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One-generated nilpotent Novikov algebras

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ABSTRACT

We give a classification of five- and six-dimensional complex one-generated nilpotent Novikov algebras.

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Introduction

The algebraic classification (up to isomorphism) of algebras of dimension n from a certain variety defined by some family of polynomial identities is a classic problem in the theory of non-associative algebras. There are many results related to the algebraic classification of low-dimensional algebras in the varieties of Jordan, Lie, Leibniz, Zinbiel and many other algebras [1–13]. Another interesting direction is a study of one-generated objects. The description of one-generated groups is well-known: there is only one one-generated group of order n . In the case of algebras, there are some similar results, such as the description of n -dimensional one-generated nilpotent associative [14], noncommutative Jordan [15], Leibniz and Zinbiel algebras [16]. It was proven that there is only one n -dimensional one-generated nilpotent algebra in these varieties. But on the other hand, as we can see in varieties of Novikov [17], assosymmetric [18], bicommutative [19], commutative [20], and terminal [21] algebras, there are more than one four-dimensional one-generated nilpotent algebra from these varieties. It is easy to see that a nilpotent one-generated algebra is a central extension of a nilpotent one-generated algebra of a smaller dimension. In the present paper, we give an algebraic classification of five- and

six-dimensional one-generated nilpotent Novikov algebras introduced by Novikov and Balinsky in 1985 [22].

The variety of Novikov algebras is defined by the following identities:

$$\begin{aligned}(xy)z &= (xz)y, \\ (xy)z - x(yz) &= (yx)z - y(xz).\end{aligned}$$

It contains the commutative associative algebras as a subvariety. On the other hand, the variety of Novikov algebras is the intersection of the variety of right commutative algebras (defined by the first Novikov identity) and the variety of left symmetric (Pre-Lie) algebras (defined by the second Novikov identity). Also, any Novikov algebra under the commutator multiplication is a Lie algebra [23,24], and Novikov algebras are related with Tortken and Novikov–Poisson algebras [25–27]. The systematic study of Novikov algebras started after the paper by Zelmanov where all complex finite-dimensional simple Novikov algebras were found [28]. The first non-trivial examples of infinite-dimensional simple Novikov algebras were constructed in [29]. Also, the simple Novikov algebras were described in infinite-dimensional case and over fields of positive characteristic [30–32]. The algebraic classification of three-dimensional Novikov algebras was given in [33], and for some classes of four-dimensional algebras, it was given in [34]; the geometric classification of three-dimensional Novikov algebras was given in [35] and of four-dimensional nilpotent Novikov algebras in [36]. Many other pure algebraic properties were studied in a series of papers by Dzhumadil'daev [37–40].

Our method to classify nilpotent Novikov algebras is based on the calculation of central extensions of smaller nilpotent algebras from the same variety. The algebraic study of central extensions of Lie and non-Lie algebras has been an important topic for years [17,41–46]. First, Skjelbred and Sund used central extensions of Lie algebras for a classification of nilpotent Lie algebras [45]. After that, using the method described by Skjelbred and Sund, all non-Lie central extensions of all four-dimensional Malcev algebras were described [43], and also all non-associative central extensions of three-dimensional Jordan algebras [42], all anticommutative central extensions of the three-dimensional anticommutative algebras [47], and all central extensions of the two-dimensional algebras [48]. Note that the method of central extensions is an important tool in the classification of nilpotent algebras, which was used to describe all four-dimensional nilpotent associative algebras [7], all five-dimensional nilpotent Jordan algebras [10], all five-dimensional nilpotent restricted Lie algebras [5], all five-dimensional nilpotent associative commutative algebras [19], all six-dimensional nilpotent Lie algebras [4,6], all six-dimensional nilpotent Malcev algebras [11], all six-dimensional nilpotent anticommutative algebras [49], all eight-dimensional dual Mock Lie algebras [50], and some others.

1. The algebraic classification of nilpotent Novikov algebras

1.1. Method of classification of nilpotent algebras

Throughout this paper, we use the notations and methods well written in [42,43,48], which we have adapted for the Novikov case with some modifications. Further in this section, we give some important definitions.

Let $(\mathbf{A}, \cdot)$ be a Novikov algebra over $\mathbb{C}$ and $\mathbb{V}$ a vector space over $\mathbb{C}$. The $\mathbb{C}$ -linear space $Z^2(\mathbf{A}, \mathbb{V})$ is defined as the set of all bilinear maps $\theta: \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{V}$ such that

$$\begin{aligned}\theta(xy, z) &= \theta(xz, y), \\ \theta(xy, z) - \theta(x, yz) &= \theta(yx, z) - \theta(y, xz).\end{aligned}$$

These elements will be called *cocycles*. For a linear map f from $\mathbf{A}$ to $\mathbb{V}$, if we define $\delta f: \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{V}$ by $\delta f(x, y) = f(xy)$, then $\delta f \in Z^2(\mathbf{A}, \mathbb{V})$. We define $B^2(\mathbf{A}, \mathbb{V}) = \{\theta = \delta f : f \in \text{Hom}(\mathbf{A}, \mathbb{V})\}$. We define the *second cohomology space* $H^2(\mathbf{A}, \mathbb{V})$ as the quotient space $Z^2(\mathbf{A}, \mathbb{V})/B^2(\mathbf{A}, \mathbb{V})$.

Let $\text{Aut}(\mathbf{A})$ be the automorphism group of $\mathbf{A}$ and let $\phi \in \text{Aut}(\mathbf{A})$. For $\theta \in Z^2(\mathbf{A}, \mathbb{V})$ define the action of the group $\text{Aut}(\mathbf{A})$ on $H^2(\mathbf{A}, \mathbb{V})$ by $\phi\theta(x, y) = \theta(\phi(x), \phi(y))$. It is easy to verify that $B^2(\mathbf{A}, \mathbb{V})$ is invariant under the action of $\text{Aut}(\mathbf{A})$. So, we have an induced action of $\text{Aut}(\mathbf{A})$ on $H^2(\mathbf{A}, \mathbb{V})$.

Let $\mathbf{A}$ be a Novikov algebra of dimension m over $\mathbb{C}$ and $\mathbb{V}$ be a $\mathbb{C}$ -vector space of dimension k . For $\theta \in Z^2(\mathbf{A}, \mathbb{V})$, define on the linear space $\mathbf{A}_\theta = \mathbf{A} \oplus \mathbb{V}$ the bilinear product $[-, -]_{\mathbf{A}_\theta}$ by $[x + x', y + y']_{\mathbf{A}_\theta} = xy + \theta(x, y)$ for all $x, y \in \mathbf{A}, x', y' \in \mathbb{V}$. The algebra $\mathbf{A}_\theta$ is called a *k-dimensional central extension* of $\mathbf{A}$ by $\mathbb{V}$. One can easily check that $\mathbf{A}_\theta$ is a Novikov algebra if and only if $\theta \in Z^2(\mathbf{A}, \mathbb{V})$.

Call the set $\text{Ann}(\theta) = \{x \in \mathbf{A} : \theta(x, \mathbf{A}) + \theta(\mathbf{A}, x) = 0\}$ the *annihilator* of θ . We recall that the *annihilator* of an algebra $\mathbf{A}$ is defined as the ideal $\text{Ann}(\mathbf{A}) = \{x \in \mathbf{A} : x\mathbf{A} + \mathbf{A}x = 0\}$. Observe that $\text{Ann}(\mathbf{A}_\theta) = (\text{Ann}(\theta) \cap \text{Ann}(\mathbf{A})) \oplus \mathbb{V}$.

The following result shows that every algebra with a non-zero annihilator is a central extension of a smaller-dimensional algebra. As a consequence, it is easy to see that a one-generated nilpotent algebra is a central extension of a one-generated nilpotent algebra of a smaller dimension.

Lemma 1.1: *Let $\mathbf{A}$ be an n -dimensional Novikov algebra such that $\dim(\text{Ann}(\mathbf{A})) = m \neq 0$. Then there exists, up to isomorphism, a unique $(n - m)$ -dimensional Novikov algebra $\mathbf{A}'$ and a bilinear map $\theta \in Z^2(\mathbf{A}, \mathbb{V})$ with $\text{Ann}(\mathbf{A}) \cap \text{Ann}(\theta) = 0$, where $\mathbb{V}$ is a vector space of dimension m , such that $\mathbf{A} \cong \mathbf{A}'_\theta$ and $\mathbf{A}/\text{Ann}(\mathbf{A}) \cong \mathbf{A}'$.*

Proof: Let $\mathbf{A}'$ be a linear complement of $\text{Ann}(\mathbf{A})$ in $\mathbf{A}$. Define a linear map $P: \mathbf{A} \longrightarrow \mathbf{A}'$ by $P(x + v) = x$ for $x \in \mathbf{A}'$ and $v \in \text{Ann}(\mathbf{A})$, and define a multiplication on $\mathbf{A}'$ by $[x, y]_{\mathbf{A}'} = P(xy)$ for $x, y \in \mathbf{A}'$. For $x, y \in \mathbf{A}$, we have

$$P(xy) = P((x - P(x) + P(x))(y - P(y) + P(y))) = P(P(x)P(y)) = [P(x), P(y)]_{\mathbf{A}'}.$$

Since P is a homomorphism $P(\mathbf{A}) = \mathbf{A}'$ is a Novikov algebra and $\mathbf{A}/\text{Ann}(\mathbf{A}) \cong \mathbf{A}'$, which gives us the uniqueness. Now, define the map $\theta: \mathbf{A}' \times \mathbf{A}' \longrightarrow \text{Ann}(\mathbf{A})$ by $\theta(x, y) = xy - [x, y]_{\mathbf{A}'}$. Thus, $\mathbf{A}'_\theta$ is $\mathbf{A}$ and therefore $\theta \in Z^2(\mathbf{A}, \mathbb{V})$ and $\text{Ann}(\mathbf{A}) \cap \text{Ann}(\theta) = 0$. ■

Definition 1.2: Let $\mathbf{A}$ be an algebra and I be a subspace of $\text{Ann}(\mathbf{A})$. If $\mathbf{A} = \mathbf{A}_0 \oplus I$ then I is called an *annihilator component* of $\mathbf{A}$.

Definition 1.3: A central extension of an algebra $\mathbf{A}$ without annihilator component is called a *non-split central extension*.

Our task is to find all central extensions of an algebra $\mathbf{A}$ by a space $\mathbb{V}$. In order to solve the isomorphism problem we need to study the action of $\text{Aut}(\mathbf{A})$ on $H^2(\mathbf{A}, \mathbb{V})$. To do that, let us fix a basis $e_1, \dots, e_s$ of $\mathbb{V}$, and $\theta \in Z^2(\mathbf{A}, \mathbb{V})$. Then θ can be uniquely written as $\theta(x, y) = \sum_{i=1}^s \theta_i(x, y)e_i$, where $\theta_i \in Z^2(\mathbf{A}, \mathbb{C})$. Moreover, $\text{Ann}(\theta) = \text{Ann}(\theta_1) \cap \text{Ann}(\theta_2) \cap \dots \cap \text{Ann}(\theta_s)$. Furthermore, $\theta \in B^2(\mathbf{A}, \mathbb{V})$ if and only if all $\theta_i \in B^2(\mathbf{A}, \mathbb{C})$. It is not difficult to prove (see [43, Lemma 13]) that given a Novikov algebra $\mathbf{A}_\theta$, if we write as above $\theta(x, y) = \sum_{i=1}^s \theta_i(x, y)e_i \in Z^2(\mathbf{A}, \mathbb{V})$ and $\text{Ann}(\theta) \cap \text{Ann}(\mathbf{A}) = 0$, then $\mathbf{A}_\theta$ has an annihilator component if and only if $[\theta_1], [\theta_2], \dots, [\theta_s]$ are linearly dependent in $H^2(\mathbf{A}, \mathbb{C})$.

Let $\mathbb{V}$ be a finite-dimensional vector space over $\mathbb{C}$. The *Grassmannian* $G_k(\mathbb{V})$ is the set of all k -dimensional linear subspaces of $\mathbb{V}$. Let $G_s(H^2(\mathbf{A}, \mathbb{C}))$ be the Grassmannian of subspaces of dimension s in $H^2(\mathbf{A}, \mathbb{C})$. There is a natural action of $\text{Aut}(\mathbf{A})$ on $G_s(H^2(\mathbf{A}, \mathbb{C}))$. Let $\phi \in \text{Aut}(\mathbf{A})$. For $W = \langle [\theta_1], [\theta_2], \dots, [\theta_s] \rangle \in G_s(H^2(\mathbf{A}, \mathbb{C}))$ define $\phi W = \langle [\phi\theta_1], [\phi\theta_2], \dots, [\phi\theta_s] \rangle$. We denote the orbit of $W \in G_s(H^2(\mathbf{A}, \mathbb{C}))$ under the action of $\text{Aut}(\mathbf{A})$ by $\text{Orb}(W)$. Given

$$W_1 = \langle [\theta_1], [\theta_2], \dots, [\theta_s] \rangle, W_2 = \langle [\vartheta_1], [\vartheta_2], \dots, [\vartheta_s] \rangle \in G_s(H^2(\mathbf{A}, \mathbb{C})),$$

we easily have that if $W_1 = W_2$, then $\bigcap_{i=1}^s \text{Ann}(\theta_i) \cap \text{Ann}(\mathbf{A}) = \bigcap_{i=1}^s \text{Ann}(\vartheta_i) \cap \text{Ann}(\mathbf{A})$, and therefore we can introduce the set

$$\mathbf{T}_s(\mathbf{A}) = \left\{ W = \langle [\theta_1], [\theta_2], \dots, [\theta_s] \rangle \in G_s(H^2(\mathbf{A}, \mathbb{C})) : \bigcap_{i=1}^s \text{Ann}(\theta_i) \cap \text{Ann}(\mathbf{A}) = 0 \right\},$$

which is stable under the action of $\text{Aut}(\mathbf{A})$.

Now, let $\mathbb{V}$ be an s -dimensional linear space and let us denote by $\mathbf{E}(\mathbf{A}, \mathbb{V})$ the set of all *non-split s -dimensional central extensions* of $\mathbf{A}$ by $\mathbb{V}$. By above, we can write

$$\mathbf{E}(\mathbf{A}, \mathbb{V}) = \left\{ \mathbf{A}_\theta : \theta(x, y) = \sum_{i=1}^s \theta_i(x, y)e_i \quad \text{and} \quad \langle [\theta_1], [\theta_2], \dots, [\theta_s] \rangle \in \mathbf{T}_s(\mathbf{A}) \right\}.$$

We also have the following result, which can be proved as in [43, Lemma 17].

Lemma 1.4: *Let $\mathbf{A}_\theta, \mathbf{A}_\vartheta \in \mathbf{E}(\mathbf{A}, \mathbb{V})$. Suppose that $\theta(x, y) = \sum_{i=1}^s \theta_i(x, y)e_i$ and $\vartheta(x, y) = \sum_{i=1}^s \vartheta_i(x, y)e_i$. Then the Novikov algebras $\mathbf{A}_\theta$ and $\mathbf{A}_\vartheta$ are isomorphic if and only if*

$$\text{Orb} \langle [\theta_1], [\theta_2], \dots, [\theta_s] \rangle = \text{Orb} \langle [\vartheta_1], [\vartheta_2], \dots, [\vartheta_s] \rangle.$$

This shows that there exists a one-to-one correspondence between the set of $\text{Aut}(\mathbf{A})$ -orbits on $\mathbf{T}_s(\mathbf{A})$ and the set of isomorphism classes of $\mathbf{E}(\mathbf{A}, \mathbb{V})$. Consequently we have a procedure that allows us, given a Novikov algebra $\mathbf{A}'$ of dimension $n-s$, to construct all non-split central extensions of $\mathbf{A}'$. This procedure is:

Procedure

- (1) For a given Novikov algebra $\mathbf{A}'$ of dimension $n-s$, determine $H^2(\mathbf{A}', \mathbb{C})$, $\text{Ann}(\mathbf{A}')$ and $\text{Aut}(\mathbf{A}')$.
- (2) Determine the set of $\text{Aut}(\mathbf{A}')$ -orbits on $T_s(\mathbf{A}')$.
- (3) For each orbit, construct the Novikov algebra associated with a representative of it.

1.2. Notations

Let $\mathcal{N}$ be a Novikov algebra with a basis $e_1, e_2, \dots, e_n$. Then by Δ_{ij} we will denote the bilinear form $\Delta_{ij}: \mathcal{N} \times \mathcal{N} \longrightarrow \mathbb{C}$ with $\Delta_{ij}(e_l, e_m) = \delta_{il}\delta_{jm}$. Then the set $\{\Delta_{ij} : 1 \leq i, j \leq n\}$ is a basis for the linear space of the bilinear forms on $\mathcal{N}$. Then every bilinear form θ can be uniquely written as $\theta = \sum_{1 \leq i, j \leq n} c_{ij} \Delta_{ij}$, where $c_{ij} \in \mathbb{C}$. Let us fix the following notations:

$$\mathcal{N}_j^i \quad - \quad j\text{th } i\text{-dimensional one-generated Novikov algebra.}$$

1.3. Low-dimensional one-generated nilpotent Novikov algebras

Thanks to [36] we have the algebraic classification of two-, three- and four-dimensional and five-dimensional with two-dimensional annihilator one-generated nilpotent algebras. Let us give a list of one-, two-, three- and four-dimensional one-generated nilpotent Novikov algebras.

$$\begin{array}{llll} \mathcal{N}_{01}^2 & : & e_1 e_1 = e_2 & \\ \mathcal{N}_{01}^3 & : & e_1 e_1 = e_2 & e_2 e_1 = e_3 \\ \mathcal{N}_{02}^3(\lambda) & : & e_1 e_1 = e_2 & e_1 e_2 = e_3 \quad e_2 e_1 = \lambda e_3 \\ \mathcal{N}_{01}^4 & : & e_1 e_1 = e_2 & e_1 e_2 = e_3 \quad e_2 e_1 = e_4 \\ \mathcal{N}_{02}^4 & : & e_1 e_1 = e_2 & e_2 e_1 = e_3 \quad e_1 e_3 = e_4 \quad e_3 e_1 = -e_4 \\ \mathcal{N}_{03}^4(\lambda) & : & e_1 e_1 = e_2 & e_1 e_2 = e_3 \quad e_1 e_3 = (2 - \lambda)e_4 \quad e_2 e_1 = \lambda e_3 \\ & & e_2 e_2 = \lambda e_4 & e_3 e_1 = \lambda e_4 \\ \mathcal{N}_{04}^4 & : & e_1 e_1 = e_2 & e_1 e_2 = e_3 \quad e_1 e_3 = 2e_4 \quad e_2 e_1 = e_4 \\ \mathcal{N}_{05}^4 & : & e_1 e_1 = e_2 & e_1 e_2 = e_3 \quad e_1 e_3 = e_4 \quad e_2 e_1 = e_3 + e_4 \\ & & e_2 e_2 = e_4 & e_3 e_1 = e_4 \end{array}$$

Remark 1.5: Note that a non-split central extension of a split algebra can not be a one-generated algebra. Hence, we will consider central extensions only of non-split one-generated nilpotent algebras.

