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The Algebraic and Geometric Classification of Nilpotent Right Commutative Algebras

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Abstract. This paper is devoted to the complete algebraic and geometric classification of complex 4-dimensional nilpotent right commutative algebras. The corresponding geometric variety has dimension 15 and decomposes into 5 irreducible components determined by the Zariski closures of four one-parameter families of algebras and a two-parameter family of algebras (see Theorem B). In particular, there are no rigid complex 4-dimensional nilpotent right commutative algebras.

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Introduction

The algebraic classification (up to isomorphism) of algebras of dimension n from a certain variety defined by a certain family of polinomial identities is a classic problem in the theory of non-associative algebras. There are many results related to the algebraic classification of small-dimensional algebras in the varieties of Jordan, Lie, Leibniz, Zinbiel and many other algebras [3,7]. Another interesting direction in the classification of algebras is the geometric classification. There are many results related to the geometric classification of Jordan, Lie, Leibniz, Zinbiel and many other algebras [4,7,9]. In the present

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paper, we give the algebraic and geometric classification of 4-dimensional nilpotent right commutative algebras. The variety of right commutative algebras contains commutative associative algebras as a subvariety and it is related to some interesting varieties of algebras whose definitions we give below.

- An algebra is called a *bicommutative algebra* [8], if it satisfies the following identities

$$x(yz) = y(xz), \ (xy)z = (xz)y.$$

- An algebra is called a *Novikov algebra* [6], if it satisfies the following identities

$$(xy)z = (xz)y, \ (xy)z - x(yz) = (yx)z - y(xz).$$

One-sided commutative algebras first appeared in the paper by Cayley in 1857 [2]. The variety of right commutative algebras is defined by the following identity:

$$(xy)z = (xz)y.$$

Our method for classifying nilpotent right commutative algebras is based on the calculation of central extensions of nilpotent algebras of smaller dimensions from the same variety. The algebraic study of central extensions of Lie and non-Lie algebras has been an important topic for years [5,10]. First, Skjelbred and Sund used central extensions of Lie algebras to obtain a classification of nilpotent Lie algebras [10]. After that, using the method described by Skjelbred and Sund, all non-Lie central extensions of all 4-dimensional Malcev algebras were described [5], and also all non-associative central extensions of 3-dimensional Jordan algebras, all anticommutative central extensions of the 3-dimensional anticommutative algebras, and all central extensions of the 2-dimensional algebras [1]. Note that the Skjelbred–Sund method of central extensions is an important tool in the classification of nilpotent algebras, which was used to describe all 4-dimensional nilpotent associative algebras, all 4-dimensional nilpotent assosymmetric algebras, all 4-dimensional nilpotent bicommutative algebras [8], all 4-dimensional nilpotent Novikov algebras [6], all 4-dimensional nilpotent terminal algebras [7], all 5-dimensional nilpotent Jordan algebras, all 5-dimensional nilpotent restricted Lie algebras, all 5-dimensional nilpotent associative commutative algebras, all 6-dimensional nilpotent Lie algebras [3], all 6-dimensional nilpotent Malcev algebras, all 6-dimensional nilpotent Binary Lie algebras, all 6-dimensional nilpotent anticommutative algebras and some others.

1. The Algebraic Classification of Nilpotent Right Commutative Algebras

1.1. Method of Classification of Nilpotent Algebras

Throughout this paper, we use the notations and methods well written in [1,5], which we have adapted for the right commutative case with some modifications. Further in this section we give some important definitions.

Let $(\mathbf{A}, \cdot)$ be complex right commutative algebra and $\mathbb{V}$ be a complex vector space. The $\mathbb{C}$-linear space $\mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ is defined as the set of all bilinear maps $\theta\colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{V}$ such that

$$\theta(xy, z) = \theta(xz, y).$$

These elements will be called *cocycles*. For a linear map f from $\mathbf{A}$ to $\mathbb{V}$, if we define $\delta f\colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{V}$ by $\delta f(x, y) = f(xy)$, then $\delta f \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$. We define $\mathrm{B}^2(\mathbf{A}, \mathbb{V}) = \{\theta = \delta f \ : f \in \mathrm{Hom}(\mathbf{A}, \mathbb{V})\}$. We define the *second cohomology space* $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$ as the quotient space $\mathrm{Z}^2(\mathbf{A}, \mathbb{V}) / \mathrm{B}^2(\mathbf{A}, \mathbb{V})$.

Let $\mathrm{Aut}(\mathbf{A})$ be the automorphism group of $\mathbf{A}$ and let $\phi \in \mathrm{Aut}(\mathbf{A})$. For $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ define the action of the group $\mathrm{Aut}(\mathbf{A})$ on $\mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ by $\phi\theta(x, y) = \theta(\phi(x), \phi(y))$. It is easy to verify that $\mathrm{B}^2(\mathbf{A}, \mathbb{V})$ is invariant under the action of $\mathrm{Aut}(\mathbf{A})$. So, we have an induced action of $\mathrm{Aut}(\mathbf{A})$ on $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$.

Let $\mathbf{A}$ be a right commutative algebra of dimension m over $\mathbb{C}$ and $\mathbb{V}$ be a $\mathbb{C}$-vector space of dimension k. For the bilinear map θ, define on the linear space $\mathbf{A}_\theta = \mathbf{A} \oplus \mathbb{V}$ the bilinear product "$[-, -]_{\mathbf{A}_\theta}$" by $[x + x', y + y']_{\mathbf{A}_\theta} = xy + \theta(x, y)$ for all $x, y \in \mathbf{A}, x', y' \in \mathbb{V}$. The algebra $\mathbf{A}_\theta$ is called a *k-dimensional central extension* of $\mathbf{A}$ by $\mathbb{V}$. One can easily check that $\mathbf{A}_\theta$ is a right commutative algebra if and only if $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$.

Call the set $\mathrm{Ann}(\theta) = \{x \in \mathbf{A} : \theta(x, \mathbf{A}) + \theta(\mathbf{A}, x) = 0\}$ the *annihilator* of θ. We recall that the *annihilator* of an algebra $\mathbf{A}$ is defined as the ideal $\mathrm{Ann}(\mathbf{A}) = \{x \in \mathbf{A} : x\mathbf{A} + \mathbf{A}x = 0\}$. Observe that $\mathrm{Ann}(\mathbf{A}_\theta) = (\mathrm{Ann}(\theta) \cap \mathrm{Ann}(\mathbf{A})) \oplus \mathbb{V}$.

The following result shows that every algebra with a non-zero annihilator is a central extension of a smaller-dimensional algebra.

Lemma 1. *Let $\mathbf{A}$ be an n-dimensional right commutative algebra such that $\dim(\mathrm{Ann}(\mathbf{A})) = m \neq 0$. Then there exists, up to isomorphism, a unique $(n - m)$-dimensional right commutative algebra $\mathbf{A}'$ and a bilinear map $\theta \in \mathrm{Z}^2(\mathbf{A}', \mathbb{V})$ with $\mathrm{Ann}(\mathbf{A}') \cap \mathrm{Ann}(\theta) = 0$, where $\mathbb{V}$ is a vector space of dimension m, such that $\mathbf{A} \cong \mathbf{A}'_\theta$ and $\mathbf{A}/\mathrm{Ann}(\mathbf{A}) \cong \mathbf{A}'$.*

Proof. Let $\mathbf{A}'$ be a linear complement of $\mathrm{Ann}(\mathbf{A})$ in $\mathbf{A}$. Define a linear map $P\colon \mathbf{A} \longrightarrow \mathbf{A}'$ by $P(x + v) = x$ for $x \in \mathbf{A}'$ and $v \in \mathrm{Ann}(\mathbf{A})$, and define a multiplication on $\mathbf{A}'$ by $[x, y]_{\mathbf{A}'} = P(xy)$ for $x, y \in \mathbf{A}'$. For $x, y \in \mathbf{A}$, we have

$$\begin{aligned} P(xy) &= P((x - P(x) + P(x))(y - P(y) + P(y))) \\ &= P(P(x)P(y)) = [P(x), P(y)]_{\mathbf{A}'}. \end{aligned}$$

Since P is a homomorphism $P(\mathbf{A}) = \mathbf{A}'$ is a right commutative algebra and $\mathbf{A}/\operatorname{Ann}(\mathbf{A}) \cong \mathbf{A}'$, which gives us the uniqueness. Now, define the map $\theta\colon \mathbf{A}' \times \mathbf{A}' \longrightarrow \operatorname{Ann}(\mathbf{A})$ by $\theta(x,y) = xy - [x,y]_{\mathbf{A}'}$. Thus, $\mathbf{A}'_\theta$ is $\mathbf{A}$ and therefore $\theta \in \mathrm{Z}^2(\mathbf{A}', \mathbb{V})$ and $\operatorname{Ann}(\mathbf{A}') \cap \operatorname{Ann}(\theta) = 0$. □

Definition 2. Let $\mathbf{A}$ be an algebra and I be a subspace of $\operatorname{Ann}(\mathbf{A})$. If $\mathbf{A} = \mathbf{A}_0 \oplus I$ then I is called an *annihilator component* of $\mathbf{A}$. A central extension of an algebra $\mathbf{A}$ without annihilator component is called a *non-split central extension.*

Our task is to find all central extensions of an algebra $\mathbf{A}$ by a space $\mathbb{V}$. In order to solve the isomorphism problem we need to study the action of $\operatorname{Aut}(\mathbf{A})$ on $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$. To do that, let us fix a basis $e_1, \ldots, e_s$ of $\mathbb{V}$, and $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$. Then θ can be uniquely written as $\theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)\, e_i$, where $\theta_i \in \mathrm{Z}^2(\mathbf{A}, \mathbb{C})$. Moreover, $\operatorname{Ann}(\theta) = \operatorname{Ann}(\theta_1) \cap \operatorname{Ann}(\theta_2) \cap \ldots \cap \operatorname{Ann}(\theta_s)$. Furthermore, $\theta \in \mathrm{B}^2(\mathbf{A}, \mathbb{V})$ if and only if all $\theta_i \in \mathrm{B}^2(\mathbf{A}, \mathbb{C})$. It is not difficult to prove (see [5, Lemma 13]) that given a right commutative algebra $\mathbf{A}_\theta$, if we write as above $\theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)\, e_i \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ and $\operatorname{Ann}(\theta) \cap \operatorname{Ann}(\mathbf{A}) = 0$, then $\mathbf{A}_\theta$ has an annihilator component if and only if $[\theta_1], [\theta_2], \ldots, [\theta_s]$ are linearly dependent in $\mathrm{H}^2(\mathbf{A}, \mathbb{C})$.

Let $\mathbb{V}$ be a finite-dimensional vector space over $\mathbb{C}$. The *Grassmannian* $G_k(\mathbb{V})$ is the set of all k-dimensional linear subspaces of $\mathbb{V}$. Let $G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right)$ be the Grassmannian of subspaces of dimension s in $\mathrm{H}^2(\mathbf{A}, \mathbb{C})$. There is a natural action of $\operatorname{Aut}(\mathbf{A})$ on $G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right)$. Let $\phi \in \operatorname{Aut}(\mathbf{A})$. For $W = \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right)$ define $\phi W = \langle [\phi\theta_1], [\phi\theta_2], \ldots, [\phi\theta_s] \rangle$. We denote the orbit of $W \in G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right)$ under the action of $\operatorname{Aut}(\mathbf{A})$ by $\operatorname{Orb}(W)$. Given

$$W_1 = \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle, W_2 = \langle [\vartheta_1], [\vartheta_2], \ldots, [\vartheta_s] \rangle \in G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right),$$

we easily have that if $W_1 = W_2$, then $\bigcap_{i=1}^{s} \operatorname{Ann}(\theta_i) \cap \operatorname{Ann}(\mathbf{A}) = \bigcap_{i=1}^{s} \operatorname{Ann}(\vartheta_i) \cap \operatorname{Ann}(\mathbf{A})$, and therefore we can introduce the set

$$\mathbf{T}_s(\mathbf{A}) = \left\{ W = \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right) : \bigcap_{i=1}^{s} \operatorname{Ann}(\theta_i) \cap \operatorname{Ann}(\mathbf{A}) = 0 \right\},$$

which is stable under the action of $\operatorname{Aut}(\mathbf{A})$.

Now, let $\mathbb{V}$ be an s-dimensional linear space and let us denote by $\mathbf{E}(\mathbf{A}, \mathbb{V})$ the set of all *non-split s-dimensional central extensions* of $\mathbf{A}$ by $\mathbb{V}$. By above, we can write

$$\mathbf{E}(\mathbf{A}, \mathbb{V}) = \left\{ \mathbf{A}_\theta : \theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)\, e_i \ \text{ and } \ \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in \mathbf{T}_s(\mathbf{A}) \right\}.$$

We also have the following result, which can be proved as in [5, Lemma 17].

