-k+3} e_j \\
&= b_0 e_2 + (a_0 + a_1 + b_1)\alpha_3 e_3 + \sum_{k=4}^{n-1}(a_0 + a_1 + b_1)\alpha_k e_k + ((a_0 + b_1)\theta + a_1\alpha_n)\, e_n \\
&\quad + \sum_{k=4}^{n}\left(\sum_{j=4}^{k} a_{j-2}\alpha_{k-j+3}\right) e_k \\
&= b_0 e_2 + (a_0 + a_1 + b_1)\alpha_3 e_3 + \sum_{k=4}^{n-1}\left((a_0 + a_1 + b_1)\alpha_k + \sum_{j=4}^{k} a_{j-2}\alpha_{k-j+3}\right) e_k \\
&\quad + \left((a_0 + b_1)\theta + a_1\alpha_n + \sum_{j=4}^{n} a_{j-2}\alpha_{n-j+3}\right) e_n.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
d([e_0, e_1]) &= d\left(\sum_{k=3}^{n-1}\alpha_k e_k + \theta e_n\right) = \sum_{k=3}^{n-1}\alpha_k d(e_k) + \theta d(e_n) \\
&= \sum_{k=3}^{n-1}\alpha_k\left((k a_0 + a_1)e_k + \sum_{j=k+1}^{n}(a_{j-k+1} + (k-1)a_1\alpha_{j-k+2})e_j\right) \\
&\quad + (n a_0 + a_1)\theta e_n \\
&= (3a_0 + a_1)\alpha_3 e_3 + \sum_{k=4}^{n-1}\alpha_k(k a_0 + a_1)e_k + (n a_0 + a_1)\theta e_n \\
&\quad + \sum_{k=4}^{n}\left(\alpha_{k-1}\sum_{j=k}^{n}(a_{j-k+2} + (k-2)a_1\alpha_{j-k+3})e_j\right) \\
&= (3a_0 + a_1)\alpha_3 e_3 + \sum_{k=4}^{n-1}\left((k a_0 + a_1)\alpha_k + \sum_{j=4}^{k} a_{j-2}\alpha_{k-j+3} + \frac{k}{2}a_1\sum_{j=4}^{k}\alpha_{j-1}\alpha_{k-j+3}\right) e_k \\
&\quad + \left((n a_0 + a_1)\theta + \sum_{j=4}^{n} a_{j-2}\alpha_{n-j+3} + \frac{n}{2}a_1\sum_{j=4}^{n}\alpha_{j-1}\alpha_{n-j+3}\right) e_n.
\end{aligned}
$$

Comparing the coefficients at the basic elements we conclude

$$
b_0 = 0 \Rightarrow b_1 = a_0 + a_1, \quad (a_0 + a_1 + b_1)\alpha_3 = (3a_0 + a_1)\alpha_3,
$$

$$(a_0 + a_1 + b_1)\alpha_k + \sum_{j=4}^{k} a_{j-2}\alpha_{k-j+3} = (ka_0 + a_1)\alpha_k + \sum_{j=4}^{k} a_{j-2}\alpha_{k-j+3} + \frac{k}{2}a_1 \sum_{j=4}^{k} \alpha_{j-1}\alpha_{k-j+3}, 4 \leq k \leq n-1,$$

$$(a_0 + b_1)\theta + a_1\alpha_n + \sum_{j=4}^{n} a_{j-2}\alpha_{n-j+3} = (na_0 + a_1)\theta + \sum_{j=4}^{n} a_{j-2}\alpha_{n-j+3} + \frac{n}{2}a_1 \sum_{j=4}^{n} \alpha_{j-1}\alpha_{n-j+3}.$$

Replacing $b_1 = a_0 + a_1$ we get

$$(a_1 - a_0)\alpha_3 = 0,$$

$$(a_1 - (k-2)a_0)\alpha_k = \frac{k}{2}a_1 \sum_{j=4}^{k} \alpha_{j-1}\alpha_{k-j+3}, \qquad 4 \leq k \leq n-1,$$

$$(2-n)a_0\theta + a_1\alpha_n = \frac{n}{2}a_1 \sum_{j=4}^{n} \alpha_{j-1}\alpha_{n-j+3}. \tag{3.2}$$

Similarly to the above argumentations we derive

$$d([e_1, e_1]) = [d(e_1), e_1] + [e_1, d(e_1)] = 2(a_0 + a_1)\alpha_3 e_3 + \sum_{k=4}^{n} \left((2a_0 + 2a_1)\alpha_k + \sum_{j=4}^{k} a_{j-2}\alpha_{k-j+3} \right) e_k.$$

On the other hand

$$d([e_1, e_1]) = d\left(\sum_{k=3}^{n} \alpha_k e_k\right) = \sum_{k=3}^{n} \alpha_k d(e_k)$$
$$= (3a_0 + a_1)\alpha_3 e_3 + \sum_{k=4}^{n} \left((ka_0 + a_1)\alpha_k + \sum_{j=4}^{k} a_{j-2}\alpha_{k-j+3} + \frac{k}{2}a_1 \sum_{j=4}^{k} \alpha_{j-1}\alpha_{k-j+3} \right) e_k.$$

Comparing the coefficients at the basic elements we obtain

$$(a_1 - a_0)\alpha_3 = 0$$

and the restriction 3.1, i.e. $(a_1 - (k-2)a_0)\alpha_k = \frac{k}{2}a_1 \sum\limits_{j=4}^{k} \alpha_{j-1}\alpha_{k-j+3}, 4 \leq k \leq n.$

From Eq. 3.1 for $k = n$ and the restriction 3.2, we have $a_0(\theta - \alpha_n) = 0$.

Considering the properties of the derivation for $d([e_i, e_2]),\ 3 \leq i \leq n-2$, we have the same restrictions. □

From Proposition 3.1 it is obvious that if there exist the pair a_0, a_1 such that $(a_0, a_1) \neq (0, 0)$ and the restriction 3.1 holds, then a filiform Leibniz algebra of the first family is non-characteristically nilpotent, otherwise is characteristically nilpotent.

From [5] and [21] it is known that the naturally graded filiform Leibniz algebra (the algebra with $\alpha_i = 0, 3 \leq i \leq n, \theta = 0$) is non-characteristically nilpotent.

