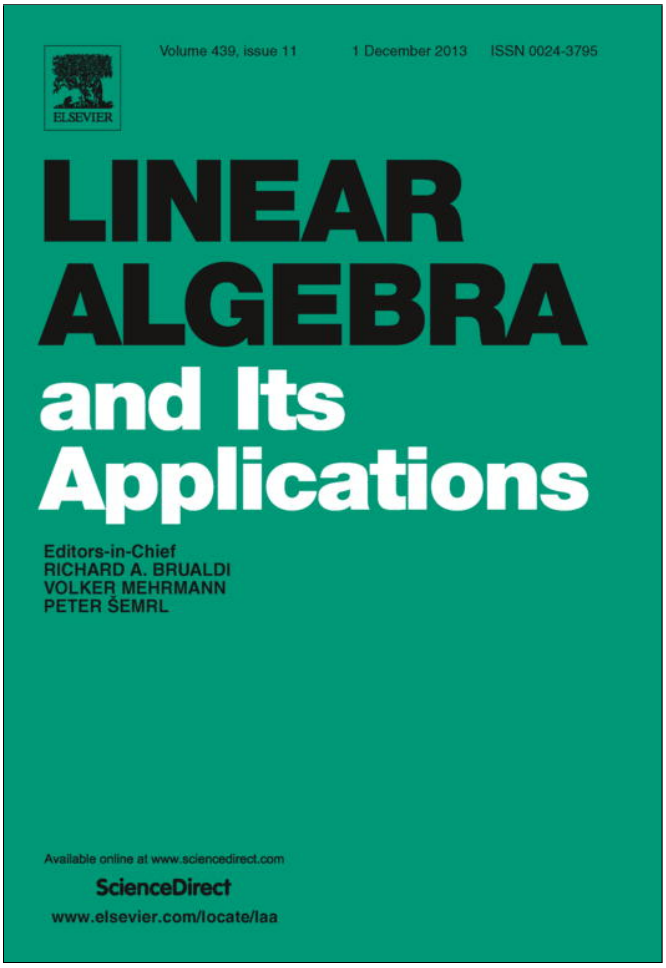
Volume 439, issue 11
1 December 2013
ISSN 0024-3795
ELSEVIER
LINEAR ALGEBRA and Its Applications
Editors-in-Chief
RICHARD A. BRUALDI
VOLKER MEHRMANN
PETER ŠEMRL
Available online at www.sciencedirect.com
ScienceDirect
www.elsevier.com/locate/laa

Linear Algebra and its Applications 439 (2013) 3460–3463

Contents lists available at ScienceDirect

Linear Algebra and its Applications

www.elsevier.com/locate/laa

The classification of algebras of level one

A.Kh. Khudoyberdiyev *, B.A. Omirov

Institute of Mathematics, National University of Uzbekistan, Tashkent, 100125, Uzbekistan

ARTICLE INFO

Article history:
Received 6 April 2013
Accepted 15 September 2013
Available online 17 October 2013
Submitted by P. Semrl

MSC:
14D06
14L30

Keywords:
Closure of orbits
Degeneration
Level of an algebra

ABSTRACT

In the present paper we obtain a list of algebras, up to isomorphism, such that the closure of any complex finite-dimensional algebra contains one of the algebras of the given list.

1. Introduction

It is known that any n-dimensional algebra L over a field F is regarded as an element λ of affine variety $\mathrm{Hom}(V \otimes V, V)$ via the bilinear mapping $\lambda : V \otimes V \to V$ on underlying vector space V of L.

Since the space $\mathrm{Hom}(V \otimes V, V)$ forms an n^3-dimensional affine space $B(V)$ over F, we will consider the Zariski topology on this space and the linear reductive group $\mathrm{GL}_n(F)$ acting on the space as follows:

$$(g * \lambda)(x, y) = g\big(\lambda\big(g^{-1}(x), g^{-1}(y)\big)\big).$$

The orbits $(\mathrm{Orb}(-))$ under this action are the isomorphism classes of algebras. It is clear that if a subvariety of $\mathrm{Hom}(V \otimes V, V)$ is specified by an identity or identities like commutativity, skew-symmetricity, nilpotency, etc. then it is invariant under the action $*$. The closures of orbits under this action play a crucial role in the description of irreducible components of a variety of algebras. Since the closure of an open set gives rise to irreducible components of a variety, those algebras whose

* Corresponding author.
E-mail addresses: khabror@mail.ru (A.Kh. Khudoyberdiyev), omirovb@mail.ru (B.A. Omirov).