Lemma 3. *Let $\mathbf{A}_\theta, \mathbf{A}_\vartheta \in \mathbf{E}(\mathbf{A}, \mathbb{V})$. Suppose that $\theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y) e_i$ and $\vartheta(x,y) = \sum_{i=1}^{s} \vartheta_i(x,y) e_i$. Then the right commutative algebras $\mathbf{A}_\theta$ and $\mathbf{A}_\vartheta$ are isomorphic if and only if*

$$\operatorname{Orb}\langle[\theta_1],[\theta_2],\ldots,[\theta_s]\rangle = \operatorname{Orb}\langle[\vartheta_1],[\vartheta_2],\ldots,[\vartheta_s]\rangle.$$

This shows that there exists a one-to-one correspondence between the set of $\operatorname{Aut}(\mathbf{A})$-orbits on $\mathbf{T}_s(\mathbf{A})$ and the set of isomorphism classes of $\mathbf{E}(\mathbf{A}, \mathbb{V})$. Consequently we have a procedure that allows us, given a right commutative algebra $\mathbf{A}'$ of dimension $n-s$, to construct all non-split central extensions of $\mathbf{A}'$. This procedure is:

(1) For a given right commutative algebra $\mathbf{A}'$ of dimension $n-s$, determine $\mathrm{H}^2(\mathbf{A}', \mathbb{C})$, $\operatorname{Ann}(\mathbf{A}')$ and $\operatorname{Aut}(\mathbf{A}')$.
(2) Determine the set of $\operatorname{Aut}(\mathbf{A}')$-orbits on $\mathbf{T}_s(\mathbf{A}')$.
(3) For each orbit, construct the right commutative algebra associated with a representative of it.

The above described method gives all (Novikov and non-Novikov) right commutative algebras. But we are interested in developing this method in such a way that it only gives non-Novikov right commutative algebras, because the classification of all Novikov algebras is given in [6]. Clearly, any central extension of a non-Novikov right commutative algebra is non-Novikov. But a Novikov algebra may have extensions which are not Novikov algebras. More precisely, let $\mathbf{N}$ be a Novikov algebra and $\theta \in \mathrm{Z}^2_{\mathrm{R}}(\mathbf{N}, \mathbb{C})$. Then $\mathbf{N}_\theta$ is a Novikov algebra if and only if

$$\theta(xy,z) - \theta(x,yz) = \theta(yx,z) - \theta(y,xz).$$

for all $x,y,z \in \mathbf{N}$. Define the subspace $\mathrm{Z}^2_{\mathrm{N}}(\mathbf{N}, \mathbb{C})$ of $\mathrm{Z}^2_{\mathrm{R}}(\mathbf{N}, \mathbb{C})$ by

$$\mathrm{Z}^2_{\mathrm{N}}(\mathbf{N}, \mathbb{C}) = \big\{\theta \in \mathrm{Z}^2_{\mathrm{R}}(\mathbf{N}, \mathbb{C}) : \theta(xy,z) - \theta(x,yz) = \theta(yx,z) - \theta(y,xz), \text{ for all } x,y,z \in \mathbf{N}\big\}.$$

Observe that $\mathrm{B}^2(\mathbf{N}, \mathbb{C}) \subseteq \mathrm{Z}^2_{\mathrm{N}}(\mathbf{N}, \mathbb{C})$. Let $\mathrm{H}^2_{\mathrm{N}}(\mathbf{N}, \mathbb{C}) = \mathrm{Z}^2_{\mathrm{N}}(\mathbf{N}, \mathbb{C}) / \mathrm{B}^2(\mathbf{N}, \mathbb{C})$. Then $\mathrm{H}^2_{\mathrm{N}}(\mathbf{N}, \mathbb{C})$ is a subspace of $\mathrm{H}^2_{\mathrm{R}}(\mathbf{N}, \mathbb{C})$. Define

$$\mathbf{R}_s(\mathbf{N}) = \big\{\mathbf{W} \in \mathbf{T}_s(\mathbf{N}) : \mathbf{W} \in G_s(\mathrm{H}^2_{\mathrm{N}}(\mathbf{N}, \mathbb{C}))\big\},$$
$$\mathbf{U}_s(\mathbf{N}) = \big\{\mathbf{W} \in \mathbf{T}_s(\mathbf{N}) : \mathbf{W} \notin G_s(\mathrm{H}^2_{\mathrm{N}}(\mathbf{N}, \mathbb{C}))\big\}.$$

Then $\mathbf{T}_s(\mathbf{N}) = \mathbf{R}_s(\mathbf{N}) \mathbin{\dot\cup} \mathbf{U}_s(\mathbf{N})$. The sets $\mathbf{R}_s(\mathbf{N})$ and $\mathbf{U}_s(\mathbf{N})$ are stable under the action of $\operatorname{Aut}(\mathbf{N})$. Thus, the right commutative algebras corresponding to the representatives of $\operatorname{Aut}(\mathbf{N})$ -orbits on $\mathbf{R}_s(\mathbf{N})$ are Novikov algebras, while those corresponding to the representatives of $\operatorname{Aut}(\mathbf{N})$-orbits on $\mathbf{U}_s(\mathbf{N})$ are not Novikov algebras. Hence, we may construct all non-split non-Novikov right commutative algebras $\mathbf{A}$ of dimension n with s-dimensional annihilator from a given right commutative algebra $\mathbf{A}'$ of dimension $n-s$ in the following way:

(1) If $\mathbf{A}'$ is non-Novikov, then apply the procedure.
(2) Otherwise, do the following:
   (a) Determine $\mathbf{U}_s(\mathbf{A}')$ and $\mathrm{Aut}(\mathbf{A}')$.
   (b) Determine the set of $\mathrm{Aut}(\mathbf{A}')$-orbits on $\mathbf{U}_s(\mathbf{A}')$.
   (c) For each orbit, construct the right commutative algebra corresponding to one of its representatives.

1.2. Notations

Let us introduce the following notations. Let $\mathbf{A}$ be a nilpotent algebra with a basis $e_1, e_2, \ldots, e_n$. Then by Δ_{ij} we will denote the bilinear form Δ_{ij} : $\mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{C}$ with $\Delta_{ij}(e_l, e_m) = \delta_{il}\delta_{jm}$. The set $\{\Delta_{ij} : 1 \leq i, j \leq n\}$ is a basis for the linear space of bilinear forms on $\mathbf{A}$, so every $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ can be uniquely written as $\theta = \sum\limits_{1\leq i,j\leq n} c_{ij}\Delta_{ij}$, where $c_{ij} \in \mathbb{C}$. Let us fix the following notations:

$\mathbf{R}^{\mathbf{i}}_{\mathbf{j}}$ — jth i-dimensional right commutative (non-Novikov) algebra.

$\mathbf{R}^{\mathbf{i}*}_{\mathbf{j}}$ — jth i-dimensional right commutative (Novikov) algebra.

1.3. The Algebraic Classification of 3-Dimensional Nilpotent Right Commutative Algebras

There are no nontrivial 1-dimensional nilpotent right commutative algebras. There is only one nonzero 2-dimensional nilpotent right commutative algebra (it is the non-split central extension of 1-dimensional algebra with zero product):

$$\mathbf{R}^{2*}_{01} : e_1e_1 = e_2.$$

It is known the classification of all non-split 3-dimensional nilpotent right commutative algebras:

$$\begin{array}{llll}
\mathbf{R}^{3*}_{02} & : e_1e_1 = e_3 & e_2e_2 = e_3 & \\
\mathbf{R}^{3*}_{03} & : e_1e_2 = e_3 & e_2e_1 = -e_3 & \\
\mathbf{R}^{3*}_{04}(\lambda) & : e_1e_1 = \lambda e_3 & e_2e_1 = e_3 & e_2e_2 = e_3 \\
\mathbf{R}^{3*}_{05} & : e_1e_1 = e_2 & e_2e_1 = e_3 & \\
\mathbf{R}^{3*}_{06}(\lambda) & : e_1e_1 = e_2 & e_1e_2 = e_3 & e_2e_1 = \lambda e_3.
\end{array}$$

1.4. Central Extensions of 3-Dimensional Nilpotent Right Commutative Algebras

1.4.1. The Description of Second Cohomology Spaces of 3-Dimensional Nilpotent Right Commutative Algebras. In the following table we give the description of the second cohomology space of 3-dimensional nilpotent right commutative algebras. where $\mathbf{R}^{3*}_{01} = \mathbf{R}^{2*}_{01} \oplus \mathbb{C}$.

$\mathbf{R}$	$\mathrm{H}^2_{\mathbf{N}}(\mathbf{R})$	$\mathrm{H}^2_{\mathbf{R}}(\mathbf{R})$
$\mathbf{R}^{3*}_{01}$	$\Big\langle [\Delta_{12}], [\Delta_{13}], [\Delta_{21}], [\Delta_{31}], [\Delta_{33}] \Big\rangle$	$\mathrm{H}^2_{\mathbf{N}}(\mathbf{R}^{3*}_{01}) \oplus \Big\langle [\Delta_{32}] \Big\rangle$
$\mathbf{R}^{3*}_{02}$	$\Big\langle [\Delta_{12}], [\Delta_{21}], [\Delta_{22}] \Big\rangle$	$\mathrm{H}^2_{\mathbf{N}}(\mathbf{R}^{3*}_{02}) \oplus \Big\langle [\Delta_{13}], [\Delta_{23}] \Big\rangle$
$\mathbf{R}^{3*}_{03}$	$\Big\langle [\Delta_{11}], [\Delta_{21}], [\Delta_{22}] \Big\rangle$	$\mathrm{H}^2_{\mathbf{N}}(\mathbf{R}^{3*}_{03}) \oplus \Big\langle [\Delta_{13}], [\Delta_{23}] \Big\rangle$
$\mathbf{R}^{3*}_{04}(\lambda)_{\lambda\neq 0}$	$\Big\langle [\Delta_{11}], [\Delta_{12}], [\Delta_{21}] \Big\rangle$	$\mathrm{H}^2_{\mathbf{N}}(\mathbf{R}^{3*}_{04}(\lambda)) \oplus \Big\langle [\Delta_{13}], [\Delta_{23}] \Big\rangle$
$\mathbf{R}^{3*}_{04}(0)$	$\Big\langle [\Delta_{11}], [\Delta_{12}], [\Delta_{21}], [\Delta_{13}-\Delta_{31}-\Delta_{32}], [\Delta_{23}] \Big\rangle$	$\mathrm{H}^2_{\mathbf{N}}(\mathbf{R}^{3*}_{04}(0)) \oplus \Big\langle [\Delta_{31}+\Delta_{32}] \Big\rangle$
$\mathbf{R}^{3*}_{05}$	$\Big\langle [\Delta_{12}], [\Delta_{13}-\Delta_{31}] \Big\rangle$	$\mathrm{H}^2_{\mathbf{N}}(\mathbf{R}^{3*}_{05}) \oplus \Big\langle [\Delta_{31}] \Big\rangle$
$\mathbf{R}^{3*}_{06}(\lambda)_{\lambda\neq 0}$	$\Big\langle [\Delta_{21}], [(2-\lambda)\Delta_{13}+ \lambda(\Delta_{22}+\Delta_{31})] \Big\rangle$	$\mathrm{H}^2_{\mathbf{N}}(\mathbf{R}^{3*}_{06}(\lambda)) \oplus \Big\langle [\Delta_{22}+\Delta_{31}-\Delta_{13}] \Big\rangle$
$\mathbf{R}^{3*}_{06}(0)$	$\Big\langle [\Delta_{21}], 2[\Delta_{13}] \Big\rangle$	$\mathrm{H}^2_{\mathbf{N}}(\mathbf{R}^{3*}_{06}(0)) \oplus \Big\langle [\Delta_{22}+\Delta_{31}-\Delta_{13}], [\Delta_{32}] \Big\rangle$

1.4.2. Central Extensions of $\mathbf{R}^{3*}_{01}$. Let us use the following notations:

$$\nabla_1=[\Delta_{12}], \quad \nabla_2=[\Delta_{13}], \quad \nabla_3=[\Delta_{21}], \quad \nabla_4=[\Delta_{31}], \quad \nabla_5=[\Delta_{33}], \quad \nabla_6=[\Delta_{32}].$$

