# A Characterization of Nilpotent Leibniz Algebras

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**Abstract** W. A. Moens proved that a Lie algebra is nilpotent if and only if it admits an invertible Leibniz-derivation. In this paper, using the definition of a Leibnizderivation from Moens (2010), we show that a similar result for non-Lie Leibniz algebras is not true. Namely, we give an example of non-nilpotent Leibniz algebra that admits an invertible Leibniz-derivation. In order to extend the results of the paper by Moens (2010) for Leibniz algebras, we introduce a definition of a Leibnizderivation of Leibniz algebras that agrees with Leibniz-derivation of the Lie algebra case. Further, we prove that a Leibniz algebra is nilpotent if and only if it admits an invertible Leibniz-derivation of Definition 3.4. Moreover, the result that a solvable radical of a Lie algebra is invariant with respect to a Leibniz-derivation was extended to the case of Leibniz algebras.

**Keywords** Lie algebra · Leibniz algebra · Derivation · Leibniz-derivation · Solvability · Nilpotency

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#### **1** Introduction

In 1955, Jacobson [11] proved that a Lie algebra over a field of characteristic zero admitting a non-singular (invertible) derivation is nilpotent. The problem of whether the inverse of this statement is correct remained open until the work of [9], where an example of a nilpotent Lie algebra whose derivations are nilpotent (and hence, singular), was constructed. Such types of Lie algebras are called "characteristically nilpotent Lie algebras".

The study of derivations of Lie algebras leads to the appearance of natural generalization—pre-derivations of Lie algebras [15]. In [2], it is proved that Jacobson's result is also true in terms of pre-derivations. Similar to the example of Dixmier and Lister [9], several examples of nilpotent Lie algebras whose pre-derivations are nilpotent were presented in [2, 4]. Such Lie algebras are called "strongly nilpotent" [4].

In paper [16], a generalized notion of derivations and pre-derivation of Lie algebras is defined as a Leibniz-derivation of order k. In fact, a Leibniz-derivation is a derivation of a Leibniz k-algebra constructed by Lie algebra [6].

In the section below, we present the characterization of nilpotency for Lie algebras in terms of Leibniz-derivations.

**Theorem 1.1** [16] *A Lie algebra over a field of characteristic zero is nilpotent if and only if it has an invertible Leibniz-derivation.* 

Leibniz algebras were introduced by Loday in [13, 14] as a non-antisymmetric version of Lie algebras. Many results of Lie algebras are extended to the case of Leibniz algebras. Since the study of derivations and automorphisms of a Lie algebra plays an essential role in the structure theory, the natural question arises as to whether the corresponding results for Lie algebras can be extended to more general objects.

In [12], it is proved that a finite dimensional complex Leibniz algebra admitting a non-singular derivation is nilpotent. Moreover, it was shown that (similar to the case of Lie algebras) the inverse of this statement does not hold, and the notion of characteristically nilpotent Lie algebra can be extended for Leibniz algebras [17].

In this paper, we show that if we define Leibniz-derivations for Leibniz algebra as in [16], then Theorem 1.1 does not hold. In order to avoid confusion, we need to modify the notion of Leibniz-derivation for Leibniz algebras.

Recall that in the definition of Leibniz-derivation of order k for Lie algebras, the k-ary bracket is defined as multiplication of k elements on the left side. For the case of Leibniz algebras, we propose the definition of Leibniz-derivation of order k as k-ary bracket on the right side. Due to anti-commutativity of multiplication in Lie algebras, this definition agrees with the case of Lie algebras.

Note that a vector space equipped with right-sided k-ary multiplication is not a Leibniz k-algebra as defined in [6]. For a Leibniz-derivation of Leibniz algebra, we prove the analogue of Theorem 1.1 for finite dimensional Leibniz algebras over a field of characteristic zero.

Throughout the paper, all spaces and algebras are assumed finite dimensional.

## 2 Preliminaries

In this section, we present some known facts about Leibniz algebras and Leibniz *n*-algebras.

**Definition 2.1** A vector space *L* over a field *F* with a binary operation [-, -] is a (right) Leibniz algebra, if for any  $x, y, z \in L$  the so-called Leibniz identity

$$[x, [y, z]] = [[x, y], z] - [[x, z], y]$$

holds.

Every Lie algebra is a Leibniz algebra, but the bracket in a Leibniz algebra does not need to be skew-symmetric.

For a Leibniz algebra L, consider the following central lower and derived sequences:

$$L^{1} = L, \quad L^{k+1} = [L^{k}, L^{1}], \quad k \ge 1,$$
  
 $L^{[1]} = 1, \quad L^{[s+1]} = [L^{[s]}, L^{[s]}], \quad s \ge 1.$ 

**Definition 2.2** A Leibniz algebra *L* is called nilpotent (respectively, solvable), if there exists  $p \in \mathbb{N}$  ( $q \in \mathbb{N}$ ) such that  $L^p = 0$  (respectively,  $L^{[q]} = 0$ ).

Levi's theorem, which has been proven for left Leibniz algebras in [3], is also true for right Leibniz algebras.

**Theorem 2.3** (Levi's Theorem) Let L be a Leibniz algebra over a field of characteristic zero and R be its solvable radical. Then there exists a semisimple subalgebra Lie S of L, such that L = S + R.

The following theorem from linear algebra characterizes the decomposition of a vector space into the direct sum of characteristic subspaces.

**Theorem 2.4** [8] Let A be a linear transformation of the vector space V. Then V decomposes into the direct sum of characteristic subspaces  $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_k}$  with respect to A, where  $V_{\lambda_i} = \{x \in V \mid (A - \lambda_i I)^k(x) = 0 \text{ for some } k \in \mathbb{N}\}$  and  $\lambda_i, 1 \leq i \leq k$ , are eigenvalues of A.

