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RESEARCH ARTICLES

Classification of Three-Dimensional Solvable p-Adic Leibniz Algebras*

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Abstract—The present paper is devoted to the study of low dimensional Leibniz algebras over the field of p-adic numbers. The classification up to isomorphism of three-dimensional Lie algebras over the integer p-adic numbers is already known [8]. Here, we extend this classification to solvable Lie and non-Lie Leibniz algebras over the field of p-adic numbers.

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1. INTRODUCTION

The notion of Leibniz algebra was introduced at the beginning of 90's of the last century as a "non-antisymmetric" generalization of Lie algebra [9]. Therefore, it emerges the problem of establishing results similar to these from the theory of Lie algebras. The investigation of nilpotent Leibniz algebras shows that many properties of nilpotent Lie algebras can be extended to nilpotent Leibniz algebras. Nevertheless, the study of inherent properties of non-Lie Leibniz algebras also attracts the attention of researchers.

From the theory of Lie algebras, we know that the description of finite-dimensional Lie algebras reduces to the description of nilpotent ones [5, 10]. Many works are devoted to the description of nilpotent Lie algebras. Namely, the classification of complex nilpotent Lie algebras up to dimension 8 is obtained [2]. Concerning Leibniz algebras, we have their classification up to dimension 4 and the classification of nilpotent ones up to dimension 5, see [1, 3] and [4].

Recall, there are only two types of algebraically closed fields of zero characteristic. One of them is a field of complex numbers and the other one is algebraic closure of p-adic numbers [6, 11, 12]. Mostly, algebras are described over the fields of real and complex numbers, or over field of finite characteristic. This is due to the relationship between algebraic and differential geometries. Only a few works are devoted to the description of algebras over the field of p-adic numbers [7, 8].

The classification up to isomorphism of any class of algebras, even in low dimensions, is a fundamental and a very difficult problem. It is one of the first problems that one encounters while trying to understand the structure of the algebra in a certain class. In this work we describe p-adic solvable three-dimensional Lie algebras and then continue the classification for the case of Leibniz algebras. It should be noted that non-Lie Leibniz algebras of dimensions less than 4 over the field of zero characteristic are solvable.

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2. PRELIMINARIES

Let $\mathbb{Q}$ be the field of rational numbers. For a fixed prime number p every rational number x can be presented in the form $x = p^r \cdot \frac{n}{m}$, where $r, n \in \mathbb{Z}$, $m \in \mathbb{N}$ and $\gcd(p,n) = 1$, $\gcd(p,m) = 1$. The p-adic norm of x is given by

$$|x|_p = \begin{cases} p^{-r} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

The completion of $\mathbb{Q}$ with respect to p-adic norm defines the p-adic field which is denoted by $\mathbb{Q}_p$. It is known that any p-adic number x $(x \neq 0)$ is uniquely represented in the canonical form $x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + ...)$, where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers such that $0 \leq x_j \leq p-1$, $x_0 > 0, (j = 0, 1, \dots)$.

p-Adic numbers, satisfying the condition $|x|_p \leq 1$ are called *integer p-adic numbers*. We denote by $\mathbb{Z}_p$ the set of all integer p-adic numbers. Integers $x \in \mathbb{Z}_p$ with the property $|x|_p = 1$ are called *units*.

Recall, that a number $a \in \mathbb{Z}_p$ is said to be *quadratic residue modulo p* if the equation $x^2 \equiv a (mod\, p)$ has a solution $x \in \mathbb{Z}_p$; otherwise a is called *quadratic non-residue modulo p*.

Let a be a non zero p-adic number with the following canonical form $a = p^{\gamma(a)}(a_0 + a_1 p + ...)$.

Lemma 2.1. *[12] The equation $x^2 = a$ has solution $x \in \mathbb{Q}_p$ if and only if the following conditions hold:*

1) $\gamma(a)$ is an even number,

2) a_0 is a quadratic residue modulo p if $p \neq 2$ and $a_1 = a_2 = 0$ if $p = 2$.

Let η be a p-adic unit not being the square of any p-adic number. Lemma 2.1 implies the following corollaries.

Corollary 2.1. *[12] Any p-adic number $(p \neq 2)$ x can be presented in one of the following forms: $x = \varepsilon_j y_j^2$, $0 \leq j \leq 3$, where $y_j \in \mathbb{Q}_p$, $\varepsilon_0 = 1$ and $\varepsilon_1 = \eta, \varepsilon_2 = p, \varepsilon_3 = p\eta$ are not squares of any p-adic numbers.*

Corollary 2.2. *[12] The numbers $\varepsilon_j \in \{1, 2, 3, 5, 6, 7, 10, 14\}$ $(1 \leq j \leq 8)$ and their mutual products are not squares of any 2-adic numbers. Therefore, any 2-adic number can be presented in the form $\varepsilon_j y_j^2$, where $y_j \in \mathbb{Q}_p$, $1 \leq j \leq 8$.*

Now we give the definitions of Lie and Leibniz algebras.

Definition 2.1. An algebra G over a field F is called a Lie algebra if for any $x, y, z \in G$ it satisfies the following identities:

$$[x,x] = 0, \quad [[x,y],z] + [[y,z],x] + [[z,x],y] = 0 - \text{Jacobi identity},$$

where $[\ ,\ ]$ is a multiplication in G.

Definition 2.2. An algebra L over a field F is said to be a Leibniz algebra if for any $x, y, z \in L$ it satisfies the Leibniz identity $[x,[y,z]] = [[x,y],z] - [[x,z],y]$, where $[\ ,\ ]$ is a multiplication in L.

