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Classification of Leibniz algebras corresponding to three dimensional solvable Lie algebras

Abror Khudoyberdiyev, Emily Olson

Abstract. In this paper we investigate Leibniz algebras whose quotient Lie algebra is a two and three dimensional Solvable Lie algebras. We provide classification of Leibniz algebras L whose $G = L/I$ is the algebra two and three dimensional Solvable Lie algebra and the action $I \times G \to I$ gives rise to a minimal faithful representation of G.

Keywords: Lie algebras, Leibniz algebras, representation, faithful representation.

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1 Introduction

It is known that Lie algebras are a special kind of Leibniz algebras, with useful additional properties and behaviors. Throughout the course of this paper, we will examine a special type of Leibniz algebras, those that associated with the two and three dimensional solvable Lie algebras. In doing so, we utilize tactics from representation theory, module theory, and parameter manipulation. With the use of thorough basis changes, and substitution, we are able to define some non-isomorphic Leibniz algebras that share certain characteristics with the beginning Lie algebra.

Recall that the variety of Leibniz algebras is defined by the fundamental identity [6]

$$[x, [y, z]] = [[x, y], z] - [[x, z], y].$$

One of the methods of classification of Leibniz algebras is the study of those with given corresponding Lie algebras. In the papers [2, 4], Leibniz algebras whose corresponding Lie algebras are naturally graded filiform Lie algebras and Heisenberg algebras are studied.

In fact, each non-Lie Leibniz algebra contains a non-trivial ideal (further denoted by I), which is the subspace spanned by squares of elements of the algebra [3]. Moreover, it is readily available to see that this ideal belongs to the right annihilator of L, that is $[L, I] = 0$. Note also that the ideal I is the minimal ideal with the property that the quotient algebra L/I is a Lie algebra.

The map $I \times L/I \to I$, $(i, \overline{x}) \mapsto [i, x]$ endows I with a structure of L/I-module [1].

Denote by $Q(L) = L/I \oplus I$, then the operation $(-,-)$ defines the Leibniz algebra structure on $Q(L)$, where

$$(\overline{x}, \overline{y}) = \overline{[x, y]}, \quad (\overline{x}, i) = [x, i], \quad (i, \overline{x}) = 0, \quad (i, j) = 0, \qquad x, y \in L, \ i, j \in I.$$

Therefore, given a Lie algebra G and a $G-$module M, we can construct a Leibniz algebra (G, M) by the above construction.

The main problem which occurs in this connection is in finding a description of Leibniz algebras L, such that the corresponding Leibniz algebra $Q(L)$ is isomorphic to an a priory given algebra (G, M).

2 Preliminaries

In this section, we lay the groundwork for the preceding sections, first establishing the fundamental identities, then the minimal faithful representation, as supplied by the paper by Ceballos et. al. [5], then utilizing the left modules that correspond to the given representations.

We will devote most of our time in this paper on the notion of solvable algebras. First, we define a sequence of ideals of a Leibniz algebra, L. The sequence of ideals, **a derived series**, is denoted

$$L^{(1)} = L, \quad L^{(k+1)} = [L^{(k)} L^{(k)}], \quad k \geq 1.$$

We call L **solvable** if there exists some n for which $L^{(n)} = 0$.

In the cases where a given Lie algebra has all zero Lie-brackets, we call these algebras **abelian**. We are considering the two and three-dimensional solvable Lie algebras, of which none are all zero Lie-brackets.

As a precursor to the problem we are investigating, we will reference the following resource, [5], for the algebras of interest.

The List of Two and Three Dimensional Solvable Lie Algebras

$$\begin{array}{lll}
S_2^2: & [e_1, e_2] = e_2, & \\
S_3^2: & [e_1, e_3] = e_2, & \\
S_3^3: & [e_1, e_3] = e_1, & [e_2, e_3] = e_2, \\
S_3^4: & [e_1, e_3] = e_2, & [e_2, e_3] = -e_1, \\
S_3^5: & [e_1, e_3] = -e_1, & [e_2, e_3] = -e_1 - e_2, \\
S_3^6: & [e_1, e_3] = -e_1. &
\end{array}$$

Given a Lie algebra G, a representation of g in n-dimensional vector space V is a homomorphism of Lie algebras $\varphi : G \to gl(V)$. The natural integer n is called dimension (or degree) of this representation. The representation is called **faithful** if $Ker(\varphi) = 0$.

Additionally, we will utilize representation theory in order to better understand these two and three dimensional Lie algebras [5].

Minimal Faithful Representations

$$\begin{array}{llll}
\varphi(S_2^2): & \varphi(e_1) = X_{1,1}, & \varphi(e_2) = X_{1,2}, & \\
\varphi(S_3^2): & \varphi(e_1) = X_{1,2}, & \varphi(e_2) = X_{1,3}, & \varphi(e_3) = X_{2,3}, \\
\varphi(S_3^3): & \varphi(e_1) = X_{1,3}, & \varphi(e_2) = X_{1,2}, & \varphi(e_3) = -X_{1,1}, \\
\varphi(S_3^4): & \varphi(e_1) = X_{1,2}, & \varphi(e_2) = iX_{1,2} + X_{1,3}, & \varphi(e_3) = iX_{2,2} + X_{2,3} - iX_{3,3}, \\
\varphi(S_3^5): & \varphi(e_1) = X_{1,3}, & \varphi(e_2) = X_{2,3}, & \varphi(e_3) = X_{1,2} - X_{3,3}, \\
\varphi(S_3^6): & \varphi(e_1) = X_{1,2}, & \varphi(e_2) = X_{1,1} + X_{2,2}, & \varphi(e_3) = -X_{2,2}.
\end{array}$$

Using the above knowledge, we also use module theoretic notions and the representation to better understand the algebras.

