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# *n*-Dimensional filiform Leibniz algebras of length (n-1) and their derivations

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#### Abstract

In this work *n*-dimensional filiform Leibniz algebras admitting a gradation of length (n - 1) are classified. Derivations of such algebras are also described. © 2007 Elsevier Inc. All rights reserved.

Keywords: Filiform Leibniz algebra; Gradation; Natural gradation; Length of Leibniz algebra; Derivation

## 1. Introduction

The well-known natural gradations of nilpotent Lie and Leibniz algebras are very helpful in investigations of properties of those algebras in the general case without restriction on the gradation. This technique provides a rather deep information on the algebra and it is more effective when the length of the natural gradation is sufficiently large. A similar approach was considered

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in [1,6,11] (and some other papers). The idea of consideration of more convenient gradations firstly was suggested in [3,4]. In particular, in the mentioned papers the authors considered a gradation which has the maximal possible number of non-zero subspaces, i.e. the length of the gradation coincides with the dimension of the algebra. Actually, gradations with a large number of non-zero subspaces enable us to describe the multiplication on the algebra more exactly. Indeed, we can always choose a homogeneous basis and thus the gradation allows to obtain more explicit conditions for the structural constants.

Moreover, such gradations are useful for the investigation of cohomologies for the considered algebras because they induce a corresponding gradation of the group of cohomologies. A similar approach was considered in [2,6,10].

In the present paper we consider gradations which have the length (n-1) for the classification of complex filiform Leibniz algebras and apply the obtained results for the description of the algebras of their derivations.

Recall [9] that an algebra L over a field F is called a *Leibniz algebra* if it satisfies the following Leibniz identity:

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

where [, ] denotes the multiplication in L.

It is not difficult to check that the variety of Leibniz algebras is a "non-antisymmetric" generalization of the Lie algebras' variety.

#### 2. Preliminaries

Let *L* be an arbitrary Leibniz algebra of dimension *n* and let  $\{e_1, \ldots, e_n\}$  be a basis of the algebra *L*. Then the multiplication on the algebra *L* is defined by the products of the basic elements, namely, as  $[e_i, e_j] = \sum_{k=1}^n \gamma_{ij}^k e_k$ , where  $\gamma_{ij}^k$  are the structural constants. Thus the problem of classification of algebras can be reduced to the problem of finding a description of the structural constants up to a non-degenerate basis transformation.

From the Leibniz identity we have the following polynomial equalities for the structural constants:

$$\sum_{l=1}^{n} \left( \gamma_{jk}^{l} \gamma_{il}^{m} - \gamma_{ij}^{l} \gamma_{lk}^{m} + \gamma_{ik}^{l} \gamma_{lj}^{m} \right) = 0.$$

But the straightforward description of structural constants is somewhat cumbersome and therefore usually one has to apply different methods of investigation.

Let *L* be a *Z*-graded Leibniz algebra with a finite number of non-zero subspaces, i.e.  $L = \bigoplus_{i \in \mathbb{Z}} V_i$ , where  $[V_i, V_j] \subseteq V_{i+j}$  for any  $i, j \in \mathbb{Z}$ .

We say that a nilpotent Leibniz algebra *L* admits a *connected gradation* if  $L = V_{k_1} \oplus V_{k_2} \oplus \cdots \oplus V_{k_t}$ , where each  $V_i$  is non-zero for  $k_1 \leq i \leq k_t$ .

The number of subspaces  $l(\bigoplus L) = k_t - k_1 + 1$  is called the length of the gradation. In the case where  $l(\bigoplus L) = \dim L$ , the gradation is called the maximum length gradation [3,4]. If  $l(\bigoplus L) = \dim L - 1$ , then we have a gradation of length (n - 1).

**Definition 2.1.** The length l(L) of a Leibniz algebra L is defined as

$$l(L) = \max \left\{ l\left(\bigoplus L\right) = k_t - k_1 + 1 \mid L = V_{k_1} \oplus V_{k_2} \oplus \cdots \oplus V_{k_t} \text{ is a connected gradation} \right\}.$$

This definition means that l(L) is the greatest number of subspaces from connected gradations which exist in L. Thus, every Leibniz algebra L has the length at least 1, because we can put  $L = V_0$ .

Given an arbitrary Leibniz algebra L we define the lower central series:

$$L^1 = L, \qquad L^{k+1} = \begin{bmatrix} L^k, L \end{bmatrix}, \quad k \ge 1.$$

**Definition 2.2.** A Leibniz algebra *L* is said to be filiform if dim  $L^i(L) = n - i$ , where  $n = \dim L$  and  $2 \le i \le n$ .

**Definition 2.3.** Given a filiform Leibniz algebra L, put  $L_i = L^i/L^{i+1}$ ,  $1 \le i \le n-1$ , and  $gr L = L_1 \oplus L_2 \oplus \cdots \oplus L_{n-1}$ . Then  $[L_i, L_j] \subseteq L_{i+j}$  and we obtain the graded algebra gr L. If gr L and L are isomorphic, denoted by gr L = L, we say that the algebra L is naturally graded.

In the following theorem we summarize the results of the works [1,11].

**Theorem 2.4.** Any complex *n*-dimensional naturally graded filiform Leibniz algebra is isomorphic to one of the following pairwise non-isomorphic algebras:

$$NGF_{1} = \begin{cases} [e_{1}, e_{1}] = e_{3}, \\ [e_{i}, e_{1}] = e_{i+1}, & 2 \leq i \leq n-1, \end{cases}$$
$$NGF_{2} = \begin{cases} [e_{1}, e_{1}] = e_{3}, \\ [e_{i}, e_{1}] = e_{i+1}, & 3 \leq i \leq n-1, \end{cases}$$
$$NGF_{3} = \begin{cases} [e_{i}, e_{1}] = -[e_{1}, e_{i}] = e_{i+1}, & 2 \leq i \leq n-1 \\ [e_{i}, e_{n+1-i}] = -[e_{n+1-i}, e_{i}] = \alpha(-1)^{i+1}e_{n}, & 2 \leq i \leq n-1 \end{cases}$$

where all omitted products are equal to zero,  $\alpha \in \{0, 1\}$  for even *n* and  $\alpha = 0$  for odd *n*.

As we can see the natural gradations in these algebras have length equal to (n - 1). Consider the algebra  $NGF_1$  and the gradation  $V_{-1} = \langle e_1 - e_2 \rangle$ ,  $V_{n-2} = \langle e_2 \rangle$ ,  $V_i = \langle e_{n-i} \rangle$ ,  $0 \le i \le n - 3$ . Then  $NGF_1 = V_{-1} \oplus V_0 \oplus \cdots \oplus V_{n-2}$ , i.e. this algebra has the maximal length.

It should be noted (see [2]) that the algebra  $NGF_1$  is the only filiform non-Lie Leibniz algebra of maximal length.