Take $\theta = \sum\limits_{i=1}^{6} \alpha_i \nabla_i \in \mathrm{H}^2_{\mathbf{R}}(\mathbf{R}^{3*}_{01})$. The automorphism group of $\mathbf{R}^{3*}_{01}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ y & x^2 & u \\ z & 0 & t \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ \alpha_3 & 0 & 0 \\ \alpha_4 & \alpha_6 & \alpha_5 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* & \alpha_2^* \\ \alpha_3^* & 0 & 0 \\ \alpha_4^* & \alpha_6^* & \alpha_5^* \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathbf{R}^{3*}_{01})$ on the subspace $\langle \sum\limits_{i=1}^{6} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{6} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{ll} \alpha_1^* = \alpha_1 x^3 + \alpha_6 x^2 z & \alpha_2^* = \alpha_1 xu + \alpha_2 xt + \alpha_5 zt + \alpha_6 zu \\ \alpha_3^* = \alpha_3 x^3 & \alpha_4^* = \alpha_3 xu + \alpha_4 xt + \alpha_5 zt + \alpha_6 yt \\ \alpha_5^* = \alpha_5 t^2 + \alpha_6 ut & \alpha_6^* = \alpha_6 x^2 t. \end{array}$$

Since $\mathrm{H}^2_{\mathbf{R}}(\mathbf{R}^{3*}_{01}) = \mathrm{H}^2_{\mathbf{N}}(\mathbf{R}^{3*}_{01}) \oplus \langle \nabla_6 \rangle$ and we are interested only in new algebras, we have $\alpha_6 \neq 0$. Then putting $z = -\frac{\alpha_1 x}{\alpha_6}$, $u = -\frac{\alpha_5 t}{\alpha_6}$ and $y = \frac{(\alpha_1\alpha_5+\alpha_3\alpha_5-\alpha_4\alpha_6)x}{\alpha_6^2}$, we have the following

$$\alpha_1^* = \alpha_4^* = \alpha_5^* = 0 \quad \alpha_2^* = \frac{(\alpha_2\alpha_6-\alpha_1\alpha_5)xt}{\alpha_6} \quad \alpha_3^* = \alpha_3 x^3 \quad \alpha_6^* = \alpha_6 x^2 t.$$

Consider the following possible cases.

(1) $\alpha_3 \neq 0$, $\alpha_2\alpha_6 - \alpha_1\alpha_5 \neq 0$, then choosing $x = \frac{\alpha_2\alpha_6-\alpha_1\alpha_5}{\alpha_6^2}$ and $t = \frac{\alpha_3(\alpha_2\alpha_6-\alpha_1\alpha_5)}{\alpha_6^3}$, we have the representative $\langle \nabla_2 + \nabla_3 + \nabla_6 \rangle$.

(2) $\alpha_3 = 0$, $\alpha_2\alpha_6 - \alpha_1\alpha_5 \neq 0$, then choosing $x = \frac{\alpha_2\alpha_5-\alpha_1\alpha_5}{\alpha_6^2}$ and $t = 1$ we have the representative $\langle \nabla_2 + \nabla_6 \rangle$.

(3) $\alpha_3 \neq 0$, $\alpha_2\alpha_6 - \alpha_1\alpha_5 = 0$, then choosing $x = 1$ and $t = \frac{\alpha_3}{\alpha_6}$, we have the representative $\langle \nabla_3 + \nabla_6 \rangle$.

(4) $\alpha_3 = 0$, $\alpha_2\alpha_6 - \alpha_1\alpha_5 = 0$, then we have the representative $\langle \nabla_6 \rangle$.

Hence, we have the following distints orbits

$$\langle \nabla_2 + \nabla_3 + \nabla_6 \rangle \ \langle \nabla_2 + \nabla_6 \rangle \ \langle \nabla_3 + \nabla_6 \rangle \ \langle \nabla_6 \rangle,$$

which give the following new algebras:

$\mathbf{R}^4_{01} : e_1e_1 = e_2 \ e_1e_3 = e_4 \ \ e_2e_1 = e_4 \ e_3e_2 = e_4$
$\mathbf{R}^4_{02} : e_1e_1 = e_2 \ e_1e_3 = e_4 \ \ e_3e_2 = e_4$
$\mathbf{R}^4_{03} : e_1e_1 = e_2 \ e_2e_1 = e_4 \ \ e_3e_2 = e_4$
$\mathbf{R}^4_{04} : e_1e_1 = e_2 \ e_3e_2 = e_4.$

1.4.3. Central Extensions of $\mathbf{R}^{3*}_{02}$. Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{22}], \quad \nabla_4 = [\Delta_{13}], \quad \nabla_5 = [\Delta_{23}].$$

Take $\theta = \sum\limits_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2_{\mathbf{R}}(\mathbf{R}^{3*}_{02})$. The automorphism group of $\mathbf{R}^{3*}_{02}$ consists of invertible matrices of the form

$$\phi_1 = \begin{pmatrix} x & -y & 0 \\ y & x & 0 \\ z & t & x^2+y^2 \end{pmatrix} \quad \text{or} \quad \phi_2 = \begin{pmatrix} x & y & 0 \\ y & -x & 0 \\ z & t & x^2+y^2 \end{pmatrix}.$$

Since

$$\phi_1^T \begin{pmatrix} 0 & \alpha_1 & \alpha_4 \\ \alpha_2 & \alpha_3 & \alpha_5 \\ 0 & 0 & 0 \end{pmatrix} \phi_1 = \begin{pmatrix} \alpha^* & \alpha_1^* & \alpha_4^* \\ \alpha_2^* & \alpha^* + \alpha_3^* & \alpha_5^* \\ 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathbf{R}^{3*}_{02})^+$ (it is the subgroup in $\mathrm{Aut}(\mathbf{R}^{3*}_{02})$ formed by all automorphisms of the first type) on the subspace $\langle \sum\limits_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by

$\langle \sum_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{aligned}
\alpha_1^* &= \alpha_1 x^2 - \alpha_2 y^2 + \alpha_3 xy + \alpha_4 xt + \alpha_5 yt\\
\alpha_2^* &= -\alpha_1 y^2 + \alpha_2 x^2 + \alpha_3 xy - \alpha_4 yz + \alpha_5 xz\\
\alpha_3^* &= -2\alpha_1 xy - 2\alpha_2 xy + \alpha_3(x^2 - y^2) - \alpha_4(yt + xz) + \alpha_5(xt - yz)\\
\alpha_4^* &= (\alpha_4 x + \alpha_5 y)(x^2 + y^2)\\
\alpha_5^* &= (-\alpha_4 y + \alpha_5 x)(x^2 + y^2).
\end{aligned}$$

Since $\mathrm{H}^2_{\mathbf{R}}(\mathbf{R}^{3*}_{02}) = \mathrm{H}^2_{\mathbf{R}}(\mathbf{R}^{3*}_{02}) \oplus \langle \nabla_4, \nabla_5 \rangle$ and we are interested only in new algebras, we have $(\alpha_4, \alpha_5) \neq (0, 0)$. Moreover, without loss of generality, one can assume $\alpha_4 \neq 0$. Then we have the following cases.

(1) $\alpha_4^2 + \alpha_5^2 \neq 0$, then choosing $y = \frac{x\alpha_5}{\alpha_4}$, $t = \frac{(\alpha_2\alpha_5^2 - \alpha_1\alpha_4^2 - \alpha_4\alpha_3\alpha_5)x}{\alpha_4(\alpha_4^2+\alpha_5^2)}$, $z = \frac{(\alpha_3(\alpha_4^2-\alpha_5^2) - 2\alpha_4\alpha_5(\alpha_1+\alpha_3))x}{\alpha_4(\alpha_4^2+\alpha_5^2)}$, we have

$$\alpha_1^* = \alpha_3^* = \alpha_5^* = 0 \ \ \alpha_2^* = \frac{(\alpha_4^2\alpha_3 - \alpha_5(\alpha_1\alpha_5 - \alpha_4\alpha_3))x^2}{\alpha_4^2} \ \ \alpha_4^* = \frac{x^3(\alpha_4^2+\alpha_5^2)^2}{\alpha_4^3}.$$

   (a) $\alpha_4^2\alpha_2 - \alpha_5(\alpha_1\alpha_5 - \alpha_4\alpha_3) = 0$, then we have the representative $\langle \nabla_4 \rangle$.
   (b) $\alpha_4^2\alpha_2 - \alpha_5(\alpha_1\alpha_5 - \alpha_4\alpha_3) \neq 0$, then choosing $x = \frac{\alpha_4(\alpha_4^2\alpha_2 - \alpha_5(\alpha_1\alpha_5 - \alpha_4\alpha_3))}{(\alpha_4^2+\alpha_5^2)^2}$, we have the representative $\langle \nabla_2 + \nabla_4 \rangle$.

(2) $\alpha_4^2 + \alpha_5^2 = 0$, i.e., $\alpha_5 = \pm i\alpha_4$, then choosing

$$t = -\frac{\alpha_1 x^2 - \alpha_2 y^2 + \alpha_3 xy}{\alpha_4(x \pm iy)},\ z = \frac{-2\alpha_1 xy - \alpha_4(yt \mp ixt) - 2\alpha_2 xy + \alpha_3(x^2 - y^2)}{\alpha_4(x \pm iy)},$$

we have

$$\begin{aligned}
&\alpha_1^* = \alpha_2^* = 0 && \alpha_2^* = \frac{(\alpha_1+\alpha_2 \pm i\alpha_3)(x^2+y^2)^2}{(x \pm iy)^2}\\
&\alpha_4^* = \alpha_4(x \pm iy)(x^2 + y^2) && \alpha_5^* = \pm\alpha_2(x \pm iy)(x^2 + y^2).
\end{aligned}$$

   (a) $\alpha_1 + \alpha_2 \pm i\alpha_3 = 0$, then we have the representative $\langle \nabla_4 \pm i\nabla_5 \rangle$.
   (b) $\alpha_1 + \alpha_2 \pm i\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_1+\alpha_2 \pm i\alpha_3}{\alpha_4}$ and $y = 0$, we have representative $\langle \nabla_2 + \nabla_4 \pm i\nabla_5 \rangle$. Since the automorphism $\phi = \mathrm{diag}(1, -1, 1)$ acts as

$$\phi(\nabla_4 + i\nabla_5) = \langle \nabla_4 - i\nabla_5 \rangle \text{ and } \phi(\nabla_2 + \nabla_4 + i\nabla_5) = \langle \nabla_2 + \nabla_4 - i\nabla_5 \rangle,$$

we have two representatives of distinct orbits $\langle \nabla_4 + i\nabla_5 \rangle$ and $\langle \nabla_2 + \nabla_4 + i\nabla_5 \rangle$.

Summarizing, we have the following distinct orbits:

$$\langle \nabla_4 \rangle \ \langle \nabla_2 + \nabla_4 \rangle \ \langle \nabla_4 + i\nabla_5 \rangle \ \langle \nabla_2 + \nabla_4 + i\nabla_5 \rangle.$$

Hence, we have the following new algebras:

$\mathbf{R}^4_{05}$	$: e_1e_1 = e_3 \ \ e_1e_3 = e_4 \ \ e_2e_2 = e_3$
$\mathbf{R}^4_{06}$	$: e_1e_1 = e_3 \ \ e_1e_3 = e_4 \ \ e_2e_1 = e_4 \ \ e_2e_2 = e_3$
$\mathbf{R}^4_{07}$	$: e_1e_1 = e_3 \ \ e_1e_3 = e_4 \ \ e_2e_2 = e_3 \ \ e_2e_3 = ie_4$
$\mathbf{R}^4_{08}$	$: e_1e_1 = e_3 \ \ e_1e_3 = e_4 \ \ e_2e_1 = e_4 \ \ e_2e_2 = e_3 \ \ e_2e_3 = ie_4$

1.4.4. Central Extensions of $\mathbf{R}^{3*}_{03}$. Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = [\Delta_{22}], \quad \nabla_4 = [\Delta_{13}], \quad \nabla_5 = [\Delta_{23}].$$

Take $\theta = \sum\limits_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2_{\mathbf{R}}(\mathbf{R}^{3*}_{03})$. The automorphism group of $\mathbf{R}^{3*}_{03}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & u & 0 \\ y & v & 0 \\ z & t & xv - yu \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & 0 & \alpha_4 \\ \alpha_2 & \alpha_3 & \alpha_5 \\ 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha^* & \alpha_4^* \\ \alpha_2^* - \alpha^* & \alpha_3^* & \alpha_5^* \\ 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathbf{R}^{3*}_{03})$ on the subspace $\langle \sum\limits_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum\limits_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{aligned}
\alpha_1^* &= \alpha_1 x^2 + \alpha_2 xy + \alpha_3 y^2 + \alpha_4 xz + \alpha_5 yz \\
\alpha_2^* &= 2\alpha_1 xu + \alpha_2(yu + xv) + 2\alpha_3 yv + \alpha_4(xt + zu) + \alpha_5(zv + yt) \\
\alpha_3^* &= \alpha_1 u^2 + \alpha_2 uv + \alpha_3 v^2 + \alpha_4 ut + \alpha_5 vt \\
\alpha_4^* &= (\alpha_4 x + \alpha_5 y)(xv - yu) \\
\alpha_5^* &= (\alpha_4 u + \alpha_5 v)(xv - yu).
\end{aligned}$$

Since $\mathrm{H}^2_{\mathbf{R}}(\mathbf{R}^{3*}_{03}) = \mathrm{H}^2_{\mathbf{N}}(\mathbf{R}^{3*}_{03}) \oplus \langle \nabla_4, \nabla_5 \rangle$, we have $(\alpha_4, \alpha_5) \neq (0, 0)$. Moreover, without loss of generality, one can assume $\alpha_4 \neq 0$. Choosing $u = -\frac{\alpha_5 v}{\alpha_4}$, $z = -\frac{\alpha_1 x^2 + \alpha_2 xy + \alpha_3 y^2}{\alpha_4 x + \alpha_5 y}$ and $t = -\frac{2\alpha_1 xu + \alpha_2(xv + yu) + 2\alpha_3 yv}{\alpha_4 x + y\alpha_5}$, we have

$$\alpha_1^* = \alpha_2^* = \alpha_5^* = 0 \;\; \alpha_3^* = \frac{(\alpha_1 \alpha_5^2 - \alpha_2 \alpha_4 \alpha_5 + \alpha_3 \alpha_4^2) v^2}{\alpha_4^2} \;\; \alpha_4^* = \frac{v(\alpha_4 x + \alpha_5 y)^2}{\alpha_4}$$

We find the following new cases.

