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Infinitesimal deformations of null-filiform Leibniz superalgebras

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ABSTRACT

In this paper we describe the infinitesimal deformations of null-filiform Leibniz superalgebras over a field of zero characteristic. It is known that up to isomorphism in each dimension there exist two such superalgebras $NF^{n,m}$. One of them is a Leibniz algebra (that is $m = 0$) and the second one is a pure Leibniz superalgebra (that is $m \neq 0$) of maximum nilindex. We show that the closure of the union of orbits of single-generated Leibniz algebras forms an irreducible component of the variety of Leibniz algebras. We prove that any single-generated Leibniz algebra is a linear integrable deformation of the algebra NF^n. Similar results for the case of Leibniz superalgebras are obtained.

1. Introduction

Deforming a given mathematical structure is a tool of fundamental importance in most parts of mathematics, mathematical physics and physics. Deformations and contractions have been investigated by researchers who had different approaches and goals. Tools such as cohomology, gradings, etc. which are utilized in the study of one concept are likely to be useful for the other concept as well.

The theory of deformations originated with the problem of classifying all possible pairwise non-isomorphic complex structures on a given differentiable real manifold. Formal deformations of arbitrary rings and associative algebras, and related cohomology questions, were first investigated by Gerstenhaber [1]. Later, the notion of deformation was applied to Lie algebras by Nijenhuis and Richardson [2]. Because various fields in mathematics and physics exist in which deformations are used, we focus on the study of Leibniz superalgebras. One-parameter deformations were studied and established connection between Lie algebra cohomology and infinitesimal deformations.

Deformation is one of the tools used to study a specific object, by deforming it into some families of "similar" structure objects. This way we get a richer picture about the original object itself [3]. But there is also another question approached via deformation. Roughly speaking, it is the question: can we equip the set of mathematical structures under consideration (may be up to certain equivalence) with the structure of a topological or geometric space.

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The theory of deformations is one of the effective approaches in investigating solvable and nilpotent Lie algebras and superalgebras [4–7], etc.

Recall that Leibniz algebras are a generalization of Lie algebras [8,9] and it is natural to apply the theory of deformations to the study of Leibniz algebras. Particularly, the problems which were studied in [4,7] and others can be considered from point of Leibniz algebras view. Due to a results of [10] we can apply the general principles of deformations theory to Leibniz algebras.

It is well known that Lie superalgebras are a generalization of Lie algebras. In the same way, the notion of Leibniz algebra can be generalized to Leibniz superalgebras. Lie superalgebras with maximal nilindex were classified in [11]. In fact, there exists a unique Lie superalgebra of maximal nilindex. This superalgebra is a filiform Lie superalgebra. For nilpotent Leibniz superalgebras the description of the maximal nilindex case (nilpotent Leibniz superalgebras distinguished by the feature of being single-generated) was easily done in [12].

Let $V = V_0 \oplus V_1$ be the underlying vector space of the Leibniz superalgebra $L = L_0 \oplus L_1$ of dimension $n + m$ (where n and m are dimensions of L_0 and L_1, respectively) and let $GL(V)$ be the group of the invertible linear mappings of the form $f = f_0 + f_1$ such that $f_0 \in GL_n(F)$ and $f_1 \in GL_m(F)$ (where $GL(V) = GL_n(F) \oplus GL_m(F)$). The action of the group $GL(V)$ on the variety of Leibniz superalgebras induces an action on the Leibniz superalgebras' variety: two laws μ_1 and μ_2 are isomorphic if there exists a linear mapping f, $f = f_0 + f_1 \in GL(V)$, such that

$$\mu_2(x, y) = f^{-1}_{\alpha+\beta}(\mu_1(f_\alpha(x), f_\beta(y))) \quad \text{for all } x \in V_\alpha, y \in V_\beta,\ \alpha, \beta \in \mathbb{Z}_2.$$

The orbit under this action, denoted by $\mathrm{Orb}(\mu)$, consists of all superalgebras isomorphic to the superalgebra μ. Therefore the description of $(n+m)$-dimensional superalgebras with dimensions of even and odd parts equal to n and m, respectively (further denoted by $\mathrm{Leib}^{n,m}$) can be reduced to a geometric problem of classification of orbits under the action of the group $GL(V)$. Note that nilpotent Leibniz superalgebras $N^{n,m}$ form also an invariant subvariety of the variety $\mathrm{Leib}^{n,m}$ under the above action. From algebraic geometry it is known that an algebraic variety is a union of irreducible components. The superalgebras with open orbits in the variety of Leibniz superalgebras are *called rigid*. The closures of these open orbits give irreducible components of the variety. Therefore studying the rigid superalgebras is a crucial problem from the geometrical point of view. The problem of finding such algebras is crucial for the description of the variety $\mathrm{Leib}^{n,m}$.

The structure of the paper is as follows: in the section Preliminaries we give the necessary definitions and results for understanding the main parts of this paper. In Section 3 we calculate the second group of cohomology of the null-filiform Leibniz algebra and show that the set of single-generated Leibniz algebras forms an irreducible component of the variety of Leibniz algebras. Moreover, it is established that any single-generated algebra is a linear integrable deformation of the null-filiform algebra. In the last section we extend the calculations of the previous section for the case of Leibniz superalgebras.

Throughout the paper we consider finite-dimensional vector spaces and superalgebras over a field of zero characteristic. Moreover, in the multiplication table of a Leibniz superalgebra the omitted products and in the expansion of 2-cocycles the omitted values are assumed to be zero.

2. Preliminaries

In this section we give necessary definitions and results for understanding the main parts of the work.

Definition 2.1 (*[12]*). A $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ is called a Leibniz superalgebra if it is equipped with a product $[-,-]$ which satisfies the following conditions:

$$[x, [y, z]] = [[x, y], z] - (-1)^{|y|\,|z|}[[x, z], y]\text{-Leibniz superidentity}$$

for all $x \in L$, $y \in L_{|y|}$, $z \in L_{|z|}$.

Let L be a Leibniz superalgebra. We call a $\mathbb{Z}_2$-graded vector space $M = M_0 \oplus M_1$ a module over L if there are two bilinear maps

$$[-,-] : L \times M \to M \quad \text{and} \quad [-,-] : M \times L \to M$$

satisfying the following three axioms:

$$[m, [x, y]] = [[m, x], y] - (-1)^{|x|\,|y|}[[m, y], x],$$
$$[x, [m, y]] = [[x, m], y] - (-1)^{|y|\,|m|}[[x, y], m],$$
$$[x, [y, m]] = [[x, y], m] - (-1)^{|m|\,|y|}[[x, m], y],$$

for any $m \in M_{|m|}$, $x \in L_{|x|}$, $y \in L_{|y|}$.