0024-3795/$ – see front matter
http://dx.doi.org/10.1016/j.laa.2013.09.020

orbits are open (they are called rigid algebras) generate the irreducible components. The method of degenerations is one of the tools of finding the rigid algebras. The degenerations of algebras can be represented by "arrows". Since there is no algebra that degenerates to a rigid algebra it can be easily concluded that the rigid algebras are located on the highest vertices of this representation scheme. Since any n-dimensional algebra degenerates to the abelian $\mathfrak{a}_n$ the lowest edges end on $\mathfrak{a}_n$. For some examples of the descriptions of varieties of algebras by means of the degeneration graphs we refer to the papers [1,4,6] and others.

In [3] Gorbatsevich describes the nearest-neighbor algebras to $\mathfrak{a}_n$ (algebras of level one) in the degeneration graphs of commutative and skew-symmetric algebras.

If the ground field F is the field of complex numbers $\mathbb{C}$, it is known from [4] that the closures of orbits with respect to the Zariski and Euclidean topologies coincide. Therefore, the fact that $\lambda \in \overline{\mathrm{Orb}(\mu)}$ can be reformulated as follows:

$$\exists \varphi : (0,1] \to \mathrm{GL}_n\big(\mathbb{C}(t)\big),\ t \mapsto g_t \quad \text{such that} \quad \lim_{t\to 0} g_t * \lambda = \mu,$$

where φ is analytic and semialgebraic mapping [2].

In this paper we ameliorate the result of the paper [3] correcting some non-accuracies made in it and give a complete list of algebras level one in the variety of finite-dimensional complex algebras.

Definition 1.1. An algebra λ is said to degenerate to an algebra μ, if $\mathrm{Orb}(\mu)$ lies in the Zariski closure of $\mathrm{Orb}(\lambda)$. We denote this by $\lambda \to \mu$.

The degeneration $\lambda \to \mu$ is called a *direct degeneration* if there is no chain of non-trivial degenerations of the form: $\lambda \to \nu \to \mu$.

Definition 1.2. A level of an algebra λ is the maximum length of chain of direct degenerations. We denote the level of an algebra λ by $lev_n(\lambda)$.

Consider the following n-dimensional algebras with the table of multiplications on a basis: $e_1, e_2, \dots, e_n$:

$$p_n^{\pm}:\quad e_1e_i = e_i, \qquad e_ie_1 = \pm e_i, \quad i \geqslant 2,$$
$$n_3^{\pm}:\quad e_1e_2 = e_3, \qquad e_2e_1 = \pm e_3.$$

Here is main result of [3] which we have mentioned above.

Theorem 1.3. *Let λ be an n-dimensional algebra. Then*:

1. *If λ is skew-commutative, then $lev_n(\lambda) = 1$ if and only if it is isomorphic to p_n^- or to the algebra $n_3^- \oplus \mathfrak{a}_{n-3}$ with $n \geqslant 3$. In particular, the λ is a Lie algebra*;
2. *If λ is commutative, then $lev_n(\lambda) = 1$ if and only if it is isomorphic to p_n^+ or to the algebra $n_3^+ \oplus \mathfrak{a}_{n-3}$ for $n \geqslant 3$. In particular, the λ is a Jordan algebra.*

2. Main result

In this section we describe all complex finite-dimensional algebras of level one.

Consider the following algebras

$$\lambda_2:\quad e_1e_1 = e_2,$$
$$\nu_n(\alpha):\quad e_1e_1 = e_1, \qquad e_1e_i = \alpha e_i, \qquad e_ie_1 = (1-\alpha)e_i, \quad 2 \leqslant i \leqslant n.$$

Let us first give some counter-arguments to Theorem 1.3.

Proposition 2.1. *The algebras $\mathfrak{p}_n^+$ and $\mathfrak{n}_3^+ \oplus \mathfrak{a}_{n-3}$ are not algebras of level one, i.e. there are the following degenerations through $\lambda_2 \oplus \mathfrak{a}_{n-2}$:*

$$p_n^+ \to \lambda_2 \oplus \mathfrak{a}_{n-2} \to \mathfrak{a}_n \quad \text{and} \quad n_3^+ \oplus \mathfrak{a}_{n-3} \to \lambda_2 \oplus \mathfrak{a}_{n-2} \to \mathfrak{a}_n.$$

Proof. The first degeneration is given by the family of transformations g_t:

$$g_t(e_1) = t^{-1}e_1 - \frac{t^{-2}}{2}e_2, \qquad g_t(e_2) = \frac{t^{-2}}{2}e_2, \qquad g_t(e_i) = t^{-2}e_i, \quad 3 \leqslant i \leqslant n.$$

The second one is given by f_t as follows:

$$f_t(e_1) = t^{-1}e_1 - t^{-2}e_3, \qquad f_t(e_2) = t^{-2}e_3,$$
$$f_t(e_3) = \frac{t^{-2}}{2}e_2, \qquad f_t(e_i) = e_i, \quad 4 \leqslant i \leqslant n. \qquad \square$$

To prove the main theorem we give the following subsidiary result.