Left Modules Corresponding to Given Representations

$$\begin{array}{lllll}
V * S_2^2: & v_1 * e_1 = v_1, & v_1 * e_2 = v_2, & & \\
V * S_3^2: & v_1 * e_1 = v_2, & v_1 * e_2 = v_3, & v_2 * e_3 = v_3, & \\
V * S_3^3: & v_1 * e_1 = v_3, & v_1 * e_2 = v_2, & v_1 * e_3 = -v_1, & \\
V * S_3^4: & v_1 * e_1 = v_2, & v_1 * e_2 = iv_2 + v_3, & v_2 * e_3 = iv_2 + v_3, & v_3 * e_3 = -iv_3, \\
V * S_3^5: & v_1 * e_1 = v_3, & v_1 * e_3 = v_2, & v_2 * e_2 = v_3, & v_3 * e_3 = -v_3, \\
V * S_3^6: & v_1 * e_1 = v_2, & v_1 * e_2 = v_1, & v_2 * e_2 = v_2, & v_2 * e_3 = -v_2.
\end{array}$$

In the present paper we restrict our attention to the cases in which the Lie algebra G is a two or three dimensional solvable Lie algebras presented above and the G-module V is the minimal faithful module.

3 Main results

Our main focus is on the different conditions of parameters that beget different algebras. At the outset, we want to examine the known solvable, 2-dimensional, non-abelian, Lie Algebra, in order to determine the association with this and another Leibniz Algebra.

3.1 Leibniz algebras associated with S_2^2

Let L be a Leibniz algebra associated with the algebra S_2^2. Since the minimal faithful representation on the algebra S_2^2 defined in the two dimensional vector space, we know that L is a four dimensional. Then there exist a basis $\{e_1, e_2, v_1, v_2\}$ of L such that

$$\begin{array}{ll}
[v_1, e_1] = v_1, & [v_1, e_2] = v_2, \\
[e_1, e_1] = \alpha_1 v_1 + \alpha_2 v_2, & [e_1, e_2] = e_2 + \beta_1 v_1 + \beta_2 v_2, \\
[e_2, e_1] = -e_2 + \gamma_1 v_1 + \gamma_2 v_2, & [e_2, e_2] = \theta_1 v_1 + \theta_2 v_2.
\end{array}$$

Now consider the following change of basis, $e_1' = e_1 - \alpha_1 v_1$, $e_2' = e_2 + \beta_1 v_1 + (\beta_2 - \alpha_1)v_2$, which we can assume that

$$[e_1, e_1] = \alpha_2 v_2, \quad [e_1, e_2] = e_2.$$

With the new basis change, we check some of the Leibniz identities.

From $[e_2, [e_2, e_1]] = [[e_2, e_2], e_1] - [[e_2, e_1], e_2]$ we obtain that $\theta_1 = 0$, $\gamma_1 = 2\theta_2$. The identity $[e_1, [e_2, e_1]] = [[e_1, e_2], e_1] - [[e_1, e_1], e_2]$ becomes $\theta_2 = 0, \gamma_2 = 0$.

Thus, the table of multiplication of L has the form:

$$[v_1, e_1] = v_1, \quad [v_1, e_2] = v_2, \quad [e_1, e_1] = \alpha_2 v_2, \quad [e_1, e_2] = e_2, \quad [e_2, e_1] = -e_2.$$

There are two cases, the first in which we assume that $\alpha_2 = 0$. In this instance, we have one algebra. In the second case, if $\alpha_2 \neq 0$, by changing the basis, $e_1' = e_1, e_2' = e_2, v_1' = \alpha_2 v_1, v_2' = \alpha_2 v_2$, we may assume that $\alpha_2 = 1$.

Therefore, we obtain the following theorem

Theorem 3.1. *Let L be a 4-dimensional Leibniz algebra such that $L/I \cong S_2^2$ and I is the L/I-module with the minimal faithful representation. Then L is isomorphic to one of the following two non isomorphic algebras*

$$L_4^1: \quad [v_1, e_1] = v_1, \quad [v_1, e_2] = v_2, \quad [e_1, e_2] = e_2, \quad [e_2, e_1] = -e_2,$$
$$L_4^2: \quad [v_1, e_1] = v_1, \quad [v_1, e_2] = v_2, \quad [e_1, e_1] = v_2, \quad [e_1, e_2] = e_2, \quad [e_2, e_1] = -e_2.$$

Proof. The algebras L_1 and L_2 were obtained from the previous consideration. Since $dim(Der(L_4^1)) = 4$ and $dim(Der(L_4^2)) = 3$ these two algebras are not isomorphic. □