2. Classification of five-dimensional one-generated nilpotent Novikov algebras

All the necessary information about coboundaries, cocycles and second cohomology spaces of five-dimensional one-generated nilpotent Novikov algebras is given in Table A1 (see Table A1 from Appendix).

From the results of [36], it is not difficult to see that there are two five-dimensional one-generated Novikov algebras with two-dimensional annihilator:

$$\begin{array}{llll} \mathcal{N}_{01}^5 & : & e_1 e_1 = e_2 & e_1 e_2 = e_4 \quad e_1 e_3 = e_5 \quad e_2 e_1 = e_3 \quad e_3 e_1 = -e_5 \\ \mathcal{N}_{02}^5(\lambda) & : & e_1 e_1 = e_2 & e_1 e_2 = e_3 \quad e_1 e_3 = (2 - \lambda)e_5 \quad e_2 e_1 = \lambda e_3 + e_4 \\ & & e_2 e_2 = \lambda e_5 & e_3 e_1 = \lambda e_5 \end{array}$$

2.1. Central extensions of $\mathcal{N}_{01}^4$

Let us use the following notations

$$\nabla_1 = [\Delta_{13}], \quad \nabla_2 = [\Delta_{14}] - [\Delta_{41}], \quad \nabla_3 = [\Delta_{22}] + [\Delta_{31}] + 2[\Delta_{41}].$$

The automorphism group of $\mathcal{N}_{01}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & xy & x^3 & 0 \\ v & xy & 0 & x^3 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_1 & \alpha_2 \\ 0 & \alpha_3 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 \\ 2\alpha_3 - \alpha_2 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha_1^* & \alpha_2^* \\ \alpha^{***} & \alpha_3^* & 0 & 0 \\ \alpha_3^* & 0 & 0 & 0 \\ 2\alpha_3^* - \alpha_2^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{01}^4)$ on the subspace $\langle \sum_{i=1}^3 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^4 \alpha_1, \quad \alpha_2^* = x^4 \alpha_2, \quad \alpha_3^* = x^4 \alpha_3.$$

2.1.1. One-dimensional central extensions of $\mathcal{N}_{01}^4$

Since we are interested only in new algebras, we have the following cases:

- (1) if $\alpha_3 \neq 0$, then we have the representative $\langle \lambda \nabla_1 + \mu \nabla_2 + \nabla_3 \rangle$;
- (2) if $\alpha_3 = 0$, then $(\alpha_1, \alpha_2) \neq (0, 0)$ and we have the representative $\langle \nabla_1 + \lambda \nabla_2 \rangle$ with $\lambda \neq 0$.

Hence, we have the following new five-dimensional algebras:

$$\begin{array}{ll} \mathcal{N}_{03}^5(\lambda, \mu) & : \quad e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = \lambda e_5 \quad e_1 e_4 = \mu e_5 \\ & \quad e_2 e_1 = e_4 \quad e_2 e_2 = e_5 \quad e_3 e_1 = e_5 \quad e_4 e_1 = (2 - \mu) e_5 \\ \mathcal{N}_{04}^5(\lambda)_{\lambda \neq 0} & : \quad e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = e_5 \quad e_1 e_4 = \lambda e_5 \\ & \quad e_2 e_1 = e_4 \quad e_4 e_1 = -\lambda e_5 \end{array}$$

2.1.2. Two-dimensional central extensions of $\mathcal{N}_{01}^4$

Consider the vector space generated by the following two cocycles

$$\begin{aligned} \theta_1 &= \alpha_1 \nabla_1 + \alpha_2 \nabla_2 + \alpha_3 \nabla_3, \\ \theta_2 &= \beta_1 \nabla_1 + \beta_2 \nabla_2. \end{aligned}$$

If $\alpha_3 = 0$, then we have the representative $\langle \nabla_1, \nabla_2 \rangle$. If $\alpha_3 \neq 0$, then

- (1) $\beta_2 = 0$, then we have the family of representatives $\langle \nabla_1, \lambda \nabla_2 + \nabla_3 \rangle$;
- (2) $\beta_2 \neq 0$, then we have the family of representatives $\langle \lambda \nabla_1 + \nabla_2, \mu \nabla_1 + \nabla_3 \rangle$.

Hence, we have the following 6-dimensional algebras:

$$\begin{array}{llll}
 \mathcal{N}_{01}^6 & : & e_1 e_1 = e_2 & e_1 e_2 = e_3 & e_1 e_3 = e_5 & e_1 e_4 = e_6 \\
 & & e_2 e_1 = e_4 & e_4 e_1 = -e_6 & & \\
 \mathcal{N}_{02}^6(\lambda) & : & e_1 e_1 = e_2 & e_1 e_2 = e_3 & e_1 e_3 = e_5 & e_1 e_4 = \lambda e_6 \\
 & & e_2 e_1 = e_4 & e_2 e_2 = e_6 & e_3 e_1 = e_6 & e_4 e_1 = (2 - \lambda) e_6 \\
 \mathcal{N}_{03}^6(\lambda, \mu) & : & e_1 e_1 = e_2 & e_1 e_2 = e_3 & e_1 e_3 = \lambda e_5 + \mu e_6 & e_1 e_4 = e_5 \\
 & & e_2 e_1 = e_4 & e_2 e_2 = e_6 & e_3 e_1 = e_6 & e_4 e_1 = -e_5 + 2e_6
 \end{array}$$

2.2. Central extensions of $\mathcal{N}_{02}^4$

Let us use the following notations

$$\nabla_1 = [\Delta_{12}], \quad \nabla_2 = 2[\Delta_{14}] - [\Delta_{23}] - [\Delta_{41}].$$

The automorphism group of $\mathcal{N}_{02}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 \\ y & 0 & x^3 & 0 \\ z & 0 & 0 & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & 0 & 2\alpha_2 \\ 0 & 0 & -\alpha_2 & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha_2 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha_1^* & 0 & 2\alpha_2^* \\ \alpha^{**} & 0 & -\alpha_2^* & 0 \\ 0 & 0 & 0 & 0 \\ -\alpha_2^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{02}^4)$ on the subspace $\langle \sum_{i=1}^2 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^2 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^3 \alpha_1, \quad \alpha_2^* = x^5 \alpha_2.$$

2.2.1. One-dimensional central extensions of $\mathcal{N}_{02}^4$

Since we are interested only in new algebras, we have assume $\alpha_2 \neq 0$, and we have the following cases:

- (1) if $\alpha_1 = 0$, then we have the representative $\langle \nabla_2 \rangle$;
- (2) if $\alpha_1 \neq 0$, then by choosing $x = \sqrt{\alpha_1 / \alpha_2}$, we have the representative $\langle \nabla_1 + \nabla_2 \rangle$.

Thus, we have the following five-dimensional algebras:

$$\begin{array}{ll}
 \mathcal{N}_{05}^5 : & e_1 e_1 = e_2 \quad e_1 e_3 = e_4 \quad e_1 e_4 = 2e_5 \quad e_2 e_1 = e_3 \quad e_2 e_3 = -e_5 \quad e_3 e_1 = -e_4 \\
 & e_4 e_1 = -e_5 \\
 \mathcal{N}_{06}^5 : & e_1 e_1 = e_2 \quad e_1 e_2 = e_5 \quad e_1 e_3 = e_4 \quad e_1 e_4 = 2e_5 \quad e_2 e_1 = e_3 \quad e_2 e_3 = -e_5 \\
 & e_3 e_1 = -e_4 \quad e_4 e_1 = -e_5
 \end{array}$$

2.2.2. Two-dimensional central extensions of $\mathcal{N}_{02}^4$

Since $\dim(H^2(\mathcal{N}_{02}^4)) = 2$, there is only one two-dimensional central extension of $\mathcal{N}_{02}^4$. Thus, we have the following six-dimensional algebra:

$$\begin{aligned} \mathcal{N}_{04}^6 : e_1 e_1 &= e_2 & e_1 e_2 &= e_5 & e_1 e_3 &= e_4 & e_1 e_4 &= 2e_6 & e_2 e_1 &= e_3 & e_2 e_3 &= -e_6 \\ e_3 e_1 &= -e_4 & e_4 e_1 &= -e_6 \end{aligned}$$

2.3. Central extensions of $\mathcal{N}_{03}^4(\lambda)_{\lambda \notin \{0,1\}}$

Let us use the following notations

$$\nabla_1 = [\Delta_{21}], \quad \nabla_2 = (3 - 2\lambda)[\Delta_{14}] + \lambda(2 - \lambda)[\Delta_{23}] + \lambda[\Delta_{32}] + \lambda[\Delta_{41}].$$

The automorphism group of $\mathcal{N}_{03}^4(\lambda)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 \\ y & 0 & x^3 & 0 \\ z & 2xy & 0 & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & (3 - 2\lambda)\alpha_2 \\ \alpha_1 & 0 & \lambda(2 - \lambda)\alpha_2 & 0 \\ 0 & \lambda\alpha_2 & 0 & 0 \\ \lambda\alpha_2 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^{**} & \alpha^* & 0 & (3 - 2\lambda)\alpha_2^* \\ \alpha_1^* + \lambda\alpha^* & 0 & \lambda(2 - \lambda)\alpha_2^* & 0 \\ 0 & \lambda\alpha_2^* & 0 & 0 \\ \lambda\alpha_2^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{03}^4(\lambda))$ on the subspace $\langle \sum_{i=1}^2 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^2 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^2(x\alpha_1 + 2\lambda(\lambda - 1)y\alpha_2), \quad \alpha_2^* = x^5\alpha_2.$$

2.3.1. One-dimensional central extensions of $\mathcal{N}_{03}^4(\lambda)_{\lambda \notin \{0,1\}}$

Since we are interested only in new algebras, we have $\alpha_2 \neq 0$. Then choosing $y = x\alpha_1/2\lambda(1 - \lambda)\alpha_2$, we have the representative $\langle \nabla_2 \rangle$. Hence, we have the following five-dimensional algebras:

$$\begin{aligned} \mathcal{N}_{07}^5(\lambda)_{\lambda \notin \{0,1\}} : \quad & e_1 e_1 = e_2 & e_1 e_2 &= e_3 & e_1 e_3 &= (2 - \lambda)e_4 \\ & e_1 e_4 = (3 - 2\lambda)e_5 & e_2 e_1 &= \lambda e_3 & & \\ & e_2 e_2 = \lambda e_4 & e_2 e_3 &= \lambda(2 - \lambda)e_5 & e_3 e_1 &= \lambda e_4 \\ & e_3 e_2 = \lambda e_5 & e_4 e_1 &= \lambda e_5 \end{aligned}$$

2.3.2. Two-dimensional central extensions of $\mathcal{N}_{03}^4(\lambda)_{\lambda \notin \{0,1\}}$

Since $\dim(H^2(\mathcal{N}_{03}^4(\lambda)_{\lambda \notin \{0,1\}})) = 2$, there is only one two-dimensional central extension of $\mathcal{N}_{03}^4(\lambda)_{\lambda \notin \{0,1\}}$. Hence, we have the following six-dimensional algebras:

$$\begin{aligned} \mathcal{N}_{05}^6(\lambda)_{\lambda \notin \{0,1\}} : \quad & e_1 e_1 = e_2 & e_1 e_2 &= e_3 & e_1 e_3 &= (2 - \lambda)e_4 \\ & e_1 e_4 = (3 - 2\lambda)e_5 & e_2 e_1 &= \lambda e_3 + e_6 & & \\ & e_2 e_2 = \lambda e_4 & e_2 e_3 &= \lambda(2 - \lambda)e_5 & e_3 e_1 &= \lambda e_4 \\ & e_3 e_2 = \lambda e_5 & e_4 e_1 &= \lambda e_5 \end{aligned}$$

2.4. Central extensions of $\mathcal{N}_{03}^4(0)$

Let us use the following notations

$$\nabla_1 = [\Delta_{14}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = 2[\Delta_{23}] - 2[\Delta_{32}] + [\Delta_{41}].$$

The automorphism group of $\mathcal{N}_{03}^4(0)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & xy & x^3 & 0 \\ t & 2xz & 2x^2y & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & \alpha_1 \\ \alpha_2 & 0 & 2\alpha_3 & 0 \\ 0 & -2\alpha_3 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^* & \alpha^{**} & \alpha^{***} & \alpha_1^* \\ \alpha_2^* & 0 & 2\alpha_3^* & 0 \\ 0 & -2\alpha_3^* & 0 & 0 \\ \alpha_3^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{03}^4(0))$ on the subspace $\langle \sum_{i=1}^3 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^5 \alpha_1, \quad \alpha_2^* = x^3 \alpha_2 + (4x^2 z - 2xy^2) \alpha_3, \quad \alpha_3^* = x^5 \alpha_3.$$

2.4.1. One-dimensional central extensions of $\mathcal{N}_{03}^4(0)$

Since we are interested only in new algebras, we have $(\alpha_1, \alpha_3) \neq (0, 0)$ and:

- (1) if $\alpha_3 = 0, \alpha_2 = 0$, then we have the representative $\langle \nabla_1 \rangle$;
- (2) if $\alpha_3 = 0, \alpha_2 \neq 0$, then choosing $x = \sqrt{\alpha_2/\alpha_1}$, we have the representative $\langle \nabla_1 + \nabla_2 \rangle$;
- (3) if $\alpha_3 \neq 0$, then choosing $y = 0, z = -\alpha_2/4\alpha_3 \sqrt[5]{\alpha_3}$, we have the family of representatives $\langle \lambda \nabla_1 + \nabla_3 \rangle$.

Hence, we have the following 5-dimensional algebras:

$$\begin{array}{lll} \mathcal{N}_{07}^5(0) & : & e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = 2e_4 \quad e_1 e_4 = 3e_5 \\ \mathcal{N}_{08}^5 & : & e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = e_4 \quad e_1 e_4 = e_5 \quad e_2 e_1 = e_5 \\ \mathcal{N}_{09}^5(\lambda) & : & e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = 2e_4 \quad e_1 e_4 = \lambda e_5 \quad e_2 e_3 = 2e_5 \\ & & e_3 e_2 = -2e_5 \quad e_4 e_1 = e_5 \end{array}$$

2.4.2. Two-dimensional central extensions of $\mathcal{N}_{03}^4(0)$

Consider the vector space generated by the following two cocycles

$$\theta_1 = \alpha_1 \nabla_1 + \alpha_2 \nabla_2 + \alpha_3 \nabla_3,$$

$$\theta_2 = \beta_1 \nabla_1 + \beta_2 \nabla_2.$$

If $\alpha_3 = 0$, then we have the representative $\langle \nabla_1, \nabla_2 \rangle$. If $\alpha_3 \neq 0$, then

- (1) if $\beta_1 = 0, \beta_2 \neq 0$, then we have the family of representatives $\langle \lambda \nabla_1 + \nabla_3, \nabla_2 \rangle$;

- (2) if $\beta_1 \neq 0, \beta_2 \neq 0$, then choosing $x = \sqrt{\beta_2/\beta_1}, y = 0, z = (x^3\alpha_1 - x\alpha_2)/4\alpha_3$, we have the representative $\langle \nabla_1 + \nabla_2, \nabla_3 \rangle$;
- (3) if $\beta_1 \neq 0, \beta_2 = 0$, then we have the representative $\langle \nabla_1, \nabla_3 \rangle$.

Hence we have the following six-dimensional algebras:

$$\begin{aligned}
 \mathcal{N}_{05}^6(0) &: e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = 2e_4 & e_1e_4 = 3e_5 & e_2e_1 = e_6 \\
 \mathcal{N}_{06}^6(\lambda) &: e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = 2e_4 & e_1e_4 = \lambda e_5 & e_2e_1 = e_6 \\
 & & e_2e_3 = 2e_5 & e_3e_2 = -2e_5 & e_4e_1 = e_5 & \\
 \mathcal{N}_{07}^6 &: e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = 2e_4 & e_1e_4 = e_5 & \\
 & & e_2e_1 = e_5 & e_2e_3 = 2e_6 & e_3e_2 = -2e_6 & e_4e_1 = e_6 \\
 \mathcal{N}_{08}^6 &: e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = 2e_4 & e_1e_4 = e_5 & \\
 & & e_2e_3 = 2e_6 & e_3e_2 = -2e_6 & e_4e_1 = e_6 &
 \end{aligned}$$

2.5. Central extensions of $\mathcal{N}_{03}^4(1)$

Let us use the following notations

$$\nabla_1 = [\Delta_{21}], \quad \nabla_2 = [\Delta_{14}] + [\Delta_{23}] + [\Delta_{32}] + [\Delta_{41}].$$

The automorphism group of $\mathcal{N}_{03}^4(1)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 \\ y & x^2 & 0 & 0 \\ z & 2xy & x^3 & 0 \\ t & 2xz + y^2 & 3x^2y & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & 0 & \alpha_2 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^{***} & \alpha^{**} & \alpha^* & \alpha_2^* \\ \alpha_1^* + \alpha^{**} & \alpha^* & \alpha_2^* & 0 \\ \alpha^* & \alpha_2^* & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{03}^4(1))$ on the subspace $\langle \sum_{i=1}^2 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^2 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^3\alpha_1, \quad \alpha_2^* = x^5\alpha_2.$$

2.5.1. One-dimensional central extensions of $\mathcal{N}_{03}^4(1)$

We may assume $\alpha_2 \neq 0$, so we have the following cases:

- (1) if $\alpha_1 = 0$, then we have the representative $\langle \nabla_2 \rangle$;
- (2) if $\alpha_1 \neq 0$, then choosing $x = \sqrt{\alpha_1/\alpha_2}$ we have the representative $\langle \nabla_1 + \nabla_2 \rangle$.