Theorem 3.2 *Let $\theta \neq \alpha_n$ and suppose that there exist $\alpha_k \neq 0,\ \ 3 \leq k \leq n$. Then a filiform Leibniz algebra of the family $F_1(\alpha_3, \alpha_4, \ldots, \alpha_n, \theta)$ is characteristically nilpotent.*

Proof Note that it is sufficient to prove $a_0 = a_1 = 0$.

Let $\theta \neq \alpha_n$, then the restriction 3.1 implies that $a_0 = 0$ and we get

$$\alpha_3 a_1 = 0, \quad a_1 \alpha_k = \frac{k}{2} a_1 \sum_{j=4}^{k} \alpha_{j-1} \alpha_{k-j+3}, \quad 4 \leq k \leq n.$$

If there exist $\alpha_k \neq 0$, then for the first non-zero $\alpha_k \neq 0$, we have $\alpha_k a_1 = 0$. Hence $a_1 = 0$. □

From the above theorem we have that an algebra of the class $F_1(0, 0, \ldots, 0, \theta),\ \theta \neq 0$, is non-characteristically nilpotent.

Remark 3.3 Note that in the notations of Theorem 2.6 putting $A = \sqrt[n-2]{\theta}$, we conclude that an algebra $F_1(0, 0, \ldots, 0, \theta),\ \theta \neq 0$, is isomorphic to the algebra $F_1(0, 0, \ldots, 0, 1)$.

Below, we present an example which shows that Theorem 2.8 is not true in general.

Example 3.4 Let L be a 6-dimensional filiform Leibniz algebra and let $\{e_0, e_1, e_2, e_3, e_4, e_5\}$ be a basis of L with the following multiplication:

$$\begin{cases} [e_0, e_0] = e_2, & \\ [e_i, e_0] = e_{i+1}, & 1 \leq i \leq 4, \\ [e_0, e_1] = e_3 - 2e_4 + 5e_5, & \\ [e_1, e_1] = e_3 - 2e_4 + 5e_5, & \\ [e_2, e_1] = e_4 - 2e_5, & \\ [e_3, e_1] = e_5, & \end{cases}$$

(omitted products are equal to zero).

Clearly, this algebra satisfies the condition of Theorem 2.8, but it is non-characteristically nilpotent, because the derivations of the algebra have the form:

$$\begin{pmatrix} a_1 & a_1 & a_3 & a_4 & a_5 & a_6 \\ 0 & 2a_1 & a_3 & a_4 & a_5 & b_6 \\ 0 & 0 & 3a_1 & a_1+a_3 & -2a_1+a_4 & 5a_1+a_5 \\ 0 & 0 & 0 & 4a_1 & 2a_1+a_3 & -4a_1+a_4 \\ 0 & 0 & 0 & 0 & 5a_1 & 3a_1+a_3 \\ 0 & 0 & 0 & 0 & 0 & 6a_1 \end{pmatrix}.$$

Let us now consider the case $\alpha_n = \theta$.

Lemma 3.5 *Let L be a non-characteristically nilpotent filiform Leibniz algebra from the family $F_1(\alpha_3, \alpha_4, \dots, \alpha_n, \alpha_n)$ and let $\alpha_s \neq 0$ be the first non-zero parameter from $\{\alpha_3, \alpha_4, \dots, \alpha_n\}$. Then*

$$\alpha_k = \begin{cases} 0, & \text{if } k \not\equiv s \pmod{(s-2)}; \\ (-1)^t C_{t+1}^{s-1} \alpha_s^{t+1}, & \text{if } k \equiv s \pmod{(s-2)}, \end{cases}$$

where $3 \leq k \leq n$, $t = \frac{k-s}{s-2}$ and $C_n^p = \frac{1}{(p-1)n+1}\binom{pn}{n}$ is the p-th Catalan number.

Proof Since $\alpha_s \neq 0$, from the equality 3.1, we obtain $(a_1 - (s-2)a_0)\alpha_s = 0$, and consequently, $a_1 = (s-2)a_0$. Replacing $a_1 = (s-2)a_0$ we have

$$(s-k)a_0\alpha_k = \frac{k}{2}(s-2)a_0 \sum_{j=4}^{k} \alpha_{j-1}\alpha_{k-j+3}, \quad k \geq s+1.$$

Since the algebra is non-characteristically nilpotent, we have $a_0 \neq 0$ and

$$\alpha_k = \frac{k(s-2)}{2(s-k)} \sum_{j=4}^{k} \alpha_{j-1}\alpha_{k-j+3}. \tag{3.3}$$

We will prove the statement of the lemma by induction on $l = \lfloor \frac{k-s}{s-2} \rfloor$, where $\lfloor x \rfloor$ is the integer part of x.

The base of induction $l = 0$ is straightforward from the condition of the lemma.

Let us suppose the induction hypothesis is true for $t < l$ and we will prove it for $l = \lfloor \frac{k-s}{s-2} \rfloor$.

From equality 3.3 we have that if $k \not\equiv s \pmod{(s-2)}$, then one of the values of $j-1$ and $k-j+3$ are not congruent by mod $(s-2)$ with s, simultaneously. Otherwise, if $j-1 \equiv s \pmod{(s-2)}$ and $k-j+3 \equiv s \pmod{(s-2)}$, then $k \equiv s \pmod{(s-2)}$, which is a contradiction. Thus, by induction hypothesis, we have $\alpha_{j-1}\alpha_{k-j+3} = 0$ for any j $(4 \leq j \leq k)$, which implies $\alpha_k = 0$ for any $k \not\equiv s \pmod{(s-2)}$ with $\lfloor \frac{k-s}{s-2} \rfloor = l$.