(1) $\alpha_1 \alpha_5^2 - \alpha_2 \alpha_4 \alpha_5 + \alpha_3 \alpha_4^2 = 0$, then choosing $v = \frac{\alpha_4}{(\alpha_4 x + \alpha_5 y)^2}$, we have the representative $\langle \nabla_4 \rangle$.

(2) $\alpha_1 \alpha_5^2 - \alpha_2 \alpha_4 \alpha_5 + \alpha_3 \alpha_4^2 \neq 0$, then choosing $v = \frac{\alpha_4 (\alpha_4 x + \alpha_5 y)^2}{\alpha_1 \alpha_5^2 - \alpha_2 \alpha_4 \alpha_5 + \alpha_3 \alpha_4^2}$, we have the representative $\langle \nabla_3 + \nabla_4 \rangle$.

Summarizing, we have the following distinct orbits

$$\langle \nabla_4 \rangle \;\; \langle \nabla_3 + \nabla_4 \rangle.$$

Hence, we have the following new algebras:

$$\begin{array}{l}
\mathbf{R}^4_{09} : e_1 e_2 = e_3 \;\; e_1 e_3 = e_4 \;\; e_2 e_1 = -e_3 \\
\hline
\mathbf{R}^4_{10} : e_1 e_2 = e_3 \;\; e_1 e_3 = e_4 \;\; e_2 e_1 = -e_3 \;\; e_2 e_2 = e_4.
\end{array}$$

1.4.5. Central Extensions of $\mathbf{R}_{04}^{3*}(\lambda)_{\lambda\neq 0}$. Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \quad \nabla_2 = [\Delta_{12}], \quad \nabla_3 = [\Delta_{21}], \quad \nabla_4 = [\Delta_{13}], \quad \nabla_5 = [\Delta_{23}].$$

Take $\theta = \sum_{i=1}^{5} \alpha_i \nabla_i \in \mathrm{H}^2_{\mathbf{R}}(\mathbf{R}_{04}^{3*}(\lambda)_{\lambda\neq 0})$. The automorphism group of $\mathbf{R}_{04}^{3*}(\lambda)_{\lambda\neq 0}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & y & 0 \\ -\lambda y & x-y & 0 \\ z & t & x^2 - xy + \lambda y^2 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_4 \\ \alpha_3 & 0 & \alpha_5 \\ 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* + \lambda\alpha^* & \alpha_2^* & \alpha_4^* \\ \alpha_3^* + \alpha^* & \alpha^* & \alpha_5^* \\ 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathbf{R}_{04}^{3*}(\lambda)_{\lambda\neq 0})$ on the subspace $\langle \sum_{i=1}^{5} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{5} \alpha_i^* \nabla_i \rangle$, where

$$\begin{aligned}
\alpha_1^* &= x^2\alpha_1 + xz\alpha_4 - \lambda x(2y(\alpha_2+\alpha_3) + t\alpha_5) \\
&\quad + \lambda y(y(-\alpha_1+\alpha_2+\alpha_3) - t\alpha_4 + t\alpha_5 - z\alpha_5) \\
\alpha_2^* &= xy(\alpha_1-\alpha_2) + x^2\alpha_2 + tx\alpha_4 - \lambda y(y\alpha_3 + t\alpha_5) \\
\alpha_3^* &= xy(\alpha_1-\alpha_2-2\alpha_3) + x^2\alpha_3 + y^2(-\alpha_1+\alpha_2-\lambda\alpha_2+\alpha_3) \\
&\quad - y(t-z)(\alpha_4-\alpha_5) + x(z-t)\alpha_5 \\
\alpha_4^* &= (\alpha_4 x - \alpha_5\lambda y)(x^2 - xy + \lambda y^2) \\
\alpha_5^* &= (\alpha_4 y + \alpha_5(x-y))(x^2 - xy + \lambda y^2).
\end{aligned}$$

Since $\mathrm{H}^2_{\mathbf{R}}(\mathbf{R}_{04}^{3*}(\lambda)_{\lambda\neq 0}) = \mathrm{H}^2_{\mathbf{N}}(\mathbf{R}_{04}^{3*}(\lambda)_{\lambda\neq 0}) \oplus \langle \nabla_4, \nabla_5 \rangle$, we are interested in $(\alpha_4, \alpha_5) \neq (0,0)$.

Then we have the following cases.

(1) $\alpha_5 = 0$, then $\alpha_4 \neq 0$ and choosing $y = 0,\ t = -\frac{\alpha_1 x}{\alpha_4},\ z = \frac{\alpha_3 \lambda x}{\alpha_4}$, we have

$$\alpha_1^* = \alpha_2^* = \alpha_5^* = 0, \quad \alpha_3^* = \alpha_3 x^2, \quad \alpha_4^* = \alpha_4 x^3.$$

  (a) $\alpha_3 = 0$, then we have the representative $\langle \nabla_4 \rangle$.

  (b) $\alpha_3 \neq 0$, then choosing $x = \frac{\alpha_3}{\alpha_4}$, we have the representative $\langle \nabla_3 + \nabla_4 \rangle$.

(2) $\alpha_5 \neq 0$ and $\alpha_4^2 - \alpha_4\alpha_5 + \alpha_5^2\lambda \neq 0$, then choosing $x = \frac{\alpha_5 - \alpha_4}{\alpha_5}$ and $y = 1$, we have $\alpha_5^* = 0$ and it is the situation considered above.

(3) $\alpha_5 \neq 0$ and $\alpha_4^2 - \alpha_4\alpha_5 + \alpha_5^2\lambda = 0$, then choosing $y = 0,\ t = \frac{x\alpha_2}{\alpha_4}$, $z = -\frac{x(\alpha_1\alpha_4 + \lambda\alpha_2\alpha_5)}{\alpha_4^2}$, we have

$$\alpha_1^* = \alpha_2^* = 0 \ \alpha_3^* = \frac{x^2(\alpha_3\alpha_4^2 - \alpha_5(\alpha_1\alpha_4 + \alpha_2(\lambda\alpha_5 - \alpha_4)))}{\alpha_4^2} \ \alpha_4^* = x^3\alpha_4 \ \alpha_5^* = x^3\alpha_5.$$

  (a) $\alpha_3\alpha_4^2 = \alpha_5(\alpha_1\alpha_4 + \alpha_2(\lambda\alpha_5 - \alpha_4))$, then we have the representative $\langle \nabla_4 + \frac{1\pm\sqrt{1-4\lambda}}{2\lambda} \nabla_5 \rangle$.

(b) $\alpha_3\alpha_4^2 \neq \alpha_5(\alpha_1\alpha_4 + \alpha_2(\lambda\alpha_5 - \alpha_4))$, then we have the representative $\langle \nabla_3 + \nabla_4 + \frac{1\pm\sqrt{1-4\lambda}}{2\lambda}\nabla_5\rangle$.

Summarizing, we have the following distinct orbits

$$\begin{array}{lll} \langle\nabla_4\rangle & \langle 2\lambda\nabla_4 + (1-\sqrt{1-4\lambda})\nabla_5\rangle & \langle 2\lambda\nabla_3 + 2\lambda\nabla_4 + (1-\sqrt{1-4\lambda})\nabla_5\rangle \\ \langle\nabla_3+\nabla_4\rangle & \langle 2\lambda\nabla_4 + (1+\sqrt{1-4\lambda})\nabla_5\rangle & \langle 2\lambda\nabla_3 + 2\lambda\nabla_4 + (1+\sqrt{1-4\lambda})\nabla_5\rangle. \end{array}$$

Hence, we have the following new algebras:

$\mathbf{R}^4_{11}(\lambda)_{\lambda\neq 0}$:	$e_1e_1 = \lambda e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$	
$\mathbf{R}^4_{12}(\lambda)_{\lambda\neq 0}$:	$e_1e_1 = \lambda e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_3$	
$\mathbf{R}^4_{13}(\lambda)_{\lambda\neq 0}$:	$e_1e_1 = \lambda e_3$	$e_1e_3 = 2\lambda e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$	$e_2e_3 = (1-\sqrt{1-4\lambda})e_4$
$\mathbf{R}^4_{14}(\lambda)_{\lambda\neq 0}$:	$e_1e_1 = \lambda e_3$	$e_1e_3 = 2\lambda e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$	$e_2e_3 = (1+\sqrt{1-4\lambda})e_4$
$\mathbf{R}^4_{15}(\lambda)_{\lambda\neq 0}$:	$e_1e_1 = \lambda e_3$	$e_1e_3 = 2\lambda e_4$	$e_2e_1 = e_3 + 2\lambda e_4$	$e_2e_2 = e_3$	$e_2e_3 = (1-\sqrt{1-4\lambda})e_4$
$\mathbf{R}^4_{16}(\lambda)_{\lambda\neq 0}$:	$e_1e_1 = \lambda e_3$	$e_1e_3 = 2\lambda e_4$	$e_2e_1 = e_3 + 2\lambda e_4$	$e_2e_2 = e_3$	$e_2e_3 = (1+\sqrt{1-4\lambda})e_4$

1.4.6. Central Extensions of $\mathbf{R}^{3*}_{04}(0)$. Let us use the following notations:

$$\nabla_1 = [\Delta_{11}], \quad \nabla_2 = [\Delta_{12}], \quad \nabla_3 = [\Delta_{21}], \quad \nabla_4 = [\Delta_{13} - \Delta_{31} - \Delta_{32}], \\ \nabla_5 = [\Delta_{23}], \quad \nabla_6 = [\Delta_{31} + \Delta_{32}].$$

Take $\theta = \sum\limits_{i=1}^{6} \alpha_i\nabla_i \in \mathrm{H}^2_{\mathbf{R}}(\mathbf{R}^{3*}_{04}(0))$. The automorphism group of $\mathbf{R}^{3*}_{04}(0)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & x-y & 0 \\ 0 & y & 0 \\ z & t & xy \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_4 \\ \alpha_3 & 0 & \alpha_5 \\ \alpha_6 - \alpha_4 & \alpha_6 - \alpha_4 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha_1^* & \alpha_2^* & \alpha_4^* \\ \alpha_3^* + \alpha^* & \alpha^* & \alpha_5^* \\ \alpha_6^* - \alpha_4^* & \alpha_6^* - \alpha_4^* & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathbf{R}^{3*}_{04}(0))$ on the subspace $\langle \sum\limits_{i=1}^{6} \alpha_i\nabla_i\rangle$ is given by $\langle \sum\limits_{i=1}^{6} \alpha_i^*\nabla_i\rangle$, where

$$\begin{array}{ll} \alpha_1^* = x(x\alpha_1 + z\alpha_6) & \alpha_2^* = x(x\alpha_1 + y(-\alpha_1 + \alpha_2) + t\alpha_4 - z\alpha_4 + z\alpha_6) \\ \multicolumn{2}{l}{\alpha_3^* = xy(\alpha_1 - \alpha_2) + y^2(-\alpha_1 + \alpha_2 + \alpha_3) + x(-t+z)\alpha_4 + y(t-z)(\alpha_4 - \alpha_5)} \\ \alpha_4^* = x^2y\alpha_4 \quad \alpha_5^* = xy((x-y)\alpha_4 + y\alpha_5) & \alpha_6^* = x^2y\alpha_6 \end{array}$$

Since $\mathrm{H}^2_{\mathbf{R}}(\mathbf{R}^{3*}_{04}(0)) = \mathrm{H}^2_{\mathbf{N}}(\mathbf{R}^{3*}_{04}(0)) \oplus \langle\nabla_6\rangle$, we are interested in $\alpha_6 \neq 0$.