Given a Leibniz superalgebra L, let $C^n(L, M)$ be the space of all super skew-symmetric F-linear homogeneous mapping $L^{\otimes n} \to M$, $n \geq 0$ and $C^0(L, M) = M$. This space is graded by $C^n(L, M) = C^n_0(L, M) \oplus C^n_1(L, M)$ with

$$C^n_p(L, M) = \bigoplus_{\substack{n_0+n_1=n \\ n_1+r\equiv p \bmod 2}} \mathrm{Hom}(L_0^{\otimes n_0} \otimes L_1^{\otimes n_1}, M_r).$$

Let $d^n : C^n(L, M) \to C^{n+1}(L, M)$ be an F-homomorphism defined by

$$(d^n f)(x_1, \ldots, x_{n+1}) := [x_1, f(x_2, \ldots, x_{n+1})] + \sum_{i=2}^{n+1}(-1)^{i+|x_i|(|f|+|x_{i+1}|+\cdots+|x_{n+1}|)}[f(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}), x_i]$$
$$+ \sum_{1\le i<j\le n+1}(-1)^{j+1+|x_j|(|x_{i+1}|+\cdots+|x_{j-1}|)} f(x_1, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \widehat{x_j}, \ldots, x_{n+1}),$$

where $f \in C^n(L, M)$ and $x_i \in L$. Since the derivative operator $d = \sum_{i\ge 0} d^i$ satisfies the property $d \circ d = 0$, the cohomology group is well defined and

$$HL^n_p(L, M) = ZL^n_p(L, M)/BL^n_p(L, M),$$

where the elements $ZL^n_0(L, M)$ $(BL^n_0(L, M))$ and $ZL^n_1(L, M)$ $(BL^n_1(L, M))$ are called *even n-cocycles* (*even n-coboundaries*) and *odd n-cocycles* (*odd n-coboundaries*), respectively.

It is a remarkable fact that the formula for d^n can be obtained from the derivative operator for color Leibniz algebras [13].

Note that the space $ZL^1(L, L)$ consists of derivations of the superalgebra L, which are defined by the condition

$$d([x, y]) = (-1)^{|d|\,|y|}[d(x), y] + [x, d(y)].$$

For a given $x \in L$, R_x denotes the map $R_x : L \to L$ such that $R_x(y) = [y, x]$, $\forall x \in L$. Note that the map R_x is a derivation.

A deformation of a Leibniz superalgebra L is a one-parameter family L_t of Leibniz superalgebras with the bracket

$$\mu_t = \mu_0 + t\varphi_1 + t^2\varphi_2 + \cdots,$$

with μ_0 being the original Leibniz bracket on L and φ_i are L-valued even 2-cochains, i.e., elements of $\mathrm{Hom}(L \otimes L, L)_0 = C^2(L, L)_0$.

Two deformations L_t, L'_t with corresponding laws μ_t, μ'_t are *equivalent* if there exists a linear automorphism $f_t = id + f_1 t + f_2 t^2 + \cdots$ of L, where f_i are elements of $C^1(L, L)_0$ such that the following equation holds:

$$\mu'_t(x, y) = f_t^{-1}(\mu_t(f_t(x), f_t(y))) \quad \text{for } x, y \in L.$$

The Leibniz superidentity for the superalgebras L_t implies that the 2-cochain φ_1 is an even 2-cocycle, i.e. $d^2\varphi_1 = 0$. If φ_1 vanishes identically, the first non-vanishing φ_i will be a 2-cocycle.

If μ'_t is an equivalent deformation with cochains φ'_i, then $\varphi'_1 - \varphi_1 = d^1 f_1$; hence every equivalence class of deformations defines uniquely an element of $HL^2(L, L)_0$.

Note that the linear integrable deformation φ satisfies the condition

$$\varphi(x, \varphi(y, z)) - \varphi(\varphi(x, y), z) + (-1)^{|y|\,|z|}\varphi(\varphi(x, z), y) = 0. \tag{2.1}$$

It should be noted that a Leibniz algebra is a superalgebra with trivial odd part and the definition of cohomology groups of Leibniz superalgebras extends the definition of cohomology groups of Leibniz algebras given in [9].

For a Leibniz superalgebra L consider the following central lower series:

$$L^1 = L, \qquad L^{k+1} = [L^k, L^1], \quad k \ge 1.$$

Definition 2.2. A Leibniz superalgebra L is said to be nilpotent if there exists $p \in \mathbb{N}$ such that $L^p = 0$.

Now we give the notion of null-filiform Leibniz superalgebra.

Definition 2.3. An n-dimensional Leibniz superalgebra is said to be null-filiform if $\dim L^i = n + 1 - i$, $1 \le i \le n + 1$.

Similarly to the case of nilpotent Leibniz algebras [14] it is easy to check that a Leibniz superalgebra is null-filiform if and only if it is single-generated. Moreover, a null-filiform superalgebra has the maximal nilindex.

Theorem 2.4 (*[12]*)**.** *Let L be a null-filiform Leibniz superalgebra of the variety* $\mathrm{Leib}^{n,m}$. *Then L is isomorphic to one of the following non-isomorphic superalgebras:*

$$NF^n : [x_i, x_1] = x_{i+1}, \quad 1 \le i \le n-1; \qquad NF^{n,m} : \begin{cases} [y_i, y_1] = x_i, & 1 \le i \le n, \\ [x_i, y_1] = \frac{1}{2}y_{i+1}, & 1 \le i \le m-1, \\ [y_j, x_1] = y_{j+1}, & 1 \le j \le m-1, \\ [x_i, x_1] = x_{i+1}, & 1 \le i \le n-1. \end{cases}$$

where $\{x_1, x_2, \ldots, x_n\}$ *and* $\{y_1, y_2, \ldots, y_m\}$ *are bases of the even and odd parts, respectively.*

Remark 2.5. Note that the first superalgebra is a null-filiform Leibniz algebra [14] and from the assertion of Theorem 2.4 we conclude that in the case of non-trivial odd part of the null-filiform Leibniz superalgebra $NF^{n,m}$ there are two possibilities for m, namely $m = n$ or $m = n + 1$.

3. Deformations of the null-filiform Leibniz algebra

In this section we calculate infinitesimal deformations of the algebra NF^n and we show that any single-generated Leibniz algebra is a linear integrable deformation of NF^n.

Note that any derivation of the null-filiform Leibniz algebra NF^n has the following form [15]:

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & 2a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & 3a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & na_1 \end{pmatrix}.$$

From this we conclude that $\dim BL^2(NF^n, NF^n) = n^2 - n$.