Hence, we have the following five-dimensional algebras:

$$\begin{aligned}
 \mathcal{N}_{07}^5(1) &: e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = e_4 & e_1e_4 = e_5 & e_2e_1 = e_3 \\
 & & e_2e_2 = e_4 & e_2e_3 = e_5 & e_3e_1 = e_4 & e_3e_2 = e_5 & e_4e_1 = e_5 \\
 \mathcal{N}_{10}^5 &: e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = e_4 & e_1e_4 = e_5 & e_2e_1 = e_3 + e_5 \\
 & & e_2e_2 = e_4 & e_2e_3 = e_5 & e_3e_1 = e_4 & e_3e_2 = e_5 & e_4e_1 = e_5
 \end{aligned}$$

2.5.2. Two-dimensional central extensions of $\mathcal{N}_{03}^4(1)$

Since $\dim(H^2(\mathcal{N}_{03}^4(1))) = 2$, there is only one two-dimensional central extension of $\mathcal{N}_{03}^4(1)$. Hence, we have the following six-dimensional algebra:

$$\begin{aligned} \mathcal{N}_{05}^6(1) : \quad & e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = e_4 & e_1e_4 = e_5 & e_2e_1 = e_3 + e_6 \\ & e_2e_2 = e_4 & e_2e_3 = e_5 & e_3e_1 = e_4 & e_3e_2 = e_5 & e_4e_1 = e_5 \end{aligned}$$

2.6. Central extensions of $\mathcal{N}_{04}^4$

Let us use the following notations

$$\nabla_1 = [\Delta_{13}], \quad \nabla_2 = 2[\Delta_{14}] + [\Delta_{22}] + [\Delta_{31}], \quad \nabla_3 = [\Delta_{14}] - 2[\Delta_{23}] + 2[\Delta_{32}] - [\Delta_{41}].$$

The automorphism group of $\mathcal{N}_{04}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & x & 1 & 0 \\ z & x + 2y & 2x & 1 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_1 & 2\alpha_2 + \alpha_3 \\ 0 & \alpha_2 & -2\alpha_3 & 0 \\ \alpha_2 & 2\alpha_3 & 0 & 0 \\ -\alpha_3 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^{***} & \alpha^{**} & 2\alpha^* + \alpha_1^* & 2\alpha_2^* + \alpha_3^* \\ \alpha^* & \alpha_2^* & -2\alpha_3^* & 0 \\ \alpha_2^* & 2\alpha_3^* & 0 & 0 \\ -\alpha_3^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{04}^4)$ on the subspace $\langle \sum_{i=1}^3 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = \alpha_1 + 2(x - 2x^2 + 4y)\alpha_3, \quad \alpha_2^* = \alpha_2, \quad \alpha_3^* = \alpha_3.$$

2.6.1. One-dimensional central extensions of $\mathcal{N}_{04}^4$

Since we are interested only in new algebras, we assume $(\alpha_2, \alpha_3) \neq (0, 0)$ and we have the following cases:

- (1) if $\alpha_2 \neq 0, \alpha_3 = 0$, then we have the family of representatives $\langle \lambda \nabla_1 + \nabla_2 \rangle$;
- (2) if $\alpha_3 \neq 0$, then by choosing $y = -(\alpha_1 + (2x - 4x^2)\alpha_3)/8\alpha_3$, we have the family of representatives $\langle \lambda \nabla_2 + \nabla_3 \rangle$.

Hence, we have the following five-dimensional algebras:

$$\begin{aligned} \mathcal{N}_{11}^5(\lambda) : \quad & e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = 2e_4 + \lambda e_5 & e_1e_4 = 2e_5 \\ & e_2e_1 = e_4 & e_2e_2 = e_5 & e_3e_1 = e_5 & \\ \mathcal{N}_{12}^5(\lambda) : \quad & e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = 2e_4 & \\ & e_1e_4 = (2\lambda + 1)e_5 & e_2e_1 = e_4 & & \\ & e_2e_2 = \lambda e_5 & e_2e_3 = -2e_5 & e_3e_1 = \lambda e_5 & \\ & e_3e_2 = 2e_5 & e_4e_1 = -e_5 & & \end{aligned}$$

2.6.2. Two-dimensional central extensions of $\mathcal{N}_{04}^4$

Consider the vector space generated by the following two cocycles

$$\theta_1 = \alpha_1 \nabla_1 + \alpha_2 \nabla_2 + \alpha_3 \nabla_3,$$

$$\theta_2 = \beta_1 \nabla_1 + \beta_2 \nabla_2.$$

If $\alpha_3 = 0$, then we have the representative $\langle \nabla_1, \nabla_2 \rangle$. If $\alpha_3 \neq 0$, then

- (1) if $\beta_2 = 0$, then we have the family of representatives $\langle \nabla_1, \lambda \nabla_2 + \nabla_3 \rangle$;
- (2) if $\beta_2 \neq 0$, then we have the family of representatives $\langle \lambda \nabla_1 + \nabla_2, \nabla_3 \rangle$.

Hence, we have the following six-dimensional algebras:

$$\begin{aligned} \mathcal{N}_{09}^6 &: \quad e_1 e_1 = e_2 & e_1 e_2 = e_3 & e_1 e_3 = 2e_4 + e_6 & e_1 e_4 = 2e_5 \\ & \quad e_2 e_1 = e_4 & e_2 e_2 = e_5 & e_3 e_1 = e_5 & \\ \mathcal{N}_{10}^6(\lambda) &: \quad e_1 e_1 = e_2 & e_1 e_2 = e_3 & e_1 e_3 = 2e_4 + e_6 & e_1 e_4 = (2\lambda + 1)e_5 \\ & \quad e_2 e_1 = e_4 & & & \\ & \quad e_2 e_2 = \lambda e_5 & e_2 e_3 = -2e_5 & e_3 e_1 = \lambda e_5 & e_3 e_2 = 2e_5 \\ & \quad e_4 e_1 = -e_5 & & & \\ \mathcal{N}_{11}^6(\lambda) &: \quad e_1 e_1 = e_2 & e_1 e_2 = e_3 & e_1 e_3 = 2e_4 + \lambda e_6 & e_1 e_4 = e_5 + 2e_6 \\ & \quad e_2 e_1 = e_4 & & & \\ & \quad e_2 e_2 = e_6 & e_2 e_3 = -2e_5 & e_3 e_1 = e_6 & e_3 e_2 = 2e_5 \\ & \quad e_4 e_1 = -e_5 & & & \end{aligned}$$

2.7. Central extensions of $\mathcal{N}_{05}^4$

Let us use the following notations

$$\nabla_1 = [\Delta_{21}], \quad \nabla_2 = -2[\Delta_{13}] + [\Delta_{14}] + [\Delta_{23}] + [\Delta_{32}] + [\Delta_{41}].$$

The automorphism group of $\mathcal{N}_{05}^4$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ y & 2x & 1 & 0 \\ z & x^2 + x + 2y & 3x & 1 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & -2\alpha_2 & \alpha_2 \\ \alpha_1 & 0 & \alpha_2 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^{***} & \alpha^{**} & -2\alpha_2^* + \alpha^* & \alpha_2^* \\ \alpha_1^* + \alpha^* + \alpha^{**} & \alpha^* & \alpha_2^* & 0 \\ \alpha^* & \alpha_2^* & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{05}^4)$ on the subspace $\langle \sum_{i=1}^2 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^2 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = \alpha_1, \quad \alpha_2^* = \alpha_2.$$

2.7.1. One-dimensional central extensions of $\mathcal{N}_{05}^4$

Assuming $\alpha_2 \neq 0$, we have the family of representatives $\langle \lambda \nabla_1 + \nabla_2 \rangle$.

Hence, we have the following five-dimensional algebras:

$$\mathcal{N}_{13}^5(\lambda) : \quad \begin{array}{llll} e_1 e_1 = e_2 & e_1 e_2 = e_3 & e_1 e_3 = e_4 - 2e_5 & e_1 e_4 = e_5 \\ e_2 e_1 = e_3 + e_4 + \lambda e_5 & e_2 e_2 = e_4 & e_2 e_3 = e_5 & e_3 e_1 = e_4 \\ e_3 e_2 = e_5 & e_4 e_1 = e_5 \end{array}$$

2.7.2. Two-dimensional central extensions of $\mathcal{N}_{05}^4$

Since $\dim(H^2(\mathcal{N}_{05}^4)) = 2$, there is only one two-dimensional central extension of $\mathcal{N}_{05}^4$.

Hence, we have the following six-dimensional algebra:

$$\mathcal{N}_{12}^6 : \quad \begin{array}{llllll} e_1 e_1 = e_2 & e_1 e_2 = e_3 & e_1 e_3 = e_4 - 2e_5 & e_1 e_4 = e_5 & e_2 e_1 = e_3 + e_4 + e_6 \\ e_2 e_2 = e_4 & e_2 e_3 = e_5 & e_3 e_1 = e_4 & e_3 e_2 = e_5 & e_4 e_1 = e_5 \end{array}$$

Summarizing the results of the present sections and using the information from Table A1 from Appendix, we have the following theorem.

Theorem 2.1: *Let $\mathcal{N}$ be a five-dimensional complex one-generated nilpotent Novikov algebra. Then $\mathcal{N}$ is isomorphic to one of the algebras $\mathcal{N}_j^5$, $j = 01, \dots, 13$ listed in Table A2 presented in Appendix.*

3. Classification of six-dimensional one-generated nilpotent Novikov algebras

The multiplication tables of five-dimensional one-generated nilpotent Novikov algebras are given in Table A2 presented in Appendix. All the necessary information about coboundaries, cocycles and second cohomology spaces of five-dimensional one-generated nilpotent Novikov algebras is given in Table A3 presented in Appendix.

3.1. Central extensions of $\mathcal{N}_{01}^5$

Let us use the following notations

$$\nabla_1 = [\Delta_{14}], \quad \nabla_2 = 2[\Delta_{13}] + [\Delta_{22}] + [\Delta_{41}], \quad \nabla_3 = 2[\Delta_{15}] - [\Delta_{23}] - [\Delta_{51}].$$

The automorphism group of $\mathcal{N}_{01}^5$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ y & x^2 & 0 & 0 & 0 \\ z & xy & x^3 & 0 & 0 \\ v & xy & 0 & x^3 & 0 \\ w & 0 & -x^2y & x^2y & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 2\alpha_2 & \alpha_1 & 2\alpha_3 \\ 0 & \alpha_2 & -\alpha_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \\ -\alpha_3 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^{****} & \alpha^{***} & 2\alpha_2^* + \alpha^* & \alpha_1^* & 2\alpha_3^* \\ \alpha^{**} & \alpha_2^* & -\alpha_3^* & 0 & 0 \\ -\alpha^* & 0 & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \\ -\alpha_3^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{01}^5)$ on the subspace $\langle \sum_{i=1}^3 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^4 \alpha_1 + 2x^3 y \alpha_3, \quad \alpha_2^* = x^4 \alpha_2 - x^3 y \alpha_3, \quad \alpha_3^* = x^5 \alpha_3.$$

We assume $\alpha_3 \neq 0$, since otherwise we obtain algebras with two-dimensional annihilator. Then we have the following cases:

- (1) if $\alpha_1 = -2\alpha_2$, then choosing $y = \alpha_2 x / \alpha_3$, we have the representative $\langle \nabla_3 \rangle$, which gives an algebra with two-dimensional annihilator;
- (2) if $\alpha_1 \neq -2\alpha_2$, then choosing $y = \alpha_2 x / \alpha_3$, $x = (\alpha_1 + 2\alpha_2) / \alpha_3$, we have the representative $\langle \nabla_1 + \nabla_3 \rangle$.

Hence, we have only one new algebra:

$$\begin{array}{llllll} \mathcal{N}_{13}^6 & : & e_1 e_1 = e_2 & e_1 e_2 = e_4 & e_1 e_3 = e_5 & e_1 e_4 = e_6 & e_1 e_5 = 2e_6 \\ & & e_2 e_1 = e_3 & e_2 e_3 = -e_6 & e_3 e_1 = -e_5 & e_5 e_1 = -e_6 & \end{array}$$

3.2. Central extensions of $\mathcal{N}_{02}^5(\lambda)_{\lambda \neq 0}$

Let us use the following notations

$$\begin{aligned} \nabla_1 &= [\Delta_{14}] - [\Delta_{41}], & \nabla_2 &= [\Delta_{13}] - \lambda[\Delta_{41}], \\ \nabla_3 &= (3 - 2\lambda)[\Delta_{15}] + \lambda(2 - \lambda)[\Delta_{23}] + \lambda[\Delta_{32}] + \lambda[\Delta_{51}]. \end{aligned}$$

The automorphism group of $\mathcal{N}_{02}^5(\lambda)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ y & x^2 & 0 & 0 & 0 \\ z & (\lambda + 1)xy & x^3 & 0 & 0 \\ v & xy & 0 & x^3 & 0 \\ w & \lambda y^2 + 2xz & (-\lambda^2 + 2\lambda + 2)x^2 y & \lambda^2(\lambda - 1)x^2 y & x^4 \end{pmatrix}.$$

Since

$$\begin{aligned} \phi^T &= \begin{pmatrix} 0 & 0 & \alpha_2 & \alpha_1 & (3 - 2\lambda)\alpha_3 \\ 0 & 0 & \lambda(2 - \lambda)\alpha_3 & 0 & 0 \\ 0 & \lambda\alpha_3 & 0 & 0 & 0 \\ -\alpha_1 - \lambda\alpha_2 & 0 & 0 & 0 & 0 \\ \lambda\alpha_3 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \phi &= \begin{pmatrix} \alpha^{****} & \alpha^{***} & \alpha_2^* + (2 - \lambda)\alpha^* & \alpha_1^* & (3 - 2\lambda)\alpha_3^* \\ \alpha^{**} & \lambda\alpha^* & \lambda(2 - \lambda)\alpha_3^* & 0 & 0 \\ \lambda\alpha^* & \lambda\alpha_3^* & 0 & 0 & 0 \\ -\alpha_1^* - \lambda\alpha_2^* & 0 & 0 & 0 & 0 \\ \lambda\alpha_3^* & 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

we have that the action of $\text{Aut}(\mathcal{N}_{02}^5(\lambda)_{\lambda \neq 0})$ on the subspace $\langle \sum_{i=1}^3 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^4 \alpha_1 + x^3 y \lambda^2 (\lambda - 1) (3 - 2\lambda) \alpha_3,$$

$$\alpha_2^* = x^4\alpha_2 + x^3y\lambda(\lambda - 1)(\lambda - 3)\alpha_3, \alpha_3^* = x^5\alpha_3.$$

We assume $\alpha_3 \neq 0$ and $(\alpha_1, \alpha_2) \neq (0, 0)$. Then we have the following cases

- (1) $\lambda = 1$. Then
 - (a) $\alpha_2 \neq 0$, then choosing $x = \alpha_2/\alpha_3$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 + \nabla_3 \rangle$;
 - (b) $\alpha_2 = 0$, then choosing $x = \alpha_1/\alpha_3$, we have the representative $\langle \nabla_1 + \nabla_3 \rangle$.
- (2) $\lambda = 3$. If $\alpha_2 \neq 0$, then choosing $x = \alpha_2/\alpha_3$, $y = x\alpha_1/54\alpha_3$ we have the representative $\langle \nabla_2 + \nabla_3 \rangle$. If $\alpha_2 = 0$, then we have an algebra with two-dimensional annihilator.
- (3) $\lambda \notin \{1, 3\}$. Then in the case of $\alpha_1 \neq \alpha_2\lambda(3 - 2\lambda)/(\lambda - 3)$, choosing $x = (\alpha_1(\lambda - 3) - \alpha_2\lambda(3 - 2\lambda))/\alpha_3(\lambda - 3)$, $y = -x\alpha_2/\alpha_3\lambda(\lambda - 1)(\lambda - 3)$, we have the representative $\langle \nabla_1 + \nabla_3 \rangle$. If $\alpha_1 = \alpha_2\lambda(3 - 2\lambda)/(\lambda - 3)$, then we have an algebra with two-dimensional annihilator.

Thus, we have three representatives of distinct orbits $\langle \alpha\nabla_1 + \nabla_2 + \nabla_3 \rangle_{\lambda=1}$, $\langle \nabla_2 + \nabla_3 \rangle_{\lambda=3}$, $\langle \nabla_1 + \nabla_3 \rangle_{(\lambda \notin \{0,3\})}$, which give the following algebras:

$$\begin{array}{ll}
 \mathcal{N}_{14}^6(\alpha) & : \quad \begin{array}{llll} e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = e_5 + e_6 & e_1e_4 = \alpha e_6 \\ e_1e_5 = e_6 & e_2e_1 = e_3 + e_4 & e_2e_2 = e_5 & e_2e_3 = e_6 \\ e_3e_1 = e_5 & e_3e_2 = e_6 & e_4e_1 = -(\alpha + 1)e_6 & e_5e_1 = e_6 \end{array} \\
 \mathcal{N}_{15}^6 & : \quad \begin{array}{llll} e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = -e_5 + e_6 & e_1e_5 = -3e_6 \\ e_2e_1 = 3e_3 + e_4 & e_2e_2 = 3e_5 & e_2e_3 = -3e_6 & e_3e_1 = 3e_5 \\ e_3e_2 = 3e_6 & e_4e_1 = -3e_6 & e_5e_1 = 3e_6 & \end{array} \\
 \mathcal{N}_{16}^6(\lambda)_{\lambda \notin \{0,3\}} & : \quad \begin{array}{llll} e_1e_1 = e_2 & e_1e_2 = e_3 & e_1e_3 = (2 - \lambda)e_5 & e_1e_4 = e_6 \\ e_1e_5 = (3 - 2\lambda)e_6 & e_2e_1 = \lambda e_3 + e_4 & e_2e_2 = \lambda e_5 & \\ e_2e_3 = \lambda(2 - \lambda)e_6 & & & \\ e_3e_1 = \lambda e_5 & e_3e_2 = \lambda e_6 & e_4e_1 = -e_6 & e_5e_1 = \lambda e_6 \end{array}
 \end{array}$$

3.3. Central extensions of $\mathcal{N}_{02}^5(0)$

Let us use the following notations

$$\begin{aligned}
 \nabla_1 &= [\Delta_{14}] - [\Delta_{41}], \quad \nabla_2 = 2[\Delta_{14}] + [\Delta_{22}] + [\Delta_{31}], \\
 \nabla_3 &= [\Delta_{15}], \quad \nabla_4 = 2[\Delta_{23}] - 2[\Delta_{32}] + [\Delta_{51}].
 \end{aligned}$$

The automorphism group of $\mathcal{N}_{02}^5(0)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ y & x^2 & 0 & 0 & 0 \\ z & xy & x^3 & 0 & 0 \\ v & xy & 0 & x^3 & 0 \\ w & 2xz & 2x^2y & 0 & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & \alpha_1 + 2\alpha_2 & \alpha_3 \\ 0 & \alpha_2 & 2\alpha_4 & 0 & 0 \\ \alpha_2 & -2\alpha_4 & 0 & 0 & 0 \\ -\alpha_1 & 0 & 0 & 0 & 0 \\ \alpha_4 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^{****} & \alpha^{***} & \alpha^{**} & \alpha_1^* + 2\alpha_2^* & \alpha_3^* \\ \alpha^* & \alpha_2^* & 2\alpha_4^* & 0 & 0 \\ \alpha_2^* & -2\alpha_4^* & 0 & 0 & 0 \\ -\alpha_1^* & 0 & 0 & 0 & 0 \\ \alpha_4^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{02}^5(0))$ on the subspace $\langle \sum_{i=1}^4 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^4 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^4 \alpha_1, \quad \alpha_2^* = x^4 \alpha_2, \quad \alpha_3^* = x^5 \alpha_3, \quad \alpha_4^* = x^5 \alpha_4.$$

We assume $(\alpha_1, \alpha_2) \neq (0, 0)$ and $(\alpha_3, \alpha_4) \neq (0, 0)$. Then we have the following cases:

- (1) if $\alpha_2 = 0, \alpha_4 = 0$, then $\alpha_1 \neq 0, \alpha_3 \neq 0$ and choosing $x = \alpha_1/\alpha_3$, we have the representative $\langle \nabla_1 + \nabla_3 \rangle$;
- (2) if $\alpha_2 = 0, \alpha_4 \neq 0$, then $\alpha_1 \neq 0$ and choosing $x = \alpha_1/\alpha_4$, we have the family of representatives $\langle \nabla_1 + \alpha \nabla_3 + \nabla_4 \rangle$;
- (3) if $\alpha_2 \neq 0, \alpha_4 = 0$, then $\alpha_3 \neq 0$ and choosing $x = \alpha_2/\alpha_3$, we have the family of representatives $\langle \alpha \nabla_1 + \nabla_2 + \nabla_3 \rangle$;
- (4) if $\alpha_2 \neq 0, \alpha_4 \neq 0$, then choosing $x = \alpha_2/\alpha_4$, we have the family of representatives $\langle \alpha \nabla_1 + \nabla_2 + \beta \nabla_3 + \nabla_4 \rangle$.