Summarizing the results from [1,5] we obtain the decomposition of all complex filiform Leibniz algebras into three disjoint classes. **Theorem 2.5.** Any complex *n*-dimensional filiform Leibniz algebra admits a basis  $\{e_1, e_2, \ldots, e_n\}$  such that the table of multiplication of the algebra have one of the following forms:

$$F_{1} = \begin{cases} [e_{1}, e_{1}] = e_{3}, \\ [e_{i}, e_{1}] = e_{i+1}, \\ [e_{1}, e_{2}] = \alpha_{4}e_{4} + \alpha_{5}e_{5} + \dots + \alpha_{n-1}e_{n-1} + \theta e_{n}, \\ [e_{j}, e_{2}] = \alpha_{4}e_{j+2} + \alpha_{5}e_{j+3} + \dots + \alpha_{n+2-j}e_{n}, \\ 2 \le j \le n-2 \end{cases}$$

(omitted products are equal to zero),

$$F_{2} = \begin{cases} [e_{1}, e_{1}] = e_{3}, \\ [e_{i}, e_{1}] = e_{i+1}, \\ [e_{1}, e_{2}] = \beta_{3}e_{4} + \beta_{4}e_{5} + \dots + \beta_{n-1}e_{n}, \\ [e_{2}, e_{2}] = \gamma e_{n}, \\ [e_{j}, e_{2}] = \beta_{3}e_{j+2} + \beta_{4}e_{j+3} + \dots + \beta_{n+1-j}e_{n}, \quad 3 \leq j \leq n-2 \end{cases}$$

(omitted products are equal to zero),

$$F_{3} = \begin{cases} [e_{i}, e_{1}] = e_{i+1}, & 2 \leq i \leq n-1, \\ [e_{1}, e_{i}] = -e_{i+1}, & 3 \leq i \leq n-1, \\ [e_{1}, e_{1}] = \theta_{1}e_{n}, & \\ [e_{1}, e_{2}] = -e_{3} + \theta_{2}e_{n}, \\ [e_{2}, e_{2}] = \theta_{3}e_{n}, & \\ [e_{i}, e_{j}] = -[e_{j}, e_{i}] \in \ln\langle e_{i+j+1}, e_{i+j+2}, \dots, e_{n} \rangle, & 2 \leq i \leq n-2, \\ & 2 \leq j \leq n-i, \\ [e_{i}, e_{n+1-i}] = -[e_{n+1-i}, e_{i}] = \alpha(-1)^{i+1}e_{n}, & 2 \leq i \leq n-1, \end{cases}$$

where all omitted products are equal to zero,  $\alpha \in \{0, 1\}$  for even *n* and  $\alpha = 0$  for odd *n*.

**Definition 2.6.** A linear transformation *d* of a Leibniz algebra *L* is called a derivation if for any  $x, y \in L$ 

$$d([x, y]) = [d(x), y] + [x, d(y)].$$

The space of all derivations of the algebra L equipped with the multiplication defined as the commutator, forms a Lie algebra which is denoted by Der(L).

It is clear that the operator of right multiplication  $R_x$  by an element x of the algebra L (i.e.  $R_x(y) = [y, x]$ ) is also a derivation. Derivations of this type are called inner derivations. Similar to the Lie algebras case the set of the inner derivation Inn(L) forms an ideal of the algebra Der(L).

## 3. The main result

**Proposition 3.1.** Let L be any filiform Leibniz algebra from the family  $F_1$  of Theorem 2.5. If L admits a gradation with the length (n - 1), then this gradation coincides with the natural gradation. In particular, L is isomorphic to the algebra NGF<sub>1</sub>.

**Proof.** Let *L* admit a gradation of length (n - 1):  $L = V_{k_1} \oplus V_{k_1+1} \oplus \cdots \oplus V_{k_1+n-2}$ . Take a homogeneous basis  $\{x_0, x_1, \ldots, x_{n-1}\}$ , where  $x_i \in V_{k_1+i-1}$ ,  $1 \le i \le n-1$ ,  $x_0 \in V_{k_1+j-1}$ , for some j  $(1 \le j \le n-1)$ .

Since the algebra *L* is generated by two elements we may suppose that for some *s* and *t* the elements  $x_s, x_t$  from the basis  $\{x_0, x_1, x_2, ..., x_{n-1}\}$  are the generators of *L*. Expressing these basic elements via the initial basis  $\{e_1, e_2, ..., e_n\}$  we obtain

$$x_s = \sum_{i=1}^n a_i e_i, \qquad x_t = \sum_{j=1}^n b_j e_j$$

where  $a_1b_2 - a_2b_1 \neq 0$ .

Without restriction of generality we may assume that  $a_1 = b_2 = 1$ .

Using the multiplication of the algebra from Theorem 2.5 we consider the products:

$$\left[\dots\left[\underbrace{[x_s, x_s], x_s}_{i\text{-times}}\right] = (1+a_2)e_{i+1} + (*)e_{i+2} + \dots + (*)e_n, \quad 2 \le i \le n-1.$$

**Case 1.** Let  $1 + a_2 \neq 0$ . Set

$$y_i = \left[ \dots \left[ \underbrace{[x_s, x_s], x_s], \dots, x_s}_{i\text{-times}} \right], \quad 1 \le i \le n-1, \qquad y_n = x_t.$$

Then we have  $\langle y_i \rangle \subseteq V_{ik_s}$ ,  $1 \leq k \leq n-1$  and from the connectedness of the gradation we obtain  $k_s = \pm 1$ .

In the case where  $k_s = 1$  we have  $L = V_1 \oplus V_2 \oplus \cdots \oplus V_{n-1}$ . The case  $k_s = -1$ , i.e.  $L = V_{-n+1} \oplus V_{-n+2} \oplus \cdots \oplus V_{-1}$ , can be reduced to the case  $k_s = 1$  by putting  $V_j = V_{-j}$ ,  $-1 \le j \le n-1$ .

Thus, we may assume that  $k_s = 1$ . Since  $l(\bigoplus L) = n - 1$ , we have the existence of some p  $(1 \le p \le n - 1)$  such that  $y_n \in V_p$ .

Consider the products:

$$[x_t, x_s] = (1+b_1)e_3 + (*)e_4 + \dots + (*)e_n,$$
  
$$[x_s, x_t] = b_1(1+a_2)e_3 + (*)e_4 + \dots + (*)e_n,$$

where the asterisks (\*) denote appropriate coefficients at the basic elements.

It is evident that the coefficient at  $e_3$  in the products  $[x_t, x_s]$  and  $[x_s, x_t]$  are not equal to zero simultaneously (otherwise  $1 - a_2b_1 = 0$ ). Therefore we have that either  $[x_t, x_s] \neq 0$  or  $[x_s, x_t] \neq 0$  and these elements belong to  $V_2$ , i.e. p = 1.

Thus, we obtain  $L = V_1 \oplus V_2 \oplus \cdots \oplus V_{n-1}$ , where  $V_1 = \langle y_1, y_n \rangle$ ,  $V_j = \langle y_j \rangle$  for  $2 \leq j \leq n-1$ , but this gradation coincides with the natural gradation. Therefore in this case we obtain the algebra *NGF*<sub>1</sub>.