It should be noted that Leibniz identity along with the identity $[x,x] = 0$ implies Jacoby identity. Thus, Lie algebras are particular case of Leibniz algebras.

For a given Leibniz algebra L we define the following sequences:

a) $L^1 = L, L^{n+1} = [L^n, L]$, $n \geq 1$;

b) $L^{[1]} = L, L^{[n+1]} = [L^{[n]}, L^{[n]}]$, $n \geq 1$.

Definition 2.3. An algebra L is said to be nilpotent (solvable) if there exists $k \in N$ $(m \in N)$ such that $L^k = 0$ $(L^{[m]} = 0)$.

The set $R(L) = \{x \in L : [y, x] = 0\}$ is called *right annihilator* of the Leibniz algebra L. Note that $R(L)$ is an ideal of L and right annihilator under an isomorphism of algebras transforms to a right annihilator.

We call a Lie algebra *perfect* if its all derivations are inner and the center of the algebra is zero.

Proposition 2.1. *[5] Let $\Re$ be an ideal of Lie algebra G and $\Re$ be a perfect algebra. Then $G = \Re \oplus B$, where B is a centralizer of $\Re$ in algebra G and B is an ideal of G.*

It is not difficult to check that the descriptions of p-adic Lie algebras and complex Lie algebras of dimension less than three coincide, i.e. if $dimG = 1$, then $[x, x] = 0$, and if $dimG = 2$,then G is either abelian or $[x, y] = x$.

Moreover, for the three-dimensional p-adic Lie algebra with the condition $dimG^2 = 1$, similar to the case of the complex Lie algebras, we obtain the algebras $[x, y] = z$ and $[x, y] = x$.

Theorem 2.1. *[8] Let G be a three-dimensional solvable Lie algebra over $\mathbb{Z}_p$ with condition $dimG^2 = 2$. Then G is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{aligned}
&[x,z] = x, && [y,z] = y, \\
&[x,z] = x + p^r dy, && [y,z] = p^r x + y, \qquad r \in \mathbb{N}, d \in \mathbb{Z}_p, \\
&[x,z] = dy, && [y,z] = x + p^r y, \qquad r \in \mathbb{N}_0, d \in \mathbb{Z}_p, \\
&[x,z] = \rho p^r y, && [y,z] = x, \quad r \in \mathbb{N}_0,
\end{aligned}$$

where ρ is a p-adic number, which is not a square of any p-adic number.

Below we present the list of complex three-dimensional solvable Lie algebras.

Theorem 2.2. *[5] An arbitrary complex three-dimensional solvable Lie algebra G with condition $dimG^2 = 2$ is isomorphic to one of the following non-isomorphic algebras:*

$$\begin{aligned}
&[x,z] = x, && [y,z] = \alpha y, \;\; \alpha \neq 0; \\
&[x,z] = x + y, && [y,z] = y,
\end{aligned}$$

where only algebras with parameters $\dfrac{1}{\alpha}$ and α of the first family are isomorphic.

However, the description of p-adic Lie algebras with condition $dimG^2 = 2$ differs from the complex ones. The next sections are devoted to the classification of such algebras.

3. CLASSIFICATION OF THREE-DIMENSIONAL p-ADIC SOLVABLE LIE ALGEBRAS

Let G be a three dimensional solvable Lie algebra over the field F. Evidently, $dimG^2 \leq 2$. Since the case of $dimG^2 = 1$ coincides with the complex one [5], further we will investigate the case of $dimG^2 = 2$.

Put $\Re = G^2$. If $\Re$ is non-abelian Lie algebra, then by Proposition 2.1 we have $G = \Re \oplus B$ and $G^2 = \Re^2 = \Re$. Due to solvability of G we obtain $\Re^2 \subsetneq \Re$, which is a contradiction with $\Re^2 = \Re$. Therefore, $\Re$ is an abelian algebra.

Let $\{x, y, z\}$ be a basis of G such that $G^2 = \{x, y\}$, i.e. $G^2 = Fx + Fy$. Then $G^2 = F[x, z] + F[y, z]$ and operator R_z (the operator of right multiplication on an element z defined as $R_z(x) = [x, z]$) induces a one-to-one linear mapping in G^2. Therefore, the algebra G has the basis $\{x, y, z\}$ with the multiplication

$[x, y] = 0, \quad [x, z] = \alpha x + \beta y, \quad [y, z] = \gamma x + \delta y$, where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is a non-degenerated matrix.

Theorem 3.1. *Let G be a three-dimensional p-adic solvable Lie algebra with condition $dimG^2 = 2$. Then G is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$I: \quad [x,z]=x, \qquad [y,z]=\delta y, \quad \delta\neq 0;$$
$$II: \quad [x,z]=x, \qquad [y,z]=x+y;$$
$$III: \quad [x,z]=x+y, \quad [y,z]=\gamma x;$$
$$IV: \quad [x,z]=x+y, \quad [y,z]=(\varepsilon_j-1)x-y,$$

where $\delta,\gamma\in\mathbb{Q}_p$; only algebras with parameters $\frac{1}{\delta}$ and δ of the family I are isomorphic; $1+4\gamma$ does not have a square root in $\mathbb{Q}_p$ and $\varepsilon_j\in\{2,3,5,6,7,10,14\}$ $(1\leq j\leq 8)$ if $p=2$, $\varepsilon_j\in\{\eta,p,p\eta\}$ if $p\neq 2$ $(1\leq j\leq 3)$.