If $k \equiv s \pmod{(s-2)}$, i.e. $k = s + (s-2)t$ then

$$\alpha_k = \frac{(s+(s-2)t)(s-2)}{2(s-s-(s-2)t)} \sum_{j=4}^{s+(s-2)t} \alpha_{j-1}\alpha_{s+(s-2)t-j+3}$$

$$= -\frac{s+(s-2)t}{2t} \sum_{j=4}^{s+(s-2)t} \alpha_{j-1}\alpha_{s+(s-2)t-j+3} = -\frac{s+(s-2)t}{2t} \sum_{j=s+1}^{(s-2)t+3} \alpha_{j-1}\alpha_{s+(s-2)t-j+3}.$$

Changing $j-1 = s+(s-2)j'$ and using the induction hypothesis, we obtain

$$\alpha_k = -\frac{s+(s-2)t}{2t} \sum_{j'=0}^{t-1} \alpha_{s+(s-2)j'}\alpha_{s+(s-2)(t-j'-1)}$$

$$= -\frac{s+(s-2)t}{2t}(-1)^{t-1}\alpha_s^{t+1} \sum_{j'=0}^{t-1} C_{j'+1}^{s-1} C_{t-j'}^{s-1}$$

$$= (-1)^t \alpha_s^{t+1} \left(\frac{s+(s-2)t}{2t} \sum_{j=1}^{t} C_j^{s-1} C_{t+1-j}^{s-1} \right).$$

Applying formula 2.2, we derive

$$\alpha_k = (-1)^t \alpha_s^{t+1} C_{t+1}^{s-1},$$

where $3 \le k \le n$, $t = \frac{k-s}{s-2}$ and C_n^p is the p-th Catalan number. □

Below, we present the classification of algebras obtained in Lemma 3.5.

Theorem 3.6 *Let L be a non-characteristically nilpotent filiform Leibniz algebra of the family $F_1(\alpha_3, \alpha_4, \ldots, \alpha_n, \alpha_n)$. Then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$F_1^s(\alpha_3, \alpha_4, \ldots, \alpha_n, \alpha_n), \qquad 3 \le s \le n,$$

where

$$\alpha_k = \begin{cases} 0, & \text{if } k \not\equiv s \pmod{(s-2)}; \\ (-1)^t C_{t+1}^{s-1}, & \text{if } k \equiv s \pmod{(s-2)} \text{ for } t = \dfrac{k-s}{s-2}, \end{cases}$$

$3 \le k \le n$ and C_n^p is the p-th Catalan number.

Proof From Lemma 3.5 we have

$$\alpha_k = \begin{cases} 0, & \text{if } k \not\equiv s \pmod{(s-2)}; \\ (-1)^t C_{t+1}^{s-1}\alpha_s^{t+1}, & \text{if } k \equiv s \pmod{(s-2)}. \end{cases}$$

From Theorem 2.6, we have the isomorphism criterion

$$\alpha_s' = \frac{1}{A^{s-1}}(A+B)\alpha_s.$$

Putting $B = \frac{A^{s-1}}{\alpha_s} - A$, we get $\alpha_s' = 1$. Thus, without loss of generality we can assume $\alpha_s = 1$, then

$$\alpha_k = \begin{cases} 0, & \text{if } k \not\equiv s \pmod{(s-2)}; \\ (-1)^t C_{t+1}^{s-1}, & \text{if } k \equiv s \pmod{(s-2)}. \end{cases}$$

□

3.2 Non-Characteristically Nilpotent Filiform Leibniz Algebras of the Family $F_2(\beta_3, \beta_4, \dots, \beta_n, \gamma)$

Now we consider algebras of the family $F_2(\beta_3, \beta_4, \dots, \beta_n, \gamma)$. Similar to the above section, firstly we describe the derivations of such algebras.

Proposition 3.7 *Any derivation of a filiform Leibniz algebra of the family $F_2(\beta_3, \beta_4, \dots, \beta_n, \gamma)$ has the form:*

$$\begin{pmatrix} a_0 & a_1 & a_2 & a_3 & \dots & a_{n-2} & a_{n-1} & a_n \\ 0 & b_1 & 0 & 0 & \dots & 0 & -a_1\gamma & b_n \\ 0 & 0 & 2a_0 & a_2 + a_1\beta_3 & \dots & a_{n-3} + a_1\beta_{n-2} & a_{n-2} + a_1\beta_{n-1} & a_{n-1} + a_1\beta_n \\ 0 & 0 & 0 & 3a_0 & \dots & a_{n-4} + 2a_1\beta_{n-3} & a_{n-3} + 2a_1\beta_{n-2} & a_{n-2} + 2a_1\beta_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & (n-2)a_0 & a_2 + (n-3)a_1\beta_3 & a_3 + (n-3)a_1\beta_4 \\ 0 & 0 & 0 & 0 & \dots & 0 & (n-1)a_0 & a_2 + (n-2)a_1\beta_3 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & na_0 \end{pmatrix}$$

where

$$\gamma(2b_1 - na_0) = 0, \qquad \beta_3(b_1 - 2a_0) = 0,$$

$$\beta_k(b_1 - (k-1)a_0) = \frac{k}{2}a_1 \sum_{j=4}^{k} \beta_{j-1}\beta_{k-j+3}, \qquad 4 \le k \le n-1, \tag{3.4}$$

$$\beta_n(b_1 - (n-1)a_0) = -a_1\gamma + \frac{n}{2}a_1 \sum_{j=4}^{n} \beta_{j-1}\beta_{n-j+3}.$$

Proof Let L be a filiform Leibniz algebra from the second family and let $d : L \to L$ be a derivation of L.

We set

$$d(e_0) = \sum_{k=0}^{n} a_k e_k, \quad d(e_1) = \sum_{k=0}^{n} b_k e_k.$$

From the property of the derivation d one has

$$d(e_2) = d([e_0, e_0]) = [d(e_0), e_0] + [e_0, d(e_0)] = \left[\sum_{k=0}^{n} a_k e_k, e_0\right] + \left[e_0, \sum_{k=0}^{n} a_k e_k\right]$$
$$= a_0 e_2 + \sum_{k=3}^{n} a_{k-1} e_k + a_0 e_2 + a_1 \sum_{k=3}^{n} \beta_k e_k = 2a_0 e_2 + \sum_{k=3}^{n} (a_{k-1} + a_1 \beta_k) e_k.$$

By induction it is not difficult to obtain

$$d(e_i) = i a_0 e_i + \sum_{k=i+1}^{n} (a_{k+1-i} + (i-1) a_1 \beta_{k+2-i})\, e_k, \quad 2 \leq i \leq n.$$

Consider the property of the derivation

$$d([e_1, e_0]) = [d(e_1), e_0] + [e_1, d(e_0)] = \left[\sum_{k=0}^{n} b_k e_k, e_0\right] + \left[e_1, \sum_{k=0}^{n} a_k e_k\right]$$
$$= b_0 e_2 + \sum_{k=3}^{n} b_{k-1} e_k + a_1 \gamma e_n.$$