In general, a 2-cocycle is a bilinear map from $NF^n \otimes NF^n$ to NF^n such that $d^2\varphi = 0$, i.e.,

$$d^2\varphi(x, y, z) = [x, \varphi(y, z)] - [\varphi(x, y), z] + [\varphi(x, z), y] + \varphi(x, [y, z]) - \varphi([x, y], z) + \varphi([x, z], y).$$

Proposition 3.1. *The following cochains*

$$\varphi_{j,k}(x_j, x_1) = x_k, \quad 1 \le j \le n,\ 2 \le k \le n,$$
$$\psi_j\ (1 \le j \le n-1) = \begin{cases} \psi_j(x_j, x_1) = x_1, \\ \psi_j(x_i, x_{j+1}) = -x_{i+1}, \quad 1 \le i \le n-1, \end{cases}$$

form a basis of $ZL^2(NF^n, NF^n)$.

Proof. Using the Leibniz 2-cocycle property $(d^2\varphi)(x_i, x_1, x_1) = 0$, we have

$$\varphi(x_i, x_2) = -[x_i, \varphi(x_1, x_1)], \quad 1 \le i \le n-1, \qquad \varphi(x_n, x_2) = 0. \tag{3.1}$$

The conditions $(d^2\varphi)(x_i, x_1, x_j) = 0$, $(d^2\varphi)(x_i, x_j, x_1) = 0$ for $1 \le i \le n$, $2 \le j \le n$ imply

$$[x_i, \varphi(x_1, x_j)] + [\varphi(x_i, x_j), x_1] - \varphi([x_i, x_1], x_j) = 0,$$
$$[x_i, \varphi(x_j, x_1)] - [\varphi(x_i, x_j), x_1] + \varphi(x_i, [x_j, x_1]) + \varphi([x_i, x_1], x_j) = 0.$$

Summarizing the above equalities, we derive

$$\begin{cases} \varphi(x_i, x_{j+1}) = -[x_i, \varphi(x_1, x_j) + \varphi(x_j, x_1)], & 1 \le i \le n-1,\ 2 \le j \le n-1, \\ \varphi(x_n, x_{j+1}) = 0, & 2 \le j \le n-1, \\ [x_i, \varphi(x_1, x_n) + \varphi(x_n, x_1)] = 0, & 1 \le i \le n. \end{cases} \tag{3.2}$$

Set $\varphi(x_j, x_1) = \sum_{k=1}^{n} a_{j,k}x_k$ for $1 \le i \le n$.

Using inductively method from equalities (3.1) and (3.2) we get $a_{n,1} = 0$ and

$$\varphi(x_i, x_{j+1}) = -a_{j,1}x_{i+1}, \quad 1 \le i \le n-1,\ 1 \le j \le n-1.$$

Therefore, we obtain that any infinitesimal deformation of NF^n has the following form:

$$\begin{cases} \varphi(x_j, x_1) = a_{j,1}x_1 + a_{j,2}x_2 + \cdots + a_{j,n}x_n, & 1 \le j \le n-1 \\ \varphi(x_n, x_1) = a_{n,2}x_2 + \cdots + a_{n,n}x_n, & \\ \varphi(x_i, x_{j+1}) = -a_{j,1}x_{i+1}, & 1 \le i \le n-1,\ 1 \le j \le n-1. \end{cases}$$

Therefore, $\varphi_{j,k}$ and ψ_j form a basis of $ZL^2(NF^n, NF^n)$. □

Corollary 3.2. $\dim(ZL^2(NF^n, NF^n)) = n^2 - 1$.

Below, we describe a basis of the subspace $BL^2(NF^n, NF^n)$ in terms of $\varphi_{j,k}$ and ψ_j.

Proposition 3.3. *The cocycles*

$$\xi_{j,k} \begin{cases} \xi_{j,1} = \psi_{j-1} - \varphi_{j,2}, & 2 \le j \le n, \\ \xi_{j,k} = \varphi_{j-1,k}, & 2 \le j \le k \le n, \\ \xi_{j,k} = \varphi_{j-1,k} - \varphi_{j,k+1}, & 2 \le k < j \le n \end{cases}$$

form a basis of $BL^2(NF^n, NF^n)$.

Proof. Consider the endomorphisms $f_{j,k}$ defined as follows:

$$f_{j,k}(x_j) = x_k, \quad 2 \le j \le n,\ 1 \le k \le n.$$

It is easy to see that $f_{j,k}$ are complement of derivations to $C^1(NF^n, NF^n)$. Therefore, the elements of the space $BL^2(NF^n, NF^n)$ are $d^1f_{j,k}$ such that $d^1f_{j,k} = f_{j,k}([x, y]) - [f_{j,k}(x), y] - [x, f_{j,k}(y)]$.

Then we obtain

$$d^1f_{j,1}\ (2 \le j \le n) = \begin{cases} d^1f_{j,1}(x_{j-1}, x_1) = x_1, \\ d^1f_{j,1}(x_j, x_1) = -x_2, \\ d^1f_{j,1}(x_i, x_j) = -x_{i+1}, \quad 2 \le i \le n-1, \end{cases}$$

$$d^1f_{j,k} \begin{pmatrix} 2 \le j \le n, \\ 2 \le k \le n-1 \end{pmatrix} = \begin{cases} d^1f_{j,k}(x_{j-1}, x_1) = x_k, \\ d^1f_{j,k}(x_j, x_1) = -x_{k+1}, \end{cases}$$

$$d^1f_{k,n}\ (2 \le k \le n) = \{d^1f_{k,n}(x_{k-1}, x_1) = x_n.$$

It should be noted that

$$\begin{cases} d^1f_{j,1} = \psi_{j-1} - \varphi_{j,2} & 2 \le j \le n, \\ d^1f_{j,k} = \varphi_{j-1,k} - \varphi_{j,k+1}, & 2 \le j \le n,\ 2 \le k \le n-1, \\ d^1f_{j,n} = \varphi_{j-1,n}, & 2 \le j \le n. \end{cases}$$

From the condition $d^1f_{k,s} + d^1f_{k+1,s+1} + \cdots + d^1f_{n+k-s,n} = \varphi_{k-1,s}$ for $2 \le k \le s \le n$, we conclude that the maps $\xi_{k,s},\ 2 \le k \le n,\ 1 \le s \le n$, form a basis of $BL^2(NF^n, NF^n)$. $\square$

Corollary 3.4. *The classes* $\overline{\varphi_{n,k}}$ $(2 \le k \le n)$ *form a basis of* $HL^2(NF^n, NF^n)$. *Consequently,* $\dim HL^2(NF^n, NF^n) = n - 1$.

In the following proposition we describe infinitesimal deformations of NF^n satisfying the equality (2.1).

