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The uniqueness condition for Gibbs measure of the XY model on Cayley trees

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Maqolada Keli daraxtida XY model qaralgan. Gibbs o‘lchovlari ketma-ketligining muvofiqlashganligini ta’minlovchi cheksiz funksional tenglamalar sistemasi olingan. Bundan tashqari, shu model uchun Gibbs o‘lchovli yagona bo‘ladigan yetarli shart topilgan.

Рассмотрена ХУ модель на дереве Кэли. Получена бесконечная система функциональных уравнений, которая описывает условие, гарантирующее выполнение условия согласованности последовательности мер Гиббса. Кроме того, найдено достаточное условие единстевенности меры Гиббса для этой модели.

The XY model (see [1, 2, 4, 5]) in several applications to real problems in physics are considered. In [6] authors studied forward quantum Markov chains (QMC) defined on Cayley tree. A construction of such QMC is provided. Using the provided construction, they investigate QMC associated with XY-model on a Cayley tree of order two. Moreover they proved uniqueness of QMC associated with such a model, this means the QMC does not depend on the boundary conditions.

The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$, where V is the set of vertices and L the set of edges. Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them. We will use the notation $l =< x, y >$. A collection of nearest neighbor pairs $< x, x_1 >, < x_1, x_2 >, \dots, < x_{d-1}, y >$ is called a *path* from x to y. The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x to y.

For a fixed $x_0 \in V$, called the root, we set

$$W_n = \{x \in V : d(x, x_0) = n\}, \quad V_n = \{x \in V : d(x, x_0) \leq n\}$$

and denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \ x \in W_n,$$

the set of *direct successors* of x.

We consider XY model where the spin σ in each site of the lattice is described by an angle from $[0, 2\pi)$.

For $A \subset V$ a configuration σ_A on A is an arbitrary function $\sigma_A : A \to [0, 2\pi)$. Denote $\Omega_A = [0, 2\pi)^A$ the set of all configurations on A. A configuration σ on V is then defined as a function $x \in V \to \sigma(x) \in [0, 2\pi)$.

The (formal) Hamiltonian of XY-model is

$$H(\sigma) = -J \sum_{<x,y>\in L} \cos(\sigma(x) - \sigma(y)) - h \sum_{x\in V} \cos(\sigma(x)). \tag{1}$$

In this paper we use the will known method (see [7]) of Markov random filed theory and recurrent equations of this theory on trees.

Given $n = 1, 2, \ldots$ consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\left(-\beta H(\sigma_n) + \sum_{x\in W_n} h_{\sigma(x),x}\right), \tag{2}$$

where $\beta = \frac{1}{T}$, $T > 0$ is temperature, $\sigma_n : x \in V_n \to \sigma_n(x)$ is a configuration in V_n and Ω_{V_n} is the set of configurations in V_n.

The probability distributions $\mu^{(n)}$ are compatible if for any $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\int_{\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \cup \omega_n)\lambda_{W_n}(d(\omega_n)) = \mu^{(n-1)}(\sigma_{n-1}). \tag{3}$$

Here $\lambda_{W_n}(d(\omega_n)) = \prod_{x\in W_n} \lambda(dx)$, λ is the Lebesgue measure.

Theorem 1. Probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \ldots$ in (2) are compatible iff for any $x \in V$ the following equation holds:

$$f(t, x) = \prod_{y\in S(x)} \frac{\int_0^{2\pi} \exp(J\beta\cos(t-u) - h\beta\cos(u)) f(u, y)du}{\int_0^{2\pi} \exp(J\beta\cos(u) - h\beta\cos(u)) f(u, y)du} \tag{4}$$

Here, and below $f(t, x) = \exp(h_{t,x} - h_{0,x})$, $t \in [0, 2\pi)$ and $du = \lambda(du)$ is the Lebesgue measure.