(1) $\alpha_4 \neq 0$, $\alpha_4 \neq \alpha_5$ and $\alpha_3\alpha_4 - (\alpha_1 - \alpha_2)\alpha_5 \neq 0$ then by choosing $x = \frac{\alpha_3\alpha_4-(\alpha_1-\alpha_2)\alpha_5}{(\alpha_4-\alpha_5)\alpha_6}$, $y = \frac{\alpha_4(\alpha_3\alpha_4-(\alpha_1-\alpha_2)\alpha_5)}{(\alpha_4-\alpha_5)^2\alpha_6}$, $z = \frac{\alpha_1(-\alpha_3\alpha_4+(\alpha_1-\alpha_2)\alpha_5)}{(\alpha_4-\alpha_5)\alpha_6^2}$ and $t = -\frac{(\alpha_3\alpha_4-(\alpha_1-\alpha_2)\alpha_5)(\alpha_1(\alpha_4-\alpha_5-\alpha_6)+\alpha_2\alpha_6)}{(\alpha_4-\alpha_5)^2\alpha_6^2}$, we have the family of representatives $\langle \nabla_3 + \alpha\nabla_4 + \nabla_6\rangle_{\alpha\neq 0}$.

(2) $\alpha_4 \neq 0$, $\alpha_4 \neq \alpha_5$ and $\alpha_3\alpha_4-(\alpha_1-\alpha_2)\alpha_5 = 0$ then by choosing $x = 1-\frac{\alpha_5}{\alpha_4}$, $y = 1$, $z = \frac{\alpha_1(\alpha_4-\alpha_5)}{\alpha_4\alpha_6}$ and $t = \frac{-\alpha_2\alpha_6+\alpha_1(-\alpha_4+\alpha_5+\alpha_6)}{\alpha_4\alpha_6}$, we have the family of representatives $\langle \alpha\nabla_4 + \nabla_6\rangle_{\alpha\neq 0}$.

(3) $\alpha_4 \neq 0$, $\alpha_4 = \alpha_5$ and $\alpha_1 - \alpha_2 - \alpha_3 \neq 0$ then by choosing $x = 1$, $y = \frac{\alpha_6}{-\alpha_1+\alpha_2+\alpha_3}$, $z = -\frac{\alpha_1}{\alpha_6}$ and $t = -\frac{\alpha_1}{\alpha_6} + \frac{(\alpha_1-\alpha_2)\alpha_6}{(-\alpha_1+\alpha_2+\alpha_3)\alpha_5}$, we have the family of representatives $\langle \nabla_3 + \alpha\nabla_4 + \alpha\nabla_5 + \nabla_6\rangle_{\alpha\neq 0}$.

(4) $\alpha_4 \neq 0$, $\alpha_4 = \alpha_5$ and $\alpha_1 = \alpha_2 + \alpha_3$ then by choosing $x = 1$, $y = 1$, $z = -\frac{\alpha_1}{\alpha_6}$ and $t = \frac{\alpha_1-\alpha_2}{\alpha_5} - \frac{\alpha_1}{\alpha_6}$, we have the family of representatives $\langle \alpha\nabla_4 + \alpha\nabla_5 + \nabla_6\rangle_{\alpha\neq 0}$.

(5) $\alpha_4 = 0$, $\alpha_5 \neq 0$ and $\alpha_1 \neq \alpha_2$, then by choosing $x = \frac{-\alpha_1+\alpha_2}{\alpha_6}$, $y = \frac{-\alpha_1+\alpha_2}{\alpha_5}$, $z = \frac{\alpha_1(\alpha_1-\alpha_2)}{\alpha_6^2}$ and $t = \frac{(\alpha_1-\alpha_2)(\alpha_1(\alpha_5^2-\alpha_5\alpha_6+\alpha_6^2)-\alpha_6(\alpha_3\alpha_6+\alpha_2(-\alpha_5+\alpha_6))}{\alpha_5^2\alpha_6^2}$ we have the representative $\langle \nabla_2 + \nabla_5 + \nabla_6\rangle$.

(6) $\alpha_4 = 0$, $\alpha_5 \neq 0$ and $\alpha_1 = \alpha_2$, then by choosing $x = \frac{\alpha_5}{\alpha_6}$, $y = 1$, $z = -\frac{\alpha_1\alpha_5}{\alpha_6^2}$ and $t = \frac{-\alpha_1+\alpha_2+\alpha_3}{\alpha_5} - \frac{\alpha_1(\alpha_5-\alpha_6)+\alpha_2\alpha_6}{\alpha_6^2}$ we have the representative $\langle \nabla_5 + \nabla_6\rangle$.

(7) $\alpha_4 = 0$, $\alpha_5 = 0$, $\alpha_1 \neq \alpha_2$ and $\alpha_1 \neq \alpha_2+\alpha_3$ then by choosing $x = \frac{-\alpha_1+\alpha_2}{\alpha_6}$, $y = -\frac{(\alpha_1-\alpha_2)^2}{(\alpha_1-\alpha_2-\alpha_3)\alpha_6}$, $z = \frac{\alpha_1(\alpha_1-\alpha_2)}{\alpha_6^2}$ and $t = 0$ we have the representative $\langle \nabla_2 + \nabla_6\rangle$.

(8) $\alpha_4 = 0$, $\alpha_5 = 0$, $\alpha_1 \neq \alpha_2$ and $\alpha_1 = \alpha_2 + \alpha_3$ then by choosing $x = \frac{\alpha_2}{\alpha_6}$, $y = 1$, $z = -\frac{\alpha_3(\alpha_2+\alpha_3)}{\alpha_6^2}$ and $t = 0$ we have the representative $\langle -\nabla_2+\nabla_3+\nabla_6\rangle$.

(9) $\alpha_4 = 0$, $\alpha_5 = 0$, $\alpha_1 = \alpha_2$ and $\alpha_3 \neq 0$, then by choosing $x = 1$, $y = \frac{\alpha_6}{\alpha_3}$, $z = -\frac{\alpha_2}{\alpha_6}$ and $t = 0$ we have the representative $\langle \nabla_3 + \nabla_6\rangle$.

(10) $\alpha_4 = 0$, $\alpha_5 = 0$, $\alpha_1 = \alpha_2$ and $\alpha_3 = 0$, then by choosing $x = 1$, $y = 1$, $z = -\frac{\alpha_2}{\alpha_6}$ and $t = 0$ we have the representative $\langle \nabla_6\rangle$.

Summarizing, we have the following distinct orbits:

$$\langle \nabla_2 + \nabla_6\rangle \quad \langle \nabla_3 + \alpha\nabla_4 + \nabla_6\rangle_{\alpha\neq 0} \quad \langle \alpha\nabla_4 + \nabla_6\rangle_{\alpha\neq 0} \quad \langle \nabla_3 + \alpha\nabla_4 + \alpha\nabla_5 + \nabla_6\rangle$$
$$\langle \alpha\nabla_4 + \alpha\nabla_5 + \nabla_6\rangle \quad \langle \nabla_2 + \nabla_5 + \nabla_6\rangle \quad \langle \nabla_5 + \nabla_6\rangle \quad \langle -\nabla_2 + \nabla_3 + \nabla_6\rangle.$$

Hence, we have the following new algebras:

$\mathbf{R}^4_{17}$	$: e_1e_2 = e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$	$e_3e_1 = e_4$	$e_3e_2 = e_4$
$\mathbf{R}^4_{18}(\alpha)_{\alpha\neq 0}$	$: e_1e_3 = \alpha e_4$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_3$	$e_3e_1 = (1-\alpha)e_4$	$e_3e_2 = (1-\alpha)e_4$
$\mathbf{R}^4_{19}(\alpha)_{\alpha\neq 0}$	$: e_1e_3 = \alpha e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$	$e_3e_1 = (1-\alpha)e_4$	$e_3e_2 = (1-\alpha)e_4$
$\mathbf{R}^4_{20}(\alpha)$	$: e_1e_3 = \alpha e_4$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_3$		
	$\alpha_2\alpha_3 = \alpha e_4$	$e_3e_1 = (1-\alpha)e_4$	$e_3e_2 = (1-\alpha)e_4$		
$\mathbf{R}^4_{21}(\alpha)$	$: e_1e_3 = \alpha e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$		
	$\alpha_2\alpha_3 = \alpha e_4$	$e_3e_1 = (1-\alpha)e_4$	$e_3e_2 = (1-\alpha)e_4$		
$\mathbf{R}^4_{22}$	$: e_1e_2 = e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$	$e_2e_3 = e_4$	$e_3e_1 = e_4$ $e_3e_2 = e_4$
$\mathbf{R}^4_{23}$	$: e_2e_1 = e_3$	$e_2e_2 = e_3$	$e_2e_3 = e_4$	$e_3e_1 = e_4$	$e_3e_2 = e_4$
$\mathbf{R}^4_{24}$	$: e_1e_2 = e_4$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_3$	$e_3e_1 = e_4$	$e_3e_2 = e_4$

1.4.7. Central Extensions of $\mathbf{R}^{3*}_{05}$. Let us use the following notations:

$$\nabla_1 = [\Delta_{12}], \quad \nabla_2 = [\Delta_{13} - \Delta_{31}], \quad \nabla_3 = [\Delta_{31}].$$

Take $\theta = \sum_{i=1}^{3} \alpha_i \nabla_i \in \mathrm{H}^2_{\mathbf{R}}(\mathbf{R}^{3*}_{05})$. The automorphism group of $\mathbf{R}^{3*}_{05}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ y & x^2 & 0 \\ z & xy & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & 0 \\ \alpha_3 - \alpha_2 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* & \alpha_2^* \\ \alpha^{**} & 0 & 0 \\ \alpha_3^* - \alpha_2^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathbf{R}^{3*}_{05})$ on the subspace $\langle \sum_{i=1}^{3} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{3} \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^2(\alpha_1 x + \alpha_2 y) \ \alpha_2^* = \alpha_2 x^4 \ \alpha_3^* = \alpha_3 x^4.$$

Since $\mathrm{H}^2_{\mathbf{R}}(\mathbf{R}^{3*}_{05}) = \mathrm{H}^2_{\mathbf{N}}(\mathbf{R}^{3*}_{05}) \oplus \langle \nabla_3 \rangle$, we are interested in $\alpha_3 \neq 0$.

(1) $\alpha_2 = 0$, then we have the representatives $\langle \nabla_3 \rangle$ and $\langle \nabla_1 + \nabla_3 \rangle$ depending on whether $\alpha_1 = 0$ or not.
(2) $\alpha_2 \neq 0$, then choosing $x = 1$ and $y = -\frac{\alpha_1}{\alpha_2}$, we have the family of representatives $\langle \alpha\nabla_2 + \nabla_3 \rangle_{\alpha\neq 0}$.