Proposition 3.5. *A* 2*-cocycle of* NF^n *satisfies the equality* (2.1) *if and only if it has the form*

$$\sum_{j,k} a_{j,k}\varphi_{j,k}.$$

Proof. It is easy to check that 2-cocycles of the form $\sum_{j,k} a_{j,k}\varphi_{j,k}$ satisfy the equality (2.1).

If $\varphi \in ZL^2(NF^n, NF^n)$, then $\varphi = \sum_{j,k} a_{j,k}\varphi_{k,s} + \sum_{j=1}^{n-1} b_j\psi_k$.

From the condition

$$\varphi(x_1, \varphi(x_1, x_1)) - \varphi(\varphi(x_1, x_1), x_1) + \varphi(\varphi(x_1, x_1), x_1) = 0,$$

we get $b_1 = 0$.

The following chain of equalities

$$\begin{aligned} &\varphi(x_i, \varphi(x_j, x_{j+1})) - \varphi(\varphi(x_i, x_j), x_{j+1}) + \varphi(\varphi(x_i, x_{j+1}), x_j) \\ &= \varphi(x_i, \psi_j(x_j, x_{j+1})) - \varphi(\psi_{j-1}(x_i, x_j), x_{j+1}) + \varphi(\psi_j(x_i, x_{j+1}), x_j) \\ &= -\psi_j(x_i, b_jx_{j+1}) + \psi_j(b_{j-1}x_{i+1}, x_{j+1}) - \psi_{j-1}(b_jx_{i+1}, x_j) \\ &= b_j^2x_{i+1} - b_jb_{j-1}x_{i+2} + b_jb_{j-1}x_{i+2} = b_j^2x_{i+1} \end{aligned}$$

implies $b_j = 0,\ 2 \le j \le n-1$. $\square$

Consider the linear integrable deformations $\mu_t = NF^n + t\sum_{j,k} a_{j,k}\varphi_{j,k}$ of NF^n.

Since every non-trivial equivalence class of deformations defines uniquely an element of $HL^2(L, L)$, due to Corollary 3.4 it is sufficient to consider $\mu_t(a_2, a_3, \dots, a_n) = NF^n + t\sum_{k=2}^n a_k\varphi_{n,k}$, where $(a_2, a_3, \dots, a_n) \ne (0, 0, \dots, 0)$.

Thus, the multiplication table of $\mu_t(a_2, a_3, \dots, a_n)$ has the form

$$\begin{cases} [x_i, x_1] = x_{i+1}, \quad 1 \le i \le n-1, \\ [x_n, x_1] = t\sum_{k=2}^{n} a_kx_k. \end{cases}$$

Putting $a'_k = ta_k$, we can assume $t = 1$.

Proposition 3.6. *An arbitrary single-generated Leibniz algebra admits a basis* $\{x_1, x_2, \dots, x_n\}$ *such that the multiplication table has the form of* $\mu_1(a_2, a_3, \dots, a_n)$.

Proof. Let L be a single-generated Leibniz algebra and x a generator of L. We put

$$x_1 = x, \qquad x_2 = [x, x], \qquad x_3 = [[x, x], x], \dots, x_n = [[x, x], \dots, x].$$

Since x is a generator, $\{x_1, x_2, \dots, x_n\}$ form a basis of L. Evidently $\{x_2, \dots, x_n\}$ belong to the right annihilator of L. Hence, we have $[x_i, x_j] = 0,\ 2 \le j \le n-1$. Let $[x_n, x_1] = \sum_{k=1}^n a_kx_k$.

From the Leibniz identity $[x_1, [x_n, x_1]] = [[x_1, x_n], x_1] - [[x_1, x_n], x_1] = 0$, we conclude that $a_1 = 0$. Therefore, we obtain the existence of a basis $\{x_1, x_2, \dots, x_n\}$ in any single-generated Leibniz algebra such that the multiplication table in this basis has the form

$$\begin{cases} [x_i, x_1] = x_{i+1}, & 1 \le i \le n-1, \\ [x_n, x_1] = \sum\limits_{k=2}^{n} a_k x_k. & \square \end{cases}$$

Let a_j be the first non-vanishing parameter in the algebra $\mu(a_2, a_3, \dots, a_n)$; then by scaling $x_i' = \frac{1}{\sqrt[n-j+1]{a_j^i}} x_i,\ 1 \le i \le n$, we can assume $a_j = 1$, i.e., the first non-vanishing parameter can be taken equal to 1.

Note that the set of single-generated Leibniz algebras is open. Indeed, if a q-generated ($q > 1$) Leibniz algebra has a basis $\{e_1, e_2, \dots, e_n\}$, then for any $e_i \in L$ the elements $e_i, e_i^2, \dots, e_i^n$ are linearly dependent. That is, determinants of the matrices $A_i,\ 1 \le i \le n$, which consist of the rows $e_i, e_i^2, \dots, e_i^n$ are zero; hence we get n-times of polynomials with structure constants of the algebra. Therefore, q-generated ($q > 1$) Leibniz algebras form a closed set. Taking into account that the set of all single-generated Leibniz algebras is a complemented set to a closed set, we conclude that the set of single-generated Leibniz algebras is open.

It is easy to see that an algebra $\mu_1(a_2, a_3, \dots, a_n)$ is a linear deformation of an algebra $\mu_1(a_2', a_3', \dots, a_n')$.

Since $\dim(\mathrm{Der}(\mu_1(a_2, a_3, \dots, a_n))) = n-1,\ (a_2, a_3, \dots, a_n) \neq (0, 0, \dots, 0)$, then by arguments used in [16] for non-isomorphic algebras $\mu_1(a_2, a_3, \dots, a_n)$ and $\mu_1(a_2', a_3', \dots, a_n')$ we derive $\mu_1(a_2, a_3, \dots, a_n) \notin \overline{\mathrm{Orb}(\mu_1(a_2', a_3', \dots, a_n'))}$.

Summarizing these results on single-generated Leibniz algebras, we obtain the following theorem.

Theorem 3.7. $\overline{\bigcup_{a_2,\dots,a_n} \mathrm{Orb}(\mu_1(a_2, a_3, \dots, a_n))}$ *is an irreducible component.*

4. Cohomology of Leibniz superalgebras

In this section we describe all infinitesimal deformations of the Leibniz superalgebra $NF^{n,m}$ and we prove similar results as in the previous section.

In the next proposition the description of even derivations of $NF^{n,m}$ is given.

Proposition 4.1. *Any derivation of* $\mathrm{Der}(NF^{n,m})_0$ *has the form*

$$d(y_j) = (2j-1)a_1 y_j + \sum_{k=2}^{m+1-j} a_k y_{j+k-1}, \quad 1 \le j \le m,$$

$$d(x_i) = 2ia_1 x_i + \sum_{k=2}^{n+1-i} a_i x_{i+k-1}, \quad 1 \le i \le n,$$

where $m = n$ *or* $m = n+1$.