In Leibniz algebras, a derivation is defined as follows

**Definition 2.5** A linear transformation *d* of a Leibniz algebra *L* is a derivation if for any  $x, y \in L$ 

$$d([x, y]) = [d(x), y] + [x, d(y)].$$

Consider that for an arbitrary element  $x \in L$  the operator of right multiplication  $R_x : L \to L$ , defined by  $R_x(z) = [z, x]$ . Operators of right multiplication are derivations of the Leibniz algebra *L*. The set  $R(L) = \{R_x \mid x \in L\}$  is a Lie algebra with respect to the commutator, and the following identity holds:

$$R_x R_y - R_y R_x = R_{[y,x]}.$$
 (2.1)

A subset *S* of an associative algebra *A* over a field *F* is called a *weakly closed subset* if for every pair  $(a, b) \in S \times S$  there is an element  $\gamma_{(a,b)} \in F$  such that  $ab + \gamma_{(a,b)} ba \in S$ . We will need the following result concerning weekly closed sets

**Theorem 2.6** [11] Let S be a weakly closed subset of the associative algebra A of linear transformations of a vector space V over F. Assume that every  $W \in S$  is nilpotent, that is,  $W^k = 0$  for some positive integer k. Then the enveloping associative algebra  $S^*$  of S is nilpotent.

The classical Engel's theorem for Lie algebras has the following analogue for Leibniz algebras.

**Theorem 2.7** [1] A Leibniz algebra L is nilpotent if and only if  $R_x$  is nilpotent for any  $x \in L$ .

The following theorem generalizes Jacobson's theorem to Leibniz algebras.

**Theorem 2.8** [12] Let L be a complex Leibniz algebra that admits a non-singular derivation. Then L is nilpotent.

The next example presents *n*-dimensional Leibniz algebra possessing only nilpotent derivations.

*Example 2.9* Let *L* be an *n*-dimensional Leibniz algebra and let  $\{e_1, e_2, ..., e_n\}$  be a basis of *L* with the following table of multiplication:

$$\begin{cases} [e_1, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, \ 2 \le i \le n-1, \\ [e_1, e_2] = e_4, \\ [e_i, e_2] = e_{i+2}, \ 2 \le i \le n-2, \end{cases}$$

(omitted products are equal to zero).

Using the derivation property, it is easy to see that every derivation of L has the following matrix form:

 $\begin{pmatrix} 0 & 0 & a_3 & a_4 & a_5 & \dots & a_{n-1} & a_n \\ 0 & 0 & a_3 & a_4 & a_5 & \dots & a_{n-1} & b_n \\ 0 & 0 & 0 & a_3 & a_4 & \dots & a_{n-2} & a_{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & a_3 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$ 

Thus, every derivation of L is nilpotent, i.e., L is characteristically nilpotent.

Let us give the definition of Leibniz *n*-algebras.

**Definition 2.10** [6] A vector space  $\mathcal{L}$  with an *n*-ary multiplication [-, -, ..., -]:  $\mathcal{L}^{\otimes n} \to \mathcal{L}$  is a Leibniz *n*-algebra if it satisfies the following identity:

$$[[x_1, x_2, \dots, x_n], y_2, \dots, y_n] = \sum_{i=1}^n [x_1, \dots, x_{i-1}, [x_i, y_2, \dots, y_n], x_{i+1}, \dots, x_n].$$
(2.2)

Let *L* be a Leibniz algebra with the product [-, -]. Then, the vector space *L* can be equipped with a Leibniz *n*-algebra structure with the following product:

 $[x_1, x_2, \dots, x_n] = [x_1, [x_2, \dots, [x_{n-1}, x_n]]].$ 

**Definition 2.11** A derivation of a Leibniz *n*-algebra  $\mathcal{L}$  is a  $\mathbb{K}$ -linear map  $d : \mathcal{L} \to \mathcal{L}$  satisfying

$$d([x_1, x_2, \dots, x_n]) = \sum_{i=1}^n [x_1, \dots, d(x_i), \dots, x_n].$$

The notion of Leibniz-derivation of Lie algebra was introduced in [16] and it generalizes the notions of derivation and pre-derivation of Lie algebra.

**Definition 2.12** A Leibniz-derivation of order n for a Lie algebra G is an endomorphism P of G satisfying the identity

$$P([x_1, [x_2, \dots, [x_{n-1}, x_n]]]) = [P(x_1), [x_2, \dots, [x_{n-1}, x_n]]]$$
  
+ [x\_1, [P(x\_2), \dots, [x\_{n-1}, x\_n]]] + \dots  
+ [x\_1, [x\_2, \dots, [x\_{n-1}, P(x\_n)]]]

for every  $x_1, x_2, \ldots, x_n \in G$ .

In other words, a Leibniz-derivation of order n for a Lie algebra G is a derivation of G viewed as a Leibniz n-algebra.

#### 3 Leibniz-Derivation of Leibniz Algebras

The following example shows that Definition 2.12 is not substantial for the case of Leibniz algebras.

*Example 3.1* Let *R* be an (n + 1)-dimensional solvable Leibniz algebra and  $\{e_1, e_2, \dots, e_n, e_{n+1}\}$  be a basis of *R* with the table of multiplication given by

$$\begin{cases} [e_1, e_1] = e_3, & [e_i, e_1] = e_{i+1}, & 2 \le i \le n-1, \\ [e_1, e_{n+1}] = e_2 + \sum_{i=4}^{n-1} \alpha_i e_i, & [e_2, e_{n+1}] = e_2 + \sum_{i=4}^{n-1} \alpha_i e_i, \\ [e_i, e_{n+1}] = e_i + \sum_{j=i+2}^n \alpha_{j-i+2} e_i, & 3 \le i \le n, \end{cases}$$

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It is easy to see that [R, [R, R]] = 0. For the identity map, d, we have

$$0 = d([x, [y, z]]) = [d(x), [y, z]] + [x, [d(y), z]] + [x, [y, d(z)]] = 0.$$

Therefore, the invertible map d satisfies the condition of Definition 2.12, but the Leibniz algebra R is not nilpotent, i.e., the analogue of Theorem 1.1 for Leibniz algebras is not true.