Summarizing, we have the following distinct orbits

$$\langle \nabla_1 + \nabla_3 \rangle \ \langle \alpha\nabla_2 + \nabla_3 \rangle.$$

Hence, we have the following new algebras:

$\mathbf{R}^4_{25}$	$: e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_3$	$e_3e_1 = e_4$
$\mathbf{R}^4_{26}(\alpha)$	$: e_1e_1 = e_2$	$e_1e_3 = \alpha e_4$	$e_2e_1 = e_3$	$e_3e_1 = (1-\alpha)e_4$

1.4.8. Central Extensions of $\mathbf{R}_{06}^{3*}(\lambda)_{\lambda\neq 0}$. Let us use the following notations:

$\nabla_1 = [\Delta_{21}], \quad \nabla_2 = [(2-\lambda)\Delta_{13} + \lambda(\Delta_{22}+\Delta_{31})], \quad \nabla_3 = [\Delta_{22}+\Delta_{31}-\Delta_{13}].$

Take $\theta = \sum\limits_{i=1}^{3} \alpha_i\nabla_i \in \mathrm{H}^2_{\mathbf{R}}(\mathbf{R}_{06}^{3*}(\lambda)_{\lambda\neq 0})$. The automorphism group of $\mathbf{R}_{06}^{3*}(\lambda)_{\lambda\neq 0}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ y & x^2 & 0 \\ z & xy(1+\lambda) & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & (2-\lambda)\alpha_2-\alpha_3 \\ \alpha_1 & \lambda\alpha_2+\alpha_3 & 0 \\ \lambda\alpha_2+\alpha_3 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & (2-\lambda)\alpha_2^*-\alpha_3^* \\ \alpha_1^*+\lambda\alpha^{**} & \lambda\alpha_2^*+\alpha_3^* & 0 \\ \lambda\alpha_2^*+\alpha_3^* & 0 & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathbf{R}_{06}^{3*}(\lambda)_{\lambda\neq 0})$ on the subspace $\langle \sum\limits_{i=1}^{3} \alpha_i\nabla_i\rangle$ is given by $\langle \sum\limits_{i=1}^{3} \alpha_i^*\nabla_i\rangle$, where

$$\alpha_1^* = x^2(x\alpha_1 + y(\lambda^3\alpha_2 + 2\alpha_3 + \lambda\alpha_3 + \lambda^2(\alpha_3-\alpha_2))) \ \alpha_2^* = \alpha_2 x^4 \ \alpha_3^* = \alpha_3 x^4.$$

Since $\mathrm{H}^2_{\mathbf{R}}(\mathbf{R}_{06}^{3*}(\lambda)_{\lambda\neq 0}) = \mathrm{H}^2_{\mathbf{N}}(\mathbf{R}_{06}^{3*}(\lambda)_{\lambda\neq 0}) \oplus \langle\nabla_3\rangle$, we are interested in $\alpha_3 \neq 0$. We have the following cases.

(1) $\lambda^3\alpha_2 + 2\alpha_3 + \lambda\alpha_3 + \lambda^2(\alpha_3-\alpha_2) \neq 0$, then we have the family of representatives $\langle \alpha\nabla_2+\nabla_3\rangle_{\alpha\neq\frac{2+\lambda+\lambda^2}{(1-\lambda)\lambda^2}}$.

(2) $\lambda^3\alpha_2 + 2\alpha_3 + \lambda\alpha_3 + \lambda^2(\alpha_3-\alpha_2) = 0, \lambda\neq 1$ then we have two representatives $\langle \frac{2+\lambda+\lambda^2}{(1-\lambda)\lambda^2}\nabla_2+\nabla_3\rangle$ and $\langle \nabla_1 + \frac{2+\lambda+\lambda^2}{(1-\lambda)\lambda^2}\nabla_2+\nabla_3\rangle$.

(3) if $\lambda = 1$, then $\lambda^3\alpha_2 + 2\alpha_3 + \lambda\alpha_3 + \lambda^2(\alpha_3-\alpha_2) = 4\alpha_3 \neq 0$, and we have a case considered above.

Summarizing, we have the following distinct orbits

$$\langle \alpha\nabla_2+\nabla_3\rangle \ \langle (1-\lambda)\lambda^2\nabla_1 + (2+\lambda+\lambda^2)\nabla_2 + (1-\lambda)\lambda^2\nabla_3\rangle_{\lambda\neq 0,1}.$$

Hence, we have the following new algebras:

$\mathbf{R}_{27}^4(\lambda,\alpha)_{\lambda\neq 0}$	: $e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = (2\alpha-\alpha\lambda-1)e_4$
	$e_2e_1 = \lambda e_3$	$e_2e_2 = (\alpha\lambda+1)e_4$	$e_3e_1 = (\alpha\lambda+1)e_4$
$\mathbf{R}_{28}^4(\lambda)_{\lambda\neq 0,1}$	: $e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = 4e_4$
	$e_2e_1 = \lambda e_3 + \lambda^2(1-\lambda)e_4$	$e_2e_2 = 2\lambda(1+\lambda)e_4$	$e_3e_1 = 2\lambda(1+\lambda)e_4$.

1.4.9. Central Extensions of $\mathbf{R}_{06}^{3*}(0)$. Let us use the following notations:

$$\nabla_1 = [\Delta_{21}], \quad \nabla_2 = 2[\Delta_{13}], \quad \nabla_3 = [\Delta_{22} + \Delta_{31} - \Delta_{13}], \quad \nabla_4 = [\Delta_{32}].$$

Take $\theta = \sum_{i=1}^{4} \alpha_i \nabla_i \in \mathrm{H}^2_{\mathbf{R}}(\mathbf{R}_{06}^{3*}(0))$. The automorphism group of $\mathbf{R}_{06}^{3*}(0)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 \\ y & x^2 & 0 \\ z & xy & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 2\alpha_2 - \alpha_3 \\ \alpha_1 & \alpha_3 & 0 \\ \alpha_3 & \alpha_4 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & 2\alpha_2^* - \alpha_3^* \\ \alpha_1^* & \alpha_3^* & 0 \\ \alpha_3^* & \alpha_4^* & 0 \end{pmatrix},$$

we have that the action of $\mathrm{Aut}(\mathbf{R}_{06}^{3*}(0))$ on the subspace $\langle \sum_{i=1}^{4} \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^{4} \alpha_i^* \nabla_i \rangle$, where

$$\begin{array}{ll} \alpha_1^* = x(x^2\alpha_1 + 2xy\alpha_3 + y^2\alpha_4) & \alpha_2^* = x^4\alpha_2 + \frac{1}{2}x^3y\alpha_4 \\ \alpha_3^* = x^3(x\alpha_3 + y\alpha_4) & \alpha_4^* = x^5\alpha_4. \end{array}$$

Since $\mathrm{H}^2_{\mathbf{R}}(\mathbf{R}_{06}^{3*}(0)) = \mathrm{H}^2_{\mathbf{N}}(\mathbf{R}_{06}^{3*}(0)) \oplus \langle \nabla_3, \nabla_4 \rangle$, we are interested in $(\alpha_3, \alpha_4) \neq (0, 0)$. We have the following cases.

(1) $\alpha_4 \neq 0$ and $2\alpha_2 \neq \alpha_3$, then by choosing $x = \frac{-2\alpha_2 + \alpha_3}{2\alpha_4}$ and $y = \frac{\alpha_3(\alpha_3 - 2\alpha_2)}{2\alpha_4^2}$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 + \nabla_4 \rangle$.
(2) $\alpha_4 \neq 0$ and $2\alpha_2 = \alpha_3$ then by choosing $y = -\frac{x\alpha_3}{\alpha_4}$, we have two representatives $\langle \nabla_1 + \nabla_4 \rangle$ and $\langle \nabla_4 \rangle$ depending of $\alpha_3^2 \neq \alpha_1\alpha_4$ or not.
(3) $\alpha_4 = 0$, $\alpha_3 \neq 0$, then by choosing $x = 1$ and $y = -\frac{\alpha_1}{2\alpha_3}$, we have the family of representatives $\langle \alpha\nabla_2 + \nabla_3 \rangle$.

Summarizing, we have the following distinct orbits

$$\langle \alpha\nabla_2 + \nabla_3 \rangle \; \langle \alpha\nabla_1 + \nabla_2 + \nabla_4 \rangle \; \langle \nabla_1 + \nabla_4 \rangle \; \langle \nabla_4 \rangle.$$

Hence, we have the following new algebras

$\mathbf{R}_{27}^4(0, \alpha)$	: $e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = (2\alpha - 1)e_4$	$e_2e_2 = e_4$	$e_3e_1 = e_4$
$\mathbf{R}_{29}^4(\alpha)$	: $e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = 2e_4$	$e_2e_1 = \alpha e_4$	$e_3e_2 = e_4$
$\mathbf{R}_{30}^4$	: $e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_2e_1 = e_4$	$e_3e_2 = e_4$	
$\mathbf{R}_{31}^4$	: $e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_3e_2 = e_4$.		

1.5. Classification Theorem

Now we are ready summarize all results related to the algebraic classification of complex 4-dimensional nilpotent right commutative algebras.

Theorem A. *Let* $\mathbf{R}$ *be a complex* 4*-dimensional nilpotent right commutative algebra. Then* $\mathbf{R}$ *is a Novikov algebra or isomorphic to one algebra from the following list:*

$\mathbf{R}^4_{01}$	: $e_1e_1 = e_2$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_3e_2 = e_4$		
$\mathbf{R}^4_{02}$	: $e_1e_1 = e_2$	$e_1e_3 = e_4$	$e_3e_2 = e_4$			
$\mathbf{R}^4_{03}$	: $e_1e_1 = e_2$	$e_2e_1 = e_4$	$e_3e_2 = e_4$			
$\mathbf{R}^4_{04}$	: $e_1e_1 = e_2$	$e_3e_2 = e_4$				
$\mathbf{R}^4_{05}$	: $e_1e_1 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_3$			
$\mathbf{R}^4_{06}$	: $e_1e_1 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_2e_2 = e_3$		
$\mathbf{R}^4_{07}$	: $e_1e_1 = e_3$	$e_1e_3 = e_4$	$e_2e_2 = e_3$	$e_2e_3 = ie_4$		
$\mathbf{R}^4_{08}$	: $e_1e_1 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_4$	$e_2e_2 = e_3$	$e_2e_3 = ie_4$	
$\mathbf{R}^4_{09}$	: $e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = -e_3$			
$\mathbf{R}^4_{10}$	: $e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_2e_1 = -e_3$	$e_2e_2 = e_4$		
$\mathbf{R}^4_{11}(\lambda)_{\lambda\neq 0}$	: $e_1e_1 = \lambda e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$		
$\mathbf{R}^4_{12}(\lambda)_{\lambda\neq 0}$	: $e_1e_1 = \lambda e_3$	$e_1e_3 = e_4$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_3$		
$\mathbf{R}^4_{13}(\lambda)_{\lambda\neq 0}$	: $e_1e_1 = \lambda e_3$	$e_1e_3 = 2\lambda e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$	$e_2e_3 = (1 - \sqrt{1-4\lambda})e_4$	
$\mathbf{R}^4_{14}(\lambda)_{\lambda\neq 0}$	: $e_1e_1 = \lambda e_3$	$e_1e_3 = 2\lambda e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$	$e_2e_3 = (1 + \sqrt{1-4\lambda})e_4$	
$\mathbf{R}^4_{15}(\lambda)_{\lambda\neq 0}$	: $e_1e_1 = \lambda e_3$	$e_1e_3 = 2\lambda e_4$	$e_2e_1 = e_3 + 2\lambda e_4$	$e_2e_2 = e_3$	$e_2e_3 = (1 - \sqrt{1-4\lambda})e_4$	
$\mathbf{R}^4_{16}(\lambda)_{\lambda\neq 0}$	: $e_1e_1 = \lambda e_3$	$e_1e_3 = 2\lambda e_4$	$e_2e_1 = e_3 + 2\lambda e_4$	$e_2e_2 = e_3$	$e_2e_3 = (1 + \sqrt{1-4\lambda})e_4$	
$\mathbf{R}^4_{17}$	: $e_1e_2 = e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$	$e_3e_1 = e_4$	$e_3e_2 = e_4$	
$\mathbf{R}^4_{18}(\alpha)_{\alpha\neq 0}$	: $e_1e_3 = \alpha e_4$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_3$	$e_3e_1 = (1-\alpha)e_4$	$e_3e_2 = (1-\alpha)e_4$	
$\mathbf{R}^4_{19}(\alpha)_{\alpha\neq 0}$	: $e_1e_3 = \alpha e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$	$e_3e_1 = (1-\alpha)e_4$	$e_3e_2 = (1-\alpha)e_4$	
$\mathbf{R}^4_{20}(\alpha)$	: $e_1e_3 = \alpha e_4$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_3$			
	$e_2e_3 = \alpha e_4$	$e_3e_1 = (1-\alpha)e_4$	$e_3e_2 = (1-\alpha)e_4$			
$\mathbf{R}^4_{21}(\alpha)$	: $e_1e_3 = \alpha e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$			
	$e_2e_3 = \alpha e_4$	$e_3e_1 = (1-\alpha)e_4$	$e_3e_2 = (1-\alpha)e_4$			
$\mathbf{R}^4_{22}$	: $e_1e_2 = e_4$	$e_2e_1 = e_3$	$e_2e_2 = e_3$	$e_2e_3 = e_4$	$e_3e_1 = e_4$	$e_3e_2 = e_4$
$\mathbf{R}^4_{23}$	: $e_2e_1 = e_3$	$e_2e_2 = e_3$	$e_2e_3 = e_4$	$e_3e_1 = e_4$	$e_3e_2 = e_4$	
$\mathbf{R}^4_{24}$	: $e_1e_2 = e_4$	$e_2e_1 = e_3 + e_4$	$e_2e_2 = e_3$	$e_3e_1 = e_4$	$e_3e_2 = e_4$	
$\mathbf{R}^4_{25}$	: $e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_2e_1 = e_3$	$e_3e_1 = e_4$		
$\mathbf{R}^4_{26}(\alpha)$	: $e_1e_1 = e_2$	$e_1e_3 = \alpha e_4$	$e_2e_1 = e_3$	$e_3e_1 = (1-\alpha)e_4$		
$\mathbf{R}^4_{27}(\lambda, \alpha)$	: $e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = (2\alpha - \alpha\lambda - 1)e_4$			
	$e_2e_1 = \lambda e_3$	$e_2e_2 = (\alpha\lambda + 1)e_4$	$e_3e_1 = (\alpha\lambda + 1)e_4$			
$\mathbf{R}^4_{28}(\lambda)_{\lambda\neq 0,1}$	: $e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = 4e_4$			
	$e_2e_1 = \lambda e_3 + \lambda^2(1-\lambda)e_4$		$e_2e_2 = 2\lambda(1+\lambda)e_4$	$e_3e_1 = 2\lambda(1+\lambda)e_4$		
$\mathbf{R}^4_{29}(\alpha)$	: $e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = 2e_4$	$e_2e_1 = \alpha e_4$	$e_3e_2 = e_4$	
$\mathbf{R}^4_{30}$	: $e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_2e_1 = e_4$	$e_3e_2 = e_4$		
$\mathbf{R}^4_{31}$	: $e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_3e_2 = e_4$			