Proof. For $d \in \mathrm{Der}(NF^{n,m})_0$ we put $d(y_1) = \sum_{k=1}^{m} a_k y_k$. Then using the properties of derivation and multiplication in the superalgebra $NF^{n,m}$ we obtain $d(x_1) = 2a_1 x_1 + \sum_{k=2}^{n} a_k x_k$.

Using induction, we deduce

$$d(y_{j+1}) = [d(y_j), x_1] + [y_j, d(x_1)] = (2j+1)a_1 y_{j+1} + \sum_{k=2}^{m-j} a_k y_{j+k},$$

$$d(x_i) = [d(y_i), y_1] + [y_i, d(y_1)] = 2ia_1 x_i + \sum_{k=2}^{n+1-i} a_k x_{i+k-1}.$$

The verification of the derivation property on other elements does not give any additional restriction on d. $\square$

Similarly, we describe odd derivations of $\mathrm{Der}(NF^{n,m})$.

Proposition 4.2. *Any derivation of* $\mathrm{Der}(NF^{n,m})_1$ *has the form*

$$d(y_j) = \sum_{k=1}^{n+1-j} b_k x_{j+k-1}, \quad 1 \le j \le n,$$

$$d(x_i) = \frac{1}{2}\left(b_1 y_{i+1} - \sum_{k=2}^{m-i} b_k x_{i+k}\right), \quad 1 \le i \le m-1,$$

where $m = n$ *or* $m = n+1$.

Now we shall consider infinitesimal deformations of the superalgebra $NF^{n,m}$, i.e., elements of the space $ZL_0^2(NF^{n,m}, NF^{n,m})$.

4.1. The case $m = n$

In this case we give the description of the infinitesimal deformations of the superalgebra $NF^{n,n}$.

Proposition 4.3. *An arbitrary infinitesimal deformation φ of $NF^{n,n}$ has the following form:*

$$\begin{cases}
\varphi(y_j, y_1) = \sum\limits_{k=1}^{n} \alpha_{j,k} x_k, & 1 \le j \le n, \\
\varphi(x_j, y_1) = \sum\limits_{k=1}^{n} \beta_{j,k} y_k, & 1 \le j \le n-1, \\
\varphi(x_n, y_1) = \sum\limits_{k=2}^{n} \beta_{n,k} y_k, & \\
\varphi(x_j, x_1) = -\alpha_{1,1} x_{i+1} + \sum\limits_{k=1}^{n} (\alpha_{j+1,k} + 2\beta_{j,k}) x_k, & 1 \le j \le n-1, \\
\varphi(x_n, x_1) = 2 \sum\limits_{k=2}^{n} \beta_{n,k} x_k, & \\
\varphi(y_j, x_1) = 2\beta_{j,1} y_1 - \alpha_{1,1} y_{j+1} + \sum\limits_{k=2}^{n} (\alpha_{j,k-1} + 2\beta_{j,k}) y_k, & 1 \le j \le n-1, \\
\varphi(y_n, x_1) = \sum\limits_{k=2}^{n} (\alpha_{n,k-1} + 2\beta_{n,k}) y_k, & \\
\varphi(x_i, x_{j+1}) = -(\alpha_{j+1,1} + 2\beta_{j,1}) x_{i+1}, & 1 \le i \le n-1,\ 1 \le j \le n-1, \\
\varphi(y_i, x_{j+1}) = -(\alpha_{j+1,1} + 2\beta_{j,1}) y_{i+1}, & 1 \le i \le n-1,\ 1 \le j \le n-1, \\
\varphi(x_i, y_{j+1}) = -\beta_{j,1} y_{i+1}, & 1 \le i \le n-1,\ 1 \le j \le n-1, \\
\varphi(y_i, y_{j+1}) = -2\beta_{j,1} x_i, & 1 \le i \le n,\ 1 \le j \le n-1.
\end{cases}$$

Proof. Let $\varphi \in ZL_0^2(NF^{n,n}, NF^{n,n})$. We set

$$\varphi(y_j, y_1) = \sum_{k=1}^{n} \alpha_{j,k} x_k, \qquad \varphi(x_j, y_1) = \sum_{k=1}^{n} \beta_{j,k} y_k, \quad 1 \le j \le n.$$

Applying the multiplication of the superalgebra and the property of cocycle for $d^2\varphi(x_j, y_1, y_1) = 0$, we obtain

$$\varphi(x_j, x_1) = -\alpha_{1,1} x_{j+1} + \sum_{k=2}^{n} (\alpha_{j+1,k-1} + 2\beta_{j,k}) x_k, \quad 1 \le j \le n-1, \qquad \varphi(x_n, x_1) = 2 \sum_{k=1}^{n} \beta_{n,k} x_k.$$

Analogously, from $d^2\varphi(y_j, y_1, y_1) = 0$ we get

$$\varphi(y_j, x_1) = 2\beta_{j,1} y_1 - \alpha_{1,1} y_{j+1} + \sum_{k=2}^{n} (\alpha_{j,k-1} + 2\beta_{j,k}) y_k, \quad 1 \le j \le n.$$

The equations $d^2\varphi(x_i, x_1, x_1) = 0$ and $d^2\varphi(y_i, x_1, x_1) = 0$ imply

$$\varphi(x_i, x_2) = -[x_i, \varphi(x_1, x_1)] = -(\alpha_{2,1} + 2\beta_{1,1}) x_{i+1}, \quad 1 \le i \le n-1,$$
$$\varphi(y_i, x_2) = -[y_i, \varphi(x_1, x_1)] = -(\alpha_{2,1} + 2\beta_{1,1}) y_{i+1}, \quad 1 \le i \le n-1.$$

Using the conditions $d^2\varphi(x_i, x_1, x_j) = 0$ and $d^2\varphi(x_i, x_j, x_1) = 0$ for $1 \le i \le n$, $2 \le j \le n$, we derive

$$[x_i, \varphi(x_1, x_j)] + [\varphi(x_i, x_j), x_1] - \varphi([x_i, x_1], x_j) = 0,$$
$$[x_i, \varphi(x_j, x_1)] - [\varphi(x_i, x_j), x_1] + \varphi(x_i, [x_j, x_1]) + \varphi([x_i, x_1], x_j) = 0.$$