*Remark 3.2* Example 3.1 can be extended for any non-nilpotent solvable Leibniz algebra L such that  $L^2$  lies in the right annihilator of L.

Let us introduce *n*-ary multiplication as follows

$$[x_1, x_2, \dots, x_n]_r = [[[x_1, x_2], x_3] \dots, x_n].$$
(3.1)

The next example shows that, in general, a vector space equipped with defined *n*-ary multiplication  $[x_1, x_2, ..., x_n]_r$  is not a Leibniz *n*-algebra.

*Example 3.3* Let *R* be a solvable Leibniz algebra and let  $\{e_1, e_2, \ldots, e_n, x\}$  be a basis of *R* such that multiplication table of *R* in this basis has the following form [7]:

$$\begin{cases} [e_i, e_1] = e_{i+1}, \ 1 \le i \le n-1, \\ [x, e_1] = e_1, \\ [e_i, x] = -ie_i, \ 1 \le i \le n. \end{cases}$$

It is not difficult to check that the vector space R with k-ary multiplication  $[x_1, x_2, \ldots, x_k]_r$  does not define Leibniz k-algebra structure. Indeed, we have

$$[[e_1, e_1, \dots, e_1]_r, x, \dots, x]_r = [\dots [[[\underbrace{\dots [[e_1, e_1], e_1], \dots, e_1}_{k-times}, ]\underbrace{x], x], \dots, x}]]$$
$$= [\dots [[e_k, \underbrace{x], x], \dots, x}_{k-1-times}] = (-k)^{k-1} e_k.$$

On the other hand,

$$\sum_{i=1}^{k} [e_1, \dots, e_1, \underbrace{[e_1, x, \dots, x]_r}_{i-th}, e_1, \dots, e_1]_r = \sum_{i=1}^{k} [e_1, \dots, e_1, \underbrace{(-1)^{k-1}e_1}_{i-th}, e_1, \dots, e_1]_r$$
$$= (-1)^{k-1} \sum_{i=1}^{k} [\underbrace{\dots [[e_1, e_1], e_1], \dots, e_1}_{k-times}]$$
$$= (-1)^{k-1} \sum_{i=1}^{k} e_k = (-1)^{k-1} ke_k.$$

Hence, identity (2.2) does not hold for  $k \ge 3$ .

Now we define the notion of Leibniz-derivation for Leibniz algebras.

**Definition 3.4** A Leibniz-derivation of order  $n \in \mathbb{N}$  for a Leibniz algebra *L* is a K-linear map  $d : L \to L$  satisfying

$$d([x_1, x_2, \dots, x_n]_r) = \sum_{i=1}^n [x_1, \dots, d(x_i), \dots, x_n]_r$$

Proposition 3.5 For Lie algebras, Definition 3.4 agrees with Definition 2.12.

*Proof* If we let *L* be a Lie algebra, then we have

$$P([x_1, [x_2, \dots, [x_{n-1}, x_n]]]) = (-1)^n P([[[x_n, x_{n-1}], \dots, x_2], x_1])$$
  
=  $(-1)^n P([x_n, x_{n-1}, \dots, x_1]_r).$ 

On the other hand,

$$\begin{aligned} [P(x_1), [x_2, \dots, [x_{n-1}, x_n]]] + [x_1, [P(x_2), \dots, [x_{n-1}, x_n]]] \\ &+ \dots + [x_1, [x_2, \dots, [x_{n-1}, P(x_n)]]] \\ &= (-1)^n [[[x_n, x_{n-1}], \dots, x_2], P(x_1)] + (-1)^n [[[x_n, x_{n-1}], \dots, P(x_2)], x_1] \\ &+ \dots + (-1)^n [[[P(x_n), x_{n-1}], \dots, x_2], x_1] \\ &= (-1)^n ([x_n, x_{n-1}, \dots, x_2, P(x_1)]_r + [x_n, x_{n-1}, \dots, P(x_2), x_1]_r \\ &+ \dots + [P(x_n), x_{n-1}, \dots, x_2, x_1]_r) \\ &= (-1)^n \sum_{i=1}^n [x_n, \dots, P(x_i), \dots, x_1]_r. \end{aligned}$$

This implies the equality

$$P([x_1, [x_2, \dots, [x_{n-1}, x_n]]]) = [P(x_1), [x_2, \dots, [x_{n-1}, x_n]]]$$
  
+ [x\_1, [P(x\_2), \dots, [x\_{n-1}, x\_n]]] + \dots  
+ [x\_1, [x\_2, \dots, [x\_{n-1}, P(x\_n)]]],

which is equivalent to

$$P([x_n, x_{n-1}, \dots, x_1]_r) = \sum_{i=1}^n [x_n, \dots, P(x_i), \dots, x_1]_r.$$

Relabeling  $x_i$  with  $x_{n+1-i}$  for  $1 \le i \le n$  we complete the proof of the proposition.

Let  $LDer_n(L)$  denote the set of all Leibniz-derivations of order *n* for a Leibniz algebra *L* and let LDer(L) be the set of all Leibniz-derivations, i.e.,  $LDer(L) = \bigcup_{n \in \mathbb{N}} LDer_n(L)$ .

Note that a Leibniz derivation of order 2 is a derivation. Moreover, any derivation is a Leibniz-derivation of any order n. Thus, the order of a Leibniz-derivation is not unique.