Proposition 2.2. *Any n-dimensional ($n \geqslant 3$) non-abelian algebra degenerates to one of the following algebras*

$$p_n^-, \qquad n_3^- \oplus \mathfrak{a}_{n-3}, \qquad \lambda_2 \oplus \mathfrak{a}_{n-2}, \qquad \nu_n(\alpha), \quad \alpha \in \mathbb{C}.$$

Proof. Let A be an n-dimensional non-abelian algebra. The case when the algebra A is skew-symmetric the result has been given in [5] (see Theorem 5.2). Therefore, we assume that the algebra A is not skew-symmetric. Hence always there exists an element x of A such that $xx \neq 0$.

Case 1. Assume that there exists an element x of A such that $xx \notin \langle x \rangle$. Then we can choose the basis $e_1 = x$, $e_2 = xx, \dots, e_n$. The degeneration $A \to \lambda_2 \oplus \mathfrak{a}_{n-2}$ is realized by the family g_t:

$$g_t(e_1) = t^{-1}e_1, \qquad g_t(e_i) = t^{-2}e_i, \quad 2 \leqslant i \leqslant n.$$

Case 2. Let $xx \in \langle x \rangle$ for all $x \in A$. Then for any $x, y \in A$ we have

$$(x+y)(x+y) = xx + xy + yx + yy \in \langle x+y \rangle.$$

Therefore, $xy + yx \in \langle x, y \rangle$.

If there exist elements x and y such that $xy \notin \langle x, y \rangle$, then we choose the basis $\{e_1 = x, e_2 = y, e_3 = xy, \dots, e_n\}$ of the algebra A. The following family of transformations

$$g_t: \quad g_t(e_1) = t^{-1}e_1, \qquad g_t(e_2) = t^{-1}e_2, \qquad g_t(e_i) = t^{-2}e_i, \quad 3 \leqslant i \leqslant n$$

gives the degeneration $A \to n_3^- \oplus \mathfrak{a}_{n-3}$.

Now we consider the case when $xy \in \langle x, y \rangle$ for all $x, y \in A$. Then for a basis $\{e_1, e_2, e_3, \dots, e_n\}$ of A we have $e_ie_i = \alpha_i e_i$, $1 \leqslant i \leqslant n$. Taking into account that the algebra A is not skew-symmetric, we can suppose $\alpha_1 \neq 0$. Also without loss of generality we can assume that $\alpha_i \neq 0$, $1 \leqslant i \leqslant k$ and $\alpha_i = 0$, $k+1 \leqslant i \leqslant n$. By scaling of basic elements we get $e_ie_i = e_i$, $1 \leqslant i \leqslant k$, $\alpha_i = 0$, $k+1 \leqslant i \leqslant n$.

The inclusion of the following products

$$(e_1 \pm e_i)(e_1 \pm e_i) = e_1 \pm e_1e_i \pm e_ie_1 + e_i \in \langle e_1 \pm e_i \rangle,$$

implies $e_1e_i + e_ie_1 = e_1 + e_i$, $1 \leqslant i \leqslant k$.

Similarly, one obtains

$$e_1e_i + e_ie_1 = e_i, \quad k+1 \leqslant i \leqslant n.$$

Making the change of basis

$$e'_i = e_i - e_1, \quad 2 \leqslant i \leqslant k, \qquad e'_i = e_i, \quad k+1 \leqslant i \leqslant n,$$

we get the following products

$$e_1 e_1 = e_1, \qquad e_i e_i = 0, \quad 2 \leqslant i \leqslant n,$$
$$e_1 e_i = \alpha_i e_i + \beta_i e_1, \qquad e_i e_1 = (1-\alpha_i) e_i - \beta_i e_1, \quad 2 \leqslant i \leqslant n,$$

for some $\alpha_i, \beta_i \in \mathbb{C}$.

The equality

$$e_1(e_i + e_j) = (\alpha_j - \alpha_i) e_j + \alpha_i (e_i + e_j) + (\beta_i + \beta_j) e_1$$

and $e_1(e_i + e_j) \in \langle e_1, e_i + e_j \rangle$ imply $\alpha_i = \alpha$, $2 \leqslant i \leqslant n$.