Summarizing these equalities, we deduce

$$\varphi(x_i, x_{j+1}) = -[x_i, \varphi(x_1, x_j) + \varphi(x_j, x_1)] = -(\alpha_{j+1,1} + 2\beta_{j,1}) x_{i+1}, \quad 1 \le i \le n-1,\ 2 \le j \le n-1,$$

and $0 = [x_i, \varphi(x_1, x_n) + \varphi(x_n, x_1)] = \beta_{n,1} y_{i+1}$, which implies $\beta_{n,1} = 0$.
Similarly from $d^2\varphi(y_i, x_1, x_j) = 0$ and $d^2\varphi(y_i, x_j, x_1) = 0$ we obtain

$$\varphi(y_i, x_{j+1}) = -(\alpha_{j+1,1} + 2\beta_{j,1}) y_{i+1}, \quad 1 \le i \le n-1,\ 2 \le j \le n-1.$$

Considering the properties $(d^2\varphi)(x_i, y_1, x_j) = 0$ and $(d^2\varphi)(x_i, x_j, y_1) = 0$ for $1 \le i, j \le n$, we have

$$[x_i, \varphi(y_1, x_j)] - [\varphi(x_i, y_1), x_j] + [\varphi(x_i, x_j), y_1] + \varphi(x_i, [y_1, x_j]) - \varphi([x_i, y_1], x_j) + \varphi([x_i, x_j], y_1) = 0,$$
$$[x_i, \varphi(x_j, y_1)] - [\varphi(x_i, x_j), y_1] + [\varphi(x_i, y_1), x_j] + \varphi(x_i, [x_j, y_1]) - \varphi([x_i, x_j], y_1) + \varphi([x_i, y_1], x_j) = 0.$$

Again, summarizing these equalities, we get $\varphi(x_i, [y_1, x_j] + [x_j, y_1]) = -[x_i, \varphi(y_1, x_j) + \varphi(x_j, y_1)]$, from which we have

$$\varphi(x_i, y_2) = -\frac{2}{3}[x_i, \varphi(y_1, x_1) + \varphi(x_1, y_1)] = -\beta_{1,1}y_{i+1}, \quad 1 \le i \le n-1,$$
$$\varphi(x_i, y_{j+1}) = -2[x_i, \varphi(y_1, x_j) + \varphi(x_j, y_1)] = -\beta_{1,1}y_{i+1}, \quad 1 \le i \le n-1,\ 2 \le j \le n-1.$$

Applying the above arguments to the equalities $(d^2\varphi)(y_i, y_1, x_j) = 0$ and $(d^2\varphi)(y_i, x_j, y_1) = 0$ for $1 \le i, j \le n$, we get

$$\varphi(y_i, y_{j+1}) = -2\beta_{j,1}x_i, \quad 1 \le i \le n,\ 1 \le j \le n-1.$$

Checking the general condition of cocycle for the other basis elements we get the already obtained restrictions. □

Using the assertion of Proposition 4.3 we indicate a basis of the space $ZL_0^2(NF^{n,n}, NF^{n,n})$.

Theorem 4.4. *The following cochains $\varphi_{j,k,}$, $\psi_{j,k}$*

$$\varphi_{1,1}: \begin{cases} \varphi_{1,1}(y_1, y_1) = x_1, \\ \varphi_{1,1}(x_i, x_1) = -x_{i+1}, & 1 \le i \le n-1, \\ \varphi_{1,1}(y_i, x_1) = -y_{i+1}, & 2 \le i \le n-1, \end{cases} \qquad \varphi_{j,1}(2 \le j \le n): \begin{cases} \varphi_{j,1}(y_j, y_1) = x_1, \\ \varphi_{j,1}(x_{j-1}, x_1) = x_1, \\ \varphi_{j,1}(y_j, x_1) = y_2, \\ \varphi_{j,1}(x_i, x_j) = -x_{i+1}, & 1 \le i \le n-1, \\ \varphi_{j,1}(y_i, x_j) = -y_{i+1}, & 1 \le i \le n-1, \end{cases}$$

$$\varphi_{1,k}(2 \le k \le n-1): \begin{cases} \varphi_{1,k}(y_1, y_1) = x_k, \\ \varphi_{1,k}(y_1, x_1) = y_{k+1}, \end{cases} \qquad \varphi_{1,n}: \{\varphi_{1,n}(y_1, y_1) = x_n.$$

$$\varphi_{j,k}\begin{pmatrix} 2 \le j \le n, \\ 2 \le k \le n-1 \end{pmatrix}: \begin{cases} \varphi_{j,k}(y_j, y_1) = x_k, \\ \varphi_{j,k}(x_{j-1}, x_1) = x_k, \\ \varphi_{j,k}(y_j, x_1) = y_{k+1}, \end{cases} \qquad \varphi_{j,n}(2 \le j \le n): \begin{cases} \varphi_{j,n}(y_j, y_1) = x_n, \\ \varphi_{j,n}(x_{j-1}, x_1) = x_n, \end{cases}$$

$$\psi_{j,1}\ (1 \le j \le n-1): \begin{cases} \psi_{j,1}(x_j, y_1) = y_1, \\ \psi_{j,1}(x_j, x_1) = 2x_1, \\ \psi_{j,1}(y_j, x_1) = 2y_1, \\ \psi_{j,1}(x_i, x_{j+1}) = -2x_{i+1}, & 1 \le i \le n-1, \\ \psi_{j,1}(y_i, x_{j+1}) = -2y_{i+1}, & 1 \le i \le n-1, \\ \psi_{j,1}(x_i, y_{j+1}) = -y_{i+1}, & 1 \le i \le n-1, \\ \psi_{j,1}(y_i, y_{j+1}) = -2x_i, & 1 \le i \le n, \end{cases}$$

$$\psi_{j,k}\begin{pmatrix} 1 \le j \le n, \\ 2 \le k \le n \end{pmatrix}: \begin{cases} \psi_{j,k}(x_j, y_1) = y_k, \\ \psi_{j,k}(x_j, x_1) = 2x_k, \\ \psi_{j,k}(y_j, x_1) = 2y_k. \end{cases}$$

form a basis of the space $ZL_0^2(NF^{n,n}, NF^{n,n})$.

Applying the same arguments as used in the proof of Proposition 3.3 we prove the following result.

Proposition 4.5. *The 2-cochains $\xi_{j,k}$ and $\zeta_{j,k}$ defined as follows*

$$\begin{cases} \xi_{j,k} = \varphi_{j,k}, & 1 \le j \le n,\ j \le k \le n, \\ \xi_{j,k} = \varphi_{j,k} - \dfrac{1}{2}\psi_{j,k+1}, & 2 \le j \le n,\ 1 \le k \le j-1, \\ \zeta_{j,k} = \psi_{j-1,k}, & 2 \le j \le n,\ j \le k \le n, \\ \zeta_{j,k} = \dfrac{1}{2}\psi_{j-1,k} - \varphi_{j,k}, & 2 \le j \le n,\ 1 \le k \le j-1, \end{cases}$$

form a basis of $BL_0^2(NF^{n,n}, NF^{n,n})$.