Hence, we have the following new algebras:

$$\begin{array}{ll} \mathcal{N}_{16}^6(0) & : \quad e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = 2e_5 \quad e_1 e_4 = e_6 \\ & \quad e_1 e_5 = 3e_6 \quad e_2 e_1 = e_4 \quad e_4 e_1 = -e_6 \\ \mathcal{N}_{17}^6(\alpha) & : \quad e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = 2e_5 \quad e_1 e_4 = e_6 \\ & \quad e_1 e_5 = \alpha e_6 \quad e_2 e_1 = e_4 \\ & \quad e_2 e_3 = 2e_6 \quad e_3 e_1 = e_6 \quad e_3 e_2 = -2e_6 \quad e_4 e_1 = -e_6 \\ & \quad e_5 e_1 = e_6 \\ \mathcal{N}_{18}^6(\alpha) & : \quad e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = 2e_5 \quad e_1 e_4 = (\alpha + 2)e_6 \\ & \quad e_1 e_5 = e_6 \\ & \quad e_2 e_1 = e_4 \quad e_2 e_2 = e_6 \quad e_3 e_1 = e_6 \quad e_4 e_1 = -\alpha e_6 \\ \mathcal{N}_{19}^6(\alpha, \beta) & : \quad e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = 2e_5 \quad e_1 e_4 = (\alpha + 2)e_6 \\ & \quad e_1 e_5 = \beta e_6 \quad e_2 e_1 = e_4 \\ & \quad e_2 e_2 = e_6 \quad e_2 e_3 = 2e_6 \quad e_3 e_1 = e_6 \quad e_3 e_2 = -2e_6 \\ & \quad e_4 e_1 = -\alpha e_6 \quad e_5 e_1 = e_6 \end{array}$$

3.4. Central extensions of $\mathcal{N}_{03}^5(\lambda, \mu)$

Let us use the following notations

$$\begin{aligned} \nabla_1 &= [\Delta_{13}], \quad \nabla_2 = [\Delta_{14}] - [\Delta_{41}], \\ \nabla_3 &= (2\mu - 1)[\Delta_{15}] + \lambda(2 - \mu)[\Delta_{23}] + \mu(2 - \mu)[\Delta_{24}] \\ &\quad + (3 - (\lambda + 1)(2 - \mu))[\Delta_{32}] + (2 - \mu)[\Delta_{42}] + (2 - \mu)[\Delta_{51}]. \end{aligned}$$

The automorphism group of $\mathcal{N}_{03}^5(\lambda, \mu)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ y & x^2 & 0 & 0 & 0 \\ z & xy & x^3 & 0 & 0 \\ v & xy & 0 & x^3 & 0 \\ w & (\lambda + 1)xz + 2xv + y^2 & (\lambda + \mu + 1)x^2y & (4 - \mu)x^2y & x^4 \end{pmatrix}.$$

Since

$$\begin{aligned} & \phi^T \begin{pmatrix} 0 & 0 & \alpha_1 & \alpha_2 & (2\mu - 1)\alpha_3 \\ 0 & 0 & \lambda(2 - \mu)\alpha_3 & \mu(2 - \mu)\alpha_3 & 0 \\ 0 & (3 - (\lambda + 1)(2 - \mu))\alpha_3 & 0 & 0 & 0 \\ -\alpha_2 & (2 - \mu)\alpha_3 & 0 & 0 & 0 \\ (2 - \mu)\alpha_3 & 0 & 0 & 0 & 0 \end{pmatrix} \phi \\ &= \begin{pmatrix} \alpha^{****} & \alpha^{***} & \alpha_1^* + \lambda\alpha^* & \alpha_2^* + \mu\alpha^* & (2\mu - 1)\alpha_3^* \\ \alpha^{**} & \alpha^* & \lambda(2 - \mu)\alpha_3^* & \mu(2 - \mu)\alpha_3^* & 0 \\ \alpha^* & (3 - (\lambda + 1)(2 - \mu))\alpha_3^* & 0 & 0 & 0 \\ (2 - \mu)\alpha^* - \alpha_2^* & (2 - \mu)\alpha_3^* & 0 & 0 & 0 \\ (2 - \mu)\alpha_3^* & 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

we have that the action of $\text{Aut}(\mathcal{N}_{03}^5(\lambda, \mu))$ on the subspace $\langle \sum_{i=1}^3 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \rangle$, where

$$\begin{aligned} \alpha_1^* &= x^4\alpha_1 + x^3y\alpha_3(\mu + 1)(2\mu - 1 + \lambda(\mu - 2)), \\ \alpha_2^* &= x^4\alpha_2 + x^3y\alpha_3(\mu - 1)(\mu - 2)^2, \quad \alpha_3^* = x^5\alpha_3. \end{aligned}$$

Assuming $\alpha_3 \neq 0$, we have the following cases:

- (1) if $\mu \notin \{1, 2\}$, then choosing $y = -x\alpha_2/(\mu - 1)(\mu - 2)^2\alpha_3$, we get $\alpha_2^* = 0$ and have the representatives $\langle \nabla_3 \rangle$ and $\langle \nabla_1 + \nabla_3 \rangle$, depending on whether $\alpha_1 - (\mu + 1)(2\mu - 1 + \lambda(\mu - 2))/(\mu - 1)(\mu - 2)^2\alpha_2 = 0$ or not;
- (2) if $\mu = 1$, then we consider following subcases
 - (a) if $\lambda \neq 1$, then choosing $y = x\alpha_1/2(\lambda - 1)\alpha_3$, we have the representatives $\langle \nabla_3 \rangle$ and $\langle \nabla_2 + \nabla_3 \rangle$, depending on whether $\alpha_2 = 0$ or not;
 - (b) if $\lambda = 1$, then we get $\alpha_1^* = x^4\alpha_1, \alpha_2^* = x^4\alpha_2, \alpha_3^* = x^5\alpha_3$ and have the representatives $\langle \nabla_3 \rangle, \langle \nabla_1 + \nabla_3 \rangle$ and $\langle v\nabla_1 + \nabla_2 + \nabla_3 \rangle$;
- (3) if $\mu = 2$, then choosing $y = -x\alpha_1/9\alpha_3$, we have the representatives $\langle \nabla_3 \rangle$ and $\langle \nabla_2 + \nabla_3 \rangle$, depending on whether $\alpha_2 = 0$ or not.

Thus, we obtain the following representatives of distinct orbits

$$\langle \nabla_3 \rangle, \langle \nabla_1 + \nabla_3 \rangle_{\mu \neq 2}, \langle \nabla_2 + \nabla_3 \rangle_{\mu=1}, \langle v\nabla_1 + \nabla_2 + \nabla_3 \rangle_{\lambda=1, \mu=1, v \neq 0}, \langle \nabla_2 + \nabla_3 \rangle_{\mu=2},$$

Hence, we have the following new algebras:

$$\begin{array}{ll}
 \mathcal{N}_{20}^6(\lambda, \mu) & : \quad \begin{array}{ll} e_1 e_1 = e_2 & e_1 e_2 = e_3 \\ e_1 e_4 = \mu e_5 & e_1 e_5 = (2\mu - 1)e_6 \\ e_2 e_1 = e_4 & e_2 e_2 = e_5 \\ e_2 e_3 = \lambda(2 - \mu)e_6 & e_2 e_4 = \mu(2 - \mu)e_6 \\ e_3 e_1 = e_5 & e_4 e_1 = (2 - \mu)e_5 \\ e_3 e_2 = (3 - (\lambda + 1)(2 - \mu))e_6 & e_4 e_2 = (2 - \mu)e_6 \\ e_5 e_1 = (2 - \mu)e_6 & \end{array} \\
 \mathcal{N}_{21}^6(\lambda, \mu)_{\mu \neq 2} & : \quad \begin{array}{ll} e_1 e_1 = e_2 & e_1 e_2 = e_3 \\ e_1 e_3 = \lambda e_5 + e_6 & e_1 e_4 = \mu e_5 \\ e_1 e_5 = (2\mu - 1)e_6 & \\ e_2 e_1 = e_4 & e_2 e_2 = e_5 \\ e_2 e_3 = \lambda(2 - \mu)e_6 & e_2 e_4 = \mu(2 - \mu)e_6 \\ e_3 e_1 = e_5 & e_4 e_1 = (2 - \mu)e_5 \\ e_3 e_2 = (3 - (\lambda + 1)(2 - \mu))e_6 & \\ e_4 e_2 = (2 - \mu)e_6 & e_5 e_1 = (2 - \mu)e_6 \end{array} \\
 \mathcal{N}_{22}^6(\lambda) & : \quad \begin{array}{ll} e_1 e_1 = e_2 & e_1 e_2 = e_3 \\ e_1 e_4 = e_5 + e_6 & e_1 e_5 = e_6 \\ e_2 e_1 = e_4 & e_2 e_2 = e_5 \\ e_2 e_3 = \lambda e_6 & e_2 e_4 = e_6 \\ e_3 e_1 = e_5 & e_4 e_1 = e_5 - e_6 \\ e_4 e_2 = e_6 & e_5 e_1 = e_6 \end{array} \\
 \mathcal{N}_{23}^6(\nu)_{\nu \neq 0} & : \quad \begin{array}{ll} e_1 e_1 = e_2 & e_1 e_2 = e_3 \\ e_1 e_4 = e_5 + e_6 & e_1 e_5 = e_6 \\ e_2 e_1 = e_4 & e_2 e_2 = e_5 \\ e_2 e_3 = e_6 & e_2 e_4 = e_6 \\ e_3 e_1 = e_5 & e_4 e_1 = e_5 - e_6 \\ e_4 e_2 = e_6 & e_5 e_1 = e_6 \end{array} \\
 \mathcal{N}_{24}^6(\lambda) & : \quad \begin{array}{ll} e_1 e_1 = e_2 & e_1 e_2 = e_3 \\ e_1 e_3 = \lambda e_5 & e_1 e_4 = 2e_5 + e_6 \\ e_2 e_1 = e_4 & e_2 e_2 = e_5 \\ e_3 e_1 = e_5 & e_3 e_2 = e_6 \end{array}
 \end{array}$$

3.5. Central extensions of $\mathcal{N}_{04}^5(\lambda)_{\lambda \neq 0}$

Let us use the following notations

$$\begin{aligned}
 \nabla_1 &= [\Delta_{14}] - [\Delta_{41}], \quad \nabla_2 = [\Delta_{22}] + [\Delta_{31}] + 2[\Delta_{41}], \\
 \nabla_3 &= -2[\Delta_{15}] + [\Delta_{23}] + \lambda[\Delta_{24}] - [\Delta_{32}] + [\Delta_{51}].
 \end{aligned}$$

The automorphism group of $\mathcal{N}_{04}^5(\lambda)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ y & x^2 & 0 & 0 & 0 \\ z & xy & x^3 & 0 & 0 \\ v & xy & 0 & x^3 & 0 \\ w & xz & (\lambda + 1)x^2y & -\lambda x^2y & x^4 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & \alpha_1 & -2\alpha_3 \\ 0 & \alpha_2 & \alpha_3 & \lambda\alpha_3 & 0 \\ \alpha_2 & -\alpha_3 & 0 & 0 & 0 \\ 2\alpha_2 - \alpha_1 & 0 & 0 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\phi = \begin{pmatrix} \alpha^{****} & \alpha^{***} & \alpha^* & \alpha_1^* + \lambda\alpha^* & -2\alpha_3^* \\ \alpha^{**} & \alpha_2^* & \alpha_3^* & \lambda\alpha_3^* & 0 \\ \alpha_2^* & -\alpha_3^* & 0 & 0 & 0 \\ 2\alpha_2^* - \alpha_1^* - \lambda\alpha^* & 0 & 0 & 0 & 0 \\ \alpha_3^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{04}^5(\lambda))$ on the subspace $\langle \sum_{i=1}^3 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^4\alpha_1 + 2x^3y\lambda(\lambda + 2)\alpha_3, \quad \alpha_2^* = x^4\alpha_2 + x^3y\lambda\alpha_3, \quad \alpha_3^* = x^5\alpha_3.$$

Assuming $\alpha_3 \neq 0$, we have the following cases:

- (1) if $\alpha_1 = 2(\lambda + 2)\alpha_2$, then choosing $y = -x\alpha_2/\lambda\alpha_3$, we have the representative $\langle \nabla_3 \rangle$;
- 2 if $\alpha_1 \neq 2(\lambda + 2)\alpha_2$, then choosing $x = (\alpha_1 - 2(\lambda + 2)\alpha_2)/\alpha_3$, $y = -x\alpha_2/\lambda\alpha_3$, we have the representative $\langle \nabla_1 + \nabla_3 \rangle$.

Hence, we have the following new algebras:

$$\mathcal{N}_{25}^6(\lambda)_{\lambda \neq 0} : \begin{array}{lllll} e_1e_1=e_2 & e_1e_2=e_3 & e_1e_3=e_5 & e_1e_4=\lambda e_5 & e_1e_5=-2e_6 \\ e_2e_3=e_6 & e_2e_4=\lambda e_6 & e_3e_2=-e_6 & e_4e_1=-\lambda e_5 & e_5e_1=e_6 \end{array}$$

$$\mathcal{N}_{26}^6(\lambda)_{\lambda \neq 0} : \begin{array}{lllll} e_1e_1=e_2 & e_1e_2=e_3 & e_1e_3=e_5 & e_1e_4=\lambda e_5 + e_6 & e_1e_5=-2e_6 \\ e_2e_3=e_6 & e_2e_4=\lambda e_6 & e_3e_2=-e_6 & e_4e_1=-\lambda e_5 - e_6 & e_5e_1=e_6 \end{array}$$

3.6. Central extensions of $\mathcal{N}_{05}^5$

Let us use the following notations

$$\nabla_1 = [\Delta_{12}], \quad \nabla_2 = 3[\Delta_{15}] - 2[\Delta_{24}] + [\Delta_{33}] - [\Delta_{51}].$$

The automorphism group of $\mathcal{N}_{05}^5$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 & 0 \\ y & 0 & x^3 & 0 & 0 \\ z & 0 & 0 & x^4 & 0 \\ v & xz & -x^2y & 0 & x^5 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & 3\alpha_2 \\ 0 & 0 & 0 & -2\alpha_2 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^{***} & \alpha_1^* & -\alpha^* & 0 & 3\alpha_2^* \\ \alpha^{**} & 0 & 0 & -2\alpha_2^* & 0 \\ \alpha^* & 0 & \alpha_2^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{05}^5)$ on the subspace $\langle \sum_{i=1}^2 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^2 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^3 \alpha_1 + 3x^2 z \alpha_2, \quad \alpha_2^* = x^6 \alpha_2.$$

It is easy to see that we have only one non-trivial orbit with the representative $\langle \nabla_2 \rangle$ and obtain the algebra:

$$\begin{aligned} \mathcal{N}_{27}^6 : \quad & e_1 e_1 = e_2 \quad e_1 e_3 = e_4 \quad e_1 e_4 = 2e_5 \quad e_1 e_5 = 3e_6 \quad e_2 e_1 = e_3 \quad e_2 e_3 = -e_5 \\ & e_2 e_4 = -2e_6 \quad e_3 e_1 = -e_4 \quad e_3 e_3 = e_6 \quad e_4 e_1 = -e_5 \quad e_5 e_1 = -e_6 \end{aligned}$$

3.7. Central extensions of $\mathcal{N}_{06}^5$

Let us use the following notations

$$\nabla_1 = [\Delta_{12}], \quad \nabla_2 = 3[\Delta_{15}] - [\Delta_{22}] - 2[\Delta_{24}] - 2[\Delta_{31}] + [\Delta_{33}] - [\Delta_{51}].$$

The automorphism group of $\mathcal{N}_{06}^5$ consists of invertible matrices of the form

$$\phi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & y & -x & 0 & 1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ x & 0 & -1 & 0 & 0 \\ y & 0 & 0 & 1 & 0 \\ z & -y & -x & 0 & -1 \end{pmatrix}.$$

Since

$$\phi_i^T \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & 3\alpha_2 \\ 0 & -\alpha_2 & 0 & -2\alpha_2 & 0 \\ -2\alpha_2 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix} \phi_i = \begin{pmatrix} \alpha^{***} & \alpha_1^* & \alpha^* & 0 & 3\alpha_2^* \\ \alpha^{**} & -2\alpha_2^* & 0 & -2\alpha_2^* & 0 \\ -2\alpha_2^* - \alpha^* & 0 & \alpha_2^* & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{06}^5)$ on the subspace $\langle \sum_{i=1}^2 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^2 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = (-1)^i \alpha_1 + 3y\alpha_2, \quad \alpha_2^* = \alpha_2.$$

It is easy to see that we have only one non-trivial orbit with the representative $\langle \nabla_2 \rangle$ and obtain the algebra:

$$\begin{array}{llllll} \mathcal{N}_{28}^6 : & e_1 e_1 = e_2 & e_1 e_2 = e_5 & e_1 e_3 = e_4 & e_1 e_4 = 2e_5 & e_1 e_5 = 3e_6 \\ & e_2 e_1 = e_3 & e_2 e_2 = -e_6 & e_2 e_3 = -e_5 & e_2 e_4 = -2e_6 & e_3 e_1 = -e_4 - 2e_6 \\ & e_3 e_3 = e_6 & e_4 e_1 = -e_5 & e_5 e_1 = -e_6 & & \end{array}$$