3.2 Leibniz algebras associated with S_3^3

Let L be a Leibniz algebra associated with the algebra S_3^3. Then L is six dimensional and there exist a basis $\{e_1, e_2, e_3, v_1, v_2, v_3\}$ of L such that

$$\begin{aligned}
&[v_1, e_1] = v_3, \quad [v_1, e_2] = v_2, && [v_1, e_3] = -v_1,\\
&[e_1, e_1] = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3, && [e_2, e_2] = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3\\
&[e_3, e_3] = \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3 && \\
&[e_1, e_2] = \omega_1 v_1 + \omega_2 v_2 + \omega_3 v_3, && [e_2, e_1] = \delta_1 v_1 + \delta_2 v_2 + \delta_3 v_3\\
&[e_1, e_3] = e_1 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3, && [e_3, e_1] = -e_1 + \theta_1 v_1 + \theta_2 v_2 + \theta_3 v_3\\
&[e_2, e_3] = e_2 + \mu_1 v_1 + \mu_2 v_2 + \mu_3 v_3, && [e_3, e_2] = -e_2 + \zeta_1 v_1 + \zeta_2 v_2 + \zeta_3 v_3
\end{aligned}$$

Considering the following basis changes

$$e_1' = e_1 - \alpha_3 v_1 + \lambda_2 v_2 + \lambda_3 v_3, \; e_2' = e_2 - \beta_2 v_1 + \mu_2 v_2 + \mu_3 v_3, \; e_3' = e_3 + \gamma_1 v_3$$

we may assume

$$[e_1, e_1] = \alpha_1 v_1 + \alpha_2 v_2, \; [e_2, e_2] = \beta_1 v_1 + \beta_3 v_3, \; [e_3, e_3] = \gamma_2 v_2 + \gamma_3 v_3,$$

$$[e_1, e_3] = e_1 + \lambda_1 v_1, \ [e_2, e_3] = e_2 + \mu_1 v_1.$$

We now look at the Leibniz identities to eliminate more parameters:

$$\begin{array}{lll}
[e_1,[e_1,e_2]] = [[e_1,e_1],e_2] - [[e_1,e_2],e_1] & \Rightarrow & \alpha_1 = \omega_1 = 0, \\
[e_1,[e_1,e_3]] = [[e_1,e_1],e_3] - [[e_1,e_3],e_1] & \Rightarrow & \alpha_2 = \lambda_1 = 0, \\
[e_1,[e_2,e_3]] = [[e_1,e_2],e_3] - [[e_1,e_3],e_2] & \Rightarrow & \omega_2 = \omega_3 = 0, \\
[e_2,[e_1,e_2]] = [[e_2,e_1],e_2] - [[e_2,e_2],e_1] & \Rightarrow & \delta_1 = \beta_1 = 0, \\
[e_2,[e_1,e_3]] = [[e_2,e_1],e_3] - [[e_2,e_3],e_1] & \Rightarrow & \delta_2 = 0, \ \mu_1 = 2\delta_3, \\
[e_2,[e_2,e_3]] = [[e_2,e_2],e_3] - [[e_2,e_3],e_2] & \Rightarrow & \delta_3 = 0, \\
[e_3,[e_1,e_2]] = [[e_3,e_1],e_2] - [[e_3,e_2],e_1] & \Rightarrow & \theta_1 = 0, \\
[e_3,[e_1,e_3]] = [[e_3,e_1],e_3] - [[e_3,e_3],e_1] & \Rightarrow & \theta_2 = \theta_3 = 0, \\
[e_3,[e_2,e_3]] = [[e_3,e_2],e_3] - [[e_3,e_3],e_2] & \Rightarrow & \zeta_1 = \zeta_2 = \zeta_3 = 0.
\end{array}$$

With these Leibniz identities, we rewrite our multiplication table without the eliminated parameters:

$$\begin{array}{lll}
[v_1, e_1] = v_3, & [v_1, e_2] = v_2, & [v_1, e_3] = -v_1 \\
[e_3, e_3] = \gamma_2 v_2 + \gamma_3 v_3, & [e_1, e_3] = e_1, & [e_3, e_1] = -e_1. \\
 & [e_2, e_3] = e_2, & [e_3, e_2] = -e_2
\end{array} \tag{3.1}$$

Therefore, we obtain the following theorem

Theorem 3.2. *Let L be a 6-dimensional Leibniz algebra such that $L/I \cong S_3^3$ and I is the L/I-module with the minimal faithful representation. Then L is isomorphic to one of the following two non isomorphic algebras*

$$L_6^1: \begin{array}{llll}
[v_1, e_1] = v_3, & [v_1, e_2] = v_2, & [v_1, e_3] = -v_1, & \\
[e_1, e_3] = e_1, & [e_3, e_1] = -e_1, & [e_2, e_3] = e_2, & [e_3, e_2] = -e_2.
\end{array}$$

$$L_6^2: \begin{array}{llll}
[v_1, e_1] = v_3, & [v_1, e_2] = v_2, & [v_1, e_3] = -v_1, & [e_3, e_3] = v_2, \\
[e_1, e_3] = e_1, & [e_3, e_1] = -e_1, & [e_2, e_3] = e_2, & [e_3, e_2] = -e_2.
\end{array}$$