**Case 2.** Let  $1 + a_2 = 0$ . Consider the multiplication

$$[x_t, x_t] = b_1(1+b_1)e_3 + (*)e_4 + \dots + (*)e_n.$$

Since  $1 - a_2b_1 \neq 0$ , one has  $b_1 \neq -1$ . If  $b_1 \neq 0$ , then similarly to Case 1 we also obtain the algebra *NGF*<sub>1</sub>. Therefore  $b_1 = 0$ , i.e.  $x_t = e_2 + b_3e_2 + \cdots + b_ne_n$ .

Consider the products:

$$\left[\dots[x_{t},\underbrace{x_{s}}],\dots,x_{s}\right] = e_{i+2} + (*)e_{i+3} + \dots + (*)e_{n}, \quad 1 \le i \le n-2.$$

Put

$$y_1 = x_s,$$
  $y_2 = x_t,$   $y_i = [\dots, [x_t, \underbrace{x_s}], \dots, x_s],$   $3 \le i \le n.$ 

Then we have  $\langle y_1 \rangle \subseteq V_{k_s}$ ,  $\langle y_2 \rangle \subseteq V_{k_t}$ ,  $\langle y_i \rangle \subseteq V_{k_t+(i-2)k_s}$ ,  $3 \le i \le n$ . From the property of the gradation we have  $k_s = \pm 1$ .

Let us consider the case where  $k_s = 1$ . Since  $l(\bigoplus L) = (n-1)$ , there exists  $p \ (0 \le p \le n-2)$  such that  $y_1 \in V_{k_t+p}$ , i.e.  $k_t = 1 - p$ .

Thus, we obtain  $L = V_{1-p} \oplus V_{2-p} \oplus \cdots \oplus V_{n-1-p}$ , where  $V_{1-p} = \langle y_2 \rangle$ ,  $V_{2-p} = \langle y_3 \rangle$ , ...,  $V_0 = \langle y_{p+1} \rangle$ ,  $V_1 = \langle y_1, y_{p+2} \rangle$ ,  $V_2 = \langle y_{p+3} \rangle$ , ...,  $V_{n-1-p} = \langle y_n \rangle$ .

Consider the product

$$[y_1, y_2] = \alpha y_3 = \alpha (e_3 + (*)e_4 + \dots + (*)e_n) = \alpha e_3 + \alpha (*)e_4 + \dots + \alpha (*)e_n$$

On the other hand,

$$[y_1, y_2] = (**)e_4 + (**)e_5 + \dots + (**)e_n.$$

Comparing the coefficients at  $e_3$  we obtain  $\alpha = 0$ , i.e.  $[y_1, y_2] = 0$ . In a similar way we obtain  $[y_2, y_2] = 0$ .

From the equality

$$[y_{i+1}, y_2] = [y_i, [y_1, y_2]] + [[y_i, y_2], y_1]$$

by induction we obtain  $[y_i, y_2] = 0, 3 \le i \le n - 2$ .

Thus, we have an algebra with the following table of multiplication:

$$[y_i, y_1] = y_{i+1}, \quad 2 \le i \le n-1,$$
  

$$[y_1, y_1] = \beta y_{p+3},$$
  

$$[y_i, y_j] = 0, \quad 1 \le i \le n, \ 2 \le j \le n.$$

Applying if necessary a change of basis

$$y_1' = y_1 - \beta y_{p+2}, \qquad y_i' = y_i, \quad 2 \leq i \leq n,$$

we may assume that  $\beta = 0$ .

Without loss of generality we may suppose that p = 0, i.e.  $V_1 = \langle y_1, y_2 \rangle$ ,  $V_j = \langle y_{j+1} \rangle$ ,  $2 \leq j \leq n-1$ . So, in the case where  $k_s = 1$  the gradation of the algebra is the natural gradation, i.e.  $L = NGF_1$ .

In the case where  $k_s = -1$  by similar arguments we obtain the natural gradation of the algebra L, i.e. the algebra  $NGF_1$ .  $\Box$ 

**Definition 3.2.** The sets  $\Re(L) = \{x \in L \mid [L, x] = 0\}$  and  $\mathfrak{L}(L) = \{x \in L \mid [x, L] = 0\}$  are called respectively the right and left annihilators of the algebra *L*.

It is not difficult to check that the right annihilator is a two-sides ideal of the Leibniz algebra L. Let L be an n-dimensional filiform Leibniz algebra and suppose that  $\{e_1, e_2, \ldots, e_n\}$  is a basis of L.

Further we will need the following lemma.

**Lemma 3.3.** (See [5].) For any  $0 \le p \le n - k$ ,  $3 \le k \le n$  the equality

$$\sum_{i=k}^{n} a(i) \sum_{j=i+p}^{n} b(i,j)e_j = \sum_{j=k+p}^{n} \sum_{i=k}^{j-p} a(i)b(i,j)e_j$$

holds.

**Proposition 3.4.** Let *L* be any filiform Leibniz algebra from the family  $F_2$  of Theorem 2.5. If *L* admits a gradation with the length (n - 1), then it is isomorphic to one of the following algebras: • for any arbitrary value of *n* 

$$NGF_{2} = \begin{cases} [y_{1}, y_{1}] = y_{3}, \\ [y_{i}, y_{1}] = y_{i+1}, & 3 \leq i \leq n-1, \end{cases}$$
$$M_{1}(k) = \begin{cases} [y_{i}, y_{1}] = y_{i+1}, & 1 \leq i \leq n-2, \\ [y_{i}, y_{n}] = y_{k+i-1}, & 1 \leq i \leq n-k, & 3 \leq k \leq n-1, \end{cases}$$

• for any odd value of n

$$M_{2} = \begin{cases} [y_{i}, y_{1}] = y_{i+1}, & 1 \leq i \leq n-2\\ [y_{i}, y_{n}] = y_{\frac{n+1}{2}+i-1}, & 1 \leq i \leq \frac{n-1}{2}, \\ [y_{n}, y_{n}] = \alpha y_{n-1}, & \alpha \neq 0, \end{cases}$$
$$M_{3} = \begin{cases} [y_{i}, y_{1}] = y_{i+1}, & 1 \leq i \leq n-2, \\ [y_{n}, y_{n}] = y_{n-1}, \end{cases}$$

where all omitted products are equal to zero and  $\{y_1, y_2, ..., y_n\}$  is a basis of the corresponding algebra.

**Proof.** Let L be an algebra satisfying the conditions of the proposition. Similar to Proposition 3.1 we can choose homogeneous generators as

$$x_s = e_1 + a_2 e_2 + \dots + a_n e_n$$
 and  $x_t = b_1 e_1 + e_2 + b_3 e_3 + \dots + b_n e_n$ .

Consider the multiplications:

$$\left[\dots \underbrace{[x_s, x_s], \dots, x_s]}_{i \text{-times}} = e_{i+1} + (*)e_{i+2} + \dots + (*)e_n, \quad 2 \le i \le n-1\right]$$

Put

$$y_i = \left[ \dots \underbrace{[x_s, x_s], \dots, x_s}_{i\text{-times}} \right], \quad 1 \le i \le n-1, \ y_n = x_t.$$

Then we have  $\langle y_i \rangle \subseteq V_{ik_s}$ ,  $1 \leq i \leq n-1$ . From the connectedness of the gradation we have that  $k_s = \pm 1$ .