3.8. Central extensions of $\mathcal{N}_{07}^5(\lambda)_{\lambda \notin \{0,1\}}$

Let us use the following notations

$$\begin{aligned} \nabla_1 &= [\Delta_{21}], \quad \nabla_2 = (4 - 3\lambda)[\Delta_{15}] + \lambda(3 - 2\lambda)[\Delta_{24}] + \lambda(2 - \lambda)[\Delta_{33}] \\ &\quad + \lambda[\Delta_{42}] + \lambda[\Delta_{51}]. \end{aligned}$$

The automorphism group of $\mathcal{N}_{07}^5(\lambda)_{\lambda \notin \{0,1\}}$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ 0 & x^2 & 0 & 0 & 0 \\ 0 & 0 & x^3 & 0 & 0 \\ y & 0 & 0 & x^4 & 0 \\ z & (3 - \lambda)xy & 0 & 0 & x^5 \end{pmatrix}.$$

Since

$$\begin{aligned} \phi^T & \begin{pmatrix} 0 & 0 & 0 & 0 & (4 - 3\lambda)\alpha_2 \\ \alpha_1 & 0 & 0 & \lambda(3 - 2\lambda)\alpha_2 & 0 \\ 0 & 0 & \lambda(2 - \lambda)\alpha_2 & 0 & 0 \\ 0 & \lambda\alpha_2 & 0 & 0 & 0 \\ \lambda\alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix} \phi \\ &= \begin{pmatrix} \alpha^{**} & \alpha^* & 0 & 0 & (4 - 3\lambda)\alpha_2^* \\ \alpha_1^* + \lambda\alpha^* & 0 & 0 & \lambda(3 - 2\lambda)\alpha_2^* & 0 \\ 0 & 0 & \lambda(2 - \lambda)\alpha_2^* & 0 & 0 \\ 0 & \lambda\alpha_2^* & 0 & 0 & 0 \\ \lambda\alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix}, \end{aligned}$$

we have that the action of $\text{Aut}(\mathcal{N}_{07}^5(\lambda)_{\lambda \notin \{0,1\}})$ on the subspace $\langle \sum_{i=1}^2 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^2 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^3 \alpha_1 - 3x^2 y \lambda (\lambda - 1) (\lambda - 2) \alpha_2, \quad \alpha_2^* = x^6 \alpha_2.$$

Assuming $\alpha_2 \neq 0$, we have the following cases:

- (1) if $\lambda \neq 2$, then choosing $y = x\alpha_1 / 3\lambda(\lambda - 1)(\lambda - 2)\alpha_2$, we have the representative $\langle \nabla_2 \rangle$;

- (2) if $\lambda = 2$, then we have two representatives $\langle \nabla_2 \rangle$ and $\langle \nabla_1 + \nabla_2 \rangle$ depending on whether $\alpha_1 = 0$ or not.

Hence, we have the following new algebras:

$$\begin{array}{l} \mathcal{N}_{29}^6(\lambda)_{\lambda \notin \{0,1\}} : \quad e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_3 = (2 - \lambda)e_4 \quad e_1 e_4 = (3 - 2\lambda)e_5 \\ \quad e_1 e_5 = (4 - 3\lambda)e_6 \quad e_2 e_1 = \lambda e_3 \quad e_2 e_2 = \lambda e_4 \quad e_2 e_3 = \lambda(2 - \lambda)e_5 \\ \quad e_2 e_4 = \lambda(3 - 2\lambda)e_6 \quad e_3 e_1 = \lambda e_4 \quad e_3 e_2 = \lambda e_5 \quad e_3 e_3 = \lambda(2 - \lambda)e_6 \\ \quad e_4 e_1 = \lambda e_5 \quad e_4 e_2 = \lambda e_6 \quad e_5 e_1 = \lambda e_6 \\ \mathcal{N}_{30}^6 : \quad e_1 e_1 = e_2 \quad e_1 e_2 = e_3 \quad e_1 e_4 = -e_5 \quad e_1 e_5 = -2e_6 \\ \quad e_2 e_1 = 2e_3 + e_6 \quad e_2 e_2 = 2e_4 \quad e_2 e_4 = -e_6 \quad e_3 e_1 = 2e_4 \\ \quad e_3 e_2 = 2e_5 \quad e_4 e_1 = 2e_5 \quad e_4 e_2 = 2e_6 \quad e_5 e_1 = 2e_6 \end{array}$$

3.9. Central extensions of $\mathcal{N}_{07}^5(0)$

Let us use the following notations

$$\nabla_1 = [\Delta_{15}], \quad \nabla_2 = [\Delta_{21}], \quad \nabla_3 = 2[\Delta_{23}] - 2[\Delta_{32}] + [\Delta_{41}].$$

The automorphism group of $\mathcal{N}_{07}^5(0)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ y & x^2 & 0 & 0 & 0 \\ z & xy & x^3 & 0 & 0 \\ v & 2xz & 2x^2y & x^4 & 0 \\ w & 3xv & 6x^2z & 3x^3y & x^5 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & 0 & \alpha_1 \\ \alpha_2 & 0 & 2\alpha_3 & 0 & 0 \\ 0 & -2\alpha_3 & 0 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^{****} & \alpha^{***} & \alpha^{**} & \alpha^* & \alpha_1^* \\ \alpha_2^* & 0 & 2\alpha_3^* & 0 & 0 \\ 0 & -2\alpha_3^* & 0 & 0 & 0 \\ \alpha_3^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{07}^5(0))$ on the subspace $\langle \sum_{i=1}^3 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^6 \alpha_1, \quad \alpha_2^* = x^3 \alpha_2 + 2x(2xz - y^2) \alpha_3, \quad \alpha_3^* = x^5 \alpha_3.$$

Assuming $\alpha_1 \neq 0$, we have the following cases:

- (1) if $\alpha_3 \neq 0$, then choosing $x = \alpha_3 / \alpha_1, y = 0, z = -x\alpha_2 / 4\alpha_3$, we have the representative $\langle \nabla_1 + \nabla_3 \rangle$;
- (2) if $\alpha_3 = 0, \alpha_2 \neq 0$, then choosing $x = \sqrt[3]{\alpha_2 / \alpha_1}$, we have the representative $\langle \nabla_1 + \nabla_2 \rangle$;
- (3) if $\alpha_3 = \alpha_2 = 0$, then we have the representative $\langle \nabla_1 \rangle$.

Hence, we have the following new algebras:

$$\begin{aligned}
 \mathcal{N}_{31}^6 : \quad & e_1e_1 = e_2 \quad e_1e_2 = e_3 \quad e_1e_3 = 2e_4 \quad e_1e_4 = 3e_5 \quad e_1e_5 = e_6 \\
 & e_2e_3 = 2e_6 \quad e_3e_2 = -2e_6 \quad e_4e_1 = e_6 \\
 \mathcal{N}_{32}^6 : \quad & e_1e_1 = e_2 \quad e_1e_2 = e_3 \quad e_1e_3 = 2e_4 \quad e_1e_4 = 3e_5 \quad e_1e_5 = e_6 \quad e_2e_1 = e_6 \\
 \mathcal{N}_{29}^6(0) : \quad & e_1e_1 = e_2 \quad e_1e_2 = e_3 \quad e_1e_3 = 2e_4 \quad e_1e_4 = 3e_5 \quad e_1e_5 = 4e_6
 \end{aligned}$$

3.10. Central extensions of $\mathcal{N}_{07}^5(1)$

Let us use the following notations

$$\nabla_1 = [\Delta_{21}], \quad \nabla_2 = [\Delta_{15}] + [\Delta_{24}] + [\Delta_{33}] + [\Delta_{42}] + [\Delta_{51}].$$

The automorphism group of $\mathcal{N}_{07}^5(1)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} x & 0 & 0 & 0 & 0 \\ y & x^2 & 0 & 0 & 0 \\ z & 2xy & x^3 & 0 & 0 \\ v & 2xz + y^2 & 3x^2y & x^4 & 0 \\ w & 2xv + 2yz & 3x^2z + 3xy^2 & 4x^3y & x^5 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & 0 & 0 & \alpha_2 \\ \alpha_1 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^{****} & \alpha^{***} & \alpha^{**} & \alpha^* & \alpha_2^* \\ \alpha_1^* + \alpha^{***} & \alpha^{**} & \alpha^* & \alpha_2^* & 0 \\ \alpha^{**} & \alpha^* & \alpha_2^* & 0 & 0 \\ \alpha^* & \alpha_2^* & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{07}^5(1))$ on the subspace $\langle \sum_{i=1}^2 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^2 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = x^3 \alpha_1, \quad \alpha_2^* = x^6 \alpha_2.$$

Assuming $\alpha_2 \neq 0$, we have the following cases:

- (1) if $\alpha_1 = 0$, then we have the representative $\langle \nabla_2 \rangle$;
- (2) if $\alpha_1 \neq 0$, then choosing $x = \sqrt[3]{\alpha_1/\alpha_2}$, we have the representative $\langle \nabla_1 + \nabla_2 \rangle$.

Hence, we have the algebras:

$$\begin{aligned}
 \mathcal{N}_{29}^6(1) : \quad & e_1e_1 = e_2 \quad e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_1e_4 = e_5 \quad e_1e_5 = e_6 \\
 & e_2e_1 = e_3 \quad e_2e_2 = e_4 \quad e_2e_3 = e_5 \quad e_2e_4 = e_6 \quad e_3e_1 = e_4 \\
 & e_3e_2 = e_5 \quad e_3e_3 = e_6 \quad e_4e_1 = e_5 \quad e_4e_2 = e_6 \quad e_5e_1 = e_6 \\
 \mathcal{N}_{33}^6 : \quad & e_1e_1 = e_2 \quad e_1e_2 = e_3 \quad e_1e_3 = e_4 \quad e_1e_4 = e_5 \quad e_1e_5 = e_6 \\
 & e_2e_1 = e_3 + e_6 \quad e_2e_2 = e_4 \quad e_2e_3 = e_5 \quad e_2e_4 = e_6 \quad e_3e_1 = e_4 \\
 & e_3e_2 = e_5 \quad e_3e_3 = e_6 \quad e_4e_1 = e_5 \quad e_4e_2 = e_6 \quad e_5e_1 = e_6
 \end{aligned}$$

3.11. Central extensions of $\mathcal{N}_{08}^5$

Let us use the following notations

$$\nabla_1 = [\Delta_{21}], \quad \nabla_2 = 2[\Delta_{15}] + [\Delta_{22}] + [\Delta_{31}], \quad \nabla_3 = [\Delta_{23}] - [\Delta_{32}] + [\Delta_{41}].$$

The automorphism group of $\mathcal{N}_{08}^5$ consists of invertible matrices of the form

$$\phi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ y & x & 1 & 0 & 0 \\ z & y & x & 1 & 0 \\ v & x+z & y & x & 1 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ y & -x & -1 & 0 & 0 \\ z & -y & x & 1 & 0 \\ v & -x-z & y & -x & -1 \end{pmatrix}.$$

Since

$$\phi_i^T \begin{pmatrix} 0 & 0 & 0 & 0 & 2\alpha_2 \\ \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 \\ \alpha_2 & -\alpha_3 & 0 & 0 & 0 \\ \alpha_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \phi_i = \begin{pmatrix} \alpha^{****} & \alpha^{***} & \alpha^{**} & \alpha^* & 2\alpha_2^* \\ \alpha_1^* + \alpha^* & \alpha_2^* & \alpha_3^* & 0 & 0 \\ \alpha_2^* & -\alpha_3^* & 0 & 0 & 0 \\ \alpha_3^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{08}^5)$ on the subspace $\langle \sum_{i=1}^3 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \rangle$, where

$$\begin{aligned} \text{for } i = 1 : \alpha_1^* &= \alpha_1 - (x^2 - 2y)\alpha_3, & \alpha_2^* &= \alpha_2, & \alpha_3^* &= \alpha_3, \\ \text{for } i = 2 : \alpha_1^* &= -\alpha_1 + (x^2 + 2y)\alpha_3, & \alpha_2^* &= \alpha_2, & \alpha_3^* &= -\alpha_3. \end{aligned}$$

Assuming $\alpha_2 \neq 0$ and using ϕ_1 we have the following cases:

- (1) if $\alpha_3 = 0$, then we have the family of representatives $\langle \lambda \nabla_1 + \nabla_2 \rangle$;
- (2) if $\alpha_3 \neq 0$, then choosing $y = (x^2 \alpha_3 - \alpha_1)/2\alpha_3$, we have the family of representatives $\langle \nabla_2 + \lambda \nabla_3 \rangle_{\lambda \neq 0}$.

Hence, we have the following new algebras:

$$\begin{aligned} \mathcal{N}_{34}^6(\lambda) & : & e_1 e_1 &= e_2 & e_1 e_2 &= e_3 & e_1 e_3 &= e_4 & e_1 e_4 &= e_5 \\ & & e_1 e_5 &= 2e_6 & e_2 e_1 &= e_5 + \lambda e_6 & e_2 e_2 &= e_6 & e_3 e_1 &= e_6 \\ \mathcal{N}_{35}^6(\lambda)_{\lambda \neq 0} & : & e_1 e_1 &= e_2 & e_1 e_2 &= e_3 & e_1 e_3 &= e_4 & e_1 e_4 &= e_5 \\ & & e_1 e_5 &= 2e_6 & e_2 e_1 &= e_5 & e_2 e_2 &= e_6 & e_2 e_3 &= \lambda e_6 \\ & & e_3 e_1 &= e_6 & e_3 e_2 &= -\lambda e_6 & e_4 e_1 &= \lambda e_6 \end{aligned}$$

Moreover, using the action of ϕ_2 we have that $\mathcal{N}_{34}^6(\lambda) \cong \mathcal{N}_{34}^6(-\lambda)$ and $\mathcal{N}_{35}^6(\lambda) \cong \mathcal{N}_{35}^6(-\lambda)$.

3.12. Central extensions of $\mathcal{N}_{10}^5$

Let us use the following notations

$$\nabla_1 = [\Delta_{12}], \quad \nabla_2 = -2[\Delta_{13}] + [\Delta_{15}] + [\Delta_{24}] + [\Delta_{33}] + [\Delta_{42}] + [\Delta_{51}].$$

The automorphism group of $\mathcal{N}_{10}^5$ consists of invertible matrices of the form

$$\phi_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ y & 2x & 1 & 0 & 0 \\ z & x^2 + 2y & 3x & 1 & 0 \\ v & 2xy + x + 2z & 3x^2 + 3y & 4x & 1 \end{pmatrix},$$

$$\phi_2 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ y & -2x & -1 & 0 & 0 \\ z & x^2 - 2y & 3x & 1 & 0 \\ v & 2xy - x - 2z & -3x^2 + 3y & -4x & -1 \end{pmatrix}.$$

Since

$$\phi_i^T \begin{pmatrix} 0 & \alpha_1 & -2\alpha_2 & 0 & \alpha_2 \\ 0 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix} \phi_i = \begin{pmatrix} \alpha^{****} & \alpha_1^* + \alpha^{***} & -2\alpha_2^* + \alpha^{**} & \alpha^* & \alpha_2^* \\ \alpha^{***} & \alpha^{**} & \alpha^* & \alpha_2^* & 0 \\ \alpha^{**} & \alpha^* & \alpha_2^* & 0 & 0 \\ \alpha^* & \alpha_2^* & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{10}^5)$ on the subspace $\langle \sum_{i=1}^2 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^2 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = (-1)^i \alpha_1 - 4x\alpha_2, \quad \alpha_2^* = \alpha_2.$$

It is easy to see that we have only one non-trivial orbit with representative $\langle \nabla_2 \rangle$. The corresponding algebra is

$$\begin{array}{llllll} \mathcal{N}_{36}^6 : & e_1 e_1 = e_2 & e_1 e_2 = e_3 & e_1 e_3 = e_4 - 2e_6 & e_1 e_4 = e_5 & e_1 e_5 = e_6 \\ & e_2 e_1 = e_3 + e_5 & e_2 e_2 = e_4 & e_2 e_3 = e_5 & e_2 e_4 = e_6 & e_3 e_1 = e_4 \\ & e_3 e_2 = e_5 & e_3 e_3 = e_6 & e_4 e_1 = e_5 & e_4 e_2 = e_6 & e_5 e_1 = e_6 \end{array}$$

3.13. Central extensions of $\mathcal{N}_{11}^5(\lambda)$

Let us use the following notations

$$\nabla_1 = [\Delta_{13}], \quad \nabla_2 = [\Delta_{15}] + [\Delta_{32}], \quad \nabla_3 = [\Delta_{14}] - 2[\Delta_{23}] + 2[\Delta_{32}] - [\Delta_{41}].$$

The automorphism group of $\mathcal{N}_{11}^5(\lambda)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ y & x & 1 & 0 & 0 \\ z & x + 2y & 2x & 1 & 0 \\ v & x^2 + (\lambda + 1)y + 2z & (\lambda + 3)x + 4y & 2x & 1 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & 0 & \alpha_1 & \alpha_3 & \alpha_2 \\ 0 & 0 & -2\alpha_3 & 0 & 0 \\ 0 & \alpha_2 + 2\alpha_3 & 0 & 0 & 0 \\ -\alpha_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\phi = \begin{pmatrix} \alpha^{****} & \alpha^{***} & \alpha_1^* + 2\alpha^{**} + \lambda\alpha^* & \alpha_3^* + 2\alpha^* & \alpha_2^* \\ \alpha^{**} & \alpha^* & -2\alpha_3^* & 0 & 0 \\ \alpha^* & \alpha_2^* + 2\alpha_3^* & 0 & 0 & 0 \\ -\alpha_3^* & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{11}^5(\lambda))$ on the subspace $\langle \sum_{i=1}^3 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^3 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = \alpha_1 + (3x + 4y - 2x^2)\alpha_2 + (8y + 2x - 4x^2)\alpha_3, \quad \alpha_2^* = \alpha_2, \quad \alpha_3^* = \alpha_3.$$

Assuming $\alpha_2 \neq 0$, we have the following cases

- (1) if $\alpha_2 \neq -2\alpha_3$, then choosing $y = -(\alpha_1 + x(3\alpha_2 + 2\alpha_3) - 2x^2(\alpha_2 + 2\alpha_3))/4(\alpha_2 + 2\alpha_3)$, we have the family of orbits with the representatives $\langle \nabla_2 + \mu \nabla_3 \rangle_{\mu \neq -1/2}$;
- (2) if $\alpha_2 = -2\alpha_3$, then choosing $x = -\alpha_1/2\alpha_2$, we have the family of orbits with the representatives $\langle \nabla_2 - \frac{1}{2} \nabla_3 \rangle$.