Proof. If in the (3.1) $\gamma_2 = \gamma_3 = 0$ then we have the algebra L_6^1. In the second case, if $(\gamma_2, \gamma_3) \neq (0, 0)$ changing the basis: $e_1' = e_1 + Ae_2$, $e_2' = \gamma_2 e_2 + \gamma_3 e_1$, $e_3' = e_3$ and $v_1' = v_1$, $v_2' = \gamma_2 v_2 + \gamma_3 v_3$, $v_3' = v_3 + Av_2$, we obtain the second algebra L_6^2

Since $dim(Der(L_6^1)) = 9$ and $dim(Der(L_6^2)) = 7$ these two algebras are not isomorphic. □

3.3. Leibniz algebras associated with S_3^4

Let L be a Leibniz algebra associated with the algebra S_3^4. Then L is six dimensional and there exist a basis $\{e_1, e_2, e_3, v_1, v_2, v_3\}$ of L such that

$$\begin{array}{llll}
[v_1, e_1] = v_2, & [v_1, e_2] = iv_2 + v_3, & [v_2, e_3] = iv_2 + v_3, & [v_3, e_3] = -iv_3, \\
[e_1, e_1] = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3, & & [e_2, e_2] = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3, & \\
[e_3, e_3] = \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3, & & & \\
[e_1, e_2] = \omega_1 v_1 + \omega_2 v_2 + \omega_3 v_3, & & [e_2, e_1] = \delta_1 v_1 + \delta_2 v_2 + \delta_3 v_3, & \\
[e_1, e_3] = e_2 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3, & & [e_3, e_1] = -e_2 + \theta_1 v_1 + \theta_2 v_2 + \theta_3 v_3, & \\
[e_2, e_3] = -e_1 + \mu_1 v_1 + \mu_2 v_2 + \mu_3 v_3, & & [e_3, e_2] = e_1 + \zeta_1 v_1 + \zeta_2 v_2 + \zeta_3 v_3 &
\end{array}$$

Now, we want to eliminate some of these parameter through change of basis and the Leibniz identities. Making the change of basis

$$e_1' = e_1 - \alpha_2 v_1, \; e_2' = e_2 - \beta_3 v_1 - \theta_2 v_2 + \theta_3 v_3, \; e_3' = e_3 + i\gamma_2 v_2 + (i\gamma_3 - \gamma_2) v_3$$

we have

$$[e_1, e_1] = \alpha_1 v_1 + \alpha_3 v_3, \; [e_2, e_2] = \beta_1 v_1 + \beta_2 v_2, \; [e_3, e_3] = \gamma_1 v_1,$$

$$[e_3, e_1] = -e_2 + \theta_1 v_1, \; [e_3, e_2] = e_1 + \zeta_1 v_1.$$

We are now ready to use the Leibniz identities to further eliminate unnecessary parameters.

$$\begin{array}{lll}
[e_1, [e_1, e_2]] = [[e_1, e_1], e_2] - [[e_1, e_2], e_1] & \Rightarrow & \alpha_1 = \omega_1 = 0, \\
[e_1, [e_1, e_3]] = [[e_1, e_1], e_3] - [[e_1, e_3], e_1] & \Rightarrow & \delta_1 = 0, \omega_2 = -\delta_2 - \lambda_1, \omega_3 = -\alpha_3 i - \delta_3, \\
[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] - [[e_1, e_3], e_2] & \Rightarrow & \beta_1 = 0, \delta_2 = \delta_3 i, \beta_2 = -\delta_3, \\
[e_2, [e_1, e_3]] = [[e_2, e_1], e_3] - [[e_2, e_3], e_1] & \Rightarrow & \delta_3 = \mu_1, \alpha_3 = 2\delta_3 i - \delta_3, \\
[e_2, [e_2, e_3]] = [[e_2, e_2], e_3] - [[e_2, e_3], e_2] & \Rightarrow & \delta_3 = \lambda_1 = 0, \\
[e_3, [e_1, e_2]] = [[e_3, e_1], e_2] - [[e_3, e_2], e_1] & \Rightarrow & \theta_1 = \zeta_1 = 0, \\
[e_3, [e_1, e_3]] = [[e_3, e_1], e_3] - [[e_3, e_3], e_1] & \Rightarrow & \mu_2 = \gamma_1, \mu_3 = 0, \\
[e_3, [e_2, e_3]] = [[e_3, e_2], e_3] - [[e_3, e_3], e_2] & \Rightarrow & \lambda_2 = \mu_2 i, \mu_2 = \lambda_3.
\end{array}$$

Putting all of these together, we have the new multiplication as follows:

$$\begin{array}{lll}
[v_1, e_1] = v_2, & [v_1, e_2] = iv_2 + v_3, & [v_2, e_3] = iv_2 + v_3, \\
[v_3, e_3] = -iv_3, & [e_3, e_1] = -e_2, & [e_1, e_3] = e_2 + \mu_2 i v_2 + \mu_2 v_3, \\
[e_3, e_3] = \mu_2 v_1, & [e_3, e_2] = e_1, & [e_2, e_3] = -e_1 - \mu_2 v_2
\end{array} \tag{3.2}$$