On the other hand

$$d([e_1, e_0]) = 0.$$

Consequently, $b_0 = b_2 = b_3 = \cdots = b_{n-2} = 0, \quad b_{n-1} = -a_1 \gamma.$
From the chain of equalities

$$n a_0 \gamma e_n = d(\gamma e_n) = d([e_1, e_1]) = [d(e_1), e_1] + [e_1, d(e_1)]$$
$$= [b_1 e_1 + b_{n-1} e_{n-1} + b_n e_n, e_1] + [e_1, b_1 e_1 + b_{n-1} e_{n-1} + b_n e_n] = 2 b_1 \gamma e_n,$$

we get $(2b_1 - n a_0)\gamma = 0$.
Using Lemma 2.5 and the derivation property, we obtain

$$d([e_0, e_1]) = [d(e_0), e_1] + [e_0, d(e_1)]$$
$$= (a_0 + b_1)\beta_3 e_3 + \sum_{k=4}^{n} (a_0 + b_1)\beta_k e_k + a_1 \gamma e_n + \sum_{k=4}^{n} \left(\sum_{j=4}^{k} a_{j-2} \beta_{k-j+3}\right) e_k.$$

On the other hand

$$d([e_0, e_1]) = d\left(\sum_{k=3}^{n} \beta_k e_k\right) = \sum_{k=3}^{n} \beta_k d(e_k)$$
$$= 3 a_0 \beta_3 e_3 + \sum_{k=4}^{n} k a_0 \beta_k e_k + \sum_{k=4}^{n} \left(\sum_{j=4}^{k} a_{j-2} \beta_{k-j+3}\right) e_k$$
$$+ a_1 \sum_{k=4}^{n} \frac{k}{2} \left(\sum_{j=4}^{k} \beta_{j-1} \beta_{k-j+3}\right) e_k.$$

Comparing the coefficients at the basic elements we deduce

$$\beta_3(b_1 - 2a_0) = 0,$$
$$\beta_k(b_1 - (k-1)a_0) = \frac{k}{2}a_1 \sum_{j=4}^{k} \beta_{j-1}\beta_{k-j+3}, \qquad 4 \le k \le n-1,$$
$$\beta_n(b_1 - (n-1)a_0) = -a_1\gamma + \frac{n}{2}a_1 \sum_{j=4}^{n} \beta_{j-1}\beta_{n-j+3}.$$

Considering the properties of the derivation for products $d([e_i, e_1]),\ 2 \le i \le n-2$, we already get the obtained restrictions. □

From Proposition 3.7 it is obvious that if there exists the pair a_0, b_1 such that $(a_0, b_1) \neq (0, 0)$ and the restriction 3.4 holds, then a filiform Leibniz algebra is non-characteristically nilpotent, otherwise is characteristically nilpotent.

It is known that a naturally graded filiform Leibniz algebra of the second family (an algebra with $\gamma = 0$ and $\beta_i = 0,\ 3 \le i \le n$) is non-characteristically nilpotent [5, 21].

Theorem 3.8 *Let $\gamma \neq 0$ and n be odd. If there there exist $\beta_i \neq 0,\ 3 \le i \le n-1$, then a filiform Leibniz algebra from $F_2(\beta_3, \beta_4, \ldots, \beta_n, \gamma)$ is characteristically nilpotent.*

Proof If $\gamma \neq 0$, then since $\gamma(2b_1 - na_0) = 0$, this implies that $b_1 = \frac{na_0}{2}$ and we get that the restrictions 3.4 have the form

$$\frac{(n-4)a_0}{2}\beta_3 = 0,$$
$$\frac{n-2k+2}{2}\beta_k a_0 = \frac{k}{2}a_1 \sum_{j=4}^{k} \beta_{j-1}\beta_{k-j+3}, \qquad 4 \le k \le n-1,$$
$$\frac{-n+2}{2}\beta_n a_0 = -a_1\gamma + \frac{n}{2}a_1 \sum_{j=4}^{n} \beta_{j-1}\beta_{n-j+3}.$$

If there exist $\beta_k \neq 0,\ 3 \le k \le n-1$, then for the first non-zero $\beta_k \neq 0$, we get

$$(n-2k+2)a_0\beta_k = 0.$$

Since n is odd, we conclude $a_0 = b_1 = 0$. □

Let us clarify the situation when $\beta_i = 0$ for $3 \le i \le n-1$.

Theorem 3.9 *Let $\gamma \neq 0$ and n be odd. Then any non-characteristically nilpotent filiform Leibniz algebra of the second class is isomorphic to the algebra*

$$F_2(0, 0, \ldots, 0, 0, 1).$$

Proof Theorem 3.8 implies that if n is odd, then a non-characteristically nilpotent filiform Leibniz algebra of the second family has the form $F_2(0, 0, \dots, 0, \beta_n, \gamma)$, where

$$\frac{-n+2}{2}\beta_n a_0 = -a_1\gamma.$$

Since $\gamma \neq 0$, then for any β_n putting $a_0 \neq 0$ and $a_1 = \frac{(n-2)\beta_n a_0}{2\gamma}$, we have a non-singular derivation.

From Theorem 2.6, we derive $\beta'_k = 0,\ 3 \leq k \leq n-1$, and the isomorphism criterion is

$$\gamma' = \frac{D^2}{A^n}\gamma, \quad \beta'_n = \frac{BD\gamma}{A^n} + \frac{D}{A^{n-1}}\beta_n.$$

Since $\gamma \neq 0$, putting $D = \sqrt{\frac{A^n}{\gamma}}$, we get

$$\gamma' = 1, \quad \beta'_n = \frac{B\gamma + A\beta_n}{\sqrt{\gamma A^n}}.$$

Setting $B = -\dfrac{A\beta_n}{\gamma}$, we have $\beta'_n = 0$ and so we obtain the algebra $F_2(0, 0, \dots, 0, 1)$. □

Now we investigate the even n case.