Thus, we obtain the family of algebras:

$$\begin{array}{llll} \mathcal{N}_{37}^6(\lambda, \mu) : & e_1 e_1 = e_2 & e_1 e_2 = e_3 & e_1 e_3 = 2e_4 + \lambda e_5 \quad e_1 e_4 = 2e_5 + \mu e_6 \\ & e_1 e_5 = e_6 & e_2 e_1 = e_4 & e_2 e_2 = e_5 \quad e_2 e_3 = -2\mu e_6 \\ & e_3 e_1 = e_5 & e_3 e_2 = (1 + 2\mu)e_6 & e_4 e_1 = -\mu e_6 \end{array}$$

3.14. Central extensions of $\mathcal{N}_{13}^5(\lambda)$

Let us use the following notations

$$\begin{aligned} \nabla_1 &= [\Delta_{21}], \quad \nabla_2 = (1 - 2\lambda)[\Delta_{13}] - 2[\Delta_{14}] + [\Delta_{15}] - [\Delta_{23}] + [\Delta_{24}] + [\Delta_{33}] \\ &\quad + [\Delta_{41}] + [\Delta_{42}] + [\Delta_{51}]. \end{aligned}$$

The automorphism group of $\mathcal{N}_{13}^5(\lambda)$ consists of invertible matrices of the form

$$\phi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ y & 2x & 1 & 0 & 0 \\ z & x^2 + x + 2y & 3x & 1 & 0 \\ v & \lambda x + 2xy - 2y + 2z & 3x^2 - 3x + 3y & 4x & 1 \end{pmatrix}.$$

Since

$$\phi^T \begin{pmatrix} 0 & \alpha_1 & (1-2\lambda)\alpha_2 & -2\alpha_2 & \alpha_2 \\ 0 & 0 & -\alpha_2 & \alpha_2 & 0 \\ 0 & 0 & \alpha_2 & 0 & 0 \\ \alpha_2 & \alpha_2 & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & 0 & 0 \end{pmatrix} \phi = \begin{pmatrix} \alpha^{****} & \alpha_1^* + \alpha^{***} & (1-2\lambda)\alpha_2^* + \alpha^{**} - 2\alpha^* & -2\alpha_2^* + \alpha^* & \alpha_2^* \\ \alpha^{***} + \alpha^{**} + \lambda\alpha^* & \alpha^{**} & -\alpha_2^* + \alpha^* & \alpha_2^* & 0 \\ \alpha^{**} & \alpha^* & \alpha_2^* & 0 & 0 \\ \alpha_2^* + \alpha^* & \alpha_2^* & 0 & 0 & 0 \\ \alpha_2^* & 0 & 0 & 0 & 0 \end{pmatrix},$$

we have that the action of $\text{Aut}(\mathcal{N}_{13}^5(\lambda))$ on the subspace $\langle \sum_{i=1}^2 \alpha_i \nabla_i \rangle$ is given by $\langle \sum_{i=1}^2 \alpha_i^* \nabla_i \rangle$, where

$$\alpha_1^* = \alpha_1 - (\lambda x - x^2 + x + y)\alpha_2, \quad \alpha_2^* = \alpha_2.$$

It is easy to see that we have only one non-trivial orbit with the representative $\langle \nabla_2 \rangle$ and obtain the family of algebras:

$$\begin{array}{lll} \mathcal{N}_{38}^6(\lambda) : & e_1 e_1 = e_2 & e_1 e_2 = e_3 \quad e_1 e_3 = e_4 - 2e_5 + (1-2\lambda)e_6 \\ & e_1 e_4 = e_5 - 2e_6 & e_1 e_5 = e_6 \\ & e_2 e_1 = e_3 + e_4 + \lambda e_5 & e_2 e_2 = e_4 \quad e_2 e_3 = e_5 - e_6 \\ & e_2 e_4 = e_6 & e_3 e_1 = e_4 \\ & e_3 e_2 = e_5 & e_3 e_3 = e_6 \quad e_4 e_1 = e_5 + e_6 \\ & e_4 e_2 = e_6 & e_5 e_1 = e_6 \end{array}$$

Summarizing the results of the present section and using the information from Tables A2 and A3 from Appendix, we have the following theorem.

Theorem 3.1: *Let $\mathcal{N}$ be a six-dimensional complex one-generated nilpotent Novikov algebra. Then $\mathcal{N}$ is isomorphic to $\mathcal{N}_j^6$, where $j = 01, \dots, 38$ (see Table A4 presented in Appendix).*

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References

- [1] Abdelwahab H, Calderón AJ, Kaygorodov I. The algebraic and geometric classification of nilpotent binary Lie algebras. *Int J Algebra Comput.* **2019**;29(6):1113–1129.
- [2] Calderón Martín A, A Fernández Ouaridi, Kaygorodov I. The classification of 2-dimensional rigid algebras. *Linear Multilinear Algebra.* **2018**:1–17. doi:10.1080/03081087.2018.1519009
- [3] Cañete E, Khudoyberdiyev A. The classification of 4-dimensional Leibniz algebras. *Linear Algebra Appl.* **2013**;439(1):273–288.
- [4] Cicalò S, De Graaf W, Schneider C. Six-dimensional nilpotent Lie algebras. *Linear Algebra Appl.* **2012**;436(1):163–189.
- [5] Darijani I, Usefi H. The classification of 5-dimensional p -nilpotent restricted Lie algebras over perfect fields, I. *J Algebra.* **2016**;464:97–140.
- [6] De Graaf W. Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2. *J Algebra.* **2007**;309(2):640–653.
- [7] De Graaf W. Classification of nilpotent associative algebras of small dimension. *Int J Algebra Comput.* **2018**;28(1):133–161.
- [8] Gorshkov I, Kaygorodov I, Kytmanov A, et al. The variety of nilpotent Tortkara algebras. *J Siberian Fed Univ Math Phys.* **2019**;12(2):173–184.
- [9] Gorshkov I, Kaygorodov I, Khrypchenko M. The algebraic classification of nilpotent Tortkara algebras. arXiv:1904.00845.
- [10] Hegazi A, Abdelwahab H. Classification of five-dimensional nilpotent Jordan algebras. *Linear Algebra Appl.* **2016**;494:165–218.
- [11] Hegazi A, Abdelwahab H, Calderón Martín A. Classification of nilpotent Malcev algebras of small dimensions over arbitrary fields of characteristic not 2. *Algebras Represent Theory.* **2018**;21(1):19–45.
- [12] Kaygorodov I, Volkov Y. The variety of 2-dimensional algebras over an algebraically closed field. *Can J Math.* **2019**;71(4):819–842.
- [13] Khudoyberdiyev A, Rakhimov I, Said Husain SK. On classification of 5-dimensional solvable Leibniz algebras. *Linear Algebra Appl.* **2014**;457:428–454.
- [14] Dekimpe K, Ongenae V. Filiform left-symmetric algebras. *Geometriae Dedicata.* **1999**;74(2):165–199.
- [15] Jumaniyozov D, Kaygorodov I, Khudoyberdiyev A. The algebraic and geometric classification of nilpotent noncommutative Jordan algebras. arXiv:1912.02691.
- [16] Masutova K, Omirov B. On some zero-filiform algebras. *Ukrainian Math J.* **2014**;66(4):541–552.
- [17] Karimjanov I, Kaygorodov I, Ladra M. Central extensions of filiform associative algebras. *Linear Multilinear Algebra.* **2019**. doi:10.1080/03081087.2019.1620674
- [18] Ismailov N, Kaygorodov I, Mashurov F. The algebraic and geometric classification of nilpotent assosymmetric algebras. *Algebras Represent Theory.* **2020**. DOI:10.1007/s10468-019-09935-y
- [19] Kaygorodov I, Pérez-Guillán P, Voronin V. The algebraic and geometric classification of nilpotent bicommutative algebras. *Algebras Represent Theory.* **2020**. DOI:10.1007/s10468-019-09944-x
- [20] Fernández Ouaridi A, Kaygorodov I, Khrypchenko M, et al. Degenerations of nilpotent algebras. arXiv:1905.05361.
- [21] Kaygorodov I, Khrypchenko M, Popov Y. The algebraic and geometric classification of nilpotent terminal algebras. arXiv:1909.00358.
- [22] Balinsky A, Novikov S. Poisson brackets of hydrodynamic type, Frobenius algebras and Lie algebras. *Sov Math Dokl.* **1985**;32(1):228–231.
- [23] Burde D, Dekimpe K. Novikov structures on solvable Lie algebras. *J Geom Phys.* **2006**;56(9):1837–1855.
- [24] Burde D, Dekimpe K, Vercammen K. Novikov algebras and Novikov structures on Lie algebras. *Linear Algebra Appl.* **2008**;429(1):31–41.
- [25] Bokut L, Chen Yu., Zhang Z. On free Gelfand-Dorfman-Novikov-Poisson algebras and a PBW theorem. *J Algebra.* **2018**;500:153–170.

- [26] Dzhumadildaev A. Novikov–Jordan algebras. *Commun Algebra*. 2002;30(11):5207–5240.
- [27] Xu X. Novikov–Poisson algebras. *J Algebra*. 1997;190(2):253–279.
- [28] Zelmanov E. A class of local translation-invariant Lie algebras. *Sov Math Doklady*. 1987;35(6):216–218.
- [29] Filippov V. A class of simple nonassociative algebras. *Math Notes*. 1989;45(1-2):68–71.
- [30] Osborn J. Infinite-dimensional Novikov algebras of characteristic 0. *J Algebra*. 1994;167(1):146–167.
- [31] Xu X. Classification of simple Novikov algebras and their irreducible modules of characteristic 0. *J Algebra*. 2001;246(2):673–707.
- [32] Xu X. On simple Novikov algebras and their irreducible modules. *J Algebra*. 1996;185(3):905–934.
- [33] Bai C, Meng D. The classification of Novikov algebras in low dimensions. *J Phys A Math Gen*. 2001;34(8):1581–1594.
- [34] Burde D, de Graaf W. Classification of Novikov algebras. *Appl Algebra Eng Commun Comput*. 2013;24(1):1–15.
- [35] Beneš T, Burde D. Classification of orbit closures in the variety of three-dimensional Novikov algebras. *J Algebra Appl*. 2014;13(2):1350081. 33 pp.
- [36] Karimjanov I, Kaygorodov I, Khudoyberdiyev A. The algebraic and geometric classification of nilpotent Novikov algebras. *J Geom Phys*. 2019;143:11–21.
- [37] Dzhumadildaev A. Codimension growth and non-Koszulity of Novikov operad. *Commun Algebra*. 2011;39(8):2943–2952.
- [38] Dzhumadildaev A, Ismailov N. S_n - and GL_n -module structures on free Novikov algebras. *J Algebra*. 2014;416:287–313.
- [39] Dzhumadildaev A, Löfwall C. Trees, free right-symmetric algebras, free Novikov algebras and identities. *Homol Homot Appl*. 2002;4(2):165–190.
- [40] Dzhumadildaev A, Tulenbaev K. Engel theorem for Novikov algebras. *Commun Algebra*. 2006;34(3):883–888.
- [41] Adashev J, Camacho L, Omirov B. Central extensions of null-filiform and naturally graded filiform non-Lie Leibniz algebras. *J Algebra*. 2017;479:461–486.
- [42] Hegazi A, Abdelwahab H. The classification of n -dimensional non-associative Jordan algebras with $(n - 3)$ -dimensional annihilator. *Commun Algebra*. 2018;46(2):629–643.
- [43] Hegazi A, Abdelwahab H, Calderón Martín A. The classification of n -dimensional non-Lie Malcev algebras with $(n - 4)$ -dimensional annihilator. *Linear Algebra Appl*. 2016;505:32–56.
- [44] Kaygorodov I, Rakhimov I, Said Husain SK. The algebraic classification of nilpotent associative commutative algebras. *J Algebra Appl*. 2019;2050220. DOI:10.1142/S0219498820502205
- [45] Skjelbred T, Sund T. Sur la classification des algèbres de Lie nilpotentes. *C R Acad Sci Paris Ser A-B*. 1978;286(5):A241–A242.
- [46] Zusmanovich P. Central extensions of current algebras. *Trans Am Math Soc*. 1992;334(1):143–152.
- [47] Calderón Martín A, Fernández Ouaridi A, Kaygorodov I. The classification of n -dimensional anticommutative algebras with $(n - 3)$ -dimensional annihilator. *Commun Algebra*. 2019;47(1):173–181.
- [48] Calderón Martín A, Fernández Ouaridi A, Kaygorodov I. On the classification of bilinear maps with radical of a fixed codimension. *Linear Multilinear Algebra*; 2020. arXiv:1806.07009.
- [49] Kaygorodov I, Khrypchenko M, Lopes S. The algebraic and geometric classification of nilpotent anticommutative algebras. *J Pure Appl Algebra*. 2020.
- [50] Camacho L, Kaygorodov I, Lopatkin V, et al. The variety of dual Mock-Lie algebras. *Commun Math*. 2020. arXiv:1910.01484.

Appendix

Table A1. The list of cohomology spaces of four-dimensional one-generated nilpotent Novikov algebras.

$Z^2(\mathcal{N}_{01}^4)$	$=$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13}, \Delta_{14} - \Delta_{41}, \Delta_{22} + \Delta_{31} + 2\Delta_{41} \rangle$
$B^2(\mathcal{N}_{01}^4)$	$=$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21} \rangle$
$H^2(\mathcal{N}_{01}^4)$	$=$	$\langle [\Delta_{13}], [\Delta_{14}] - [\Delta_{41}], [\Delta_{22}] + [\Delta_{31}] + 2[\Delta_{41}] \rangle$
$Z^2(\mathcal{N}_{02}^4)$	$=$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13} - \Delta_{31}, \Delta_{21}, 2\Delta_{14} - \Delta_{23} - \Delta_{41} \rangle$
$B^2(\mathcal{N}_{02}^4)$	$=$	$\langle \Delta_{11}, \Delta_{13} - \Delta_{31}, \Delta_{21} \rangle$
$H^2(\mathcal{N}_{02}^4)$	$=$	$\langle [\Delta_{12}], 2[\Delta_{14}] - [\Delta_{23}] - [\Delta_{41}] \rangle$
$Z^2(\mathcal{N}_{03}^4(\lambda)_{\lambda \notin \{0,1\}})$	$=$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, (2 - \lambda)\Delta_{13} + \lambda\Delta_{22} + \lambda\Delta_{31}, (3 - 2\lambda)\Delta_{14} + (2 - \lambda)\lambda\Delta_{23} + \lambda\Delta_{32} + \lambda\Delta_{41} \rangle$
$B^2(\mathcal{N}_{03}^4(\lambda)_{\lambda \notin \{0,1\}})$	$=$	$\langle \Delta_{11}, \Delta_{12} + \lambda\Delta_{21}, (2 - \lambda)\Delta_{13} + \lambda\Delta_{22} + \lambda\Delta_{31} \rangle$
$H^2(\mathcal{N}_{03}^4(\lambda)_{\lambda \notin \{0,1\}})$	$=$	$\langle [\Delta_{21}], (3 - 2\lambda)\Delta_{14} + (2 - \lambda)\lambda\Delta_{23} + \lambda\Delta_{32} + \lambda\Delta_{41} \rangle$
$Z^2(\mathcal{N}_{03}^4(0))$	$=$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{21}, 2\Delta_{23} - 2\Delta_{32} + \Delta_{41} \rangle$
$B^2(\mathcal{N}_{03}^4(0))$	$=$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13} \rangle$
$H^2(\mathcal{N}_{03}^4(0))$	$=$	$\langle [\Delta_{14}], [\Delta_{21}], 2[\Delta_{23}] - 2[\Delta_{32}] + [\Delta_{41}] \rangle$
$Z^2(\mathcal{N}_{03}^4(1))$	$=$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \Delta_{22} + \Delta_{31}, \Delta_{14} + \Delta_{23} + \Delta_{32} + \Delta_{41} \rangle$
$B^2(\mathcal{N}_{03}^4(1))$	$=$	$\langle \Delta_{11}, \Delta_{12} + \Delta_{21}, \Delta_{13} + \Delta_{22} + \Delta_{31} \rangle$
$H^2(\mathcal{N}_{03}^4(1))$	$=$	$\langle [\Delta_{21}], [\Delta_{14}] + [\Delta_{23}] + [\Delta_{32}] + [\Delta_{41}] \rangle$
$Z^2(\mathcal{N}_{04}^4)$	$=$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{21}, 2\Delta_{14} + \Delta_{22} + \Delta_{31}, \Delta_{14} - 2\Delta_{23} + 2\Delta_{32} - \Delta_{41} \rangle$
$B^2(\mathcal{N}_{04}^4)$	$=$	$\langle \Delta_{11}, \Delta_{12}, 2\Delta_{13} + \Delta_{21} \rangle$
$H^2(\mathcal{N}_{04}^4)$	$=$	$\langle [\Delta_{13}], 2[\Delta_{14}] + [\Delta_{22}] + [\Delta_{31}], [\Delta_{14}] - 2[\Delta_{23}] + 2[\Delta_{32}] - [\Delta_{41}] \rangle$
$Z^2(\mathcal{N}_{05}^4)$	$=$	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \Delta_{22} + \Delta_{31}, -2\Delta_{13} + \Delta_{14} + \Delta_{23} + \Delta_{32} + \Delta_{41} \rangle$
$B^2(\mathcal{N}_{05}^4)$	$=$	$\langle \Delta_{11}, \Delta_{12} + \Delta_{21}, \Delta_{13} + \Delta_{21} + \Delta_{22} + \Delta_{31} \rangle$
$H^2(\mathcal{N}_{05}^4)$	$=$	$\langle [\Delta_{21}], -2[\Delta_{13}] + [\Delta_{14}] + [\Delta_{23}] + [\Delta_{32}] + [\Delta_{41}] \rangle$

Table A2. The list of five-dimensional one-generated nilpotent Novikov algebras.