Proof. *Necessity.* Suppose that (3) holds; we want to prove (4). Substituting (2) into (3), obtain that for any configurations $\sigma_{n-1} : x \in V_{n-1} \mapsto \sigma_{n-1}(x) \in [0, 2\pi)$:

$$\frac{Z_{n-1}}{Z_n} \int\limits_{\Omega_{W_n}} \exp\left(\sum_{x\in W_{n-1}} \sum_{y\in S(x)} J\beta \left(\cos(\sigma_{n-1}(x) - \sigma_n(y)) - h\beta\cos(\sigma_n(y)) + h_{\sigma_n(y),y}\right)\right) = \exp\left(\sum_{x\in W_{n-1}} h_{\sigma_{n-1}(x),x}\right), \tag{5}$$

where $\sigma_n : x \in W_n \to \sigma_n(x)$.

From (5) we get

$$\frac{Z_{n-1}}{Z_n} \int\limits_{\Omega_{W_n}} \prod_{x\in W_{n-1}} \prod_{y\in S(x)} \exp\left(J\beta \left(\cos(\sigma_{n-1}(x) - \sigma_n(y)) - h\beta\cos(\sigma_n(y)) + h_{\sigma_n(y),y}\right)\right) = \exp\left(\sum_{x\in W_{n-1}} h_{\sigma_{n-1}(x),x}\right) \tag{6}$$

Rewrite (6) for $\sigma_{n-1}(x) = t$, and denoting $\sigma_n(y) = u$. Then for any $t, u \in [0, 2\pi)$

$$\prod_{y\in S(x)} \frac{\int_0^{2\pi} \exp\left(J\beta\cos(t-u) - h\beta\cos(u) + h_{u,y}\right) du}{\int_0^{2\pi} \exp(J\beta\cos(u) - h\beta\cos(u) + h_{u,y}) du} = \exp(h_{t,x} - h_{0,x}), \tag{7}$$

which implies (5).

Sufficiency. Assume that (3) holds. It is equivalent to the representations

$$\prod_{y\in S(x)} \int_0^{2\pi} \exp(J\beta\cos(t-u) - h\beta\cos\sigma(u) + h_{u,y}) du = a(x)\exp(h_{t,x}), \tag{8}$$

where $t,u\in[0,1]$, for some function $a(x)>0,\ x\in V$. We have

$$\text{LHS of (3)} = \frac{1}{Z_n}\exp(-\beta H(\sigma_{n-1}))\lambda_{V_{n-1}}(d(\sigma_n))\times\prod_{x\in W_{n-1}}\prod_{y\in S(x)}\int_0^{2\pi}\exp(J\beta(\cos(\sigma_{n-1}(x)-\sigma_n(y))-h\beta\cos\sigma_n(y)+h_{\sigma_{n(y)},y}))du \quad (9)$$

substituting (8) into (9) and denoting $A_n(x)=\prod_{x\in W_{n-1}} a(x)$ we get

$$\text{RHS of (9)} = \frac{A_{n-1}}{Z_n}\exp(-\beta H(\sigma_{n-1}))\lambda_{V_{n-1}}(d_{n-1}\sigma)\prod_{x\in W_{n-1}}\exp h_{\sigma_{n-1}(x),x}. \quad (10)$$

Since $\mu^{(n)},\ n\geq 1$ is a probability, we should have

$$\int_{\Omega_{V_{n-1}}}\lambda_{V_{n-1}}(d\sigma_{n-1})\int_{\Omega_{W_n}}(d\omega_n)\mu^{(n)}(\sigma_{n-1},\omega_n)=1. \quad (11)$$

Hence from (10) we get $Z_{n-1}A_{n-1}=Z_n$ and (3).□

We are going to solve equation (4) in the class of translational-invariant functions $f(t,x)$, i.e., $f(t,x))=f(t)$, for any $x\in V$. We consider the following change of variable $\bar{t}=\frac{t}{2\pi}$. For such functions equation (4) can be written as

$$\bar{f}(t)=\left(\frac{\int_0^1\bar{K}(t,u)\bar{f}(u)du}{\int_0^1\bar{K}(0,u)\bar{f}(u)du}\right)^k, \quad (12)$$

where $\bar{f}(t)=f(2\pi t)$, $\bar{K}(t,u)=\exp\left(J\beta\cos(2\pi(t-u))-h\beta\cos(2\pi u)\right)>0$, $t,u\in[0,1]$.