By the same arguments as above without loss of generality we may take  $k_s = 1$ . Since  $l(\bigoplus L) = n - 1$  there exists p  $(1 \le p \le n - 1)$  such that  $y_n \in V_p$ . Consider the products

$$[y_n, y_1] = b_1 e_3 + \sum_{i=4}^n b_{i-1} e_i + a_2 b_1 \sum_{i=4}^n \beta_{i-1} e_i + a_2 \gamma e_n + a_2 \sum_{i=3}^{n-2} b_i \sum_{j=i+2}^n \beta_{j-i+1} e_j,$$
  
$$[y_1, y_n] = b_1 e_3 + b_1 \sum_{i=4}^n a_{i-1} e_i + \sum_{i=4}^n \beta_{i-1} e_i + a_2 \gamma e_n + \sum_{i=3}^{n-2} a_i \sum_{j=i+2}^n \beta_{j-i+1} e_j,$$
  
$$[y_n, y_n] = b_1^2 e_3 + b_1 \sum_{i=4}^n b_{i-1} e_i + b_1 \sum_{i=4}^n \beta_{i-1} e_i + \gamma e_n + \sum_{i=3}^{n-2} b_i \sum_{j=i+2}^n \beta_{j-i+1} e_j.$$

If  $b_1 \neq 0$  then p = 1 and the gradation of the algebra is natural, i.e. we obtain the algebra  $NGF_2$ . Further we may suppose that  $b_1 = 0$ .

Thus, using Lemma 3.3 we have

$$[y_n, y_1] = a_2 \gamma e_n + \sum_{i=4}^n b_{i-1} e_i + a_2 \sum_{i=3}^{n-2} b_i \sum_{j=i+2}^n \beta_{j-i+1} e_j$$
  
$$= a_2 \gamma e_n + b_3 e_4 + \sum_{j=5}^n \left( b_{j-1} + a_2 \sum_{i=3}^{j-2} b_i \beta_{j-i+1} \right) e_j,$$
  
$$[y_1, y_n] = a_2 \gamma e_n + \sum_{i=4}^n \beta_{i-1} e_i + \sum_{i=3}^{n-2} a_i \sum_{j=i+2}^n \beta_{j-i+1} e_j$$
  
$$= a_2 \gamma e_n + \beta_3 e_4 + \sum_{j=5}^n \left( \beta_{j-1} + \sum_{i=3}^{j-2} a_i \beta_{j-i+1} \right) e_j,$$
  
$$[y_n, y_n] = \gamma e_n + \sum_{i=3}^{n-2} b_i \sum_{j=i+2}^n \beta_{j-i+1} e_j = \gamma e_n + \sum_{j=5}^n \left( \sum_{i=3}^{j-2} b_i \beta_{j-i+1} \right) e_j.$$

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**Case 1.** There exists some i  $(3 \le i \le n-2)$  such that  $\beta_i \ne 0$ . Let  $\beta_k$  be the first non-zero coefficient from the set  $\{\beta_3, \beta_4, \dots, \beta_{n-2}\}$ .

Then

$$[y_n, y_1] = a_2 \gamma e_n + \sum_{j=4}^{k+1} b_{j-1} e_j + \sum_{j=k+2}^n \left( b_{j-1} + a_2 \sum_{i=3}^{j+1-k} b_i \beta_{j-i+1} \right) e_j,$$
  
$$[y_1, y_n] = a_2 \gamma e_n + \beta_k e_{k+1} + \sum_{j=k+2}^n \left( \beta_{j-1} + \sum_{i=3}^{j+1-k} a_i \beta_{j-i+1} \right) e_j,$$
  
$$[y_n, y_n] = \gamma e_n + \sum_{j=k+2}^n \left( \sum_{i=3}^{j+1-k} b_i \beta_{j-i+1} \right) e_j.$$

Since  $\langle y_p, y_n \rangle = V_p$ , from the condition  $[V_1, V_p] \subseteq V_{p+1} = \langle y_{p+1} \rangle$  it follows that

$$[y_1, y_n] = \delta y_{p+1} = \delta (e_{p+2} + (*)e_{p+3} + \dots + (*)e_n).$$

On the other hand,

$$[y_1, y_n] = a_2 \gamma e_n + \beta_k e_{k+1} + \sum_{j=k+2}^n \left(\beta_{j-1} + \sum_{i=3}^{j+1-k} a_i \beta_{j-i+1}\right) e_j.$$

Comparing the coefficients we obtain  $\delta = \beta_k$  and p = k - 1. Thus we have  $V_i = \langle y_i \rangle$  for  $1 \leq i \leq n - 1$ ,  $i \neq k - 1$ ,  $V_{k-1} = \langle y_{k-1}, y_n \rangle$ . Therefore

$$[y_n, y_1] = a_2 \gamma e_n + \sum_{j=4}^{k+1} b_{j-1} e_j + \sum_{j=k+2}^n \left( b_{j-1} + a_2 \sum_{i=3}^{j+1-k} b_i \beta_{j-i+1} \right) e_j,$$

but, on the other hand, from  $[y_n, y_1] \in V_k$ , it follows

$$[y_n, y_1] = \lambda y_k = \lambda (e_{k+1} + (*)e_{k+2} + \dots + e_n).$$

Therefore  $b_i = 0, 3 \leq i \leq k - 1$  and  $\lambda = b_k$ , i.e.  $[y_n, y_1] = b_k y_k$ .

From the equality

$$[y_1, [y_1, y_n]] = [[y_1, y_1], y_n] - [[y_1, y_n], y_1]$$

and the fact that  $[y_1, y_n] \in \Re(L)$  we have  $[[y_1, y_1], y_n] = [[y_1, y_n], y_1]$ . Therefore  $[y_2, y_n] = [[y_1, y_n], y_1] = \beta_k y_{k+1}$ .

From the equalities

$$[y_i, [y_1, y_n]] = [[y_i, y_1], y_n] - [[y_i, y_n], y_1]$$

by induction we obtain that  $[y_i, y_n] = \beta_k y_{k+i-1}, 2 \leq i \leq n-k$ .

Therefore we have the following multiplications in the algebra L

$$\begin{cases} [y_i, y_1] = y_{i+1}, & 1 \le i \le n-2, \\ [y_n, y_1] = b_k y_k, \\ [y_i, y_n] = \beta_k y_{k+i-1}, & 1 \le i \le n-k, \\ [y_n, y_n] = \delta y_{2k-2}. \end{cases}$$

**Case 1.1.** Let 2k - 2 < n - 1. Then  $[y_n, y_n] = b_k \beta_k y_{2k-2}$ . Taking a change of the basic element as follows  $y'_n = \frac{1}{\beta_k}(y_n - b_k y_{k-1})$ , we obtain  $b_k = 0$ ,  $\beta_k = 1$ . Thus we obtain the algebra  $M_1(k)$ ,  $3 \le k \le n - 2$ ,  $k \ne \frac{n+1}{2}$ .