#### **Lemma 3.6** The following statements are true

(1) If  $s, t \in \mathbb{N}$  and s|t, then  $LDer_{s+1}(L) \subset LDer_{t+1}(L)$ ; (2) for any  $k, l \in \mathbb{N}$ ,  $LDer_k(L) \cap LDer_l(L) \subset LDer_{k+l-1}$ .

*Proof* Since Definition 3.4 for the Leibniz-derivation of order *n* coincides with the Leibniz-derivation of order n - 1, defined in [16], the proof of the lemma can be carried out similarly to the proof of Lemma 1 in [16].

Similar to the case of Lie algebras, we call a Leibniz-derivation of order 3 a prederivation of Leibniz algebra. A nilpotent Leibniz algebra is called *strongly nilpotent* if all its Leibniz pre-derivations are nilpotent.

Note that a strongly nilpotent Leibniz algebra is characteristically nilpotent, but the inverse is not true in general.

*Example 3.7* Any pre-derivation of the characteristically nilpotent Leibniz algebra in Example 2.9 with n = 6 has the matrix form:

$$\begin{pmatrix} a_1 & a_1 & a_3 & a_4 & a_5 & a_6 \\ 0 & 2a_1 & a_3 & a_4 & b_5 & b_6 \\ 0 & 0 & 3a_1 - a_1 + a_3 & c_5 & c_6 \\ 0 & 0 & 0 & 4a_1 & 2a_1 + a_3 & a_4 \\ 0 & 0 & 0 & 0 & 5a_1 & a_1 + a_3 \\ 0 & 0 & 0 & 0 & 0 & 6a_1 \end{pmatrix}$$

Thus, this Leibniz algebra is not strongly nilpotent.

**Proposition 3.8** The Leibniz algebra L in Example 2.9 is strongly nilpotent if n > 6.

Proof Let  $d: L \to L$  be a pre-derivation of L. Put

$$d(e_1) = \sum_{i=1}^n a_i e_i, \ d(e_2) = \sum_{i=1}^n b_i e_i, \ d(e_3) = \sum_{i=1}^n c_i e_i.$$

Consider the property of pre-derivation

$$d(e_4) = d([e_1, e_1, e_1]_r) = (3a_1 + a_2)e_4 + (a_3 + 2a_2)e_5 + \sum_{i=4}^{n-2} a_i e_{i+2}.$$

On the other hand,

$$d(e_4) = d([e_2, e_1, e_1]_r) = (2a_1 + b_1 + b_2)e_4 + (b_3 + 2a_2)e_5 + \sum_{i=4}^{n-2} b_i e_{i+2}.$$

Comparing coefficients of basis elements, we have

$$b_1 + b_2 = a_1 + a_2, \quad b_i = a_i, \quad 3 \le i \le n - 2.$$

The equality  $d([e_1, e_1, e_3]_r) = 0$  implies  $0 = c_1e_4 + c_2e_5$ , hence  $c_1 = c_2 = 0$ .

The chain of equalities

$$b_1e_4 + (2a_1 + a_2 + b_2)e_5 + (a_3 + a_2)e_6 + \sum_{i=4}^{n-3} a_ie_{i+3}$$
$$= d([e_1, e_2, e_1]_r) = d(e_5) = d([e_3, e_1, e_1]_r)$$
$$= (2a_1 + c_3)e_5 + (2a_2 + c_4)e_6 + \sum_{i=5}^{n-2} c_ie_{i+2}.$$

deduce

$$b_1 = 0$$
,  $c_3 = a_2 + b_2$ ,  $c_4 = a_3 - a_2$ ,  $c_i = a_{i-1}$ ,  $4 \le i \le n-2$ .

From the equalities

$$(3a_1 + 3a_2)e_6 + \sum_{i=3}^{n-4} a_i e_{i+4} = d([e_1, e_2, e_2]_r) = d(e_6)$$
  
=  $d([e_4, e_1, e_1]_r)$   
=  $(5a_1 + a_2)e_6 + (4a_2 + a_3)e_7 + \sum_{i=4}^{n-4} a_i e_{i+4}$ 

we get  $a_1 = a_2 = 0$ .

Since  $b_2 = a_1 + a_2$  and  $c_3 = a_2 + b_2$ , we have  $b_2 = c_3 = 0$ . Thus we obtain

$$d(e_1) = \sum_{i=3}^{n} a_i e_i, \ d(e_2) = \sum_{i=3}^{n-2} a_i e_i + b_{n-1} e_{n-1} + b_n e_n,$$
$$d(e_3) = \sum_{i=3}^{n-3} a_i e_{i+1} + c_{n-1} e_{n-1} + c_n e_n.$$

Finally, from the expression  $d([e_{i-2}, e_1, e_1]_r)$ , we derive

 $d(e_i) = a_3 e_{i+1} + a_4 e_{i+2} + \dots + a_{n+2-i} e_n, \ i \ge 4$ 

which completes the proof of the proposition.

Following the proofs of lemmas in [10] and [5] for derivations of Lie and Leibniz n-algebras, respectively, we get the following statement for Leibniz-derivations of order n of Leibniz algebras.