We will find positive continuous solutions to (12), i.e., such that

$$\bar{f}\in C^+[0,1]=\{\bar{f}\in C[0,1]:\bar{f}(x)>0\}.$$

It is easy to see equation (12) is not linear for any $k\geq 1$.

Define the linear operator $W : C[0,1] \to C[0,1]$ by

$$\left(W\bar{f}\right)(t) = \int_0^1 \bar{K}(t,u)\bar{f}(u)du \tag{13}$$

and the linear functional $\omega : C[0,1] \to R$ by

$$\omega(\bar{f}) \equiv (W\bar{f})(0) = \int_0^1 \bar{K}(0,u)\bar{f}(u)du. \tag{14}$$

Then equation (12) can be written as

$$\bar{f}(t) = (A_k\bar{f})(t) = \left(\frac{(W\bar{f})(t)}{(W\bar{f})(0)}\right)^k, \ \bar{f} \in C^+[0,1], \ k \geq 1. \tag{15}$$

For every $k \in N$, we consider an integral operator H_k acting in $C^+[0,1]$ as follows:

$$(H_k)(f)(t) = \int_0^1 \bar{K}(t,u)f^k(u)du. \tag{16}$$

If $k \geq 2$ then the operator H_k is a non-linear operator which is called Hammerstein's operator of order k. Moreover the linear operator equation $H_1 f = f$ has a unique positive solution f in $C^+[0,1]$.

Denote

$$C_0^+ = \{h \in C^+[0,1] : h(t) \geq \frac{\kappa^{\min}}{\kappa_0^{\max}}\}.$$

where $\kappa^{\min} = \min\limits_{t,u\in[0,1]} \bar{K}(t,u)$, $\kappa_0^{\max} = \max\limits_{u\in[0,1]} \bar{K}(0,u)$.

The following is known

Theorem 2. [3]. Let $k \geq 2$. If the kernel $K(t,u)$ satisfies the condition

$$\left(\frac{M}{m}\right)^k - \left(\frac{m}{M}\right)^k < \frac{1}{k}, \tag{17}$$

then the operator H_k has a unique fixed point in $C_0^+[0,1]$, where $m = \kappa^{\min}$, $M = \max\limits_{t,u\in[0,1]} \bar{K}(t,u)$.

Theorem 3. If for the temperature we have

$$T > \frac{2k(|J|+|h|)}{\ln \frac{1+\sqrt{1+4k^2}}{2k}} \tag{18}$$

then the model (1) has a unique Gibbs measure.

Proof. The condition (17) can be reformulated with respect to temperature, or with respect to interaction parameter J. Indeed we have

$$\frac{M}{m} = \frac{\max\limits_{t,u\in[0,1]} \bar{K}(t,u)}{\min\limits_{t,u\in[0,1]} \bar{K}(t,u)} = \frac{\max\limits_{t,u\in[0,1]} \exp\left(J\beta\cos(2\pi(t-u)) - h\beta\cos(2\pi u)\right)}{\min\limits_{t,u\in[0,1]} \exp\left(J\beta\cos(2\pi(t-u)) - h\beta\cos(2\pi u)\right)}.$$