Theorem 3.3. *Let L be a 6-dimensional Leibniz algebra such that $L/I \cong S_3^4$ and I is the L/I-module with the minimal faithful representation. Then L is isomorphic to one of the following two non isomorphic algebras*

$$L_6^3 : \begin{array}{llll}
[v_1, e_1] = v_2, & [v_1, e_2] = iv_2 + v_3, & [v_2, e_3] = iv_2 + v_3, & [v_3, e_3] = -iv_3, \\
[e_1, e_3] = e_2, & [e_3, e_1] = -e_2, & [e_2, e_3] = -e_1, & [e_3, e_2] = e_1.
\end{array}$$

$$L_6^4: \begin{array}{lll} [v_1,e_1]=v_2, & [v_1,e_2]=iv_2+v_3, & [v_2,e_3]=iv_2+v_3, \\ [v_3,e_3]=-iv_3, & [e_3,e_3]=v_1, & [e_3,e_1]=-e_2, \\ [e_3,e_2]=e_1, & [e_1,e_3]=e_2+iv_2+v_3, & [e_2,e_3]=-e_1-v_2. \end{array}$$

Proof. If in the (3.2) $\mu_2=0$ then we have the algebra L_6^3. In the second case, if $\mu_2\neq 0$ changing the basis: $v_1'=\mu_2v_1$, $v_2'=\mu_2v_2$, $v_3'=\mu_2v_3$, we obtain the algebra L_6^4.

Since $dim(Der(L_6^3))=4$ and $dim(Der(L_6^4))=3$ these two algebras are not isomorphic. □

3.4 Leibniz algebras associated with S_3^5

Let L be a Leibniz algebra associated with the algebra S_3^5. Then L is six dimensional and there exist a basis $\{e_1,e_2,e_3,v_1,v_2,v_3\}$ of L such that

$$\begin{array}{ll} [v_1,e_1]=v_3, \quad [v_1,e_3]=v_2, & [v_2,e_2]=v_3, \quad [v_3,e_3]=-v_3, \\ [e_1,e_1]=\alpha_1v_1+\alpha_2v_2+\alpha_3v_3, & [e_2,e_2]=\beta_1v_1+\beta_2v_2+\beta_3v_3, \\ [e_3,e_3]=\gamma_1v_1+\gamma_2v_2+\gamma_3v_3, & \\ [e_1,e_2]=\omega_1v_1+\omega_2v_2+\omega_3v_3, & [e_2,e_1]=\delta_1v_1+\delta_2v_2+\delta_3v_3, \\ [e_1,e_3]=-e_1+\lambda_1v_1+\lambda_2v_2+\lambda_3v_3, & [e_3,e_1]=e_1+\theta_1v_1+\theta_2v_2+\theta_3v_3, \\ [e_2,e_3]=-e_1-e_2+\mu_1v_1+\mu_2v_2+\mu_3v_3, & [e_3,e_2]=e_1+e_2+\zeta_1v_1+\zeta_2v_2+\zeta_3v_3 \end{array}$$

Making the following changes of bases

$$e_1'=e_1-\alpha_3v_1+\theta_2v_2+(\theta_3-\gamma_2)v_3,$$
$$e_2'=e_2+(\alpha_3+\zeta_1)v_1-\beta_3v_2+(\zeta_3+\gamma_2-\theta_3)v_3,$$
$$e_3'=e_3-\gamma_2v_1+\gamma_3v_3$$

we may assume

$$[e_1,e_1]=\alpha_1v_1+\alpha_2v_2,\ [e_2,e_2]=\beta_1v_1+\beta_2v_2,\ [e_3,e_3]=\gamma_1v_1$$
$$[e_3,e_1]=e_1+\theta_1v_1,\ [e_3,e_2]=e_1+e_2+\zeta_2v_2.$$

We are now ready to examine the Leibniz identities to further eliminate parameters.

$$\begin{array}{lll} [e_1,[e_1,e_2]]=[[e_1,e_1],e_2]-[[e_1,e_2],e_1] & \Rightarrow & \alpha_2=\omega_1, \\ [e_1,[e_1,e_3]]=[[e_1,e_1],e_3]-[[e_1,e_3],e_1] & \Rightarrow & \alpha_1=\lambda_1=\omega_1=0, \\ [e_1,[e_2,e_3]]=[[e_1,e_2],e_3]-[[e_1,e_3],e_2] & \Rightarrow & \delta_2=0,\lambda_2=\omega_3 \\ [e_2,[e_1,e_2]]=[[e_2,e_1],e_2]-[[e_2,e_2],e_1] & \Rightarrow & \beta_1=\delta_2, \\ [e_2,[e_1,e_3]]=[[e_2,e_1],e_3]-[[e_2,e_3],e_1] & \Rightarrow & \delta_1=\delta_2=0,\delta_3=\mu_1, \\ [e_2,[e_2,e_3]]=[[e_2,e_2],e_3]-[[e_2,e_3],e_2] & \Rightarrow & \mu_2=\mu_1+\omega_3,\beta_2=0, \\ [e_3,[e_1,e_2]]=[[e_3,e_1],e_2]-[[e_3,e_2],e_1] & \Rightarrow & \omega_3=\mu_1, \\ [e_3,[e_1,e_3]]=[[e_3,e_1],e_3]-[[e_3,e_3],e_1] & \Rightarrow & \theta_1=\omega_3=0,\lambda_3=\gamma_1, \\ [e_3,[e_2,e_3]]=[[e_3,e_2],e_3]-[[e_3,e_3],e_2] & \Rightarrow & \lambda_3=-\mu_3,\zeta_2=0. \end{array}$$