Then the degeneration $A \to \nu_n(\alpha)$ is realized by using the family

$$g_t\colon \quad g_t(e_1) = e_1, \qquad g_t(e_i) = t^{-1} e_i, \quad 2 \leqslant i \leqslant n,$$

which completes the proof of proposition. □

Theorem 2.3. *An n-dimensional ($n \geqslant 3$) algebra is algebra of level one if and only if it is isomorphic to one of the following algebras*:

$$\mathfrak{p}_n^-, \qquad \mathfrak{n}_3^- \oplus \mathfrak{a}_{n-3}, \qquad \lambda_2 \oplus \mathfrak{a}_{n-2}, \qquad \nu_n(\alpha), \quad \alpha \in \mathbb{C}.$$

Proof. Due to Proposition 2.2 it is sufficient to prove that these four algebras do not degenerate to each other. From [5, Theorem 5.2] we have $\overline{\mathrm{Orb}(\mathfrak{p}_n^-)} = \{\mathfrak{p}_n^-, \mathfrak{a}_n\}$ and $\overline{\mathrm{Orb}(\mathfrak{n}_3^- \oplus \mathfrak{a}_{n-3})} = \{\mathfrak{n}_3^- \oplus \mathfrak{a}_{n-3}, \mathfrak{a}_n\}$. Therefore, we only need to prove that $\overline{\mathrm{Orb}(\lambda_2 \oplus \mathfrak{a}_{n-2})} = \{\lambda_2 \oplus \mathfrak{a}_{n-2}, \mathfrak{a}_n\}$ and $\overline{\mathrm{Orb}(\nu_n(\alpha))} = \{\nu_n(\alpha), \mathfrak{a}_n\}$.

Since $\lambda_2 \oplus \mathfrak{a}_{n-2}$ is commutative, $\mathfrak{p}_n^-$ and $\mathfrak{n}_3^- \oplus \mathfrak{a}_{n-3}$ are skew-symmetric algebras, we obtain $\mathfrak{p}_n^-, \mathfrak{n}_3^- \oplus \mathfrak{a}_{n-3} \notin \overline{\mathrm{Orb}(\lambda_2 \oplus \mathfrak{a}_{n-2})}$. Moreover, the algebra $\lambda_2 \oplus \mathfrak{a}_{n-2}$ is nilpotent, but $\nu_n(\alpha)$ is not nilpotent. Therefore, $\overline{\mathrm{Orb}(\lambda_2 \oplus \mathfrak{a}_{n-2})} = \{\lambda_2 \oplus \mathfrak{a}_{n-2}, \mathfrak{a}_n\}$.

To prove that $\overline{\mathrm{Orb}(\nu_n(\alpha))} = \{\nu_n(\alpha), \mathfrak{a}_n\}$, we make use the following fact from [1]: if λ degenerates μ then $\dim \mathrm{Der}(\lambda) < \dim \mathrm{Der}(\mu)$. Computing the dimensions of the derivation algebras as follows

$$\dim\bigl(\mathrm{Der}(\nu_n(\alpha))\bigr) = n^2 - n, \qquad \dim\bigl(\mathrm{Der}(\lambda_2 \oplus \mathfrak{a}_{n-2})\bigr) = n^2 - 2n + 2,$$
$$\dim\bigl(\mathrm{Der}(\mathfrak{p}_n^-)\bigr) = n^2 - n, \qquad \dim\bigl(\mathrm{Der}(\mathfrak{n}_3^- \oplus \mathfrak{a}_{n-3})\bigr) = n^2 - 3n + 6,$$

we conclude that $\nu_n(\alpha)$ is an algebra of level one. □

It is observed that the list of two-dimensional algebras of level one is as follows

$$\mathfrak{p}_2^-, \qquad \lambda_2, \qquad \nu_2(\alpha).$$

Acknowledgements

The authors are grateful to the anonymous referee for useful remarks. This work is supported by the Grant (RGA) No. 11-018 RG/Math/AS_I–UNESCO FR: 3240262715.

References

[1] D. Burde, Degenerations of 7-dimensional nilpotent Lie algebras, Comm. Algebra 33 (4) (2005) 1259–1277.
[2] J.M. Casas, A.Kh. Khudoyberdiyev, M. Ladra, B.A. Omirov, On the degenerations of solvable Leibniz algebras, Linear Algebra Appl. 439 (2) (2013) 472–487.
[3] V.V. Gorbatsevich, Contractions and degenerations of finite-dimensional algebras, Soviet Math. (Iz. VUZ) 35 (10) (1991) 17–24.
[4] F. Grunewald, J. O'Halloran, Varieties of nilpotent Lie algebras of dimension less than six, J. Algebra 112 (2) (1988) 315–325.
[5] J. Lauret, Degenerations of Lie algebras and geometry of Lie groups, Differential Geom. Appl. 18 (2) (2003) 177–194.
[6] C. Seeley, Degenerations of 6-dimensional nilpotent Lie algebras over ℂ, Comm. Algebra 18 (10) (1990) 3493–3505.