Proof. Let G be a Lie algebra satisfying the conditions of the theorem. Then there exists a basis $\{x,y,z\}$ with the multiplication: $[x,z]=\alpha x+\beta y, \quad [y,z]=\gamma x+\delta y$.

If $(\alpha,\delta)\neq(0,0)$ then due to symmetry property of the elements x and y we can assume $\alpha\neq 0$. If $(\alpha,\delta)=(0,0)$, then $\beta\neq 0,\gamma\neq 0$. So, we have multiplications $[x,z]=\beta y,[y,z]=\gamma x$. Making the change of the basic elements in the form $\overline{x}=x+y,\ \overline{y}=y,\ \overline{z}=z$ we obtain $[\overline{x},\overline{z}]=\overline{\alpha}\,\overline{x}+\overline{\beta}\overline{y}$, where $\overline{\alpha}\neq 0$. Thus, one can always assume $\alpha\neq 0$. Putting $\overline{z}=\frac{z}{\alpha}$ into the products $[x,z]=\alpha x+\beta y, \quad [y,z]=\gamma x+\delta y$ yields $\alpha=1$ and $[x,z]=\ x+\beta y, \quad [y,z]=\gamma x+\delta y$.

Case 1. Let $\beta=0$. Then multiplication has the form $[x,z]=x, \quad [y,z]=\gamma x+\delta y$ $(\delta\neq 0)$. Now let us check an isomorphism inside this family. For this purpose we consider two algebras from the family:

$$[x,z]=x, \quad [y,z]=\gamma x+\delta y; \quad [\overline{x},\overline{z}]=\overline{x}, \quad [\overline{y},\overline{z}]=\overline{\gamma}\ \overline{x}+\overline{\delta}\overline{y}$$

and take the general transformation of the basis in the form $\overline{x}=a_1x+a_2y+a_3z,\ \overline{y}=b_1x+b_2y+b_3z,\ \overline{z}=c_1x+c_2y+c_3z$, where $a_1b_2-b_1a_2\neq 0$.

It is not difficult to check that $a_3=b_3=c_1=c_2=0$.

Consider the multiplication $[\overline{x},\overline{z}]=[a_1x+a_2y,z]=(a_1+a_2\gamma)x+a_2\delta y$. Since $[\overline{x},\overline{z}]=\overline{x}=a_1x+a_2y$, by comparing coefficients at the corresponding basic elements, we obtain $a_1=a_1+a_2\gamma, a_2=a_2\delta$.

Consider the product $[\overline{y},\overline{z}]=[b_1x+b_2y,z]=(b_1+b_2\gamma)x+b_2\delta y$. Similarly, $[\overline{y},\overline{z}]=\overline{\gamma}\ \overline{x}+\overline{\delta}\overline{z}=\overline{\gamma}(a_1x+a_2y)+\overline{\delta}(b_1x+b_2y)$ which implies $\overline{\gamma}a_1+\overline{\delta}b_1=b_1+b_2\gamma,\ \overline{\gamma}a_2+\overline{\delta}b_2=b_2\delta$.

Thus, we obtain the following restrictions

$$\begin{cases} \gamma a_2=0 \\ a_2=\delta a_2 \\ \overline{\gamma}a_1+\overline{\delta}b_1=b_1+\gamma b_2 \\ \overline{\gamma}a_2+\overline{\delta}b_2=\delta b_2. \end{cases} \tag{1}$$

Let $\gamma\neq 0$. Then $a_2=0$ and from (1) we derive

$$\begin{cases} \overline{\gamma}a_1+\overline{\delta}b_1=b_1+\gamma b_2 \\ \overline{\delta}b_2=\delta b_2. \end{cases}$$

Hence, $\overline{\gamma}=\dfrac{b_2\gamma+(1-\delta)b_1}{a_1}$.

If $\delta\neq 1$ then putting $b_1=\dfrac{b_2\gamma}{\delta-1}$ yields $\overline{\gamma}=0$. As a result we obtain the algebra: $I':[x,z]=x, \quad [y,z]=\delta y,(\delta\neq 1)$. If $\delta=1$, then $\overline{\delta}=1$. Taking $a_1=b_2\gamma$ implies $\overline{\gamma}=1$, i.e. we obtain the following algebra: $II:[x,z]=x, \quad [y,z]=x+y$.

Let $\gamma = 0$. Then restriction (1) transforms to

$$\begin{cases} a_2(\delta-1)=0 \\ \overline{\gamma}a_1+\overline{\delta}b_1=b_1 \\ \overline{\gamma}a_2+\overline{\delta}b_2=\delta b_2. \end{cases} \tag{2}$$

If $\delta \neq 1$, then $a_2 = 0$. Consequently, we have $b_2 \neq 0, a_1 \neq 0$ and $\overline{\gamma} = \dfrac{(1-\delta)b_1}{a_1}$. If $b_1 = 0$ then $\overline{\gamma} = 0$ and we obtain the algebra I. If $\delta = 1$, then the equalities in (2) are

$$\begin{cases} \overline{\gamma}a_1+\overline{\delta}b_1=b_1 \\ \overline{\gamma}a_2+\overline{\delta}b_2=b_2. \end{cases} \tag{3}$$

Multiplying the second equation of (3) by a_1 and the first one by a_2, then subtracting the first obtained equation from the second one yields $\overline{\delta}(a_1b_2 - b_1a_2) = a_1b_2 - a_2b_1$. Hence, $\overline{\delta} = 1$.