$\mathcal{N}_{01}^5$	:	$e_1e_1 = e_2$	$e_1e_2 = e_4$	$e_1e_3 = e_5$	$e_2e_1 = e_3$	$e_3e_1 = -e_5$
$\mathcal{N}_{02}^5(\lambda)$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = (2 - \lambda)e_5$		
		$e_2e_1 = \lambda e_3 + e_4$	$e_2e_2 = \lambda e_5$	$e_3e_1 = \lambda e_5$		
$\mathcal{N}_{03}^5(\lambda, \mu)$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = \lambda e_5$	$e_1e_4 = \mu e_5$	
		$e_2e_1 = e_4$	$e_2e_2 = e_5$	$e_3e_1 = e_5$	$e_4e_1 = (2 - \mu)e_5$	
$\mathcal{N}_{04}^5(\lambda)_{\lambda \neq 0}$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_5$	$e_1e_4 = \lambda e_5$	
		$e_2e_1 = e_4$	$e_4e_1 = -\lambda e_5$			
$\mathcal{N}_{05}^5$	:	$e_1e_1 = e_2$	$e_1e_3 = e_4$	$e_1e_4 = 2e_5$	$e_2e_1 = e_3$	
		$e_2e_3 = -e_5$	$e_3e_1 = -e_4$	$e_4e_1 = -e_5$		
$\mathcal{N}_{06}^5$	:	$e_1e_1 = e_2$	$e_1e_2 = e_5$	$e_1e_3 = e_4$	$e_1e_4 = 2e_5$	
		$e_2e_1 = e_3$	$e_2e_3 = -e_5$	$e_3e_1 = -e_4$	$e_4e_1 = -e_5$	
$\mathcal{N}_{07}^5(\lambda)$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = (2 - \lambda)e_4$	$e_1e_4 = (3 - 2\lambda)e_5$	$e_2e_1 = \lambda e_3$
		$e_2e_2 = \lambda e_4$	$e_2e_3 = \lambda(2 - \lambda)e_5$	$e_3e_1 = \lambda e_4$	$e_3e_2 = \lambda e_5$	$e_4e_1 = \lambda e_5$
$\mathcal{N}_{08}^5$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	$e_2e_1 = e_5$
$\mathcal{N}_{09}^5(\lambda)$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = 2e_4$	$e_1e_4 = \lambda e_5$	
		$e_2e_3 = 2e_5$	$e_3e_2 = -2e_5$	$e_4e_1 = e_5$		
$\mathcal{N}_{10}^5$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4$	$e_1e_4 = e_5$	$e_2e_1 = e_3 + e_5$
		$e_2e_2 = e_4$	$e_2e_3 = e_5$	$e_3e_1 = e_4$	$e_3e_2 = e_5$	$e_4e_1 = e_5$
$\mathcal{N}_{11}^5(\lambda)$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = 2e_4 + \lambda e_5$	$e_1e_4 = 2e_5$	
		$e_2e_1 = e_4$	$e_2e_2 = e_5$	$e_3e_1 = e_5$		
$\mathcal{N}_{12}^5(\lambda)$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = 2e_4$	$e_1e_4 = (2\lambda + 1)e_5$	$e_2e_1 = e_4$
		$e_2e_2 = \lambda e_5$	$e_2e_3 = -2e_5$	$e_3e_1 = \lambda e_5$	$e_3e_2 = 2e_5$	$e_4e_1 = -e_5$
$\mathcal{N}_{13}^5(\lambda)$	:	$e_1e_1 = e_2$	$e_1e_2 = e_3$	$e_1e_3 = e_4 - 2e_5$	$e_1e_4 = e_5$	$e_2e_1 = e_3 + e_4 + \lambda e_5$
		$e_2e_2 = e_4$	$e_2e_3 = e_5$	$e_3e_1 = e_4$	$e_3e_2 = e_5$	$e_4e_1 = e_5$

$$\mathcal{N}_{04}^5(0) \cong \mathcal{N}_{02}^5(0).$$

Table A3. The list of cohomology spaces of five-dimensional one-generated nilpotent Novikov algebras.

$Z^2(\mathcal{N}_{01}^5)$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13} - \Delta_{31}, \Delta_{14}, \Delta_{21}, 2\Delta_{13} + \Delta_{22} + \Delta_{41}, 2\Delta_{15} - \Delta_{23} - \Delta_{51} \rangle$
$B^2(\mathcal{N}_{01}^5)$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13} - \Delta_{31}, \Delta_{21} \rangle$
$H^2(\mathcal{N}_{01}^5)$	=	$\langle [\Delta_{14}], 2[\Delta_{13}] + [\Delta_{22}] + [\Delta_{41}], 2[\Delta_{15}] - [\Delta_{23}] - [\Delta_{51}] \rangle$
$Z^2(\mathcal{N}_{02}^5(\lambda)_{\lambda \neq 0})$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{14} - \Delta_{41}, \Delta_{13} - \lambda\Delta_{41}, \Delta_{22} + \Delta_{31} + (2 - \lambda)\Delta_{41}, (3 - 2\lambda)\Delta_{15} + \lambda(2 - \lambda)\Delta_{23} + \lambda\Delta_{32} + \lambda\Delta_{51} \rangle$
$B^2(\mathcal{N}_{02}^5(\lambda)_{\lambda \neq 0})$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, (2 - \lambda)\Delta_{13} + \lambda\Delta_{22} + \lambda\Delta_{31} \rangle$
$H^2(\mathcal{N}_{02}^5(\lambda)_{\lambda \neq 0})$	=	$\langle [\Delta_{14}] - [\Delta_{41}], [\Delta_{13}] - \lambda[\Delta_{41}], (3 - 2\lambda)[\Delta_{15}] + \lambda(2 - \lambda)[\Delta_{23}] + \lambda[\Delta_{32}] + \lambda[\Delta_{51}] \rangle$
$Z^2(\mathcal{N}_{02}^5(0))$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{14} - \Delta_{41}, \Delta_{13}, 2\Delta_{14} + \Delta_{22} + \Delta_{31}, \Delta_{15}, 2\Delta_{23} - 2\Delta_{32} + \Delta_{51} \rangle$
$B^2(\mathcal{N}_{02}^5(0))$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} \rangle$
$B^2(\mathcal{N}_{02}^5(0))$	=	$\langle [\Delta_{14}] - [\Delta_{41}], 2[\Delta_{14}] + [\Delta_{22}] + [\Delta_{31}], [\Delta_{15}], 2[\Delta_{23}] - 2[\Delta_{32}] + [\Delta_{51}] \rangle$
$Z^2(\mathcal{N}_{03}^5(\lambda, \mu))$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13}, \Delta_{14} - \Delta_{41}, \Delta_{22} + \Delta_{31} + 2\Delta_{41}, (2\mu - 1)\Delta_{15} + \lambda(2 - \mu)\Delta_{23} + \mu(2 - \mu)\Delta_{24} + (3 - (\lambda + 1)(2 - \mu))\Delta_{32} + (2 - \mu)\Delta_{42} + (2 - \mu)\Delta_{51} \rangle$
$B^2(\mathcal{N}_{03}^5(\lambda, \mu))$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \lambda\Delta_{13} + \mu\Delta_{14} + \Delta_{22} + \Delta_{31} + (2 - \mu)\Delta_{41} \rangle$
$H^2(\mathcal{N}_{03}^5(\lambda, \mu))$	=	$\langle [\Delta_{13}], [\Delta_{14}] - [\Delta_{41}], (2\mu - 1)[\Delta_{15}] + \lambda(2 - \mu)[\Delta_{23}] + \mu(2 - \mu)[\Delta_{24}] + (3 - (\lambda + 1)(2 - \mu))[\Delta_{32}] + (2 - \mu)[\Delta_{42}] + (2 - \mu)[\Delta_{51}] \rangle$
$Z^2(\mathcal{N}_{04}^5(\lambda)_{\lambda \neq 0})$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{21}, \Delta_{14} - \Delta_{41}, \Delta_{22} + \Delta_{31} + 2\Delta_{41}, -2\Delta_{15} + \Delta_{23} + \lambda\Delta_{24} - \Delta_{32} + \Delta_{51} \rangle$
$B^2(\mathcal{N}_{04}^5(\lambda)_{\lambda \neq 0})$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \lambda(\Delta_{14} - \Delta_{41}) \rangle$
$H^2(\mathcal{N}_{04}^5(\lambda)_{\lambda \neq 0})$	=	$\langle [\Delta_{14}] - [\Delta_{41}], [\Delta_{22}] + [\Delta_{31}] + 2[\Delta_{41}] - 2[\Delta_{15}] + [\Delta_{23}] + \lambda[\Delta_{24}] - [\Delta_{32}] + [\Delta_{51}] \rangle$
$Z^2(\mathcal{N}_{04}^5(0))$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{15}, \Delta_{21}, \Delta_{14} - \Delta_{41}, 2\Delta_{14} + \Delta_{22} + \Delta_{31}, 2\Delta_{23} - 2\Delta_{32} + \Delta_{51} \rangle$
$B^2(\mathcal{N}_{04}^5(0))$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{21} \rangle$
$H^2(\mathcal{N}_{04}^5(0))$	=	$\langle [\Delta_{15}], [\Delta_{14}] - [\Delta_{41}], 2[\Delta_{14}] + [\Delta_{22}] + [\Delta_{31}], 2[\Delta_{23}] - 2[\Delta_{32}] + [\Delta_{51}] \rangle$
$Z^2(\mathcal{N}_{05}^5)$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13} - \Delta_{31}, 2\Delta_{14} - \Delta_{23} - \Delta_{41}, \Delta_{21}, 3\Delta_{15} - 2\Delta_{24} + \Delta_{33} - \Delta_{51} \rangle$
$B^2(\mathcal{N}_{05}^5)$	=	$\langle \Delta_{11}, \Delta_{21}, \Delta_{13} - \Delta_{31}, 2\Delta_{14} - \Delta_{23} - \Delta_{41} \rangle$
$H^2(\mathcal{N}_{05}^5)$	=	$\langle [\Delta_{12}], 3[\Delta_{15}] - 2[\Delta_{24}] + [\Delta_{33}] - [\Delta_{51}] \rangle$
$Z^2(\mathcal{N}_{06}^5)$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13} - \Delta_{31}, \Delta_{21}, 2\Delta_{14} - \Delta_{23} - \Delta_{41}, 3\Delta_{15} - \Delta_{22} - 2\Delta_{24} - 2\Delta_{31} + \Delta_{33} - \Delta_{51} \rangle$
$B^2(\mathcal{N}_{06}^5)$	=	$\langle \Delta_{11}, \Delta_{21}, \Delta_{13} - \Delta_{31}, \Delta_{12} + 2\Delta_{14} - \Delta_{23} - \Delta_{41} \rangle$
$H^2(\mathcal{N}_{06}^5)$	=	$\langle [\Delta_{12}], 3[\Delta_{15}] - [\Delta_{22}] - 2[\Delta_{24}] - 2[\Delta_{31}] + [\Delta_{33}] - [\Delta_{51}] \rangle$
$Z^2(\mathcal{N}_{07}^5(\lambda)_{\lambda \neq 0})$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, (2 - \lambda)\Delta_{13} + \lambda\Delta_{22} + \lambda\Delta_{31}, (3 - 2\lambda)\Delta_{14} + \lambda(2 - \lambda)\Delta_{23} + \lambda\Delta_{32} + \lambda\Delta_{41}, (4 - 3\lambda)\Delta_{15} + \lambda(3 - 2\lambda)\Delta_{24} + \lambda(2 - \lambda)\Delta_{33} + \lambda\Delta_{42} + \lambda\Delta_{51} \rangle$
$B^2(\mathcal{N}_{07}^5(\lambda)_{\lambda \neq 0})$	=	$\langle \Delta_{11}, \Delta_{12} + \lambda\Delta_{21}, (2 - \lambda)\Delta_{13} + \lambda\Delta_{22} + \lambda\Delta_{31}, (3 - 2\lambda)\Delta_{14} + \lambda(2 - \lambda)\Delta_{23} + \lambda\Delta_{32} + \lambda\Delta_{41} \rangle$
$H^2(\mathcal{N}_{07}^5(\lambda)_{\lambda \neq 0})$	=	$\langle [\Delta_{21}], (4 - 3\lambda)[\Delta_{15}] + \lambda(3 - 2\lambda)[\Delta_{24}] + \lambda(2 - \lambda)[\Delta_{33}] + \lambda[\Delta_{42}] + \lambda[\Delta_{51}] \rangle$
$Z^2(\mathcal{N}_{07}^5(0))$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{15}, \Delta_{21}, 2\Delta_{23} - 2\Delta_{32} + \Delta_{41} \rangle$
$B^2(\mathcal{N}_{07}^5(0))$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14} \rangle$
$H^2(\mathcal{N}_{07}^5(0))$	=	$\langle [\Delta_{15}], [\Delta_{21}], 2[\Delta_{23}] - 2[\Delta_{32}] + [\Delta_{41}] \rangle$
$Z^2(\mathcal{N}_{08}^5)$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{21}, 2\Delta_{15} + \Delta_{22} + \Delta_{31}, \Delta_{23} - \Delta_{32} + \Delta_{41} \rangle$
$B^2(\mathcal{N}_{08}^5)$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14} + \Delta_{21} \rangle$
$H^2(\mathcal{N}_{08}^5)$	=	$\langle [\Delta_{21}], 2[\Delta_{15}] + [\Delta_{22}] + [\Delta_{31}], [\Delta_{23}] - [\Delta_{32}] + [\Delta_{41}] \rangle$
$Z^2(\mathcal{N}_{09}^5(\lambda))$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{14}, \Delta_{21}, 2\Delta_{23} - 2\Delta_{32} + \Delta_{41} \rangle$
$B^2(\mathcal{N}_{09}^5(\lambda))$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \lambda\Delta_{14} + 2\Delta_{23} - 2\Delta_{32} + \Delta_{41} \rangle$

(continued).

Table A3. Continued.

$Z^2(\mathcal{N}_{01}^5)$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13} - \Delta_{31}, \Delta_{14}, \Delta_{21}, 2\Delta_{13} + \Delta_{22} + \Delta_{41}, 2\Delta_{15} - \Delta_{23} - \Delta_{51} \rangle$
$H^2(\mathcal{N}_{09}^5(\lambda))$	=	$\langle [\Delta_{14}], [\Delta_{21}] \rangle$
$Z^2(\mathcal{N}_{10}^5)$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \Delta_{22} + \Delta_{31}, \Delta_{14} + \Delta_{23} + \Delta_{32} + \Delta_{41}, -2\Delta_{13} + \Delta_{15} + \Delta_{24} + \Delta_{33} + \Delta_{42} + \Delta_{51} \rangle$
$B^2(\mathcal{N}_{10}^5)$	=	$\langle \Delta_{11}, \Delta_{12} + \Delta_{21}, \Delta_{13} + \Delta_{22} + \Delta_{31}, \Delta_{14} + \Delta_{21} + \Delta_{23} + \Delta_{32} + \Delta_{41} \rangle$
$H^2(\mathcal{N}_{10}^5)$	=	$\langle [\Delta_{12}], -2[\Delta_{13}] + [\Delta_{15}] + [\Delta_{24}] + [\Delta_{33}] + [\Delta_{42}] + [\Delta_{51}] \rangle$
$Z^2(\mathcal{N}_{11}^5(\lambda))$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{15} + \Delta_{32}, \Delta_{21}, 2\Delta_{14} + \Delta_{22} + \Delta_{31}, \Delta_{14} - 2\Delta_{23} + 2\Delta_{32} - \Delta_{41} \rangle$
$B^2(\mathcal{N}_{11}^5(\lambda))$	=	$\langle \Delta_{11}, \Delta_{12}, 2\Delta_{13} + \Delta_{21}, \lambda\Delta_{13} + 2\Delta_{14} + \Delta_{22} + \Delta_{31} \rangle$
$H^2(\mathcal{N}_{11}^5(\lambda))$	=	$\langle [\Delta_{13}], [\Delta_{15}] + [\Delta_{32}], [\Delta_{14}] - 2[\Delta_{23}] + 2[\Delta_{32}] - [\Delta_{41}] \rangle$
$Z^2(\mathcal{N}_{12}^5(\lambda))$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{13}, \Delta_{21}, \Delta_{14} - 2\Delta_{23} + 2\Delta_{32} - \Delta_{41}, \Delta_{22} + \Delta_{31} + 2\Delta_{14} \rangle$
$B^2(\mathcal{N}_{12}^5(\lambda))$	=	$\langle \Delta_{11}, \Delta_{12}, 2\Delta_{13} + \Delta_{21}, (1 + 2\lambda)\Delta_{14} + \lambda\Delta_{22} - 2\Delta_{23} + \lambda\Delta_{31} + 2\Delta_{32} - \Delta_{41} \rangle$
$H^2(\mathcal{N}_{12}^5(\lambda))$	=	$\langle [\Delta_{13}], [\Delta_{14}] - 2[\Delta_{23}] + 2[\Delta_{32}] - [\Delta_{41}] \rangle$
$Z^2(\mathcal{N}_{13}^5(\lambda))$	=	$\langle \Delta_{11}, \Delta_{12}, \Delta_{21}, \Delta_{13} + \Delta_{22} + \Delta_{31}, -2\Delta_{13} + \Delta_{14} + \Delta_{23} + \Delta_{32} + \Delta_{41}, (1 - 2\lambda)\Delta_{13} - 2\Delta_{14} + \Delta_{15} - \Delta_{23} + \Delta_{24} + \Delta_{33} + \Delta_{41} + \Delta_{42} + \Delta_{51} \rangle$
$B^2(\mathcal{N}_{13}^5(\lambda))$	=	$\langle \Delta_{11}, \Delta_{12} + \Delta_{21}, \Delta_{13} + \Delta_{21} + \Delta_{22} + \Delta_{31}, -2\Delta_{13} + \Delta_{14} + \lambda\Delta_{21} + \Delta_{23} + \Delta_{32} + \Delta_{41} \rangle$
$H^2(\mathcal{N}_{13}^5(\lambda))$	=	$\langle [\Delta_{12}], (1 - 2\lambda)[\Delta_{13}] - 2[\Delta_{14}] + [\Delta_{15}] - [\Delta_{23}] + [\Delta_{24}] + [\Delta_{33}] + [\Delta_{41}] + [\Delta_{42}] + [\Delta_{51}] \rangle$

Table A4. The list of six-dimensional one-generated nilpotent Novikov algebras.