**Lemma 3.9** For a Leibniz-derivation  $d : L \to L$  of order n of a Leibniz algebra L over a field of characteristic zero, the following formula holds for any  $k \in \mathbb{N}$ :

$$d^{k}([x_{1},\ldots,x_{n}]_{r}) = \sum_{i_{1}+i_{2}+\cdots+i_{n}=k} \frac{k!}{i_{1}!i_{2}!\ldots i_{n}!} \left[ d^{i_{1}}(x_{1}), d^{i_{2}}(x_{2}),\ldots, d^{i_{n}}(x_{n}) \right]_{r}.$$
 (3.2)

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## 4 Nilpotent Leibniz Algebras

Starting with a Leibniz algebra L, we call its underlying vector space equipped with multiplication  $[-, -, ..., -]_r$  as Eq. 3.1 an *n*-ary algebra, and denote it by  $\mathcal{L}_n(L)$ . A subalgebra I of the Leibniz algebra L is called an *n*-ideal of L or an ideal of  $\mathcal{L}_n(L)$ , if it satisfies

$$[L, \ldots, \underbrace{I}_{i-th \ place}, \ldots, L]_r \subseteq I \quad \text{for any } i \ (1 \le i \le n).$$

Let M be any Leibniz subalgebra of L. Consider the following sequences:

$$\mathcal{L}_n^1(M) = M, \quad \mathcal{L}_n^{k+1}(M) = \left[\mathcal{L}_n^k(M), M, \dots, M\right]_r, \quad k \ge 1,$$

$$\mathcal{L}_{n}^{[1]}(M) = M, \quad \mathcal{L}_{n}^{[s+1]}(M) = \left[\mathcal{L}_{n}^{[s]}(M), \mathcal{L}_{n}^{[s]}(M) \dots, \mathcal{L}_{n}^{[s]}(M)\right]_{r}, \quad s \ge 1.$$

**Definition 4.1** A Leibniz algebra *L* is called *n*-nilpotent (*n*-solvable) if there exists a natural number  $p \in \mathbb{N}$  ( $q \in \mathbb{N}$ ) such that  $\mathcal{L}_n^p(L) = 0$  ( $\mathcal{L}_n^{[q]}(L) = 0$ ).

Lemma 4.2 Let M be an ideal of L. The following inclusions are true

$$\mathcal{L}_n^{[k]}(M) \subseteq M^{[k]}, \qquad \mathcal{L}_n^k(M) \subseteq M^k.$$

*Proof* It is easy to check that  $M^k$  and  $M^{[k]}$  are also ideals of L for any k. We shall prove the first embedding by induction on k for any n.

If k = 2, then

$$\mathcal{L}_{n}^{[2]}(M) = [M, M, M, \dots, M]_{r} = [[[M, M], M], \dots, M] = [[M^{[2]}, M], \dots, M] \subseteq M^{[2]}.$$

Suppose that the statement holds for some k and we will prove it for k + 1.

$$\mathcal{L}_{n}^{[k+1]}(M) = \left[ \left[ \left[ \mathcal{L}_{n}^{[k]}(M), \mathcal{L}_{n}^{[k]}(M) \right], \mathcal{L}_{n}^{[k]}(M) \right], \dots, \mathcal{L}_{n}^{[k]}(M) \right] \\ \subseteq \left[ \left[ \left[ M^{[k]}, M^{[k]} \right], M^{[k]} \right], \dots, M^{[k]} \right] \\ = \left[ \left[ M^{[k+1]}, M^{[k]} \right], \dots, M^{[k]} \right] \subseteq M^{[k+1]}.$$

The second inclusion is established in a similar way.

**Lemma 4.3**  $M^{[tk+1]} \subseteq \mathcal{L}_n^{[k+1]}(M)$ , where  $k \in \mathbb{N}$  and t is a natural number such that  $2^t \ge n$ .

*Proof* Since  $M^{[p]} \subseteq M^{[p+q]}$  for any  $p, q \in \mathbb{N}$ , it is sufficient to prove embedding for the minimal *t* such that  $2^t \ge n$ .

We shall use induction. If n = 3, then t = 2. For k = 1, we have

$$M^{[3]} = [M^{[2]}, M^{[2]}] = [M^{[2]}, [M^{[1]}, M^{[1]}]]$$
$$\subseteq [[M^{[2]}, M^{[1]}], M^{[1]}] \subseteq [M^{[1]}, M^{[1]}, M^{[1]}]_r \subseteq \mathcal{L}_3^{[2]}(M).$$

Suppose that the statement holds for some k and we will prove it for k + 1.

$$\begin{split} M^{[2(k+1)+1]} &= M^{[2k+1+2]} = \left[ \left[ M^{[2k+1]}, M^{[2k+1]} \right], \left[ M^{[2k+1]}, M^{[2k+1]} \right] \right] \\ &\subseteq \left[ \left[ M^{[2k+1]}, M^{[2k+1]} \right], M^{[2k+1]} \right] \\ &\subseteq \left[ \mathcal{L}_{3}^{[k+1]}(M), \mathcal{L}_{3}^{[k+1]}(M), \mathcal{L}_{3}^{[k+1]}(M) \right]_{r} = \mathcal{L}_{3}^{[k+2]}(M). \end{split}$$

Let us prove the statement for any *n*. Since  $2^t \ge n$  for k = 1 we get

$$M^{[t+1]} \subseteq M^{2^{t}} = \underbrace{[[M^{[1]}, M^{[1]}], \dots, M^{[1]}]}_{2^{t} - times} \subseteq \underbrace{[[M^{[1]}, M^{[1]}], \dots, M^{[1]}]}_{n - times} = \mathcal{L}_{n}^{[2]}(M).$$

The following chain equalities and inclusions

$$M^{[t(k+1)+1]} = M^{[tk+1+t]} = (M^{[tk+1]})^{[t+1]} \subseteq (M^{[tk+1]})^{2^{t}}$$
  
=  $\underbrace{[[[M^{[tk+1]}, M^{[tk+1]}], \dots, M^{[tk+1]}]}_{2^{t}-times}$   
 $\subseteq \underbrace{[[M^{[tk+1]}, M^{[tk+1]}], \dots, M^{[tk+1]}]}_{n-times}$   
 $\subseteq \underbrace{[[\mathcal{L}_{n}^{[k+1]}(M), \mathcal{L}_{n}^{[k+1]}(M)], \dots, \mathcal{L}_{n}^{[k+1]}(M)]}_{n-times}$   
 $= \mathcal{L}_{n}^{[k+2]}(M)$ 

complete the proof of the lemma.