Similarly, multiplying the second equation of (3) by b_1 and the first one by b_2, then subtracting the first equation from the second one, we obtain $\overline{\gamma}(a_2b_1 - a_1b_2) = 0$. The last one implies $\overline{\gamma} = 0$. So, we obtain the algebra $I'' : [x,z] = x, \quad [y,z] = y$.

It should be noted that the algebra I'' is the algebra of the family I when $\delta = 1$.

Summarizing the family of algebras I' and algebra I'', we deduce the family of algebras I.

Note that, the algebras I and II are not isomorphic even in the case of complex one [5] and two algebras from the family I are isomorphic if and only if $\frac{1}{\delta} = \overline{\delta}$.

Case 2. Let $\beta \neq 0$. Applying $\overline{y} = \beta y$ we obtain $\beta = 1$ and therefore, we can assume that the multiplication is in the form $[x,z] = x + y, \quad [y,z] = \gamma x + \delta y, \quad \gamma \neq \delta$.

Taking the change $\overline{x} = a_1x + a_2y$, consider the multiplication $[\overline{x}, z] = [a_1x + a_2y, z] = a_1(x + y) + a_2(\gamma x + \delta y) = (a_1 + a_2\gamma)x + (a_1 + a_2\delta)y$. Note that, if the equality $(a_1 + a_2\gamma)a_2 = (a_1 + a_2\delta)a_1$, holds, i.e., the quadratic equation $a_1^2 + a_1(\delta - 1)a_2 - a_2^2\gamma = 0$ has a solution in $\mathbb{Q}_p$, then choosing $a_2 = 1, \quad a_1 = \dfrac{-(\delta-1)+\sqrt{(\delta-1)^2+4\gamma}}{2}$ we have that $[\overline{x}, \overline{z}] = \dfrac{1+\delta+\sqrt{(\delta-1)^2+4\gamma}}{2}\,\overline{x}$. According to the condition $\gamma \neq \delta$, it is not difficult to deduce $1 + \delta + \sqrt{(\delta-1)^2 + 4\gamma} \neq 0$.

Therefore, we conclude that, if $(\delta - 1)^2 + 4\gamma$ has a square root in $\mathbb{Q}_p$, then we can assume $[\overline{x}, \overline{z}] = \overline{x}$ which implies that case 2 is reduces to the case 1.

Let us check the isomorphism inside the family $[x,z] = x + y, \quad [y,z] = \gamma x + \delta y$, when $\gamma \neq \delta$ and $(\delta - 1)^2 + 4\gamma$ does not have a square root in $\mathbb{Q}_p$.

Consider the general change in the following form: $\overline{x} = a_1x + a_2y + a_3z,\ \overline{y} = b_1x + b_2y + b_3z,\ \overline{z} = c_1x + c_2y + c_3z$.

Similarly to the case 1 from the multiplications $[\overline{x}, \overline{x}],\ [\overline{y}, \overline{x}],\ [\overline{x}, \overline{y}],\ [\overline{y}, \overline{y}]$ and $[\overline{z}, \overline{z}]$, we obtain $a_3 = b_3 = c_1 = c_2 = 0$.

Consider the product $[\overline{x}, \overline{z}] = [a_1x + a_2y, c_1x + c_2y + c_3z] = (a_1c_3 + a_2c_3\gamma)x + (a_1c_3 + a_2c_3\delta)y$. On the other hand, $[\overline{x}, \overline{z}] = \overline{x} + \overline{y} = (a_1 + b_1)x + (a_2 + b_2)y$. Comparing the coefficients at the corresponding basic elements we obtain

$$\begin{cases} a_1+b_1=a_1c_3+a_2c_3\gamma \\ a_2+b_2=a_1c_3+a_2c_3\delta. \end{cases}$$

Similarly, from the products $[\overline{y}, \overline{z}] = [b_1x + b_2y, c_1x + c_2y + c_3z] = (b_1c_3 + b_2c_3\gamma)x + (b_1c_3 + b_2c_3\delta)y$, $[\overline{y}, \overline{z}] = \overline{\gamma}\,\overline{x} + \overline{\delta}\overline{y} = \overline{\gamma}(a_1x + a_2y) + \overline{\delta}(b_1x + b_2y)$, we derive

$$\begin{cases} \overline{\gamma}a_1+\overline{\delta}b_1=b_1c_3+b_2c_3\gamma \\ \overline{\gamma}a_2+\overline{\delta}b_2=b_1c_3+b_2c_3\delta. \end{cases}$$

Thus, we obtain the following restrictions:

$$\begin{cases} a_1 + b_1 = a_1c_3 + a_2c_3\gamma \\ a_2 + b_2 = a_1c_3 + a_2c_3\delta \\ \overline{\gamma}a_1 + \overline{\delta}b_1 = b_1c_3 + b_2c_3\gamma \\ \overline{\gamma}a_2 + \overline{\delta}b_2 = b_1c_3 + b_2c_3\delta. \end{cases} \tag{4}$$

Now if we subtract the fourth equation multiplied by a_1 from the third one multiplied by a_2, we obtain $\overline{\delta}(a_1b_2 - a_2b_1) = c_3(a_1b_1 - a_2b_1 - \gamma a_2b_2 + \delta a_1b_2)$.

Analogously, if we subtract now the second equation multiplied by b_1 from the first one multiplied by b_2 we obtain $a_1b_2 - a_2b_1 = c_3(a_1b_2 - a_1b_1 + \gamma a_2b_2 - \delta a_2b_1)$. Summarizing these equalities we conclude, that $(\overline{\delta} + 1)(a_1b_2 - a_2b_1) = c_3(a_1b_2 - a_2b_1 + \delta a_1b_2 - \delta a_2b_1)$, $\quad(\overline{\delta} + 1)(a_1b_2 - a_2b_1) = c_3(a_1b_2 - a_2b_1)(\delta + 1)$, $\quad(\overline{\delta} + 1) = c_3(\delta + 1)$.