**Case 1.2.** Let 2k - 2 = n - 1. Then  $[y_n, y_n] = (b_k \beta_k + \gamma) y_{n-1}$  and changing the basic element  $y_n$  by  $y'_n = \frac{1}{\beta_k}(y_n - b_k y_{k-1})$ , we obtain  $b_k = 0$ ,  $\beta_k = 1$ , i.e.  $[y_n, y_n] = \gamma y_{n-1}$ . If  $\gamma = 0$ , we obtain the algebra  $M_1(\frac{n+1}{2})$ . If  $\gamma \neq 0$ , then we get the algebra  $M_2$ .

**Case 2.** Let  $\beta_i = 0$  for any  $i \ (3 \le i \le n-2)$ . Then

$$[y_n, y_1] = a_2 \gamma e_n + \sum_{i=4}^n b_{i-1} e_i,$$
  
$$[y_1, y_n] = (a_2 \gamma + \beta_{n-1}) e_n,$$
  
$$[y_n, y_n] = \gamma e_n.$$

**Case 2.1.** There exists i ( $3 \le i \le n-2$ ) such that  $b_i \ne 0$ . Let  $b_k$  be the first non-zero element from the set  $\{b_3, b_4, \ldots, b_{n-2}\}$ . Then by the same arguments as in Case 1 we obtain an algebra with the following table of multiplication:

$$\begin{cases} [y_i, y_1] = y_{i+1}, & 1 \le i \le n-2, \\ [y_n, y_1] = b_k y_k, \\ [y_i, y_n] = 0, & 1 \le i \le n-1, \\ [y_n, y_n] = \gamma y_{n-1}. \end{cases}$$

It is not difficult to see that taking  $y'_n = y_n - b_k y_{k-1}$  we obtain  $b_k = 0$ . In the case  $\gamma = 0$  we have the algebra  $NGF_2$ . If  $\gamma \neq 0$ , then by scaling the basis we obtain  $\gamma = 1$ , i.e. the algebra  $M_3$ .

**Case 2.2.** Let  $b_i = 0$  for any  $i (3 \le i \le n - 2)$ . Then we have

$$[y_n, y_1] = (a_2\gamma + b_{n-1})e_n,$$
  

$$[y_1, y_n] = (a_2\gamma + \beta_{n-1})e_n,$$
  

$$[y_n, y_n] = \gamma e_n.$$

Suppose that  $\gamma = 0$ , then in the case  $(b_{n-1}, \beta_{n-1}) = (0, 0)$  we have the algebra  $NGF_2$ . If  $(b_{n-1}, \beta_{n-1}) \neq (0, 0)$  then by a change of basis (similar to Case 2.1) we obtain either the algebra  $NGF_2$  or  $M_1(n-1)$ .

Let  $\gamma \neq 0$ , then *n* is odd and  $p = \frac{n-1}{2}$ . If  $a_2\gamma + \beta_{n-1} \neq 0$  or  $a_2\gamma + \beta_{n-1} \neq 0$  then from the conditions  $[y_1, y_n], [y_n, y_n] \in V_{n-1}$  we have p = 1, n = 3, i.e. a three-dimensional naturally graded algebra. So, we have to consider the case  $a_2\gamma + \beta_{n-1} = a_2\gamma + \beta_{n-1} = 0$ . Taking the following change of basis:  $y'_n = \frac{y_1}{\sqrt{\gamma}}$  we may suppose that  $\gamma = 1$ , i.e. we obtain the algebra  $M_3$ .  $\Box$ 

From [2] we know that the family  $F_2$  of Theorem 2.5 contains no algebra which admits a gradation of length n. Therefore in Proposition 3.4 we obtain the classification of filiform algebras of length (n - 1) from the family  $F_2$ .

**Proposition 3.5.** Let L be any filiform Leibniz algebra from the family  $F_3$  in Theorem 2.5. If L admits a gradation with length (n - 1), then it is isomorphic either to the algebra

$$NGF_{3} = \begin{cases} [y_{i}, y_{1}] = -[y_{1}, y_{i}] = y_{i+1}, & 2 \leq i \leq n-1, \\ [y_{i}, y_{n+1-i}] = -[y_{n+1-i}, y_{i}] = \alpha(-1)^{i+1}y_{n}, & 2 \leq i \leq n-1, \end{cases}$$

or to the algebra

$$M_4 = \begin{cases} [y_1, y_1] = y_n, \\ [y_i, y_1] = y_{i+1}, & 2 \le i \le n-1, \\ [y_1, y_i] = -y_{i+1}, & 2 \le i \le n-1, \end{cases}$$

where all omitted products are equal to zero,  $\alpha \in \{0, 1\}$  for even *n* and  $\alpha = 0$  for odd *n*, and  $\{y_1, y_2, \ldots, y_n\}$  is a basis of the corresponding algebra.

**Proof.** As in the above propositions we choose generators from the homogeneous basis

 $x_s = e_1 + a_2 e_2 + \dots + a_n e_n$  and  $x_t = b_1 e_1 + e_2 + b_3 e_3 + \dots + b_n e_n$ .

Consider the products

$$[x_{s}, x_{s}] = (\theta_{1} + a_{2}\theta_{2} + a_{2}^{2}\theta_{3})e_{n},$$

$$[x_{t}, x_{t}] = (b_{1}^{2}\theta_{1} + b_{1}\theta_{2} + \theta_{3})e_{n},$$

$$[\dots [x_{t}, \underbrace{x_{s}], \dots, x_{s}}_{i \text{-times}}] = (1 - a_{2}b_{1})e_{i+2} + (*)e_{i+3} + \dots + (*)e_{n}, \quad 1 \le i \le n-2.$$

Since  $1 - a_2 b_1 \neq 0$ , we can choose

$$y_1 = x_s,$$
  $y_2 = x_t,$   $y_i = \left[ \dots \left[ x_t, \underbrace{x_s}, \dots, x_s \right] \right],$   $3 \le i \le n.$ 

Therefore we have

 $\langle y_1 \rangle \subseteq V_{k_s}, \quad \langle y_2 \rangle \subseteq V_{k_t}, \quad \langle y_i \rangle \subseteq V_{k_t+(i-2)k_s}, \quad 3 \leqslant i \leqslant n.$ 

From the connectedness of the gradation we have that  $k_s = \pm 1$ . Moreover, as in the previous cases we can assume that  $k_s = 1$ . So, we have  $[y_i, y_1] = y_{i+1}, 2 \le i \le n-1$ .

Consider the products

$$[y_3, y_2] = b_1(1 - a_2b_1)e_4 + (*)e_5 + \dots + (*)e_n.$$

If  $b_1 \neq 0$ , then  $[y_3, y_2] = b_1 y_4$  and therefore  $k_t = 1$ , i.e. the gradation coincides with the natural gradation. The natural gradation of this kind of algebra was considered in [7]. It will be denoted here by *NGF*<sub>3</sub>. Hence we may assume that  $b_1 = 0$ .

Since  $l(\bigoplus L) = n - 1$ , there exists p  $(1 \le p \le n - 2)$  such that  $V_1 = V_{k_t+p}$ , therefore  $k_t = 1 - p$ .