Further, we shall need the following lemma.

**Lemma 4.4**  $M^{nk-k+1} = \mathcal{L}_n^{k+1}(M).$ 

*Proof* The proof goes again by induction on k for any n. If k = 1, then

$$M^{n} = \underbrace{[\dots [[M, M], M], \dots, M]}_{n-times} = [M, M, \dots, M]_{r} = \mathcal{L}^{2}_{n}(M).$$

Applying induction in the equalities

$$M^{n(k+1)-k-1+1} = M^{nk-k+1+n-1} = [\dots [[M^{nk-k+1}, \underbrace{M], M], \dots, M]}_{n-1-times}$$
$$= [M^{nk-k+1}, M, \dots, M]_r = [\mathcal{L}_n^{k+1}(M), M, \dots, M]_r = \mathcal{L}_n^{k+2}(M)$$

we complete the proof of the lemma.

Let us denote by

- *R* solvable radical of *L*, i.e., the maximal solvable ideal of the Leibniz algebra *L*;
- $R_n$  *n*-solvable radical of *L*, i.e., the maximal *n*-solvable ideal of the *n*-ary algebra  $\mathcal{L}_n(L)$ ;
- *N* nilradical of *L*, i.e., the maximal nilpotent ideal of the Leibniz algebra *L*;
- $N_n$  *n*-nilradical of L, i.e., the maximal *n*-nilpotent ideal of the *n*-ary algebra  $\mathcal{L}_n(L)$ .

**Proposition 4.5** For a Leibniz algebra L, we have  $R = R_n$ .

*Proof* It is obvious that the ideal R of L is also ideal of the *n*-ary algebra  $\mathcal{L}_n(L)$ . Lemma 4.2 implies that any solvable ideal of the Leibniz algebra L is also *n*-solvable. Therefore, it is sufficient to prove the inclusion  $R_n \subseteq R$ .

Since  $R_n$  is an *n*-solvable radical of L, so  $R_n$  is a subalgebra of L and

$$[L, \ldots, \underbrace{R_n}_{i-th}, \ldots, L]_r \subseteq R_n$$
 for any  $i \ (1 \le i \le n)$ .

From Lemma 4.3, it follows that  $R_n$  is a solvable subalgebra of the Leibniz algebra L. Thus, we need to prove that  $R_n$  is an ideal of L, i.e.,  $[L, R_n], [R_n, L] \subseteq L$ . According to Theorem 2.3, we can write  $L = S \oplus R$ , where S is a simple Lie algebra, R is a solvable ideal of L. Let  $\pi : L \to S$  be the natural quotient map.

Since  $\pi$  is a morphism of L, we get

$$\pi([L, L, \dots, L, R_n]_r) = \pi([[[[L, L], L], \dots, L], R_n])$$
  
= [[[[\pi(L), \pi(L)], \pi(L)], \pi(L)], \pi(R\_n)]  
= [[[[S, S], S], \pi, S], \pi(R\_n)] = [S, \pi(R\_n)].

On the other hand, from  $[L, L, ..., L, R_n]_r \subseteq R_n$  we derive

$$\pi([L, L, \ldots, L, R_n]_r) \subseteq \pi(R_n).$$

Hence,  $[S, \pi(R_n)] \subseteq \pi(R_n)$ . Taking into account that *S* is a Lie algebra we obtain  $[\pi(R_n), S] \subseteq \pi(R_n)$ . Therefore,  $\pi(R_n)$  is an ideal of *S*. Since  $R_n$  is an *n*-solvable ideal of *L*,  $\pi(R_n)$  is an *n*-solvable ideal of the Lie algebra *S*, consequently  $\pi(R_n)$  is a solvable ideal of *S*.

Due to semisimplicity of S, we get  $\pi(R_n) = 0$ , which implies  $R_n \subseteq R$ .

**Lemma 4.6** Let I be an ideal of the Leibniz algebra L and  $d \in LDer_n(L)$  a Leibnizderivation for some  $n \in \mathbb{N}$ . Then

$$\mathcal{L}_n^{[k]}(d(I)) \subseteq I + d^{n^{k-1}} \left( \mathcal{L}_n^{[k]}(I) \right)$$

for all  $k \in \mathbb{N}$ .

*Proof* Evidently,  $d(I) \subseteq I + d(I)$  holds. For k = 2, using Eq. 3.2, we have

$$\mathcal{L}_{n}^{[2]}(d(I)) = [d(I), d(I), \dots, d(I)]_{r} \subseteq d([I, I, \dots, I]_{r})$$

$$+ \sum_{\substack{i_{1} + i_{2} + \dots + i_{n} = n \\ \exists i_{j} = 0}} \frac{n!}{i_{1}!i_{2}! \dots i_{n}!}$$

$$\leq [d^{i_{1}}(I), \dots, d^{i_{j}-1}(I), I, d^{i_{j}+1}(I), \dots, d^{i_{n}}(I)]_{r}$$

$$\subseteq I + d^{n} \left(\mathcal{L}_{n}^{[2]}(I)\right).$$

Assume that  $\mathcal{L}_n^{[k]}(d(I)) \subseteq I + d^{n^{k-1}}(\mathcal{L}_n^{[k]}(I))$ . Again using Eq. 3.2, we verify the inclusion for k + 1:

$$\begin{aligned} \mathcal{L}_{n}^{[k+1]}(d(I)) &= [\mathcal{L}_{n}^{[k]}(d(I)), \mathcal{L}_{n}^{[k]}(d(I)), \dots, \mathcal{L}_{n}^{[k]}(d(I))]_{r} \\ &\subseteq [I + d^{n^{k-1}}(\mathcal{L}_{n}^{[k]}(I)), I + d^{n^{k-1}}(\mathcal{L}_{n}^{[k]}(I)), \dots, I + d^{n^{k-1}}(\mathcal{L}_{n}^{[k]}(I))]_{r} \\ &\subseteq I + d^{n^{k}}([\mathcal{L}_{n}^{[k]}(I), \mathcal{L}_{n}^{[k]}(I), \dots, \mathcal{L}_{n}^{[k]}(I)]_{r}) \\ &\subseteq I + d^{n^{k}}(\mathcal{L}_{n}^{[k+1]}(I))). \end{aligned}$$

**Theorem 4.7** Let R be the solvable radical of a Leibniz algebra L over a field of characteristic zero. Then  $d(R) \subseteq R$  for any  $d \in LDer_n(L)$ .