Now subtracting the fourth equation of (4) multiplied by b_1 from the third one multiplied by b_2 we obtain $\overline{\gamma}(a_1b_2 - a_2b_1) = c_3((1 - \delta)b_1b_2 + b_2^2\gamma - b_1^2)$.

By putting $a_1c_3 + a_2c_3\gamma - a_1$, and $a_1c_3 + a_2c_3\delta - a_1$ instead of b_1 and b_2, respectively, we obtain $\overline{\gamma} = c_3^2(\gamma - \delta) + c_3(1 + \delta) - 1)$.

Therefore, we have the following restrictions

$$\begin{cases} \overline{\delta} + 1 = c_3(\delta + 1) \\ \overline{\gamma} + 1 = c_3^2(\gamma - \delta) + c_3(\delta + 1). \end{cases}$$

Note that, the equality $(\overline{\delta} - 1)^2 + 4\overline{\gamma} = c_3^2((\delta - 1)^2 + 4\gamma)$ holds. It means that, if $(\delta - 1)^2 + 4\gamma$ does not have a square root in $\mathbb{Q}_p$, then $(\overline{\delta} - 1)^2 + 4\overline{\gamma}$ also does not have a square root in $\mathbb{Q}_p$.

Case 2.1. Let $\delta \neq -1$. Then taking $c_3 = \dfrac{1}{\delta + 1}$ we obtain $\overline{\delta} = 0$, $\overline{\gamma} = \dfrac{\gamma - \delta}{(1 + \delta)^2}$ and we have the parametric algebra $III : [x, z] = x + y, \;\; [y, z] = \gamma x$, where $\gamma \in \mathbb{Q}_p$ such that $1 + 4\gamma$ does not have a square root in $\mathbb{Q}_p$.

Case 2.2. Let $\delta = -1$. Then $\overline{\delta} = -1$ and $\overline{\gamma} + 1 = c_3^2(\gamma + 1)$.

If $\delta \neq \gamma$, then $\gamma \neq -1$ and $\gamma + 1$ does not have a square root in $\mathbb{Q}_p$. Then, taking into account Corollary 2.1 and 2.2 $\gamma + 1$ is presented in the form $\gamma + 1 = \varepsilon_j y_j^2$. Taking $c_3 = \dfrac{1}{y_j}$, we have $\overline{\gamma} = \varepsilon_j - 1$, where $\varepsilon_j \in \{2, 3, 5, 6, 7, 10, 14\}$ $(1 \leq j \leq 8)$ if $p = 2$, and $\varepsilon_j \in \{\eta, p, p\eta\}$ if $p \neq 2$ $(1 \leq j \leq 3)$. Therefore, we obtain the algebra with the multiplication $[x, z] = x + y, \;\; [y, z] = (\varepsilon_j - 1)x - y$. □

Note that, the list of algebras of Theorem 2.1 is included into the list of algebras of Theorem 3.1.

Also, the first and the second families of Theorem 3.1 coincide with the corresponding families of algebras of Theorem 2.2. The algebras III and IV are different from the complex one. Thus, the list of algebras in the case of the p-adic numbers is larger then respective list in the case of complex field.

4. CLASSIFICATION OF THREE-DIMENSIONAL p-ADIC NON-LIE LEIBNIZ ALGEBRAS

Let L be a three-dimensional p-adic non-Lie Leibniz algebra and let $\{x, y, z\}$ be a basis of L. Since L is non-Lie, then $R(L) \neq 0$. In the case $R(L) = L$ the algebra L is abelian. It remains to consider the cases $dimR(L) = 1$ and $dimR(L) = 2$.

The following theorem gives the classification of three-dimensional p-adic non-Lie Leibniz algebras with the condition $dimR(L) = 1$.

Theorem 4.1. *Let L be a three-dimensional p-adic non-Lie Leibniz algebra with the condition $dimR(L) = 1$. Then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{array}{lllll} L_1: & [x,z]=-2x, & [y,y]=x, & [z,y]=y, & [y,z]=-y; \\ L_2(\alpha): & [x,z]=\alpha x, & [z,z]=x, & [z,y]=y, & [y,z]=-y; \\ L_3(\alpha): & [y,y]=x, & [z,z]=\alpha x, & [y,z]=x, & (\alpha\neq 0); \\ L_4(\varepsilon): & [y,y]=x, & [z,z]=\varepsilon x, & & \end{array}$$

where $\varepsilon \in \{1, \eta, p, p\eta\}$ if $p \neq 2$ and $\varepsilon \in \{1, 2, 3, 5, 6, 7, 10, 14\}$ if $p = 2$, $\alpha \in \mathbb{Q}_p$.

Proof. Let $dimR(L) = 1$. Then without loss of generality we can assume that $R(L) = \langle x\rangle$ and taking into account that $R(L)$ is an ideal in L we obtain the following multiplication in the algebra L:

$$\begin{array}{ll} [x,y]=\alpha_1 x, & [x,z]=\alpha_3 x, \\ [y,y]=\alpha_2 x, & [z,z]=\alpha_4 x, \\ [z,y]=\beta_1 x+\beta_2 y+\beta_3 z, & [y,z]=\gamma_1 x-\beta_2 y-\beta_3 z \end{array}$$

(omitted products are equal to zero).