With these Leibniz identities, we rewrite our multiplication table without the eliminated parameters:

$$\begin{array}{lll} [v_1, e_1] = v_3, & [v_1, e_3] = v_2, & [v_2, e_2] = v_3, \\ [v_3, e_3] = -v_3, & [e_1, e_3] = -e_1 + \gamma_1 v_3, & [e_2, e_3] = -e_1 - e_2 - \gamma_1 v_3, \\ [e_3, e_3] = \gamma_1 v_1, & [e_3, e_1] = e_1, & [e_3, e_2] = e_1 + e_2. \end{array} \tag{3.3}$$

Theorem 3.4. *Let L be a 6-dimensional Leibniz algebra such that $L/I \cong S_3^5$ and I is the L/I-module with the minimal faithful representation. Then L is isomorphic to one of the following two non isomorphic algebras*

$$L_6^5: \begin{array}{llll} [v_1, e_1] = v_3, & [v_1, e_3] = v_2, & [e_3, e_1] = e_1, & [e_3, e_2] = e_1 + e_2, \\ [v_2, e_2] = v_3, & [v_3, e_3] = -v_3, & [e_1, e_3] = -e_1, & [e_2, e_3] = -e_1 - e_2. \end{array}$$

$$L_6^6: \begin{array}{lll} [v_1, e_1] = v_3, & [v_1, e_3] = v_2, & [v_2, e_2] = v_3, \\ [v_3, e_3] = -v_3, & [e_1, e_3] = -e_1 + v_3, & [e_2, e_3] = -e_1 - e_2 - v_3, \\ [e_3, e_3] = v_1, & [e_3, e_1] = e_1, & [e_3, e_2] = e_1 + e_2. \end{array}$$

Proof. If in the (3.3) $\gamma_1 = 0$ then we have the algebra L_6^5. In the second case, if $\gamma_1 \neq 0$ changing the basis: $v_1' = \delta_2 v_1$, $v_2' = \delta_2 v_2$, $v_3' = \delta_2 v_3$, we obtain the algebra L_6^6.

Since $dim(Der(L_6^5)) = 3$ and $dim(Der(L_6^6)) = 2$ these two algebras are not isomorphic. □

3.5 Leibniz algebras associated with S_3^6

Since the dimension of the vector space V in the faithful representation on the algebra S_3^6 is two dimensional we conclude that Leibniz algebra L associated with S_3^6 is five-dimensional. Then there exists a basis $\{e_1, e_2, e_3, v_1, v_2\}$ of L such that

$$[v_1, e_1] = v_2, \quad [v_1, e_2] = v_1, \quad [v_2, e_2] = v_2, \quad [v_2, e_3] = -v_2,$$

$$\begin{array}{lll} [e_1, e_1] = \alpha_1 v_1 + \alpha_2 v_2, & [e_1, e_2] = \omega_1 v_1 + \omega_2 v_2, & [e_1, e_3] = -e_1 + \lambda_1 v_1 + \lambda_2 v_2 \\ [e_2, e_1] = \delta_1 v_1 + \delta_2 v_2, & [e_2, e_2] = \beta_1 v_1 + \beta_2 v_2, & [e_2, e_3] = \mu_1 v_1 + \mu_2 v_2, \\ [e_3, e_1] = e_1 + \theta_1 v_1 + \theta_2 v_2, & [e_3, e_2] = \zeta_1 v_1 + \zeta_2 v_2, & [e_3, e_3] = \gamma_1 v_1 + \gamma_2 v_2. \end{array}$$

Theorem 3.5. *Let L be a 5-dimensional Leibniz algebra such that $L/I \cong S_3^6$ and I is the L/I-module with the minimal faithful representation. Then L is isomorphic to the following algebra*

$$L_5^1: \begin{array}{llll} [v_1, e_1] = v_2, & [v_1, e_2] = v_1, & [v_2, e_2] = v_2, & [v_2, e_3] = -v_2, \\ [e_1, e_3] = -e_1, & [e_3, e_1] = e_1. & & \end{array}$$

Proof. Taking the change of basis

$$e_1' = e_1 - \alpha_2 v_1 + (\theta_2 - \zeta_1)v_2,\ e_2' = e_2 - \beta_1 v_1 - \beta_2 v_2,\ e_3' = e_3 - \zeta_1 v_1 + \gamma_2 v_2,$$

we may assume

$$[e_1, e_1] = \alpha_1 v_1,\ [e_2, e_2] = 0,\ [e_3, e_3] = \gamma_1 v_1,$$

$$[e_3, e_1] = e_1 + \theta_1 v_1,\ [e_3, e_2] = \zeta_2 v_2.$$

Now, we consider the Leibniz identities to further eliminate parameters.