2. The Geometric Classification of Nilpotent Right Commutative Algebras

2.1. Definitions and Notation

Given an n-dimensional vector space $\mathbb{V}$, the set $\mathrm{Hom}(\mathbb{V}\otimes\mathbb{V},\mathbb{V}) \cong \mathbb{V}^*\otimes\mathbb{V}^*\otimes\mathbb{V}$ is a vector space of dimension n^3. This space has the structure of the affine variety $\mathbb{C}^{n^3}$. Indeed, let us fix a basis $e_1,\ldots,e_n$ of $\mathbb{V}$. Then any $\mu \in \mathrm{Hom}(\mathbb{V}\otimes\mathbb{V},\mathbb{V})$ is determined by n^3 structure constants $c_{ij}^k \in \mathbb{C}$ such that $\mu(e_i\otimes e_j) = \sum\limits_{k=1}^{n} c_{ij}^k e_k$. A subset of $\mathrm{Hom}(\mathbb{V}\otimes\mathbb{V},\mathbb{V})$ is *Zariski-closed* if it can be defined by a set of polynomial equations in the variables c_{ij}^k $(1 \le i,j,k \le n)$.

Let T be a set of polynomial identities. The set of algebra structures on $\mathbb{V}$ satisfying polynomial identities from T forms a Zariski-closed subset of the variety $\mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$. We denote this subset by $\mathbb{L}(T)$. The general linear group $\mathrm{GL}(\mathbb{V})$ acts on $\mathbb{L}(T)$ by conjugations:

$$(g * \mu)(x \otimes y) = g\mu(g^{-1}x \otimes g^{-1}y)$$

for $x, y \in \mathbb{V}$, $\mu \in \mathbb{L}(T) \subset \mathrm{Hom}(\mathbb{V} \otimes \mathbb{V}, \mathbb{V})$ and $g \in \mathrm{GL}(\mathbb{V})$. Thus, $\mathbb{L}(T)$ is decomposed into $\mathrm{GL}(\mathbb{V})$-orbits that correspond to the isomorphism classes of algebras. Let $O(\mu)$ denote the orbit of $\mu \in \mathbb{L}(T)$ under the action of $\mathrm{GL}(\mathbb{V})$ and $\overline{O(\mu)}$ denote the Zariski closure of $O(\mu)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two n-dimensional algebras satisfying the identities from T, and let $\mu, \lambda \in \mathbb{L}(T)$ represent $\mathcal{A}$ and $\mathcal{B}$, respectively. We say that $\mathcal{A}$ degenerates to $\mathcal{B}$ and write $\mathcal{A} \to \mathcal{B}$ if $\lambda \in \overline{O(\mu)}$. Note that in this case we have $\overline{O(\lambda)} \subset \overline{O(\mu)}$. Hence, the definition of a degeneration does not depend on the choice of μ and λ. If $\mathcal{A} \not\cong \mathcal{B}$, then the assertion $\mathcal{A} \to \mathcal{B}$ is called a *proper degeneration*. We write $\mathcal{A} \not\to \mathcal{B}$ if $\lambda \notin \overline{O(\mu)}$.

Let $\mathcal{A}$ be represented by $\mu \in \mathbb{L}(T)$. Then $\mathcal{A}$ is *rigid* in $\mathbb{L}(T)$ if $O(\mu)$ is an open subset of $\mathbb{L}(T)$. Recall that a subset of a variety is called irreducible if it cannot be represented as a union of two non-trivial closed subsets. A maximal irreducible closed subset of a variety is called an *irreducible component*. It is well known that any affine variety can be represented as a finite union of its irreducible components in a unique way. The algebra $\mathcal{A}$ is rigid in $\mathbb{L}(T)$ if and only if $\overline{O(\mu)}$ is an irreducible component of $\mathbb{L}(T)$.

Given the spaces U and W, we write simply $U > W$ instead of $\dim U > \dim W$.

2.2. Method of the Description of Degenerations of Algebras

In the present work we use the methods applied to Lie algebras in [4,9]. First of all, if $\mathcal{A} \to \mathcal{B}$ and $\mathcal{A} \not\cong \mathcal{B}$, then $\mathfrak{Der}(\mathcal{A}) < \mathfrak{Der}(\mathcal{B})$, where $\mathfrak{Der}(\mathcal{A})$ is the Lie algebra of derivations of $\mathcal{A}$. We compute the dimensions of algebras of derivations and check the assertion $\mathcal{A} \to \mathcal{B}$ only for such $\mathcal{A}$ and $\mathcal{B}$ that $\mathfrak{Der}(\mathcal{A}) < \mathfrak{Der}(\mathcal{B})$.

To prove degenerations, we construct families of matrices parametrized by t. Namely, let $\mathcal{A}$ and $\mathcal{B}$ be two algebras represented by the structures μ and λ from $\mathbb{L}(T)$ respectively. Let $e_1, \ldots, e_n$ be a basis of $\mathbb{V}$ and c_{ij}^k $(1 \le i, j, k \le n)$ be the structure constants of λ in this basis. If there exist $a_i^j(t) \in \mathbb{C}$ $(1 \le i, j \le n,$ $t \in \mathbb{C}^*)$ such that $E_i^t = \sum\limits_{j=1}^{n} a_i^j(t) e_j$ $(1 \le i \le n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{C}^*$, and the structure constants of μ in the basis $E_1^t, \ldots, E_n^t$ are such rational functions $c_{ij}^k(t) \in \mathbb{C}[t]$ that $c_{ij}^k(0) = c_{ij}^k$, then $\mathcal{A} \to \mathcal{B}$. In this case $E_1^t, \ldots, E_n^t$ is called a *parametrized basis* for $\mathcal{A} \to \mathcal{B}$. To simplify our equations, we will use the notation $A_i = \langle e_i, \ldots, e_n \rangle,\ i = 1, \ldots, n$ and write simply $A_p A_q \subset A_r$ instead of $c_{ij}^k = 0$ $(i \ge p,\ j \ge q,\ k \ge r)$.

Since the variety of 4-dimensional nilpotent right commutative algebras contains infinitely many non-isomorphic algebras, we have to do some additional work. Let $\mathcal{A}(*) := \{\mathcal{A}(\alpha)\}_{\alpha \in I}$ be a series of algebras, and let $\mathcal{B}$ be another algebra. Suppose that for $\alpha \in I$, $\mathcal{A}(\alpha)$ is represented by the structure $\mu(\alpha) \in \mathbb{L}(T)$ and $B \in \mathbb{L}(T)$ is represented by the structure λ. Then we say that $\mathcal{A}(*) \to \mathcal{B}$ if $\lambda \in \overline{\{O(\mu(\alpha))\}_{\alpha\in I}}$, and $\mathcal{A}(*) \not\to \mathcal{B}$ if $\lambda \notin \overline{\{O(\mu(\alpha))\}_{\alpha\in I}}$.

Let $\mathcal{A}(*)$, $\mathcal{B}$, $\mu(\alpha)$ $(\alpha \in I)$ and λ be as above. To prove $\mathcal{A}(*) \to \mathcal{B}$ it is enough to construct a family of pairs $(f(t), g(t))$ parametrized by $t \in \mathbb{C}^*$, where $f(t) \in I$ and $g(t) \in \mathrm{GL}(\mathbb{V})$. Namely, let $e_1, \dots, e_n$ be a basis of $\mathbb{V}$ and c_{ij}^k $(1 \le i, j, k \le n)$ be the structure constants of λ in this basis. If we construct $a_i^j : \mathbb{C}^* \to \mathbb{C}$ $(1 \le i, j \le n)$ and $f : \mathbb{C}^* \to I$ such that $E_i^t = \sum\limits_{j=1}^{n} a_i^j(t)e_j$ $(1 \le i \le n)$ form a basis of $\mathbb{V}$ for any $t \in \mathbb{C}^*$, and the structure constants of $\mu_{f(t)}$ in the basis $E_1^t, \dots, E_n^t$ are such rational functions $c_{ij}^k(t) \in \mathbb{C}[t]$ that $c_{ij}^k(0) = c_{ij}^k$, then $\mathcal{A}(*) \to \mathcal{B}$. In this case $E_1^t, \dots, E_n^t$ and $f(t)$ are called a parametrized basis and a *parametrized index* for $\mathcal{A}(*) \to \mathcal{B}$, respectively.

We now explain how to prove $\mathcal{A}(*) \not\to \mathcal{B}$. Note that if $\mathfrak{Der}\ \mathcal{A}(\alpha) > \mathfrak{Der}\ \mathcal{B}$ for all $\alpha \in I$ then $\mathcal{A}(*) \not\to \mathcal{B}$. One can also use the following Lemma, whose proof is the same as the proof of Lemma 1.5 from [4].

Lemma 4. *Let $\mathfrak{B}$ be a Borel subgroup of* $\mathrm{GL}(\mathbb{V})$ *and $\mathcal{R} \subset \mathbb{L}(T)$ be a $\mathfrak{B}$-stable closed subset. If $\mathcal{A}(*) \to \mathcal{B}$ and for any $\alpha \in I$ the algebra $\mathcal{A}(\alpha)$ can be represented by a structure $\mu(\alpha) \in \mathcal{R}$, then there is $\lambda \in \mathcal{R}$ representing $\mathcal{B}$.*

2.3. The Geometric Classification of 4-Dimensional Nilpotent Right Commutative Algebras

The main result of the present section is the following theorem.

Theorem B. *The variety of* 4*-dimensional nilpotent right commutative algebras has dimension* 15 *and it has five irreducible components defined by infinite families of algebras* $\mathbf{R}_{12}^4(\lambda)$, $\mathbf{R}_{18}^4(\alpha)$, $\mathbf{R}_{27}^4(\lambda, \alpha)$, $\mathbf{R}_{29}^4(\alpha)$ *and* $\mathcal{N}_{20}^4(\alpha)$.