Theorem 3.10 *Let $\gamma \neq 0$ and n be even. If there exist $\beta_k \neq 0,\ 3 \leq k \leq n-1,\ k \neq \frac{n+2}{2}$, then an algebra of the family $F_2(\beta_3, \beta_4, \dots, \beta_n, \gamma)$ is characteristically nilpotent.*

Proof Analogously to the proof of Theorem 3.8 □

Theorem 3.11 *Let L be a non-characteristically nilpotent filiform Leibniz algebra from the family $F_2(\beta_3, \beta_4, \dots, \beta_n, \gamma)$. If $\gamma \neq 0$ and n is even, then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$F_2(0, \dots, 0, \beta_{\frac{n+2}{2}}, 0, \dots, 0, 0, 1).$$

Proof Let L be a non-characteristically nilpotent filiform Leibniz algebra, then by Theorem 3.10 we have $\beta_k = 0, 3 \leq k \leq n-1,\ k \neq \frac{n+2}{2}$ and

$$\frac{-n+2}{2}\beta_n a_0 = -a_1\gamma + \frac{n}{2}a_1\beta^2_{\frac{n+2}{2}}. \tag{3.5}$$

Since $\gamma \neq 0$, then for any values of $\beta_{\frac{n+2}{2}}$ and β_n there exist a_0, a_1 ($a_0 \neq 0$) such that the restriction 3.5 is held. Therefore, a non-characteristically nilpotent filiform Leibniz algebra of the second family has the form

$$F_2(0, \dots, 0, \beta_{\frac{n+2}{2}}, 0, \dots, 0, \beta_n, \gamma).$$

By Theorem 2.6 we have the isomorphism criterion

$$\beta'_k = 0, \quad 3 \leq k \leq n-1, \ \ k \neq \frac{n+2}{2},$$

$$\gamma' = \frac{D^2}{A^n}\gamma, \quad \beta'_{\frac{n+2}{2}} = \frac{D}{A^{\frac{n}{2}}}\beta_{\frac{n+2}{2}}, \quad \beta'_n = \frac{BD\gamma}{A^n} + \frac{D}{A^{n-1}}\left(\beta_n - \frac{nB}{2A}\beta^2_{\frac{n+2}{2}}\right).$$

Putting $D = \frac{A^{\frac{n}{2}}}{\sqrt{\gamma}}$, we get

$$\gamma' = 1, \quad \beta'_{\frac{n+2}{2}} = \frac{\beta_{\frac{n+2}{2}}}{\sqrt{\gamma}}, \quad \beta'_n = \frac{1}{\sqrt{A^n\gamma}}\left((\gamma - \frac{n}{2}\beta^2_{\frac{n+2}{2}})B + A\beta_n\right).$$

It is not difficult to check that $\gamma' - \frac{n}{2}\beta'^2_{\frac{n+2}{2}} = \frac{D^2}{A^n}(\gamma - \frac{n}{2}\beta^2_{\frac{n+2}{2}})$.

If $\gamma \neq \frac{n}{2}\beta^2_{\frac{n+2}{2}}$, then putting $B = -\frac{2A\beta_n}{2\gamma - n\beta^2_{\frac{n+2}{2}}}$, we have $\beta'_n = 0$, and so obtain the algebra

$$F_2(0, \ldots, 0, \beta_{\frac{n+2}{2}}, 0, \ldots, 0, 0, 1), \quad \beta_{\frac{n+2}{2}} \neq \sqrt{\frac{2}{n}}.$$

If $\gamma = \frac{n}{2}\beta^2_{\frac{n+2}{2}}$, then we have $\frac{-n+2}{2}\beta_n a_0 = 0, \quad \beta'_{\frac{n+2}{2}} = \sqrt{\frac{2}{n}}, \quad \beta'_n = \frac{\beta_n}{\sqrt{A^{n-2}\gamma}}$.

Since $a_0 \neq 0$, we have $\beta_n = 0$, $\beta'_n = 0$ and obtain the algebra $F_2(0, \ldots, 0, \sqrt{\frac{2}{n}}, 0, \ldots, 0, 0, 1)$. □

Let us investigate the case $\gamma = 0$.

Theorem 3.12 *Let L be a non-characteristically nilpotent filiform Leibniz algebra from $F_2(\beta_3, \beta_4, \ldots, \beta_n, \gamma)$. If $\gamma = 0$, then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$F_2^j(0, \ldots, 0, \overset{j}{1}, 0 \ldots, 0, 0), \qquad 3 \leq j \leq n.$$

Proof The restrictions 3.4 under the conditions of the theorem have the form

$$\beta_3(b_1 - 2a_0) = 0,$$
$$\beta_k(b_1 - (k-1)a_0) = \frac{k}{2}a_1\sum_{j=4}^{k}\beta_{j-1}\beta_{k-j+3}, \qquad 4 \leq k \leq n.$$

Let β_j be the first non-zero parameter, i.e. $\beta_i = 0$ for $3 \leq i \leq j-1$ and $\beta_j \neq 0$. Then we have $\beta_j(b_1 - (j-1)a_0) = 0$, which implies $b_1 = (j-1)a_0$. Since the algebra is non-characteristically nilpotent, the coefficient $a_0 \neq 0$. Therefore, from the restrictions 3.4, we derive

$$
\begin{aligned}
\beta_k &= 0, && j+1 \le k \le 2j-3,\\
(2-j)a_0\beta_{2j-2} &= (j-1)a_1\beta_j^2, && \qquad (3.6)\\
\beta_k &= 0, && k \neq t(j-2)+2,\\
(t-1)(2-j)a_0\beta_{t(j-2)+2} &= \frac{t(j-2)+2}{2}a_1\sum_{i=1}^{t-1}\beta_{i(j-2)+2}\beta_{(t-i)(j-2)+2},
\end{aligned}
$$

where $t \ge 3, k \le n$.

From Theorem 2.6 we get the isomorphism criterion

$$
\begin{aligned}
\beta'_k &= 0, \quad 3 \le k \le j-1, && \beta'_j = \frac{D}{A^{j-1}}\beta_j,\\
\beta'_k &= 0, \quad j+1 \le k \le 2j-3, && \beta'_{2j-2} = \frac{D}{A^{2j-3}}\left(\beta_{2j-2} - \frac{(j-1)B}{A}\beta_j^2\right).
\end{aligned}
$$

Putting $D = \frac{A^{j-1}}{\beta_j}$, $B = \frac{A\beta_{2j-2}}{(j-1)\beta_j^2}$, we obtain $\beta'_j = 1$, $\beta'_{2j-2} = 0$. Thus, if $\beta_i = 0$ for $3 \le i \le j-1$ and $\beta_j \neq 0$, then, without loss of generality, we can suppose $\beta_j = 1$ and $\beta_{2j-2} = 0$.