Consider the case $dimL^2 = 2$. Then $(\beta_2, \beta_3) \neq (0, 0)$. Since y and z are symmetric, without loss of generality, one can set $\beta_2 \neq 0$. If we substitute $\overline{y} = \beta_2 y + \beta_3 z$, $\overline{z} = \dfrac{z}{\beta_2}$, then we obtain $\beta_2 = 1, \beta_3 = 0$, which imply the following multiplications:

$$\begin{array}{ll} [x,y]=\alpha_1 x, & [x,z]=\alpha_3 x, \\ [y,y]=\alpha_2 x, & [z,z]=\alpha_4 x, \\ [z,y]=\beta_1 x+y, & [y,z]=\gamma_1 x-y. \end{array}$$

Checking the Leibniz identity for the given brackets, we obtain the following relations for the structural constants:

$$\begin{cases} 2\alpha_2 = \alpha_1\gamma_1 - \alpha_2\alpha_3 \\ \alpha_1 = 0 \\ \alpha_1\alpha_4 = \beta_1 + \alpha_3\beta_1 + \gamma_1. \end{cases} \tag{5}$$

Case 1. Let $\alpha_2 \neq 0$. Then (5) yields $\alpha_3 = -2, \gamma_1 = \beta_1$ and therefore, multiplication is in the following form:

$$\begin{array}{ll} [x,y]=0, & [x,z]=-2x, \\ [y,y]=\alpha_2 x, & [z,z]=\alpha_4 x, \\ [z,y]=\beta_1 x+y, & [y,z]=\beta_1 x-y. \end{array}$$

Now substituting $\overline{z} = z - \dfrac{\beta_1}{\alpha_2}y + \dfrac{\alpha_2\alpha_4 - \beta_1^2}{2\alpha_2}x$, $\overline{x} = \alpha_2 x$, $\overline{y} = y$ we obtain the algebra $L_1: \ [x,z] = -2x, \ [y,y] = x, \ [z,y] = y, \ [y,z] = -y.$

Case 2. Let $\alpha_2 = 0$. Then multiplication is as follows: $[x,z] = \alpha_3 x, \quad [z,z] = \alpha_4 x, \quad [z,y] = \beta_1 x + y, \quad [y,z] = -\beta_1(1+\alpha_3)x - y.$

Since we are in non-Lie case, then $(\alpha_3, \alpha_4) \neq (0, 0)$. If $\alpha_4 = 0$, then $\alpha_3 \neq 0$ and setting $\overline{z} = \dfrac{x}{\alpha_3} + z$, we can assume $[z,z] = x$. And for the case $\alpha_4 \neq 0$, substituting $\overline{x} = \alpha_4 x$, we obtain $[z,z] = x$. Thus, we can state that $\alpha_4 = 1$ and the multiplication becomes as follows: $[x, z] = x, \quad [z, y] = \beta x + y, \quad [y, z] = -\beta(1+\alpha)x - y, \quad [z, z] = x.$

If in this multiplication we replace $\overline{x} = x$, $\overline{y} = \beta x + y$, $\overline{z} = z$, we obtain one-parametric algebra: $L_2(\alpha): \quad [x, z] = \alpha x, \quad [z, z] = x, \quad [z, y] = y, \quad [y, z] = -y.$

Now we check whether the following algebras are isomorphic:

$$L_1: \qquad [x, z] = -2x, \ [y, y] = x, \ [z, y] = y, \ [y, z] = -y;$$
$$L_2(\alpha): \ [\overline{x}, \overline{z}] = \alpha\overline{x}, \quad [\overline{z}, \overline{z}] = \overline{x}, \ [\overline{z}, \overline{y}] = \overline{y}, \ [\overline{y}, \overline{z}] = -\overline{y}, \ (\alpha \neq 0).$$

Let $\varphi: L_1 \to L_2(\alpha)$ be the isomorphism defined as follows: $\varphi(x) = a_1x + a_2y + a_3z, \quad \varphi(y) = b_1x + b_2y + b_3z, \quad \varphi(z) = c_1x + c_2y + c_3z$, where $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ is a non-singular matrix.

For the sake of convenience, we introduce the notation $\overline{u} := \varphi(u)$ for $u \in L$ and suppose that the base $\overline{x}, \overline{y}, \overline{z}$ satisfies to the multiplication in $L_2(\alpha)$.

Now consider the products $[\overline{z}, \overline{z}] = (c_2^2 - 2c_1c_3)x, [\overline{z}, \overline{y}] = b_2c_2x + b_2c_3y - 2b_3c_1x - b_3c_2y, [\overline{y}, \overline{y}] = b_2^2x$. Since $[\overline{z}, \overline{z}] = \overline{x} = a_1x + a_2y + a_3z$, then comparing coefficients at the corresponding basic elements, we obtain $a_1 = c_2^2 - 2c_1c_3, \ a_2 = 0, \ a_3 = 0$.

Similarly we obtain $b_1 = b_2c_2 - 2b_3c_1, b_2 = b_2c_3 - b_3c_2, b_3 = 0, b_2 = 0$. But these equalities contradict to non-singularity of the considered matrix. Hence, algebras L_1 and $L_2(\alpha)$ are not isomorphic.

Now let us check isomorphism between the algebras of the family $L_2(\alpha)$.