*Proof* Let *d* be a Leibniz-derivation of order *n*. Due to Proposition 4.5,  $R = R_n$ , so it is enough to prove the assertion of the Theorem for  $R_n$ .

Since  $R_n$  is a *n*-solvable radical, there exists  $s \in \mathbb{N}$  such that  $\mathcal{L}_n^{[s]}(R_n) = 0$ . Then, by Lemma 4.6,  $\mathcal{L}_n^{[s]}(d(R_n)) \subseteq R_n + d^{n^{s-1}}(\mathcal{L}_n^{[s]}(R_n)) = R_n$ . Thus, we have  $\mathcal{L}_n^{[s]}(R_n + d(R_n)) \subseteq R_n$ .

Further,

$$\mathcal{L}_n^{[2s-1]}(R_n+d(R_n))\subseteq \mathcal{L}_n^{[s]}\left(\mathcal{L}_n^{[s]}(R_n+d(R_n))\right)\subseteq \mathcal{L}_n^{[s]}(R_n)=0.$$

The *n*-ideal property of  $R_n + d(R_n)$  follows from the following equalities:

$$[l_1, \dots, l_i + d(l_i), \dots, l_n]_r = [l_1, \dots, l_i, \dots, l_n]_r + [l_1, \dots, d(l_i), \dots, l_n]_r$$
$$= [l_1, \dots, l_i, \dots, l_n]_r + d([l_1, \dots, l_i, \dots, l_n]_r)$$
$$- \sum_{j=1, j \neq i}^n [l_1, \dots, d(l_j), \dots, l_n]_r.$$

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Hence,  $R_n + d(R_n)$  is an *n*-solvable ideal of the Leibniz algebra *L*. Since  $R_n$  is the *n*-solvable radical of *L*, it follows that  $R_n + d(R_n) \subseteq R_n$ , therefore  $d(R_n) \subseteq R_n$ .  $\Box$ 

**Lemma 4.8** Let I be an ideal of the Leibniz algebra L and  $d \in LDer_n(L)$  a Leibnizderivation for some  $n \in \mathbb{N}$ . Then

$$\mathcal{L}_{n}^{k}(d(I)) \subseteq I + d^{kn-k+1} \left( \mathcal{L}_{n}^{k}(I) \right)$$

for all  $k \in \mathbb{N}$ .

*Proof* For k = 1 the assertion of the lemma is obvious. Let k = 2, then using the formula 3.2 we have

$$\mathcal{L}_{n}^{2}(d(I)) = [d(I), d(I), \dots, d(I)]_{r}$$

$$\subseteq d([I, I, \dots, I]_{r}) + \sum_{\substack{i_{1} + i_{2} + \dots + i_{n} = n \\ \exists i_{j} = 0}} \frac{n!}{i_{1}!i_{2}!\dots i_{n}!}$$

$$\times \left[d^{i_{1}}(I), \dots, d^{i_{j}-1}(I), I, d^{i_{j}-1}(I), \dots, d^{i_{n}}(I)\right]_{r}$$

$$\subseteq I + d^{n} \left(\mathcal{L}_{n}^{2}(I)\right).$$

Assume that  $\mathcal{L}_n^k(d(I)) \subseteq I + d^{kn-k+1}(\mathcal{L}_n^k(I))$ . Applying the formula 3.2, we prove the inclusion for k + 1:

$$\begin{aligned} \mathcal{L}_{n}^{k+1}(d(I)) &= \left[\mathcal{L}_{n}^{k}(d(I)), (d(I)), \dots, (d(I))\right]_{r} \\ &\subseteq \left[I + d^{kn-k+1}(\mathcal{L}_{n}^{k}(I)), d(I), \dots, d(I)\right]_{r} \\ &\subseteq \left[I + d^{kn-k+1+n-1}\left(\left[\mathcal{L}_{n}^{[k]}(I), I, \dots, I\right]_{r}\right)\right] \\ &\subseteq I + d^{(k+1)n-k}\left(\mathcal{L}_{n}^{k+1}(I)\right). \end{aligned}$$

The invariant property of nilradical of a Leibniz algebra under a Leibnizderivation is presented in the following theorem.

**Theorem 4.9** Let N be the nilradical of a Leibniz algebra L over a field of characteristic zero. Then,  $d(N) \subseteq N$  for any  $d \in LDer_n(L)$ .

*Proof* The proof is similar to the proof of Theorem 4.7.

The next result establishes the properties of weight spaces with respect to a Leibniz-derivation of a Leibniz algebra.