To find extremum points of $\bar{K}(t,u)$ we denote

$$\varphi(t,u;J,h) = J\cos(2\pi(t-u)) - h\cos(2\pi u)$$

. Consider partial derivations of φ:

$$\begin{cases} \varphi'_t(t,u;J,h) = -2\pi J\sin(2\pi(t-u)) = 0; \\ \varphi'_u(t,u;J,h) = 2\pi J\sin(2\pi(t-u) + 2\pi h\sin(2\pi u) = 0. \end{cases} \tag{19}$$

From (19) we get

$$\begin{cases} t-u = \frac{n}{2}; \\ u = \frac{k}{2}, \end{cases} \quad n,k\in\mathbb{Z}, \text{ i.e. } (t,u) = \left(\frac{k+n}{2}, \frac{k}{2}\right), \text{where } \ t,u\in[0,1]. \tag{20}$$

From (20) we obtain the following points of extremums:

$$(0,0);\ (\tfrac{1}{2},0);\ (1,0);\ (\tfrac{1}{2},\tfrac{1}{2});$$
$$(1,\tfrac{1}{2});\ (0,\tfrac{1}{2});\ (1,1);\ (\tfrac{1}{2},1);\ (0,1).$$

Case (0,0):

$$\varphi''_{tt}(0,0;J,h) = -4\pi^2 J = a_{11},\ \varphi''_{tu}(0,0;J,h) = 4\pi^2 J = a_{12} = a_{21},$$
$$\varphi''_{uu}(0,0;J,h) = 4\pi^2(J-h) = a_{22},$$

It is known that if $a_{11} > 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$ then, (0,0) is a local minimum.

Solving these inequalities we get $J < 0,\ h > 0$ thus $\varphi_{\min}(0,0;J,h) = J - h$.

To get maximum at (0,0) we should have $a_{11} < 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} > 0$ which gives $J > 0,\ h < 0$ thus $\varphi_{\max}(0,0;J,h) = J - h$.

Similarly, we obtain

$$\varphi_{\min}(t,u;J,h) = \begin{cases} -J-h, & \text{if } J>0, h>0,\ t=\frac{1}{2}, u=0;\ t=\frac{1}{2}, u=1; \\ J-h, & \text{if } J<0, h>0,\ t=0, u=0;\ t=1, u=0; \\ & t=1, u=1;\ t=0, u=1; \\ J+h, & \text{if } J<0, h<0\ t=1, u=\frac{1}{2};\ t=0, u=\frac{1}{2}; \\ h-J, & \text{if } J>0, h<0,\ t=\frac{1}{2}, u=\frac{1}{2}. \end{cases} \tag{21}$$

$$\varphi_{\max}(t,u;J,h) = \begin{cases} J+h, & \text{if } J>0, h>0,\ t=1, u=\frac{1}{2};\ t=0, u=\frac{1}{2}; \\ J-h, & \text{if } J>0, h<0,\ t=0, u=0;\ t=1, u=0; \\ & t=1, u=1;\ t=0, u=1; \\ -(J+h), & \text{if } J<0, h<0\ t=\frac{1}{2}, u=0;\ t=\frac{1}{2}, u=1; \\ h-J, & \text{if } J<0, h>0\ t=\frac{1}{2}, u=\frac{1}{2}. \end{cases} \tag{22}$$

Using these formulas we set

$$\frac{M}{m} = \frac{\max\limits_{t,u\in[0,1]} e^{\beta\varphi(t,u;J,h)}}{\min\limits_{t,u\in[0,1]} e^{\beta\varphi(t,u;J,h)}} = \frac{e^{\beta(|J|+|h|)}}{e^{-\beta(|J|+|h|)}} = e^{2\beta(|J|+|h|)}$$

Using this formula from (17) we get

$$T = \beta^{-1} > \frac{2k(|J|+|h|)}{\ln\frac{1+\sqrt{1+4k^2}}{2k}} \tag{23}$$

i.e., if the temperature is greater than $\frac{2k(|J|+|h|)}{\ln\frac{1+\sqrt{1+4k^2}}{2k}}$ then there exists unique translation invariant Gibbs measure.□

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