$$\begin{array}{lcl}
[e_1,[e_1,e_2]] = [[e_1,e_1],e_2] - [[e_1,e_2],e_1] & \Rightarrow & \alpha_1 = \omega_1 = 0,\\
[e_1,[e_1,e_3]] = [[e_1,e_1],e_3] - [[e_1,e_3],e_1] & \Rightarrow & \lambda_1 = 0,\\
[e_1,[e_2,e_3]] = [[e_1,e_2],e_3] - [[e_1,e_3],e_2] & \Rightarrow & \lambda_2 = 0,\\
[e_2,[e_1,e_2]] = [[e_2,e_1],e_2] - [[e_2,e_2],e_1] & \Rightarrow & \delta_1 = \mu_2 = 0,\\
[e_2,[e_1,e_3]] = [[e_2,e_1],e_3] - [[e_2,e_3],e_1] & \Rightarrow & \mu_1 = 0,\\
[e_2,[e_2,e_3]] = [[e_2,e_2],e_3] - [[e_2,e_3],e_2] & \Rightarrow & \mu_2 = 0,\\
[e_3,[e_1,e_2]] = [[e_3,e_1],e_2] - [[e_3,e_2],e_1] & \Rightarrow & \zeta_2 = \delta_2, \theta_1 = 0,\\
[e_3,[e_1,e_3]] = [[e_3,e_1],e_3] - [[e_3,e_3],e_1] & \Rightarrow & \gamma_1 = 0,\\
[e_3,[e_2,e_3]] = [[e_3,e_2],e_3] - [[e_3,e_3],e_2] & \Rightarrow & \omega_2 = 0.
\end{array}$$

Thus the multiplication table implies that we have the usual 5-dimensional Leibniz algebra L_5^1.

□

3.6 Leibniz algebras associated with S_3^2

Similarly to the previous sections Leibniz algebra L associated with the algebra S_3^2 is six dimensional and there exist a basis $\{e_1, e_2, e_3, v_1, v_2, v_3\}$ of L such that

$$\begin{array}{lll}
[v_1, e_1] = v_2, & [v_1, e_2] = v_3, & [v_2, e_3] = v_3,\\
[e_1, e_1] = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3, & & [e_2, e_2] = \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3,\\
[e_3, e_3] = \gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3, & & \\
[e_1, e_2] = \omega_1 v_1 + \omega_2 v_2 + \omega_3 v_3, & & [e_2, e_1] = \delta_1 v_1 + \delta_2 v_2 + \delta_3 v_3,\\
[e_1, e_3] = e_2 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3, & & [e_3, e_1] = -e_2 + \theta_1 v_1 + \theta_2 v_2 + \theta_3 v_3,\\
[e_2, e_3] = \mu_1 v_1 + \mu_2 v_2 + \mu_3 v_3, & & [e_3, e_2] = \zeta_1 v_1 + \zeta_2 v_2 + \zeta_3 v_3.
\end{array}$$

Taking the change of basis

$$e_1' = e_1 - \alpha_2 v_1 - (\lambda_3 + \delta_3)v_2,\ e_2' = e_2\lambda_1 v_1 + \lambda_2 v_2 - \theta_3 v_3,\ e_3' = e_3 - (\lambda_2 + \theta_2)v_1 - \gamma_3 v_2$$

we obtain

$$[e_1, e_1] = \alpha_1 v_1 + \alpha_3 v_3, \quad [e_3, e_3] = \gamma_1 v_1 + \gamma_2 v_2,$$

$$[e_1, e_3] = e_2, \quad [e_3, e_1] = -e_2 + \theta_1 v_1.$$

We now work on the Leibniz identities:

$$\begin{array}{lll}
[e_1,[e_1,e_2]] = [[e_1,e_1],e_2] - [[e_1,e_2],e_1] & \Rightarrow & \alpha_1 = \theta_1 = 0, \\
[e_1,[e_1,e_3]] = [[e_1,e_1],e_3] - [[e_1,e_1],e_3] & \Rightarrow & \theta_2 = -\delta_2, \omega_3 = -\delta_3, \delta_1 = 0, \\
[e_1,[e_2,e_3]] = [[e_1,e_2],e_3] - [[e_1,e_3],e_2] & \Rightarrow & \beta_1 = \beta_2 = 0 \text{ and } \beta_3 = -\delta_2, \\
[e_2,[e_1,e_3]] = [[e_2,e_1],e_3] - [[e_2,e_3],e_1] & \Rightarrow & \mu_1 = \delta_2 = 0, \\
[e_3,[e_1,e_2]] = [[e_3,e_1],e_2] - [[e_3,e_2],e_1] & \Rightarrow & \theta_1 = \zeta_1 = 0, \\
[e_3,[e_1,e_3]] = [[e_3,e_1],e_3] - [[e_3,e_3],e_1] & \Rightarrow & \mu_2 = -\gamma_1 - \zeta_2, \mu_3 = -\zeta_3, \\
[e_3,[e_2,e_3]] = [[e_3,e_2],e_3] - [[e_3,e_3],e_2] & \Rightarrow & \gamma_1 = \zeta_2.
\end{array}$$