The restrictions 3.6 imply $a_1 = 0$ and $(t-1)(2-j)a_0\beta_{t(j-2)+2} = 0,\ t \ge 3$, which leads to $\beta_k = 0$ for all $k \neq j$. Thus, we obtain the algebras $F_2^j(0,\ldots,0,\overset{j}{1},0,\ldots,0,0),\ 3 \le j \le n$. □

3.3 Non-Characteristically Nilpotent Filiform Leibniz Algebras of the Family $F_3(\theta_1,\theta_2,\theta_3)$

Since non-characteristically nilpotent filiform Lie algebras are described in [11], we will classify them only in the non-Lie case.

Let L be a filiform non-Lie Leibniz algebra of the third family. Put

$$[e_i, e_1] = -[e_1, e_i] = \beta_{i,i+2}e_{i+2} + \beta_{i,i+3}e_{i+3} + \cdots + \beta_{i,n}e_n, \quad 2 \le i \le n-2.$$

Using the Leibniz identity it is not difficult to obtain the following equality:

$$[e_i, e_k] = \sum_{j=i+k+1}^{n}\left(\sum_{t=0}^{k-1}(-1)^t\binom{k-1}{t}\beta_{i+t,j+1-k+t}\right)e_j, \quad 2 \le k, \quad k \le i \le n-k-1.$$

Since $[e_k, e_k] = 0$, we have $\sum\limits_{t=0}^{k-1}(-1)^t\binom{k-1}{t}\beta_{k+t,k+t+2} = 0,\ 2 \le k \le n$.

Proposition 3.13 *Let L be a non-characteristically nilpotent non-Lie filiform Leibniz algebra from the family $F_3(\theta_1,\theta_2,\theta_3)$. Then*

$$\beta_{i,j} = 0, \quad 2 \le i \le n-2,\ i+2 \le j \le n,$$

i.e. $[e_i, e_j] = 0$, for $1 \le i < j < n-1$.

Proof Let L be a non-characteristically nilpotent non-Lie filiform Leibniz algebra from the family $F_3(\theta_1, \theta_2, \theta_3)$ and let d be a derivation of L.

Put

$$d(e_0) = \sum_{k=0}^{n} a_k e_k, \quad d(e_1) = \sum_{k=0}^{n} b_k e_k.$$

Similarly as above we establish $b_0 = 0$.

From the property of the derivation we have

$$d(e_i) = ((i-1)a_0 + b_1)\, e_i \ (\text{mod } L^{i+1}), \quad 2 \le i \le n.$$

Consider the equalities

$$\begin{aligned} d([e_0, e_0]) = [d(e_0), e_0] + [e_0, d(e_0)] &= \left[\sum_{k=0}^{n} a_k e_k, e_0\right] + \left[e_0, \sum_{k=0}^{n} a_k e_k\right] \\ &= a_0[e_0, e_0] + a_1[e_1, e_0] + a_0[e_0, e_0] + a_1[e_0, e_1] = (2a_0\theta_1 + a_1\theta_2)e_n. \end{aligned}$$

On the other hand,

$$d([e_0, e_0]) = \theta_1 d(e_n) = \theta_1\, ((n-1)a_0 + b_1)\, e_n.$$

Consequently,

$$\theta_1\, ((n-3)a_0 + b_1) = a_1\theta_2.$$

Consider the property of the derivation for the product $[e_0, e_1]$,

$$\begin{aligned} d([e_0, e_1]) &= [d(e_0), e_1] + [e_0, d(e_1)] = a_0[e_0, e_1] + a_1[e_1, e_1] \\ &\quad + \left[\sum_{k=2}^{n} a_k e_k, e_1\right] + b_1[e_0, e_1] + \left[e_0, \sum_{k=2}^{n} b_k e_k\right] \\ &= -a_0[e_1, e_0] - a_1[e_1, e_1] - \left[e_1, \sum_{k=2}^{n} a_k e_k\right] + a_0\theta_2 e_n \\ &\quad + 2a_1\theta_3 e_n - b_1[e_1, e_0] - \left[\sum_{k=2}^{n} b_k e_k, e_0\right] + b_1\theta_2 e_n \\ &= -\left[e_1, \sum_{k=0}^{n} a_k e_k\right] - \left[\sum_{k=1}^{n} b_k e_k, e_0\right] + (a_0\theta_2 + b_1\theta_2 \\ &\quad + 2a_1\theta_3)e_n = -d(e_2) + (a_0\theta_2 + b_1\theta_2 + 2a_1\theta_3)e_n. \end{aligned}$$

On the other hand,

$$d([e_0, e_1]) = d(-e_2 + \theta_2 e_n) = -d(e_2) + \theta_2 d(e_n) = -d(e_2) + \theta_2\, ((n-1)a_0 + b_1)\, e_n.$$

Therefore, $2a_1\theta_3 = (n-2)a_0\theta_2$.

Similarly, from

$$d([e_1, e_1]) = \left[\sum_{k=1}^{n} b_k e_k, e_1\right] + \left[e_1, \sum_{k=1}^{n} b_k e_k\right] = [b_1 e_1, e_1] + [e_1, b_1 e_1] = 2b_1\theta_3 e_n$$

and

$$d([e_1, e_1]) = d(\theta_3 e_n) = \theta_3\left((n-1)a_0 + b_1\right) e_n$$

we conclude that $\theta_3\left((n-1)a_0 - b_1\right) = 0$.

Thus, we obtain

$$\theta_1\left((n-3)a_0 + b_1\right) = a_1\theta_2, \quad 2a_1\theta_3 = (n-2)a_0\theta_2, \quad \theta_3\left((n-1)a_0 - b_1\right) = 0.$$

Note that for a non-Lie Leibniz algebra we have $(\theta_1, \theta_2, \theta_3) \neq (0, 0, 0)$.