$\mathcal{N}_{01}^6$	: $e_1e_1 = e_2$ $e_2e_1 = e_4$	$e_1e_2 = e_3$ $e_4e_1 = -e_6$	$e_1e_3 = e_5$	$e_1e_4 = e_6$	
$\mathcal{N}_{02}^6(\lambda)$	: $e_1e_1 = e_2$ $e_2e_1 = e_4$	$e_1e_2 = e_3$ $e_2e_2 = e_6$	$e_1e_3 = e_5$ $e_3e_1 = e_6$	$e_1e_4 = \lambda e_6$ $e_4e_1 = (2 - \lambda)e_6$	
$\mathcal{N}_{03}^6(\lambda, \mu)$	: $e_1e_1 = e_2$ $e_2e_1 = e_4$	$e_1e_2 = e_3$ $e_2e_2 = e_6$	$e_1e_3 = \lambda e_5 + \mu e_6$ $e_3e_1 = e_6$	$e_1e_4 = e_5$ $e_4e_1 = -e_5 + 2e_6$	
$\mathcal{N}_{04}^6$	: $e_1e_1 = e_2$ $e_2e_1 = e_3$	$e_1e_2 = e_5$ $e_2e_3 = -e_6$	$e_1e_3 = e_4$ $e_3e_1 = -e_4$	$e_1e_4 = 2e_6$ $e_4e_1 = -e_6$	
$\mathcal{N}_{05}^6(\lambda)$	: $e_1e_1 = e_2$ $e_2e_2 = \lambda e_4$	$e_1e_2 = e_3$ $e_2e_3 = \lambda(2 - \lambda)e_5$	$e_1e_3 = (2 - \lambda)e_4$ $e_3e_1 = \lambda e_4$	$e_1e_4 = (3 - 2\lambda)e_5$ $e_3e_2 = \lambda e_5$	$e_2e_1 = \lambda e_3 + e_6$ $e_4e_1 = \lambda e_5$
$\mathcal{N}_{06}^6(\lambda)$	: $e_1e_1 = e_2$ $e_2e_3 = 2e_5$	$e_1e_2 = e_3$ $e_3e_2 = -2e_5$	$e_1e_3 = 2e_4$ $e_4e_1 = e_5$	$e_1e_4 = \lambda e_5$	$e_2e_1 = e_6$
$\mathcal{N}_{07}^6$	: $e_1e_1 = e_2$ $e_2e_1 = e_5$	$e_1e_2 = e_3$ $e_2e_3 = 2e_6$	$e_1e_3 = 2e_4$ $e_3e_2 = -2e_6$	$e_1e_4 = e_5$ $e_4e_1 = e_6$	
$\mathcal{N}_{08}^6$	: $e_1e_1 = e_2$ $e_2e_3 = 2e_6$	$e_1e_2 = e_3$ $e_3e_2 = -2e_6$	$e_1e_3 = 2e_4$ $e_4e_1 = e_6$	$e_1e_4 = e_5$	
$\mathcal{N}_{09}^6$	: $e_1e_1 = e_2$ $e_2e_1 = e_4$	$e_1e_2 = e_3$ $e_2e_2 = e_5$	$e_1e_3 = 2e_4 + e_6$ $e_3e_1 = e_5$	$e_1e_4 = 2e_5$	
$\mathcal{N}_{10}^6(\lambda)$	: $e_1e_1 = e_2$ $e_2e_2 = \lambda e_5$	$e_1e_2 = e_3$ $e_2e_3 = -2e_5$	$e_1e_3 = 2e_4 + e_6$ $e_3e_1 = \lambda e_5$	$e_1e_4 = (2\lambda + 1)e_5$ $e_3e_2 = 2e_5$	$e_2e_1 = e_4$ $e_4e_1 = -e_5$
$\mathcal{N}_{11}^6(\lambda)$	: $e_1e_1 = e_2$ $e_2e_2 = e_6$	$e_1e_2 = e_3$ $e_2e_3 = -2e_5$	$e_1e_3 = 2e_4 + \lambda e_6$ $e_3e_1 = e_6$	$e_1e_4 = e_5 + 2e_6$ $e_3e_2 = 2e_5$	$e_2e_1 = e_4$ $e_4e_1 = -e_5$
$\mathcal{N}_{12}^6$	: $e_1e_1 = e_2$ $e_2e_2 = e_4$	$e_1e_2 = e_3$ $e_2e_3 = e_5$	$e_1e_3 = e_4 - 2e_5$ $e_3e_1 = e_4$	$e_1e_4 = e_5$ $e_3e_2 = e_5$	$e_2e_1 = e_3 + e_4 + e_6$ $e_4e_1 = e_5$
$\mathcal{N}_{13}^6$	: $e_1e_1 = e_2$ $e_2e_1 = e_3$	$e_1e_2 = e_4$ $e_2e_3 = -e_6$	$e_1e_3 = e_5$ $e_3e_1 = -e_5$	$e_1e_4 = e_6$ $e_5e_1 = -e_6$	$e_1e_5 = 2e_6$
$\mathcal{N}_{14}^6(\alpha)$	: $e_1e_1 = e_2$ $e_1e_5 = e_6$	$e_1e_2 = e_3$ $e_2e_1 = e_3 + e_4$	$e_1e_3 = e_5 + e_6$ $e_2e_2 = e_5$	$e_1e_4 = \alpha e_6$ $e_2e_3 = e_6$	
$\mathcal{N}_{15}^6$	: $e_1e_1 = e_2$ $e_2e_1 = 3e_3 + e_4$ $e_3e_2 = 3e_6$	$e_1e_2 = e_3$ $e_2e_2 = 3e_5$ $e_4e_1 = -3e_6$	$e_1e_3 = -e_5 + e_6$ $e_2e_3 = -3e_6$ $e_5e_1 = 3e_6$	$e_1e_5 = -3e_6$ $e_3e_1 = 3e_5$	
$\mathcal{N}_{16}^6(\lambda)_{\lambda \neq 3}$	: $e_1e_1 = e_2$ $e_1e_5 = (3 - 2\lambda)e_6$ $e_3e_1 = \lambda e_5$	$e_1e_2 = e_3$ $e_2e_1 = \lambda e_3 + e_4$ $e_3e_2 = \lambda e_6$	$e_1e_3 = (2 - \lambda)e_5$ $e_2e_2 = \lambda e_5$ $e_4e_1 = -e_6$	$e_1e_4 = e_6$ $e_2e_3 = \lambda(2 - \lambda)e_6$ $e_5e_1 = \lambda e_6$	
$\mathcal{N}_{17}^6(\alpha)$	: $e_1e_1 = e_2$ $e_1e_5 = \alpha e_6$ $e_3e_2 = -2e_6$	$e_1e_2 = e_3$ $e_2e_1 = e_4$ $e_4e_1 = -e_6$	$e_1e_3 = 2e_5$ $e_2e_3 = 2e_6$ $e_5e_1 = e_6$	$e_1e_4 = e_6$ $e_3e_1 = e_6$	
$\mathcal{N}_{18}^6(\alpha)$	: $e_1e_1 = e_2$ $e_2e_1 = e_4$	$e_1e_2 = e_3$ $e_2e_2 = e_6$	$e_1e_3 = 2e_5$ $e_3e_1 = e_6$	$e_1e_4 = (\alpha + 2)e_6$ $e_4e_1 = -\alpha e_6$	$e_1e_5 = e_6$
$\mathcal{N}_{19}^6(\alpha, \beta)$	: $e_1e_1 = e_2$ $e_1e_5 = \beta e_6$ $e_3e_1 = e_6$	$e_1e_2 = e_3$ $e_2e_1 = e_4$ $e_3e_2 = -2e_6$	$e_1e_3 = 2e_5$ $e_2e_2 = e_6$ $e_4e_1 = -\alpha e_6$	$e_1e_4 = (\alpha + 2)e_6$ $e_2e_3 = 2e_6$ $e_5e_1 = e_6$	
$\mathcal{N}_{20}^6(\lambda, \mu)$	: $e_1e_1 = e_2$ $e_2e_1 = e_4$ $e_3e_1 = e_5$	$e_1e_2 = e_3$ $e_2e_2 = e_5$ $e_4e_1 = (2 - \mu)e_5$	$e_1e_3 = \lambda e_5$ $e_2e_3 = \lambda(2 - \mu)e_6$ $e_3e_2 = (3 - (\lambda + 1)(2 - \mu))e_6$	$e_1e_4 = \mu e_5$ $e_2e_4 = \mu(2 - \mu)e_6$ $e_4e_2 = (2 - \mu)e_6$	$e_1e_5 = (2\mu - 1)e_6$ $e_5e_1 = (2 - \mu)e_6$

(continued)

Table A4. Continued

$\mathcal{N}_{21}^6(\lambda, \mu)_{\mu \neq 2}$	$e_1 e_1 = e_2$ $e_2 e_1 = e_4$ $e_3 e_1 = e_5$	$e_1 e_2 = e_3$ $e_2 e_2 = e_5$ $e_4 e_1 = (2 - \mu)e_5$	$e_1 e_3 = \lambda e_5 + e_6$ $e_2 e_3 = \lambda(2 - \mu)e_6$ $e_3 e_2 = (3 - (\lambda + 1)(2 - \mu))e_6$	$e_1 e_4 = \mu e_5$ $e_2 e_4 = \mu(2 - \mu)e_6$ $e_4 e_2 = (2 - \mu)e_6$	$e_1 e_5 = (2\mu - 1)e_6$ $e_5 e_1 = (2 - \mu)e_6$
$\mathcal{N}_{22}^6(\lambda)$	$e_1 e_1 = e_2$ $e_2 e_1 = e_4$ $e_3 e_1 = e_5$	$e_1 e_2 = e_3$ $e_2 e_2 = e_5$ $e_4 e_1 = e_5 - e_6$	$e_1 e_3 = \lambda e_5$ $e_2 e_3 = \lambda e_6$ $e_3 e_2 = (2 - \lambda)e_6$	$e_1 e_4 = e_5 + e_6$ $e_2 e_4 = e_6$ $e_4 e_2 = e_6$	$e_1 e_5 = e_6$ $e_5 e_1 = e_6$
$\mathcal{N}_{23}^6(\nu)_{\nu \neq 0}$	$e_1 e_1 = e_2$ $e_2 e_1 = e_4$ $e_3 e_1 = e_5$	$e_1 e_2 = e_3$ $e_2 e_2 = e_5$ $e_4 e_1 = e_5 - e_6$	$e_1 e_3 = e_5 + \nu e_6$ $e_2 e_3 = e_6$ $e_3 e_2 = e_6$	$e_1 e_4 = e_5 + e_6$ $e_2 e_4 = e_6$ $e_4 e_2 = e_6$	$e_1 e_5 = e_6$ $e_5 e_1 = e_6$
$\mathcal{N}_{24}^6(\lambda)$	$e_1 e_1 = e_2$ $e_2 e_1 = e_4$	$e_1 e_2 = e_3$ $e_2 e_2 = e_5$	$e_1 e_3 = \lambda e_5$ $e_3 e_1 = e_5$	$e_1 e_4 = 2e_5 + e_6$ $e_3 e_2 = e_6$	$e_1 e_5 = e_6$
$\mathcal{N}_{25}^6(\lambda)_{\lambda \neq 0}$	$e_1 e_1 = e_2$ $e_2 e_1 = e_4$ $e_4 e_1 = -\lambda e_5$	$e_1 e_2 = e_3$ $e_2 e_3 = e_6$ $e_5 e_1 = e_6$	$e_1 e_3 = e_5$ $e_2 e_4 = \lambda e_6$	$e_1 e_4 = \lambda e_5$ $e_3 e_2 = -e_6$	$e_1 e_5 = -2e_6$
$\mathcal{N}_{26}^6(\lambda)_{\lambda \neq 0}$	$e_1 e_1 = e_2$ $e_2 e_1 = e_4$ $e_4 e_1 = -\lambda e_5 - e_6$	$e_1 e_2 = e_3$ $e_2 e_3 = e_6$ $e_5 e_1 = e_6$	$e_1 e_3 = e_5$ $e_2 e_4 = \lambda e_6$	$e_1 e_4 = \lambda e_5 + e_6$ $e_3 e_2 = -e_6$	$e_1 e_5 = -2e_6$
$\mathcal{N}_{27}^6$	$e_1 e_1 = e_2$ $e_2 e_1 = e_3$ $e_3 e_3 = e_6$	$e_1 e_3 = e_4$ $e_2 e_3 = -e_5$ $e_4 e_1 = -e_5$	$e_1 e_4 = 2e_5$ $e_2 e_4 = -2e_6$ $e_5 e_1 = -e_6$	$e_1 e_5 = 3e_6$ $e_3 e_1 = -e_4$	
$\mathcal{N}_{28}^6$	$e_1 e_1 = e_2$ $e_2 e_1 = e_3$ $e_3 e_3 = e_6$	$e_1 e_2 = e_5$ $e_2 e_2 = -e_6$ $e_4 e_1 = -e_5$	$e_1 e_3 = e_4$ $e_2 e_3 = -e_5$ $e_5 e_1 = -e_6$	$e_1 e_4 = 2e_5$ $e_2 e_4 = -2e_6$	$e_1 e_5 = 3e_6$ $e_3 e_1 = -e_4 - 2e_6$
$\mathcal{N}_{29}^6(\lambda)$	$e_1 e_1 = e_2$ $e_2 e_1 = \lambda e_3$ $e_3 e_2 = \lambda e_5$	$e_1 e_2 = e_3$ $e_2 e_2 = \lambda e_4$ $e_3 e_3 = \lambda(2 - \lambda)e_6$	$e_1 e_3 = (2 - \lambda)e_4$ $e_2 e_3 = \lambda(2 - \lambda)e_5$ $e_4 e_1 = \lambda e_5$	$e_1 e_4 = (3 - 2\lambda)e_5$ $e_2 e_4 = \lambda(3 - 2\lambda)e_6$ $e_4 e_2 = \lambda e_6$	$e_1 e_5 = (4 - 3\lambda)e_6$ $e_3 e_1 = \lambda e_4$ $e_5 e_1 = \lambda e_6$
$\mathcal{N}_{30}^6$	$e_1 e_1 = e_2$ $e_2 e_1 = 2e_3 + e_6$ $e_3 e_2 = 2e_5$	$e_1 e_2 = e_3$ $e_2 e_2 = 2e_4$ $e_4 e_1 = 2e_5$	$e_1 e_4 = -e_5$ $e_2 e_4 = -e_6$ $e_4 e_2 = 2e_6$	$e_1 e_5 = -2e_6$ $e_3 e_1 = 2e_4$ $e_5 e_1 = 2e_6$	
$\mathcal{N}_{31}^6$	$e_1 e_1 = e_2$ $e_1 e_5 = e_6$	$e_1 e_2 = e_3$ $e_2 e_3 = 2e_6$	$e_1 e_3 = 2e_4$ $e_3 e_2 = -2e_6$	$e_1 e_4 = 3e_5$ $e_4 e_1 = e_6$	
$\mathcal{N}_{32}^6$	$e_1 e_1 = e_2$ $e_1 e_4 = 3e_5$	$e_1 e_2 = e_3$ $e_1 e_5 = e_6$	$e_1 e_3 = 2e_4$ $e_2 e_1 = e_6$		
$\mathcal{N}_{33}^6$	$e_1 e_1 = e_2$ $e_2 e_1 = e_3 + e_6$ $e_3 e_2 = e_5$	$e_1 e_2 = e_3$ $e_2 e_2 = e_4$ $e_3 e_3 = e_6$	$e_1 e_3 = e_4$ $e_2 e_3 = e_5$ $e_4 e_1 = e_5$	$e_1 e_4 = e_5$ $e_2 e_4 = e_6$ $e_4 e_2 = e_6$	$e_1 e_5 = e_6$ $e_3 e_1 = e_4$ $e_5 e_1 = e_6$
$\mathcal{N}_{34}^6(\lambda)$	$e_1 e_1 = e_2$ $e_1 e_5 = 2e_6$	$e_1 e_2 = e_3$ $e_2 e_1 = e_5 + \lambda e_6$	$e_1 e_3 = e_4$ $e_2 e_2 = e_6$	$e_1 e_4 = e_5$ $e_3 e_1 = e_6$	
$\mathcal{N}_{35}^6(\lambda)_{\lambda \neq 0}$	$e_1 e_1 = e_2$ $e_1 e_5 = 2e_6$ $e_3 e_1 = e_6$	$e_1 e_2 = e_3$ $e_2 e_1 = e_5$ $e_3 e_2 = -\lambda e_6$	$e_1 e_3 = e_4$ $e_2 e_2 = e_6$ $e_4 e_1 = \lambda e_6$	$e_1 e_4 = e_5$ $e_2 e_3 = \lambda e_6$	
$\mathcal{N}_{36}^6$	$e_1 e_1 = e_2$ $e_2 e_1 = e_3 + e_5$ $e_3 e_2 = e_5$	$e_1 e_2 = e_3$ $e_2 e_2 = e_4$ $e_3 e_3 = e_6$	$e_1 e_3 = e_4 - 2e_6$ $e_2 e_3 = e_5$ $e_4 e_1 = e_5$	$e_1 e_4 = e_5$ $e_2 e_4 = e_6$ $e_4 e_2 = e_6$	$e_1 e_5 = e_6$ $e_3 e_1 = e_4$ $e_5 e_1 = e_6$
$\mathcal{N}_{37}^6(\lambda, \mu)$	$e_1 e_1 = e_2$ $e_1 e_5 = e_6$ $e_3 e_1 = e_5$	$e_1 e_2 = e_3$ $e_2 e_1 = e_4$ $e_3 e_2 = (1 + 2\mu)e_6$	$e_1 e_3 = 2e_4 + \lambda e_5$ $e_2 e_2 = e_5$ $e_4 e_1 = -\mu e_6$	$e_1 e_4 = 2e_5 + \mu e_6$ $e_2 e_3 = -2\mu e_6$	
$\mathcal{N}_{38}^6(\lambda)$	$e_1 e_1 = e_2$ $e_2 e_1 = e_3 + e_4 + \lambda e_5$ $e_3 e_2 = e_5$	$e_1 e_2 = e_3$ $e_2 e_2 = e_4$ $e_3 e_3 = e_6$	$e_1 e_3 = e_4 - 2e_5 + (1 - 2\lambda)e_6$ $e_2 e_3 = e_5 - e_6$ $e_4 e_1 = e_5 + e_6$	$e_1 e_4 = e_5 - 2e_6$ $e_2 e_4 = e_6$ $e_4 e_2 = e_6$	$e_1 e_5 = e_6$ $e_3 e_1 = e_4$ $e_5 e_1 = e_6$

$$\begin{array}{llll}
 \mathcal{N}_{16}^6(3) \cong \mathcal{N}_{05}^6(3), & \mathcal{N}_{21}^6(\lambda, 2) \cong \mathcal{N}_{20}^6(\lambda, 2), & \mathcal{N}_{23}^6(0) \cong \mathcal{N}_{22}^6(1), & \mathcal{N}_{25}^6(0) \cong \mathcal{N}_{06}^6(-2), \\
 \mathcal{N}_{26}^6(0) \cong \mathcal{N}_{17}^6(-2), & \mathcal{N}_{35}^6(0) \cong \mathcal{N}_{34}^6(0), & \mathcal{N}_{34}^6(\lambda) \cong \mathcal{N}_{34}^6(-\lambda), & \mathcal{N}_{35}^6(\lambda) \cong \mathcal{N}_{35}^6(-\lambda).
 \end{array}$$