Thus

$$L = V_{1-p} \oplus V_{2-p} \oplus \cdots \oplus V_{n-1-p}$$

where  $V_{1-p} = \langle y_2 \rangle$ ,  $V_{2-p} = \langle y_3 \rangle$ ,...,  $V_0 = \langle y_{p+1} \rangle$ ,  $V_1 = \langle y_1, y_{p+2} \rangle$ ,  $V_2 = \langle y_{p+3} \rangle$ ,...,  $V_{n-1-p} = \langle y_n \rangle$ .

Therefore the table of multiplication in the algebra L is as follows:

$$\begin{cases} [y_1, y_1] = \alpha y_{p+3}, \\ [y_i, y_1] = y_{i+1}, & 2 \le i \le n-1, \\ [y_1, y_i] = -y_{i+1}, & 2 \le i \le n-1, \\ [y_i, y_j] = 0, & 2 \le i \le n, \ 2 \le j \le n. \end{cases}$$

In the case where  $\alpha = 0$  without loss of generality we may suppose that p = 0 and therefore we have the natural gradation and thus we obtain the algebra NGF<sub>3</sub>.

If  $\alpha \neq 0$  then p = n - 3 and taking the change of basis

$$y'_1 = y_1, \qquad y'_i = \alpha y_i, \quad 2 \leq i \leq n,$$

we can assume that  $\alpha = 1$ , i.e. we obtain the algebra  $M_4$ .  $\Box$ 

From [3] it follows that the algebra  $NGF_3$  only for  $\alpha = 0$  admits a gradation with the length n, i.e. this has the maximal length. From [2] we have that  $M_4$  does not admit any gradation with the length n. Therefore the algebras  $NGF_3$  for  $\alpha = 1$  and  $M_4$  are algebras of length (n - 1).

Summarizing Propositions 3.1–3.5 we obtain the following result.

**Theorem 3.6.** Any *n*-dimensional complex filiform Leibniz algebra of length (n - 1) is isomorphic to one of the following pairwise non-isomorphic algebras:

$$NGF_2$$
,  $NGF_3$  ( $\alpha = 1$ ),  $M_1(k)$ ,  $M_2$ ,  $M_3$ ,  $M_4$ .

**Proof.** The algebra  $M_2$  is not isomorphic to the algebra  $M_1(k)$  for any k ( $3 \le k \le n - 2$ ), because dim  $\mathfrak{L}(M_1(k)) = 2$  but dim  $\mathfrak{L}(M_2) = 1$ . The fact that the algebras  $M_1(k)$ ,  $M_2$ ,  $M_3$ ,  $M_4$  are pairwise non-isomorphic follows from the criteria of isomorphism of two filiform algebras of the class  $F_2$  [5, Theorem 4.4]. Since the families of algebras  $F_1$ ,  $F_2$ ,  $F_3$  are disjoint,  $NGF_2$  is a split non-Lie Leibniz algebra, and  $NGF_3$  ( $\alpha = 1$ ) is a Lie algebra, the proof is complete.  $\Box$ 

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#### 4. Derivations of *n*-dimensional filiform Leibniz algebras of the length (n - 1)

In this section we apply the above classification of *n*-dimensional filiform Leibniz algebras of length (n - 1) to the description of their derivations.

Let *L* be a Z-graded Leibniz algebra, i.e.  $L = \bigoplus_{i \in Z} V_i$ . This gradation induces a gradation of the algebra  $Der(L) = \bigoplus_{i \in Z} W_i$  in the following way:

$$W_i = \left\{ d \in Der(L) \mid d(x) \in W_{i+j} \text{ for any } x \in V_j \right\}.$$

If the gradation of *L* is finite, then the gradation of Der(L) is also finite. In particular, if for an *n*-dimensional filiform Leibniz algebra *L* we have  $L = V_{k_1} \oplus V_{k_1+1} \oplus \cdots \oplus V_{k_1+n-2}$  then it is easy to see that  $Der(L) = W_{2-n} \oplus W_{3-n} \oplus \cdots \oplus W_{n-3} \oplus W_{n-2}$ .

Since derivations of the algebras  $NGF_1$ - $NGF_3$  have already been described in [6,10] we only have to consider the remaining algebras from Theorem 3.6.

First, consider the family of algebras

$$M_1(k) = \begin{cases} [y_i, y_1] = y_{i+1}, & 1 \le i \le n-2, \\ [y_i, y_n] = y_{k+i-1}, & 1 \le i \le n-k, & 3 \le k \le n-2. \end{cases}$$

**Proposition 4.1.** The linear maps  $h_i$ ,  $d_j$   $(1 \le i \le 2, 0 \le j \le n-2)$  on  $M_1(K)$  defined as follows:

$$d_0(y_i) = iy_i, \quad 1 \le i \le n - 1, \qquad d_0(y_n) = (k - 1)y_n,$$
  
$$d_j(y_i) = y_{j+i}, \quad 1 \le j \le n - 2, \ 1 \le i \le n - j - 1,$$
  
$$h_1(y_n) = y_{n-1},$$

in the case  $2k - 2 \leq n - 1$  and as

$$d_0(y_i) = iy_i, \quad 1 \le i \le n - 1, \qquad d_0(y_n) = (k - 1)y_n,$$
  

$$d_j(y_i) = y_{i+j}, \quad 1 \le j \le n - 2, \quad 1 \le i \le n - j - 1,$$
  

$$h_1(y_n) = y_{n-1},$$
  

$$h_2(y_1) = y_n, \qquad h_2(y_i) = (i - 1)y_{k+i-2}, \quad 2 \le i \le n - k + 1,$$

in the case  $2k - 2 \ge n$  form a basis of the space  $Der(M_1(k))$ .

**Proof.** It is easy to see that the given maps are derivations and they are linearly independent.

From Proposition 3.4 we have  $M_1(K) = V_1 \oplus V_2 \oplus \cdots \oplus V_{n-1}$ , where  $V_i = \langle y_i \rangle$   $1 \leq i \leq n-1$ ,  $i \neq k-1$  and  $V_{k-1} = \langle y_{k-1}, y_n \rangle$ ,  $3 \leq k \leq n-1$ .

Let  $d \in Der(M_1(k))$ , then we have the decomposition  $d = \sum_{i=2-n}^{n-2} d_i$ . It is clear that  $d_{2-n}(y_i) = 0$  for  $1 \le i \le n-2$ ,  $d_{2-n}(y_n) = 0$  and  $d_{2-n}(y_{n-1}) = \alpha y_1$ ,  $\alpha \in C$ . The property of derivations implies

$$d_{2-n}(y_{n-1}) = d_{2-n}([y_{n-2}, y_1]) = [d_{2-n}(y_{n-2}), y_1] + [y_{n-2}, d_{2-n}(y_1)] = 0,$$

i.e.  $d_{2-n} = 0$ .

Similarly we obtain that  $d_j = 0, j \leq 1 - k$ .