We now rewrite the multiplication table without the unnecessary parameters:

$$L_6^7(\alpha_3, \gamma_1, \gamma_2, \delta_3, \zeta_3): \begin{cases}
[v_1,e_1] = v_2, & [v_1,e_2] = v_3, & [v_2,e_3] = v_3, \\
[e_1,e_1] = \alpha_3 v_3, & & [e_3,e_3] = \gamma_1 v_1 + \gamma_2 v_2, \\
[e_1,e_2] = -\delta_3 v_3, & [e_1,e_3] = e_2, & [e_2,e_3] = -2\gamma_1 v_2 - \zeta_3 v_3, \\
[e_2,e_1] = \delta_3 v_3, & [e_3,e_1] = -e_2, & [e_3,e_2] = \gamma_1 v_2 + \zeta_3 v_3
\end{cases}$$

Theorem 3.6. *Let L be a 6-dimensional Leibniz algebra such that $L/I \cong S_3^2$ and I is the L/I-module with the minimal faithful representation. Then L is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{array}{llll}
L_6^7(0,0,0,0,0) & L_6^7(0,0,0,0,1) & L_6^7(0,0,0,1,0) & L_6^7(0,0,0,1,1) \\
L_6^7(0,0,1,0,0) & L_6^7(0,0,1,0,1) & L_6^7(0,0,1,1,\zeta_3) & L_6^7(0,1,0,0,0) \\
L_6^7(0,1,0,0,1) & L_6^7(0,1,0,1,0) & L_6^7(0,1,0,1,1) & L_6^7(0,1,1,0,0) \\
L_6^7(0,1,1,0,1) & L_6^7(0,1,1,1,\zeta_3) & L_6^7(1,0,0,0,0) & L_6^7(1,0,0,0,1) \\
L_6^7(1,0,0,1,0) & L_6^7(1,0,0,1,1) & L_6^7(1,0,1,0,0) & L_6^7(1,0,1,0,1) \\
L_6^7(1,0,1,1,\zeta_3) & L_6^7(1,1,0,0,\zeta_3) & L_6^7(1,1,0,1,\zeta_3) & L_6^7(1,1,1,\delta_3,\zeta_3)
\end{array}$$

Proof. Let L be an 6-dimensional Leibniz algebra given by $L_6^7(\alpha_3, \gamma_1, \gamma_2, \delta_3, \zeta_3)$. We make the following change of basis:

$$\begin{aligned}
e_1' &= A_1 e_1 + A_2 e_2 + A_3 e_3 + A_4 v_1 + A_5 v_2 + A_6 v_3 \\
e_3' &= C_1 e_1 + C_2 e_2 + C_3 e_3 + C_4 v_1 + C_5 v_2 + C_6 v_3 \\
v_1' &= D_1 v_1 + D_2 v_2 + D_3 v_3
\end{aligned}$$

while the other elements of the new basis are obtained as products of the above elements.

From $[v_1', e_1'] = v_2'$, $[v_2', e_3'] = v_3'$, $[v_1', e_3'] = 0$ and $[v_2', e_1'] = 0$, we obtain that $C_1 = A_3 = 0$, $D_2 = -\frac{C_2 D_1}{C_3}$ and

$$v_2' = D_1 A_1 v_2 + (D_1 A_2 + D_2 A_3) v_3, \qquad v_3' = D_1 A_1 C_3 v_3,$$

Now from $[e'_1, e'_3] = e'_2$, we have that

$$e'_2 = A_1C_3e_2 - 2A_2C_3\gamma_1 v_2 + (-A_1C_2\delta_3 - A_2C_3\zeta_3 + A_4C_2 + A_5C_3)v_3.$$

The table of multiplication in this new basis implies the following restrictions on the coefficients

$$A_4 = 0, \quad C_4 = \frac{A_2C_3\gamma_1}{A_1}, \quad A_5 = \frac{A_2^2\gamma_1}{A_1}, \quad C_5 = \frac{C_3D_3\gamma_1}{D_1} + \frac{A_2C_3\gamma_2}{A_1} - \frac{A_2C_2\gamma_1}{A_1}$$

and new parameters are

$$\alpha'_3 = \frac{A_1\alpha_3}{D_1C_3}, \quad \gamma'_1 = \frac{C_3^2\gamma_1}{D_1}, \quad \gamma'_2 = \frac{C_3^2\gamma_2}{A_1D_1}, \quad \delta'_3 = \frac{A_1\psi_3}{D_1} \quad \delta'_3 = \frac{C_3\zeta_3}{D_1}.$$

Considering all the possible cases we obtain the families of algebras listed in the theorem.

□

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Abror Khudoyberdiyev
National University of Uzbekistan, 4, University street, 100174,
Tashkent, Uzbekistan,
e-mail: khabror@mail.ru
Emily Olson
St. Olaf College, 1520, St. Olaf Avenue, Northfield, Minnesota,
55057, United States of America,
e-mail: olson13@stolaf.edu