Proof. Recall that the description of all irreducible components of 4-dimensional nilpotent Novikov algebras was given in [6]. Using the cited result, we can see that the variety of 4-dimensional Novikov algebras has two irreducible components given by the following families of algebras:

$\mathcal{N}_{20}^4(\alpha)$:	$e_1e_2 = e_3$	$e_1e_1 = \alpha e_4$	$e_1e_3 = e_4$	$e_2e_2 = e_4$	$e_2e_3 = e_4$	$e_3e_2 = -e_4$
$\mathcal{N}_{22}^4(\lambda)$:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = (2-\lambda)e_4$	$e_2e_1 = \lambda e_4$	$e_2e_2 = \lambda e_4$	$e_3e_1 = \lambda e_4$

Now we can prove that the variety of 4-dimensional nilpotent right commutative algebras has five irreducible components. One can easily compute that

$$\mathfrak{Der}\ \mathcal{N}_{20}^4(\alpha) = 3\ \ \mathfrak{Der}\ \mathbf{R}_{12}^4(\lambda) = 2\ \ \mathfrak{Der}\ \mathbf{R}_{18}^4(\alpha) = 2\ \ \mathfrak{Der}\ \mathbf{R}_{27}^4(\lambda, \alpha) = 3\ \ \mathfrak{Der}\ \mathbf{R}_{29}^4(\alpha) = 2$$

Hence, algebras $\mathbf{R}_{12}^4(\lambda)$, $\mathbf{R}_{18}^4(\alpha)$, $\mathbf{R}_{27}^4(\lambda, \alpha)$ and $\mathbf{R}_{29}^4(\alpha)$ give components of the same dimension. It is easy to see that algebras $\mathbf{R}_{12}^4(\lambda)$, $\mathbf{R}_{18}^4(\alpha)$, $\mathbf{R}_{27}^4(\lambda, \alpha)$ and $\mathbf{R}_{29}^4(\alpha)$ are satisfying the following condition $\mathcal{R} = \{\mu \,|\, A_2A_3 = 0\}$, but the algebras $\mathcal{N}_{20}^4$ do not satisty it. Hence, the family of algebras $\mathcal{N}_{20}^4(\alpha)$ gives an irreducible component. The list of all necessary degenerations is given below:

$\mathbf{R}_{27}^4(\lambda+t, -\frac{t+2}{t^2+\lambda t}) \to \mathcal{N}_{22}^4(\lambda)$	$E_1^t = te_1$	$E_2^t = t^2 e_2$	$E_3^t = t^3 e_3$	$E_4^t = -\frac{2t^3}{\lambda+t} e_4$
$\mathbf{R}_{29}^4(t^{-1}) \to \mathbf{R}_{01}^4$	$E_1^t = 2e_1$	$E_2^t = 4e_2$	$E_3^t = 2t^{-1} e_3$	$E_4^t = 8t^{-1} e_4$
$\mathbf{R}_{29}^4(0) \to \mathbf{R}_{02}^4$	$E_1^t = 2e_1$	$E_2^t = 4e_2$	$E_3^t = t^{-1} e_3$	$E_4^t = 4t^{-1} e_4$
$\mathbf{R}_{29}^4(t^{-3}) \to \mathbf{R}_{03}^4$	$E_1^t = t^{-1} e_1$	$E_2^t = t^{-2} e_2$	$E_3^t = t^{-4} e_3$	$E_4^t = 4t^{-6} e_4$
$\mathbf{R}_{29}^4(0) \to \mathbf{R}_{04}^4$	$E_1^t = t^{-1} e_1$	$E_2^t = t^{-2} e_2$	$E_3^t = t^{-4} e_3$	$E_4^t = t^{-6} e_4$
$\mathbf{R}_{06}^4 \to \mathbf{R}_{05}^4$	$E_1^t = t^{-1} e_1$	$E_2^t = t^{-1} e_2$	$E_3^t = t^{-2} e_3$	$E_4^t = t^{-4} e_4$
$\mathbf{R}_{12}^4(\frac{1}{t^2}-2) \to \mathbf{R}_{06}^4$	$E_1^t = te_1 + te_2$	$E_2^t = e_2$	$E_3^t = e_3$	$E_4^t = te_4$
$\mathbf{R}_{16}^4(\frac{1+it}{t^2}) \to \mathbf{R}_{07}^4$	$E_1^t = e_1$	$E_2^t = -t^{-1} e_2$	$E_3^t = t^{-2} e_3$	$E_4^t = 2t^{-4} e_4$
$\mathbf{R}_{16}^4(\frac{t^2+1}{4t^2}) \to \mathbf{R}_{08}^4$	$E_1^t = -2te_1 + te_2 + te_3$	$E_2^t = e_2$	$E_3^t = e_3$	$E_4^t = -\frac{it+1}{t} e_4$
$\mathbf{R}_{27}^4(-1, 1) \to \mathbf{R}_{09}^4$	$E_1^t = te_1$	$E_2^t = te_2$	$E_3^t = t^2 e_3$	$E_4^t = 2t^3 e_4$
$\mathbf{R}_{27}^4(-1, 1+t) \to \mathbf{R}_{10}^4$	$E_1^t = te_1$	$E_2^t = -t(3t+2)e_2$	$E_3^t = -t^2(3t+2)e_3$	$E_4^t = -t^3(3t+2)^2 e_4$
$\mathbf{R}_{12}^4(\lambda) \to \mathbf{R}_{11}^4(\lambda)$	$E_1^t = t^{-1} e_1$	$E_2^t = t^{-1} e_2$	$E_3^t = t^{-2} e_3$	$E_4^t = t^{-3} e_4$
$\mathbf{R}_{15}^4(\lambda) \to \mathbf{R}_{13}^4(\lambda)$	$E_1^t = t^{-1} e_1$	$E_2^t = t^{-1} e_2$	$E_3^t = t^{-2} e_3$	$E_4^t = t^{-3} e_4$
$\mathbf{R}_{16}^4(\lambda) \to \mathbf{R}_{14}^4(\lambda)$	$E_1^t = t^{-1} e_1$	$E_2^t = t^{-1} e_2$	$E_3^t = t^{-2} e_3$	$E_4^t = t^{-3} e_4$
$\mathbf{R}_{12}^4(\lambda) \to \mathbf{R}_{15}^4(\lambda)$	$E_1^t = \frac{t}{t+\lambda} e_1 - \frac{t(1-\sqrt{1-4\lambda})}{2(t+\lambda)} e_2 - \frac{t(1-\sqrt{1-4\lambda})(2t+\lambda)}{2(t+\lambda)^2} e_3$	$E_2^t = \frac{t(1-\sqrt{1-4\lambda})}{2\lambda(t+\lambda)} e_1 - \frac{4t^2-t(1-\sqrt{1-4\lambda})^2}{4\lambda(t+\lambda)} e_2 - \frac{t(\lambda+1)(1-\sqrt{1-4\lambda}-2\lambda)}{2\lambda(t+\lambda)^2} e_3$	$E_3^t = \frac{t^3}{\lambda(t+\lambda)^2} e_3 + \frac{t^3(1-\sqrt{1-4\lambda})}{2\lambda(t+\lambda)^3} e_4$	$E_4^t = \frac{t^4}{2\lambda^2(t+\lambda)^3} e_4$
$\mathbf{R}_{12}^4(\lambda) \to \mathbf{R}_{16}^4(\lambda)$	$E_1^t = \frac{t}{t+\lambda} e_1 - \frac{t(1+\sqrt{1-4\lambda})}{2(t+\lambda)} e_2 - \frac{t(1+\sqrt{1-4\lambda})(2t+\lambda)}{2(t+\lambda)^2} e_3$	$E_2^t = \frac{t(1+\sqrt{1-4\lambda})}{2\lambda(t+\lambda)} e_1 - \frac{4t^2-t(1+\sqrt{1-4\lambda})^2}{4\lambda(t+\lambda)} e_2 - \frac{t(\lambda+1)(1+\sqrt{1-4\lambda}-2\lambda)}{2\lambda(t+\lambda)^2} e_3$	$E_3^t = \frac{t^3}{\lambda(t+\lambda)^2} e_3 + \frac{t^3(1+\sqrt{1-4\lambda})}{2\lambda(t+\lambda)^3} e_4$	$E_4^t = \frac{t^4}{2\lambda^2(t+\lambda)^3} e_4$
$\mathbf{R}_{20}^4(\alpha) \to \mathbf{R}_{17}^4$	$E_1^t = (1-t)e_1$	$E_2^t = (1-t)e_2 - \frac{(t-1)^2}{t} e_3$	$E_3^t = (t-1)^2 e_3 + \frac{(t-1)^3}{t} e_4$	$E_4^t = (t-1)^2 e_4$
$\mathbf{R}_{18}^4(\alpha) \to \mathbf{R}_{19}^4(\alpha)$	$E_1^t = t^{-1} e_1$	$E_2^t = t^{-1} e_2$	$E_3^t = t^{-2} e_3$	$E_4^t = t^{-3} e_4$
$\mathbf{R}_{18}^4(\alpha) \to \mathbf{R}_{20}^4(\alpha)$	$E_1^t = \frac{t}{\alpha-1+t} e_1$	$E_2^t = \frac{t^2}{(\alpha-1)(\alpha-1+t)} e_2$	$E_3^t = \frac{t^3}{(\alpha-1)(\alpha-1+t)^2} e_3 + \frac{t^3}{(\alpha-1)(\alpha-1+t)+3} e_4$	$E_4^t = \frac{t^4}{(\alpha-1)(\alpha-1+t)^3} e_4$
$\mathbf{R}_{20}^4(\alpha) \to \mathbf{R}_{21}^4(\alpha)$	$E_1^t = t^{-1} e_1$	$E_2^t = t^{-1} e_2$	$E_3^t = t^{-2} e_3$	$E_4^t = t^{-3} e_4$
$\mathbf{R}_{18}^4(t) \to \mathbf{R}_{22}^4$	$E_1^t = \frac{1}{2t-1} e_2 + \frac{1}{t} e_3$	$E_2^t = \frac{t-1}{t-2t^2} e_1 + \frac{1}{t} e_2 + \frac{t-1}{t} e_3$	$E_3^t = \frac{1}{t(2t-1)} e_3 - \frac{t-1}{t^2} e_4$	$E_4^t = \frac{1-t}{t(1-2t)^2} e_4$
$\mathbf{R}_{22}^4 \to \mathbf{R}_{23}^4$	$E_1^t = t^{-1} e_1$	$E_2^t = t^{-1} e_2$	$E_3^t = t^{-2} e_3$	$E_4^t = t^{-3} e_4$
$\mathbf{R}_{20}^4(\alpha) \to \mathbf{R}_{24}^4$	$E_1^t = \frac{1}{2}(t-1)^4 e_1 - \frac{1}{2}(t-1)^3 te_2 - \frac{1}{4}(t-1)^3\left(t^2-2t-1\right) e_3$	$E_2^t = -\frac{1}{2}(t-1)^3 e_2 + \frac{(t-1)^3\left(t^2+1\right)}{4t} e_3$	$E_3^t = \frac{1}{4}(t-1)^6 e_3 - \frac{(t-1)^6\left(t^2+1\right)}{8t} e_4$	$E_4^t = \frac{1}{8}(t-1)^8 e_4$
$\mathbf{R}_{27}^4(\frac{1}{t}, \frac{t}{2t-1}) \to \mathbf{R}_{25}^4$	$E_1^t = 2t^2(2t-1)e_1 - 2(1-2t)^2 t^2 e_2 - 2t^2(2t-1)^3 e_3$	$E_2^t = 4(1-2t)^2 t^4 e_2 - 4t^3(t+1)(2t-1)^3 e_3$	$E_3^t = 8t^5(2t-1)^3 e_3 - 16t^6(2t-1)^3(2t+1)e_4$	$E_4^t = 32t^8(2t-1)^3 e_4$
$\mathbf{R}_{27}^4(\frac{1-\alpha}{t}, \frac{t}{(1-\alpha)(2t-1)}) \to \mathbf{R}_{26}^4(\alpha)$	$E_1^t = t(2t-1)e_1$	$E_2^t = (1-2t)^2 t^2 e_2$	$E_3^t = (\alpha-1)t^2(1-2t)^3 e_3$	$E_4^t = 2t^4(2t-1)^3 e_4$
$\mathbf{R}_{27}^4(\frac{2\lambda^2+2\lambda+4+t}{\lambda(2\lambda-2\lambda^2-t)}, \lambda) \to \mathbf{R}_{28}^4(\lambda)$	$E_1^t = e_1 + \frac{\lambda^2-\lambda+t}{t(\lambda+1)} e_2$	$E_2^t = e_2 + \frac{\lambda^2-\lambda+t}{t} e_3 - \frac{4(\lambda^2-\lambda+t)}{t^2(2\lambda^3+\lambda(t-2)+t)} e_4$	$E_3^t = e_3 - \frac{2(2\lambda^3+\lambda^2(t+2)+\lambda(t-4)+4t)}{\lambda^2 t(2\lambda^2-2\lambda+t)} e_4$	$E_4^t = \frac{2}{\lambda(2\lambda-2\lambda^2-t)} e_4$
$\mathbf{R}_{29}^4(t^{-2}) \to \mathbf{R}_{30}^4$	$E_1^t = t^{-1} e_1 - 2t^{-1} e_3$	$E_2^t = t^{-2} e_2 + t^{-2} e_3 - 4t^{-2} e_4$	$E_3^t = t^{-3} e_3$	$E_4^t = t^{-5} e_4$
$\mathbf{R}_{29}^4(0) \to \mathbf{R}_{31}^4$	$E_1^t = t^{-1} e_1$	$E_2^t = t^{-2} e_2$	$E_3^t = t^{-3} e_3$	$E_4^t = t^{-5} e_4$

□

Compliance with ethical standard

Conflict of interest There is no potential conflict of ethical approval, conflict of interest, and ethical standards.

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