Consider two algebras:

$$L_2(\alpha): \ [x, z] = \alpha x, \ [z, z] = x, \ [z, y] = y, \ [y, z] = -y.$$
$$L_2(\overline{\alpha}): \ [\overline{x}, \overline{z}] = \overline{\alpha x}, \ [\overline{z}, \overline{z}] = \overline{x}, \ [\overline{z}, \overline{y}] = \overline{y}, \ [\overline{y}, \overline{z}] = -\overline{y}.$$

By similar argumentation as above, we obtain the following restrictions on coefficients for the matrix of isomorphism φ: $a_1 = \alpha c_1c_3 + c_3^2$, $a_2 = 0$, $a_3 = 0$, $\overline{\alpha}a_1 = \alpha a_1c_3$, $b_2 = b_2c_3$, $b_1 = 0$, $b_3 = 0$.

From these relations we obtain $c_3 = 1, \ \overline{\alpha} = \alpha, \ a_1 = \alpha c_1 + 1$. Thus, for different values of the parameter α we obtain non-isomorphic algebras.

Now we investigate the case when $dimL^2 = 1$. Then multiplication in algebra is as follows:

$$[x, y] = \alpha_1x, \ [x, z] = \alpha_3x,$$
$$[y, y] = \alpha_2x, \ [z, z] = \alpha_4x,$$
$$[z, y] = \beta_1x, \ [y, z] = \gamma_1x.$$

Checking the Leibniz identity for the given brackets, we obtain the following identities for structural constants:

$$\begin{cases} \alpha_1\gamma_1 = \alpha_2\alpha_3 \\ \alpha_1\alpha_4 = \alpha_3\beta_1. \end{cases} \qquad (6)$$

Case 1. Let $(\alpha_1, \alpha_3) \neq (0, 0)$. Obviously, we can assume $\alpha_1 \neq 0$. Indeed, if $\alpha_1 = 0$ then $\alpha_3 \neq 0$ and substituting the basic elements $\overline{y} = z, \overline{z} = y$ we obtain $[x, y] = \alpha_3x$.

Since $\alpha_1 \neq 0$ then (6) implies $\gamma_1 = \frac{\alpha_2\alpha_3}{\alpha_1}$ and $\alpha_4 = \frac{\alpha_3\beta_1}{\alpha_1}$. Therefore, multiplication has the form:

$$[x,y] = \alpha_1 x, \ [x,z] = \alpha_3 x,$$
$$[y,y] = \alpha_2 x, \ [z,z] = \frac{\alpha_3\beta_1}{\alpha_1}x,$$
$$[z,y] = \beta_1 x, \ [y,z] = \frac{\alpha_2\alpha_3}{\alpha_1}x.$$

It is not difficult to see that $z - \frac{\alpha_3}{\alpha_1}y$ belongs to the right annihilator $R(L)$ which contradicts to the condition $dimR(L) = 1$.

Case 2. Let $(\alpha_1, \alpha_3) = (0,0)$. Then multiplication has the following form:

$$[x,y] = 0, \quad [x,z] = 0,$$
$$[y,y] = \alpha_2 x, \ [z,z] = \alpha_4 x,$$
$$[z,y] = \beta_1 x, \ [y,z] = \gamma_1 x.$$

It is clear that $(\alpha_2, \alpha_4, \beta_1 + \gamma_1) \neq (0,0,0)$ (otherwise, the algebra is Lie algebra). Putting $\overline{y} = Ay + Bz$, we obtain $[\overline{y}, \overline{y}] = [A^2\alpha_2 + B^2\alpha_4 + AB(\beta_1 + \gamma_1)]x$. One can show that there exist A, B such that $A^2\alpha_2 + B^2\alpha_4 + AB(\beta_1 + \gamma_1) \neq 0$. Thus, without loss of generality we can assume $\alpha_2 \neq 0$.

Substituting $\overline{x} = \alpha_2 x, \overline{z} = z - \frac{\beta_1}{\alpha_2}y$ we obtain $\alpha_1 = 1$ and $\beta_1 = 0$. Then multiplication has the form: $[y,y] = x, \ [z,z] = \alpha x, \ [y,z] = \beta x$, where $(\alpha, \beta) \neq (0,0)$ (if $(\alpha,\beta) = (0,0)$ then L is a decomposable algebra).

If $\alpha = 0$ then $\beta \neq 0$ and one can show that the element $\beta y - z$ belongs to the right annihilator $R(L)$, which contradicts to the condition $dimR(L) = 1$.

Hence, $\alpha \neq 0$. If $\beta \neq 0$ then substituting $\overline{z} = \frac{z}{\beta}$ we obtain $\beta = 1$ and as a result the following algebra:

$L_3(\alpha): \quad [y,y] = x, \ [z,z] = \alpha x, \ [y,z] = x \ (\alpha \neq 0).$

If $\beta = 0$ then we have the algebra $L_4(\alpha): \quad [y,y] = x, \ [z,z] = \alpha x \ (\alpha \neq 0)$.

Now we show that the obtained two families of algebras do not intersect. Suppose the contrary that they intersect. Then there exists an isomorphism $\varphi : L_3(\alpha) \to L_4(\overline{\alpha})$ defined by $\varphi(x) = a_1 x + a_2 y + a_3 z, \quad \varphi(y) = b_1 x + b_2 y + b_3 z, \quad \varphi(z) = c_1 x + c_2 y + c_3 z$, where $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ is a non-singular matrix and $\varphi(x)$, $\varphi(y)$, $\varphi(z)$ satisfy the multiplication in $L_4(\overline{\alpha})$. Then the restrictions to the coefficients of the matrix of isomorphism are the following:

$$\begin{cases} a_2 = 0 \\ a_3 = 0 \\ a_1 = b_2^2 + b_2 b_3 + \alpha b_3^2 \\ b_2 c_2 + b_3 c_2 - \alpha b_3 c_3 = 0 \\ b_2 c_2 + b_2 c_3 - \alpha b_3 c_3 = 0. \end{cases}$$

Subtracting from the third equation the last one, we obtain $b_2 c_3 - b_3 c_2 = 0$ which contradicts to non-singularity of the matrix. Hence, families of algebras $L_3(\alpha)$ and $L_4(\overline{\alpha})$ do not intersect.