Corollary 4.6. $\{\overline{\psi_{n,2}}, \overline{\psi_{n,3}}, \dots, \overline{\psi_{n,n}}\}$ *form a basis of $HL_0^2(NF^n, NF^n)$.*

Consequently,

$$\dim ZL_0^2(NF^{n,n}, NF^{n,n}) = 2n^2 - 1,$$
$$\dim BL_0^2(NF^{n,n}, NF^{n,n}) = 2n^2 - n,$$
$$\dim HL_0^2(NF^n, NF^n) = n - 1.$$

In the next proposition we clarify that the basis element of $ZL_0^2(NF^{n,n}, NF^{n,n})$ satisfies the condition (2.1).

Proposition 4.7. *The infinitesimal deformations $\varphi_{j,k}$ $(1 \le j \le n, 2 \le k \le n)$ and $\psi_{j,k}$ $(1 \le j \le n, 2 \le k \le n)$ satisfy the condition (2.1), but the 2-cocycles $\varphi_{j,1}$ $(1 \le j \le n)$ and $\psi_{j,1}$ $(1 \le j \le n-1)$ do not satisfy the condition (2.1).*

Proof. The proof of this proposition is carried out by applying similar arguments as in the proof of Proposition 3.5. $\square$

4.2. The case $m = n + 1$

The results of this case we give without proofs since they can be easily proven similarly to those of case above.

Proposition 4.8. *Any 2-cocycle $\varphi \in ZL_0^2(NF^{n,n+1}, NF^{n,n+1})$ has the following form:*

$$\begin{cases}
\varphi(y_j, y_1) = \sum\limits_{k=1}^{n} \alpha_{j,k} x_k, & 1 \le j \le n+1,\\
\varphi(x_j, y_1) = \sum\limits_{k=1}^{n+1} \beta_{j,k} y_k, & 1 \le j \le n-1,\\
\varphi(x_n, y_1) = -\dfrac{\alpha_{n+1,1}}{2} y_1 + \sum\limits_{k=2}^{n} \beta_{n,k} y_k, & \\
\varphi(x_j, x_1) = -\alpha_{1,1} x_{i+1} + \sum\limits_{k=1}^{n} (\alpha_{j+1,k} + 2\beta_{j,k}) x_k, & 1 \le j \le n-1,\\
\varphi(x_n, x_1) = \sum\limits_{k=2}^{n} (\alpha_{n+1,k} + 2\beta_{n,k}) x_k, & \\
\varphi(y_j, x_1) = 2\beta_{j,1} y_1 - \alpha_{1,1} y_{j+1} + \sum\limits_{k=2}^{n+1} (\alpha_{j,k-1} + 2\beta_{j,k}) y_k, & 1 \le j \le n-1,\\
\varphi(y_n, x_1) = \alpha_{n+1,1} y_1 - \alpha_{1,1} y_{n+1} + \sum\limits_{k=2}^{n+1} (\alpha_{n,k-1} + 2\beta_{n,k}) y_k, & \\
\varphi(y_{n+1}, x_1) = \sum\limits_{k=2}^{n+1} \alpha_{n+1,k-1} y_k, & \\
\varphi(x_i, x_{j+1}) = -(\alpha_{j+1,1} + 2\beta_{j,1}) x_{i+1}, & 1 \le i \le n-1,\ 1 \le j \le n-1,\\
\varphi(y_i, x_{j+1}) = -(\alpha_{j+1,1} + 2\beta_{j,1}) y_{i+1}, & 1 \le i \le n,\ 1 \le j \le n-1,\\
\varphi(x_i, y_{j+1}) = -\beta_{j,1} y_{i+1}, & 1 \le i \le n,\ 1 \le j \le n-1,\\
\varphi(x_i, y_{n+1}) = -\dfrac{\alpha_{n+1,1}}{2} y_{i+1}, & 1 \le i \le n,\\
\varphi(y_i, y_{j+1}) = -2\beta_{j,1} x_i, & 1 \le i \le n,\ 1 \le j \le n-1,\\
\varphi(y_i, y_{n+1}) = -\alpha_{n+1,1} x_i, & 1 \le i \le n.
\end{cases}$$

Using the assertion of Proposition 4.8 we indicate a basis of the space $ZL_0^2(NF^{n,n+1}, NF^{n,n+1})$.

Theorem 4.9. *The following cochains $\varphi_{j,k,}$, $\psi_{j,k}$*

$$\varphi_{1,1} : \begin{cases}
\varphi_{1,1}(y_1, y_1) = x_1, & \\
\varphi_{1,1}(x_i, x_1) = -x_{i+1}, & 1 \le i \le n-1,\\
\varphi_{1,1}(y_i, x_1) = -y_{i+1}, & 2 \le i \le n,
\end{cases}
\qquad
\varphi_{j,1}(2 \le j \le n) : \begin{cases}
\varphi_{j,1}(y_j, y_1) = x_1, & \\
\varphi_{j,1}(x_{j-1}, x_1) = x_1, & \\
\varphi_{j,1}(y_j, x_1) = y_2, & \\
\varphi_{j,1}(x_i, x_j) = -x_{i+1}, & 1 \le i \le n-1,\\
\varphi_{1,1}(y_i, x_j) = -y_{i+1}, & 1 \le i \le n,
\end{cases}$$

$$\varphi_{n+1,1} : \begin{cases}
\varphi_{n+1,1}(y_{n+1}, y_1) = x_1, & \\
\varphi_{n+1,1}(x_n, y_1) = -\dfrac{1}{2} y_1, & \\
\varphi_{n+1,1}(y_n, x_1) = -y_1, & \\
\varphi_{n+1,1}(y_{n+1}, x_1) = y_2, & \\
\varphi_{n+1,1}(x_i, y_{n+1}) = -\dfrac{1}{2} y_{i+1}, & 1 \le i \le n,\\
\varphi_{n+1,1}(y_i, y_{n+1}) = -x_i, & 1 \le i \le n,
\end{cases}$$

$$\varphi_{1,k}(2 \le k \le n):\begin{cases}\varphi_{1,k}(y_1, y_1) = x_k,\\ \varphi_{1,k}(y_1, x_1) = y_{k+1},\end{cases}\qquad \varphi_{j,k}\begin{pmatrix}2 \le j \le n+1,\\ 2 \le k \le n\end{pmatrix}:\begin{cases}\varphi_{j,k}(y_j, y_1) = x_k,\\ \varphi_{j,k}(x_{j-1}, x_1) = x_k,\\ \varphi_{j,k}(y_j, x_1) = y_{k+1},\end{cases}$$