Consider  $d_{2-k} \in Der(M_1(k))$ . It is easy to see that  $d_{2-k}(y_i) = 0$  for  $1 \le i \le k-2$ , and  $d_{2-k}(y_n) = \alpha y_1$  for some  $\alpha \in C$ .

From

$$d_{2-k}(y_i) = d_{2-k}([y_{i-1}, y_1]) = [d_{2-k}(y_{i-1}), y_1] + [y_{i-1}, d_{2-k}(y_1)] = 0, \quad 2 \le i \le n-1,$$

we have  $d_{2-k}(y_i) = 0, 2 \le i \le n - 1$ .

By the property of derivations one has

$$0 = d_{2-k}([y_n, y_1]) = [d_{2-k}(y_n), y_1] + [y_n, d_{2-k}(y_1)] = [\alpha y_1, y_1]$$
  
=  $\alpha y_2 \Rightarrow \alpha = 0$ , i.e.  $d_{2-k} = 0$ .

Analogously we obtain  $d_{t-k} = 0, 2 \le t \le k-1$ . Thus we have  $d_i = 0, 2-n \le i \le -1$ . Consider  $d_0 \in Der(M_1(k))$ . Then

$$d_0(y_i) = \begin{cases} \beta_{0,i} y_i, & 1 \le i \le n-1, \ i \ne k-1, \\ \gamma_{0,1} y_n + \gamma_{0,2} y_{k-1}, & i = n, \\ \gamma_{0,3} y_n + \gamma_{0,4} y_{k-1}, & i = k-1, \end{cases}$$

where  $\beta_{0,i}$ ,  $\gamma_{0,j}$  are complex numbers.

Further we have

$$d_0(y_2) = d_0([y_1, y_1]) = [d_0(y_1), y_1] + [y_1, d_0(y_1)] = [\beta_{0,1}y_1, y_1] + [y_1, \beta_{0,1}y_1]$$
  
=  $2\beta_{0,1}y_2 \implies \beta_{0,2} = 2\beta_{0,1}.$ 

From the chain of equalities

$$d_0(y_i) = d_0([y_{i-1}, y_1]) = [d_0(y_{i-1}), y_1] + [y_{i-1}, d_0(y_1)]$$
  
=  $[\beta_{0,i-1}y_{i-1}, y_1] + [y_{i-1}, \beta_{0,1}y_1] = (\beta_{0,i-1} + \beta_{0,1}) = i\beta_{0,1}y_i,$ 

by induction we obtain  $\beta_{0,i} = i\beta_{0,1}, 2 \le i \le n-1, i \ne k-1$ , and  $\gamma_{0,3} = 0, \gamma_{0,4} = (k-1)\beta_{0,1}$ .

The property of derivations implies

$$0 = d_0([y_n, y_1]) = [d_0(y_n), y_1] + [y_n, d_0(y_1)] = [\gamma_{0,1}y_n + \gamma_{0,2}y_{k-1}, y_1] + [y_n, \beta_{0,1}y_1]$$
  
=  $\gamma_{0,2}y_k \implies \gamma_{0,2} = 0.$ 

Further we have

$$d_0([y_1, y_n]) = [d_0(y_1), y_n] + [y_1, d_0(y_n)] = [\beta_{0,1}y_1, y_n] + [y_1, \gamma_{0,1}y_n] = (\beta_{0,1} + \gamma_{0,1})y_k$$

On the other hand,

$$d_0([y_1, y_n]) = d_0(y_k) = k\beta_{0,1}y_k.$$

Therefore  $\gamma_{0,1} = (k - 1)\beta_{0,1}$ .

Thus we have

$$d_0(y_i) = \begin{cases} iy_i, & 1 \le i \le n-1, \\ (k-1)y_n, & i = n. \end{cases}$$

Consider  $d_j \in Der(M_1(k)), 1 \leq j \leq n-2$ . It is clear that

$$d_{j}(y_{i}) = \begin{cases} \beta_{j,i} y_{i+j}, & 1 \leq i \leq n-1, \ i+j \neq k-1, \\ \gamma_{j,1} y_{n} + \gamma_{j,2} y_{k-1}, & i+j = k-1, \\ \gamma_{j,3} y_{k-1+j}, & i = n. \end{cases}$$
(1)

We have

$$0 = d_j([y_n, y_1]) = [d_j(y_n), y_1] + [y_n, d_j(y_1)] = [\gamma_{j,3}y_{k-1+j}, y_1] + [y_n, d_j(y_1)] = \gamma_{j,3}y_{k+j}.$$

If  $j \leq n - k - 1$ , then  $\gamma_{j,3} = 0$  and if j = n - k, then  $\gamma_{j,3}$  is arbitrary. In the case where  $j \geq n - k + 1$  one has  $d_j(y_n) = 0$ .

Consider the derivation  $d_{k-2}$ . From (1) we have

$$\begin{aligned} d_{k-2}(y_1) &= \gamma_{k-2,1} y_n + \gamma_{k-2,2} y_{k-1}, \\ d_{k-2}(y_2) &= d_{k-2} ([y_1, y_1]) = [d_{k-2}(y_1), y_1] + [y_1, d_{k-2}(y_1)] \\ &= [\gamma_{k-2,1} y_n + \gamma_{k-2,2} y_{k-1}, y_1] + [y_1, \gamma_{k-2,1} y_n + \gamma_{k-2,2} y_{k-1}] \\ &= (\gamma_{k-2,1} + \gamma_{k-2,2}) y_k, \\ d_{k-2}(y_3) &= d_{k-2} ([y_2, y_1]) = [d_{k-2}(y_2), y_1] + [y_2, d_{k-2}(y_1)] \\ &= [(\gamma_{k-2,1} + \gamma_{k-2,2}) y_k, y_1] + [y_2, \gamma_{k-2,1} y_n + \gamma_{k-2,2} y_{k-1}] \\ &= (2\gamma_{k-2,1} + \gamma_{k-2,2}) y_{k+1}. \end{aligned}$$

By induction we obtain

$$d_{k-2}(y_i) = d_{k-2}([y_{i-1}, y_1]) = [d_{k-2}(y_{i-1}), y_1] + [y_{i-1}, d_{k-2}(y_1)]$$
  
= [((i-2)\gamma\_{k-2,1} + \gamma\_{k-2,2})y\_{k+i-3}, y\_1] + [y\_{i-1}, \gamma\_{k-2,1}y\_n + \gamma\_{k-2,2}y\_{k-1}]  
= ((i-1)\gamma\_{k-2,1} + \gamma\_{k-2,2})y\_{k+i-2}, \quad 2 \le i \le n-k+1. (2)

On the other hand,

$$d_{k-2}(y_k) = d_{k-2}([y_1, y_n]) = [d_{k-2}(y_1), y_n] + [y_1, d_{k-2}(y_n)]$$
  
=  $[\gamma_{k-2,1}y_n + \gamma_{k-2,2}y_{k-1}, y_1] + [y_1, \gamma_{k-2,3}y_{2k-3}] = \gamma_{k-2,2}y_{2k-2}.$  (3)

Comparing the coefficients in (2) for i = k with the coefficients in (3) we obtain  $(k - 1)\gamma_{k-2,1} + \gamma_{k-2,2} = \gamma_{k-2,1}$  for  $1 \le 2k - 2 \le n - 1$ , therefore  $\gamma_{k-2,1} = 0$ . For 2k - 2 > n - 1, we have  $d_{k-2}(y_i) = ((i - 1)\gamma_{k-2,1} + \gamma_{k-2,2})y_{k+i-2}$ ,  $2 \le i \le n - k + 1$ .