**Lemma 4.10** Let *L* be a complex Leibniz algebra with a given Leibniz-derivation *d* of order *n* and  $L = L_{\alpha} \oplus L_{\beta} \oplus \cdots \oplus L_{\gamma}$  the decomposition of *L* into weight spaces with respect to *d* (i.e.  $L_{\alpha} = \{x \in L : (d - \alpha 1)^k x = 0 \text{ for some } k\}$ ). Then

$$[L_{\alpha_1}, L_{\alpha_2}, \dots, L_{\alpha_n}]_r = \begin{cases} 0 & \text{if } \alpha_1 + \alpha_2 + \dots + \alpha_n \text{ is not a root of } d \\ L_{\alpha_1 + \alpha_2 + \dots + \alpha_n} & \text{if } \alpha_1 + \alpha_2 + \dots + \alpha_n \text{ is a root of } d. \end{cases}$$

#### *Proof* First observe that

$$(d - (\alpha_1 + \alpha_2 + \dots + \alpha_n) \cdot 1)[x_1, x_2, \dots, x_n]_r$$
  
=  $\sum_{i=1}^n [x_1, \dots, d(x_i), \dots, x_n]_r - \sum_{i=1}^n [x_1, \dots, \alpha_i x_i, \dots, x_n]_r$   
=  $\sum_{i=1}^n [x_1, \dots, (d - \alpha_i \cdot 1)x_i, \dots, x_n]_r.$ 

Similar to Lemma 3.9, by induction on k we get the following equality:

$$(d - (\alpha_1 + \alpha_2 + \dots + \alpha_n) \cdot 1)^k [x_1, x_2, \dots, x_n]_r$$
  
=  $\sum_{i_1 + i_2 + \dots + i_n = k} \frac{k!}{i_1! i_2! \cdots i_n!} [(d - \alpha_1 \cdot 1)^{i_1} x_1, (d - \alpha_2 \cdot 1)^{i_2} x_2, \dots, (d - \alpha_n \cdot 1)^{i_n} x_n]_r$   
(4.1)

for any  $x_i \in L_{\alpha_i}$ .

Consider  $x_i \in L_{\alpha_i}$ ,  $1 \le i \le n$ . Then there exist natural numbers  $k_i$  such that  $(d - \alpha_i \cdot 1)^{k_i}(x) = 0$ . Taking  $k = \sum_{i=1}^n k_i$  in Eq. 4.1, we have

$$(d - (\alpha_1 + \alpha_2 + \dots + \alpha_n) \cdot 1)^k [x_1, x_2, \dots, x_n]_r = 0$$

which completes the proof.

Similar to [16], we have the existence of an invertible Leibniz-derivation of nilpotent Leibniz algebra.

**Proposition 4.11** Every nilpotent Leibniz algebra with a nilindex equal to s has an invertible Leibniz-derivation of order  $\left[\frac{s}{2}\right] + 1$ .

*Proof* Let *L* be a Leibniz algebra with a nilindex equal to *s* and set  $q = \lfloor \frac{s}{2} \rfloor + 1$ . Consider the vector subspace *W* of *L* complementary to  $L^q$ , i.e.  $L = W + L^q$ . Define the map *P* by the following way:

$$P(x) = \begin{cases} x & \text{if } x \in W, \\ qx & \text{if } x \in L^q. \end{cases}$$

It is easy to check that *P* is a Leibniz-derivation for *L* of order *q*.

Below we present one of the main theorems of the paper.

**Theorem 4.12** Let L be a complex Leibniz algebra that admits an invertible Leibnizderivation. Then L is nilpotent.

*Proof* Let d be an invertible Leibniz-derivation of order n of the Leibniz algebra L and

$$L = L_{\rho_1} \oplus L_{\rho_2} \oplus \cdots \oplus L_{\rho_s}$$

be the decomposition of L into characteristic spaces with respect to d.

Let  $\alpha, \beta \in spec(d)$ . Then, by Lemma 4.10, we have

$$[L_{\alpha}, \underbrace{L_{\beta}, L_{\beta}, \dots, L_{\beta}]_{r}}_{n-1-times} = [\dots [[L_{\alpha}, \underbrace{L_{\beta}], L_{\beta}, ]\dots, L_{\beta}]}_{n-1-times} \subseteq L_{\alpha+(n-1)\beta}.$$

Considering k-times of the n-ary multiplication, we have

$$[\dots[[L_{\alpha}, \underbrace{L_{\beta}, L_{\beta}, \dots, L_{\beta}]_{r}}_{n-1-times}, \underbrace{L_{\beta}, L_{\beta}, \dots, L_{\beta}]_{r}}_{k-times}, \dots, \underbrace{L_{\beta}, L_{\beta}, \dots, L_{\beta}]_{r}}_{k-times}$$

$$= [\dots[[L_{\alpha}, \underbrace{L_{\beta}], L_{\beta}, ]\dots, L_{\beta}}_{k(n-1)-times}] \subseteq L_{\alpha+k(n-1)\beta}.$$

Since for sufficiently large  $k \in \mathbb{N}$  we obtain  $\alpha + k(n-1)\beta \notin spec(d)$ , by Lemma 4.10 we obtain  $[\ldots [[L_{\alpha}, L_{\beta}], L_{\beta}, ]\ldots, L_{\beta}] = 0$ .

$$k(n-1)$$
 -times

Thus, any operator of right multiplication  $R_x : L \to L$ , where  $x \in L_\beta$ , is nilpotent and, due to the fact that  $\alpha$ ,  $\beta$  were taken arbitrary, it follows that every operator from  $\bigcup_{i=1}^{k} R(L_{\rho_i})$  is nilpotent.

Now from identity (2.1) and Lemma 4.10 it follows that  $\bigcup_{i=1}^{k} R(L_{\rho_i})$  is a weekly closed set of an associative algebra End(L). Hence, by Theorem 2.6 it follows that every operator from R(L) is nilpotent.

Hence,  $R_x$  is nilpotent for any  $x \in L$ . Now by Engel's theorem (Theorem 2.7), we conclude that L is nilpotent.

Finally, from the Theorem 4.12 and Proposition 4.11, we get the analogue of Theorem 1.1 for Leibniz algebras.

**Theorem 4.13** A Leibniz algebra over a field of characteristic zero is nilpotent if and only if it has an invertible Leibniz-derivation.

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