$$\psi_{j,1}(1 \le j \le n-1):\begin{cases}\psi_{j,1}(x_j, y_1) = y_1,\\ \psi_{j,1}(x_j, x_1) = 2x_1,\\ \psi_{j,1}(y_j, x_1) = 2y_1,\\ \psi_{j,1}(x_i, x_{j+1}) = -2x_{i+1}, & 1 \le i \le n-1,\\ \psi_{j,1}(y_i, x_{j+1}) = -2y_{i+1}, & 1 \le i \le n,\\ \psi_{j,1}(x_i, y_{j+1}) = -y_{i+1}, & 1 \le i \le n,\\ \psi_{j,1}(y_i, y_{j+1}) = -2x_i, & 1 \le i \le n,\end{cases}$$

$$\psi_{j,k}\begin{pmatrix}1 \le j \le n,\\ 2 \le k \le n\end{pmatrix}:\begin{cases}\psi_{j,k}(x_j, y_1) = y_k,\\ \psi_{j,k}(x_j, x_1) = 2x_k,\\ \psi_{j,k}(y_j, x_1) = 2y_k,\end{cases}\qquad \psi_{j,n+1}\ (1 \le j \le n):\begin{cases}\psi_{j,n+1}(x_j, y_1) = y_{n+1},\\ \psi_{j,n+1}(y_j, x_1) = 2y_{n+1},\end{cases}$$

form a basis of $ZL_0^2(NF^{n,n+1}, NF^{n,n+1})$.

Proposition 4.10. *The cochains* $\xi_{j,k}$ *and* $\zeta_{j,k}$ *defined as*

$$\begin{cases}\xi_{j,k} = \varphi_{j,k}, & 1 \le j \le n,\ j \le k \le n,\\ \xi_{j,k} = \varphi_{j,k} - \frac{1}{2}\psi_{j,k+1}, & 2 \le j \le n,\ 1 \le k \le j-1,\\ \zeta_{j,1} = \frac{1}{2}\psi_{j-1,1} - \varphi_{j,1}, & 2 \le j \le n,\\ \zeta_{n+1,1} = -\varphi_{n+1,1}, & \\ \zeta_{j,k} = \psi_{j-1,k}, & 2 \le j \le n+1,\ j \le k \le n+1,\\ \zeta_{j,k} = \frac{1}{2}\psi_{j-1,k} - \varphi_{j,k}, & 2 \le j \le n+1,\ 2 \le k \le j-1,\end{cases}$$

form a basis of $BL_0^2(NF^{n,n+1}, NF^{n,n+1})$.

Corollary 4.11. $\{\overline{\varphi_{n+1,2}}, \overline{\varphi_{n+1,3}}, \dots, \overline{\varphi_{n+1,n}}\}$ *form a basis of* $HL_0^2(NF^{n,n+1}, NF^{n,n+1})$.

Therefore

$$\dim ZL_0^2(NF^{n,n+1}, NF^{n,n+1}) = 2n^2 + 2n - 1,$$
$$\dim BL_0^2(NF^{n,n+1}, NF^{n,n+1}) = 2n^2 + n,$$
$$\dim HL_0^2(NF^{n,n+1}, NF^{n,n+1}) = n - 1.$$

The proposition below specifies basic infinitesimal deformations satisfying the condition (2.1).

Proposition 4.12. *The infinitesimal deformations* $\varphi_{j,k}$ $(1 \le j \le n+1, 2 \le k \le n)$ *and* $\psi_{j,k}$ $(1 \le j \le n, 2 \le k \le n+1)$ *satisfy the condition* (2.1), *but the* 2*-cocycles* $\varphi_{j,1}$ $(1 \le j \le n+1)$ *and* $\psi_{j,1}$ $(1 \le j \le n-1)$ *do not satisfy the condition* (2.1).

Since $\sum_{k=2}^{n} b_k\psi_{n,k}$ and $\sum_{k=2}^{n} c_k\varphi_{n+1,k}$ define linear integrable deformations of $NF^{n,n}$ and $NF^{n,n+1}$, respectively, we consider two families of superalgebras $\nu_t(b_2, b_3, \dots, b_n) = NF^{n,n} + t\sum_{k=2}^{n} b_k\psi_{n,k}$ and $\eta_t(c_2, c_3, \dots, c_n) = NF^{n,n+1} + t\sum_{k=2}^{n} c_k\varphi_{n+1,k}$ with the multiplication tables

$$\begin{cases}[y_i, y_1] = x_i, & 1 \le i \le n,\\ [x_i, y_1] = \frac{1}{2}y_{i+1}, & 1 \le i \le n-1,\\ [x_n, y_1] = t\sum_{k=2}^{n} b_k y_k, & \\ [y_i, x_1] = y_{i+1}, & 1 \le i \le n-1,\\ [y_n, x_1] = 2t\sum_{k=2}^{n} b_k y_k, & \\ [x_i, x_1] = x_{i+1}, & 1 \le i \le n-1,\\ [x_n, x_1] = 2t\sum_{k=2}^{n} b_k x_k, & \end{cases}\quad\text{and}\quad \begin{cases}[x_i, x_1] = x_{i+1}, & 1 \le i \le n-1,\\ [x_n, x_1] = t\sum_{k=2}^{n} c_k x_k, & \\ [y_i, x_1] = y_{i+1}, & 1 \le j \le n,\\ [y_{n+1}, x_1] = t\sum_{k=2}^{n} c_k y_{k+1}, & \\ [y_i, y_1] = x_i, & 1 \le i \le n,\\ [y_{n+1}, y_1] = t\sum_{k=2}^{n} c_k x_k, & \\ [x_i, y_1] = \frac{1}{2}y_{i+1}, & 1 \le i \le n\end{cases}$$

respectively.

Putting $b'_k = tb_k$ and $c'_k = tc_k$, we can assume in both multiplications $t = 1$.

From the description of single-generated Leibniz superalgebras it is deduced that they have multiplication tables of the form of the superalgebras $\nu_1(b_2, b_3, \ldots, b_n)$ and $\eta_1(c_2, c_3, \ldots, c_n)$.

Similarly to the case of Leibniz algebras for superalgebras we obtain the following theorem.

Theorem 4.13. $\overline{\bigcup_{b_2,\ldots,b_n} \mathrm{Orb}(\nu_1(b_2, b_3, \ldots, b_n))}$ *and* $\overline{\bigcup_{c_2,\ldots,c_n} \mathrm{Orb}(\eta_1(c_2, c_3, \ldots, c_n))}$ *are irreducible components of the varieties* $\mathrm{Leib}^{n,n}$ *and* $\mathrm{Leib}^{n,n+1}$*, respectively.*

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