**Case 1.**  $2k - 2 \le n - 1$ . In this case  $d_{k-2}(y_i) = \gamma_{k-2,2}y_{k+i-2}, 1 \le i \le n - k + 1$ . Consider  $d_j(y_1) = \beta_{j,1}y_{j+1}, 2 \le j \le n - k + 1, j \ne k - 2$ . Then

$$d_j(y_2) = d_j([y_1, y_1]) = [d_j(y_1), y_1] + [y_1, d_j(y_1)]$$
$$= [\beta_{j,1}y_{j+1}, y_1] + [y_1, \beta_{j,1}y_{j+1}] = \beta_{j,1}y_{j+2}.$$

By induction we obtain

$$d_j(y_i) = d_j([y_{i-1}, y_1]) = [d_j(y_{i-1}), y_1] + [y_{i-1}, d_j(y_1)]$$
$$= [\beta_{j,1}y_{j+i-1}, y_1] + [y_{i-1}, \beta_{j,1}y_{j+1}] = \beta_{j,1}y_{j+i},$$

i.e.  $d_j(y_i) = \beta_{j,1}y_{j+i}, 1 \le j \le n-2, 1 \le i \le n-1-j, j \ne n-k.$ 

By the above consideration we can write  $d_{n-k}$  in the following form  $d_{n-k} = \beta_{n-k,1}d'_{n-k} + \gamma_{n-k,3}h_1$ , where  $d'_{n-k}(y_i) = y_{n-k+i}$ ,  $h_1(y_n) = y_{n-1}$ .

Denoting  $d'_{n-k}$  by  $d_{n-k}$  we obtain  $d = h_1 + d_0 + d_1 + \dots + d_{n-2}$ .

**Case 2.** Let  $2k - 2 \ge n$ , then

$$d_{k-2}(y_1) = \gamma_{k-2,1} y_n + \gamma_{k-2,2} y_{k-1},$$
  

$$d_{k-2}(y_i) = ((i-1)\gamma_{k-2,1} + \gamma_{k-2,2}) y_{k+i-2}, \quad 2 \le i \le n-k+1, \quad \text{i.e.}$$
  

$$d_j(y_i) = \beta_{j,i} y_{j+i}, \quad 1 \le j \le n-2, \ 1 \le i \le n-1-j, \ j \notin \{n-k,k-2\}.$$

As above one has  $d_{k-2} = \gamma_{k-2,2}d'_{k-2} + \gamma_{k-2,1}h_2$ , where  $d'_{k-2}(y_i) = y_{k-2+i}$ ,  $h_2(y_1) = y_n$  and  $h_2(y_i) = (i-1)y_{k+i-2}$ .

Similarly  $d_{n-k} = \beta_{n-k,1}d'_{n-k} + \gamma_{k-2,3}h_1$ , where  $d'_{n-k}(y_i) = y_{n-k+i}, h_1(y_n) = y_{n-1}$ . Denoting  $d'_{k-2}$  by  $d_{k-2}$  and  $d'_{n-k}$  by  $d_{n-k}$  we obtain that  $d = h_1 + h_2 + d_0 + d_1 + \dots + d_{n-2}$ .  $\Box$ 

Similarly one can prove the following propositions.

**Proposition 4.2.** The linear maps  $h_1$ ,  $d_j$  ( $0 \le j \le n-2$ ) on  $M_2$  (respectively on  $M_3$ ) defined as

$$d_0(y_n) = (k-1)y_n,$$
  

$$d_0(y_i) = iy_i, \quad 1 \le i \le n-1,$$
  

$$d_j(y_i) = y_{i+j}, \quad 1 \le j \le n-2, \ 1 \le i \le n-j-1,$$
  

$$h_1(y_n) = y_{n-1},$$

form a basis of the space  $Der(M_2)$  (respectively  $Der(M_3)$ ).

**Proposition 4.3.** The linear maps  $h_0$ ,  $h_1$ ,  $d_j$   $(3 - n \le j \le n - 2)$  on  $M_4$  defined as

$$d_{j-n}(y_1) = y_{j-1}, \quad 3 \le j \le n,$$
  

$$d_j(y_i) = y_{i+j}, \quad 1 \le j \le n-2, \ 2 \le i \le n-j,$$
  

$$h_0(y_1) = y_n,$$
  

$$h_0(y_i) = (2-n+i)y_i, \quad 2 \le i \le n,$$
  

$$h_1(y_1) = y_n.$$

form a basis of the space  $Der(M_4)$ .

Recall [8] that the 1-cocycle space  $Z^1(L, L)$  of the algebra L with the values in L is the space Der(L) of derivations of L, while the 1-coboundary space  $B^1(L, L)$  is the space Inn(L) of inner derivations. The first cohomology group  $H^1(L, L)$  is the quotient space  $Z^1(L, L)/B^1(L, L)$ . Thus the above results imply

**Corollary 4.4.** The first cohomology groups of Leibniz algebras of length (n - 1) have the following dimensions:

$$Dim H^{1}(M_{1}(k), M_{1}(k)) = n - 2, \quad for \ 2k - 2 \leq n - 1,$$
$$Dim H^{1}(M_{1}(k), M_{1}(k)) = n - 1, \quad for \ 2k - 2 \geq n,$$
$$Dim H^{1}(M_{i}, M_{i}) = n - 2, \quad for \ 2 \leq i \leq 3,$$
$$Dim H^{1}(M_{4}, M_{4}) = n - 3.$$

**Remark 4.5.** The cases of the algebras  $NGF_2$  and  $NGF_3$  ( $\alpha = 1$ ) have been already considered in [10] and [6], respectively.

Further recall [8] that the 2-coboundary space for the algebra *L* is  $B^2(L, L) = \{f : L \otimes L \rightarrow L \mid f(x, y) = [d(x), y] + [x, d(y)] - d([x, y])$  for some linear transformation *d* of *L*}. For the algebras *NGF*<sub>2</sub> and *NGF*<sub>3</sub> ( $\alpha = 1$ ) 2-coboundary spaces were considered in [10] and [6]. For the rest of Leibniz algebras of length (n - 1) we have

## Corollary 4.6.

$$Dim B^{2}(M_{1}(k), M_{1}(k)) = n^{2} - n + 2, \quad for \ 2k - 2 \le n - 1,$$
$$Dim B^{2}(M_{1}(k), M_{1}(k)) = n^{2} - n + 1, \quad for \ 2k - 2 \ge n,$$
$$Dim B^{2}(M_{i}, M_{i}) = n^{2} - n + 2, \quad for \ 2 \le i \le 3,$$
$$Dim B^{2}(M_{4}, M_{4}) = n^{2} - n + 3.$$

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