If $\theta_3 \neq 0$, then $b_1 = (n-1)a_0$, $a_1 = \frac{(n-2)a_0\theta_2}{2\theta_3}$ and $(2n-4)a_0\theta_1 = \frac{(n-2)a_0\theta_2^2}{2\theta_3}$.

Note that $a_0 \neq 0$ (since L is non-characteristically nilpotent). Therefore, we deduce

$$\theta_1 = \frac{\theta_2^2}{4\theta_3}.$$

If $\theta_3 = 0$, then $\theta_1\left((n-3)a_0 + b_1\right) = a_1\theta_2$, $(n-2)a_0\theta_2 = 0$.

If $\theta_2 \neq 0$, then $a_0 = 0$, and $\theta_1 b_1 = a_1\theta_2$.

If $\theta_2 = 0$, then $\theta_1 \neq 0$, and $b_1 = -(n-3)a_0$.

Thus, on the behavior of the parameters θ_1, θ_2 and θ_3 we obtain the following equalities:

$$b_1 = (n-1)a_0, \quad a_0 = 0, \quad b_1 = -(n-3)a_0.$$

Now we shall prove $\beta_{i,j} = 0, \quad 4 \leq j \leq n,\ 2 \leq i \leq j-2$, by induction on j for any values of i.

Consider the property of the derivation for the product $[e_2, e_1]$,

$$\begin{aligned} d([e_2, e_1]) &= [d(e_2), e_1] + [e_2, d(e_1)] = [(a_0 + b_1)e_2 + x_3, e_1] + \left[e_2, \sum_{k=1}^{n} b_k e_k\right] \\ &= (a_0 + 2b_1)\beta_{2,4}e_4 + x_5. \end{aligned}$$

On the other hand,

$$d([e_2, e_1]) = \beta_{2,4}d(e_4) + \beta_{2,5}d(e_5) + \cdots + \beta_{2,n}d(e_n) = (3a_0 + b_1)\beta_{2,4}e_4 + y_5,$$

where $x_3 \in L^3$ and $x_5, y_5 \in L^5$.

Comparing the coefficients at the basic element e_4, we obtain

$$\beta_{2,4}(b_1 - 2a_0) = 0.$$

Since $b_1 = (n-1)a_0$ or $a_0 = 0$ or $b_1 = -(n-3)a_0$ and $(a_0, b_1) \neq (0, 0)$, we have $\beta_{2,4} = 0$. Thus, we proved the statement of the proposition for $j = 4$.

Let the induction hypothesis be true for $j \leq k \leq n-1$, i.e. $\beta_{i,j} = 0$ for $4 \leq j \leq k$, $2 \leq i \leq j-2$, which implies $[L^{i_0}, e_1] \subseteq L^{k+1}$, $[L^{i_0+1}, e_1] \subseteq L^{k+2}$. We will prove $\beta_{i,k+1} = 0$ for $2 \leq i \leq k-1$.

Let us suppose the contrary, i.e. there exists i such that $\beta_{i,k+1} \neq 0$. Let i_0 be the greatest number among indexes such that $\beta_{i,k+1} \neq 0$.

Again, consider the property of the derivation

$$d([e_{i_0}, e_1]) = [d(e_{i_0}), e_1] + [e_{i_0}, d(e_1)] = \left((i_0-1)a_0 + 2b_1\right)\beta_{i_0,k+1}e_{k+1} + x_{k+2}.$$

On the other hand,

$$d([e_{i_0}, e_1]) = \sum_{j=k+1}^{n} \beta_{i_0,j} d(e_j) = (ka_0 + b_1)\beta_{i_0,k+1} e_{k+1} + y_{k+2},$$

where $x_{k+2}, y_{k+2} \in L^{k+2}$.

Comparing the coefficients at the basic element e_{k+1}, we get $\beta_{i_0,k+1}(b_1 - (k+1-i_0)a_0) = 0$. Since $2 \le i_0 \le k-1$, we have $2 \le k+1-i_0 \le k-1$.

Taking into account $k+1 \le n$ and the correctness of one of the following conditions:

$$b_1 = (n-1)a_0, \quad a_0 = 0, \quad b_1 = -(n-3)a_0,$$

we deduce $\beta_{i_0,k+1} = 0$ for $2 \le i \le k-1$, which is a contradiction with the assumption $\beta_{i,k+1} \neq 0$. Thus, $\beta_{i,k+1} = 0$ and we have proved that $\beta_{i,j} = 0$ for all i, j. □

Remark 3.14 The proof of Proposition 3.13 shows that the cases $\alpha = 0$ and $\alpha = 1$ are proved analogously.

Proposition 3.13 implies that the table of multiplication of a non-characteristically nilpotent filiform Leibniz algebra from the third family has the form:

$$F_3(\theta_1, \theta_2, \theta_3) = \begin{cases} [e_i, e_0] = e_{i+1}, & 1 \le i \le n-1, \\ [e_0, e_i] = -e_{i+1}, & 2 \le i \le n-1, \\ [e_0, e_0] = \theta_1 e_n, & \\ [e_0, e_1] = -e_2 + \theta_2 e_n, & \\ [e_1, e_1] = \theta_3 e_n, & \\ [e_i, e_{n-i}] = -[e_{n-i}, e_i] = \alpha(-1)^i e_n, & 1 \le i \le n-1. \end{cases}$$

For such algebras in [22] it is obtained the isomorphism criterion:

$$\theta_1' = \frac{A_0^2\theta_1 + A_0A_1\theta_2 + A_1^2\theta_3}{A_0^{n-1}B_1}, \qquad \theta_2' = \frac{A_0\theta_2 + 2A_1\theta_3}{A_0^{n-1}}, \qquad \theta_3' = \frac{B_1\theta_3}{A_0^{n-1}}. \tag{3.7}$$

Theorem 3.15 *Let L be a non-characteristically nilpotent filiform Leibniz algebra from $F_3(\theta_1, \theta_2, \theta_3)$. Then it is isomorphic to one of the following pairwise non-isomorphic algebras*

$$F_3^1(1, 0, 0), \quad F_3^2(0, 1, 0), \quad F_3^3(0, 0, 1).$$

Proof Consider several cases.

Case 1 Let $\theta_3 = 0$ and $\theta_2 = 0$. Then $\theta_1 \neq 0$ and

$$\theta_1' = \frac{\theta_1}{A_0^{n-3}B_1}, \quad \theta_2' = \theta_3' = 0.$$

Putting $B_1 = \frac{\theta_1}{A_0^{n-3}}$, we have $\theta_1' = 1$ and obtain the algebra $F_3^1(1, 0, 0)$.

Case 2 Let $\theta_3 = 0$ and $\theta_2 \neq 0$. Then we have

$$\theta_1' = \frac{A_0\theta_1 + A_1\theta_2}{A_0^{n-2}B_1}, \quad \theta_2' = \frac{\theta_2}{A_0^{n-2}}, \quad \theta_3' = 0.$$

Putting $A_0 = \sqrt[n-2]{\theta_2}$, $A_1 = -\frac{A_0\theta_1}{\theta_2}$, we have $\theta_1' = 0, \theta_2' = 1$ and obtain the algebra $F_3^2(0, 1, 0)$.

Case 3 Let $\theta_3 \neq 0$. Then similarly as in the proof of Proposition 3.13 we conclude $\theta_1 = \frac{\theta_2^2}{4\theta_3}$. Then in the isomorphism criterion 3.7 putting $B_1 = \frac{A_0^{n-1}}{\theta_3}$, $A_1 = -\frac{A_0\theta_2}{2\theta_3}$, we get

$$\theta_1' = \theta_2' = 0, \quad \theta_3' = 1.$$

Thus, in this case we have the algebra $F_3^3(0, 0, 1)$. □

Acknowledgements The second and third authors were supported by MICINN, grant MTM 2009-14464-C02 (Spain) (European FEDER support included), and by Xunta de Galicia, grant Incite09 207 215PR.

References

1. Albeverio, S., Ayupov, S.A., Omirov, B.A., Khudoyberdiyev, A.K.: n-dimensional filiform Leibniz algebras of length $(n-1)$ and their derivations. J. Algebra **319**(6), 2471–2488 (2008)
2. Ayupov, S.A., Omirov, B.A.: On Leibniz algebras. Algebra and Operators Theory. Proceedings of the Colloquium in Tashkent 1997, pp. 1–13. Kluwer (1998)
3. Ayupov, S.A., Omirov, B.A.: On some classes of nilpotent Leibniz algebras, Siberian Math. J. **42**(1), 15–24 (2001)
4. Casas, J.M., Ladra, M., Omirov, B.A., Karimjanov, I.K.: Classification of solvable Leibniz algebras with null-filiform nilradical. Linear Multilinear Algebra **61**(6), 758–774 (2013)
5. Casas, J.M., Ladra, M., Omirov, B.A., Karimjanov, I.K.: Classification of solvable Leibniz algebras with naturally graded filiform nilradical. Linear Algebra Appl. **438**, 2976–3000 (2013)
6. Castro-Jiménez, F.J., Valdés, J.N.: On characteristically nilpotent filiform Lie algebras of dimension 9. Comm. Algebra **23**(8), 3059–3071 (1995)
7. Dixmier, J., Lister, W.G.: Derivations of nilpotent Lie algebras. Proc. Amer. Math. Soc. **8**, 155–158 (1957)
8. Gómez, J.R., Jimenéz-Merchán, A., Khakimdjanov, Y.: Low-dimensional filiform Lie algebras. J. Pure Appl. Algebra **130**(2), 133–158 (1998)
9. Gómez, J.R., Omirov, B.A.: On Classification of Complex Filiform Leibniz Algebras. arXiv:0612735v1. Accepted to Algebra Colloquium (2013)
10. Gould, H.W.: Combinatorial Identities: Table II: Advanced Techniques for Summing Finite Series (2010)
11. Goze, M., Hakimjanov, Y.: Sur les algèbres de Lie nilpotentes admettant un tore de dérivations. Manuscripta Math. **84**(2), 115–124 (1994)
12. Hilton, P., Pedersen, J.: Catalan numbers, their generalization, and their uses. Math. Intelligencer **13**(2), 64–75 (1991)
13. Jacobson, N.: A note on automorphisms and derivations of Lie algebras. Proc. Amer. Math. Soc. **6**, 281–283 (1955)
14. Khakimdzhanov, Y.B.: Characteristically nilpotent Lie algebras, Mat. USSR, Sb. **70**(1), 65–78 (1991)
15. Khakimdzhanov, Y.B.: Variété des lois d'algèbres de Lie nilpotentes. Geom. Dedicata **40**(3), 269–295 (1991)
16. Ladra, M., Rikhsiboev, I.M., Turdibaev, R.M.: Automorphisms and derivations of Leibniz algebras. arXiv:1103.4721
17. Leger, G., Tôgô, S.: Characteristically nilpotent Lie algebras. Duke Math. J. **26**, 623–628 (1959)

18. Loday, J.-L.: Une version non commutative des algèbres de Lie: les algèbres de Leibniz. Enseign. Math. (2) **39**(3–4), 269–293 (1993)
19. J.Ndogmo, C., Winternitz, P.: Solvable Lie algebras with abelian nilradicals. J. Phys. A **27**(2), 405–423 (1994)
20. Omirov, B.A.: Classification of eight-dimensional complex filiform Leibniz algebras. Uzbek. Mat. Zh. **3**, 63–71 (2005)
21. Omirov, B.A.: On derivations of filiform Leibniz algebras. Math. Notes **77**(5–6), 677–685 (2005)
22. Omirov, B.A., Rakhimov, I.S.: On Lie-like complex filiform Leibniz algebras. Bull. Aust. Math. Soc. **79**(3), 391–404 (2009)
23. Rakhimov, I.S., Sozan, J.: Description of nine dimensional complex filiform Leibniz algebras arising from naturally graded non Lie filiform Leibniz algebras. Int. J. Algebra 3 (17-20) (2009) 969–980.
24. Rikhsiboev, I.M.: Classification of seven-dimensional complex filiform Leibniz algebras (Russian). Uzbek. Mat. Zh. **3**, 57–61 (2004)
25. Šnobl, L., Winternitz, P.: A class of solvable Lie algebras and their Casimir invariants. J. Phys. A **38**(12), 2687–2700 (2005)
26. Wang, Y., Lin, J., Deng, S.: Solvable Lie algebras with quasifiliform nilradicals. Comm. Algebra **36**(11), 4052–4067 (2008)