Now we investigate the class of algebras $L_3(\alpha)$. Take two algebras from this class

$$[y, y] = x, \ [z, z] = \alpha x, \ [y, z] = x;$$
$$[\overline{y}, \overline{y}] = \overline{x}, \ [\overline{z}, \overline{z}] = \overline{\alpha x}, \ [\overline{y}, \overline{z}] = \overline{x}$$

corresponding to different parameters α and $\overline{\alpha}$.

Applying similar arguments as above, we obtain the following restrictions to the coefficients of the matrix of isomorphism for the given algebras:

$$\begin{cases} a_1 = b_2^2 + b_2b_3 + \alpha b_3^2 \\ a_1 = b_2c_2 + b_2c_3 + \alpha b_3c_3 \\ 0 = b_2c_2 + b_3c_2 + \alpha b_3c_3 \\ \overline{\alpha}a_1 = c_2^2 + c_2c_3 + \alpha c_3^2. \end{cases}$$

Subtracting the second equation from the third one, we obtain $a_1 = b_2c_3 - b_3c_2$. Subtracting the first equation multiplied by c_3 from the second one multiplied by b_3 we obtain $b_2 = c_3 - b_3$. Subtracting the first equation multiplied by c_2 from the second one multiplied by b_2, we obtain $c_2 = -\alpha b_3$. Doing the same with the third and fourth equalities, we obtain $\overline{\alpha}b_3 = -c_2$, $\overline{\alpha}b_2 = c_2 + \alpha c_3$.

Thus, we obtain the following:

$$\begin{cases} \overline{\alpha}b_2 = c_2 + \alpha c_3, \\ \overline{\alpha}b_3 = -c_2, \\ c_2 = -\alpha b_3, \\ b_2 = c_3 - b_3. \end{cases} \Rightarrow \begin{cases} \overline{\alpha}b_2 = -\alpha b_3 + \alpha c_3, \\ \overline{\alpha}b_3 = \alpha b_3, \\ b_2 = c_3 - b_3. \end{cases} \Rightarrow \begin{cases} \overline{\alpha}b_2 = \alpha b_2, \\ \overline{\alpha}b_3 = \alpha b_3. \end{cases}$$

Since $(b_2, b_3) \neq (0, 0)$ the above mentioned equalities imply $\overline{\alpha} = \alpha$ and hence, for different values of the parameter we obtain mutually non-isomorphic algebras.

Now check isomorphism inside the family of algebras $L_4(\alpha)$. Consider two algebras

$$[y, y] = x, \ [z, z] = \alpha x;$$
$$[\overline{y}, \overline{y}] = \overline{x} \quad [\overline{z}, \overline{z}] = \overline{\alpha x}.$$

Analogously, we obtain the following restrictions on coefficients for the matrix of isomorphism φ:

$$\begin{cases} a_1 = b_2^2 + \alpha b_3^2 \\ \overline{\alpha}a_1 = c_2^2 + \alpha c_3^2 \\ 0 = b_2c_2 + \alpha b_3c_3 \\ a_2 = a_3 = b_1 = c_1 = 0. \end{cases}$$

If $b_2 \neq 0$, then putting $c_2 = -\dfrac{\alpha b_3 c_3}{b_2}$, we obtain $\overline{\alpha} = \frac{\alpha c_3^2}{b_2^2}$.

From Corollaries 2.1 and 2.2 the parameter α can be presented in the form of $\alpha = \varepsilon y_j^2$. If we take $b_2 = y_jc_3$, then we have $\overline{\alpha} = \varepsilon$, where $\varepsilon \in \{1, \eta, p, p\eta\}$ if $p \neq 2$ and $\varepsilon \in \{1, 2, 3, 5, 6, 7, 10, 14\}$ if $p = 2$.

If $b_2 = 0$ then $c_3 = 0$ and $\overline{\alpha} = \dfrac{c_2^2}{\alpha b_3^2}$. By Corollaries 2.1 and 2.2, the parameter can be presented in the form of $\alpha = \varepsilon y_j^2$. If we take $c_2 = \varepsilon y_j b_3$, then we obtain $\overline{\alpha} = \varepsilon$, where $\varepsilon \in \{1, \eta, p, p\eta\}$ if $p \neq 2$; $\varepsilon \in \{1, 2, 3, 5, 6, 7, 10, 14\}$ if $p = 2$. Thus, we proved that an algebra from the class $L_4(\alpha)$ is isomorphic to one of the algebras from $L_4(\varepsilon)$ and also for different ε non-isomorphic algebras are obtained. □

The following theorem gives classification of three-dimensional p-adic non-Lie Leibniz algebras with the condition $dim R(L) = 2$.

Theorem 4.2. *Let L be a three-dimensional p-adic non-Lie Leibniz algebra and let $dim R(L) = 2$. Then it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{array}{ll} L_5: & [y,z] = x; \\ L_6: & [x,z] = x, \ [y,z] = x; \\ L_7(\varepsilon): & [x,z] = y, \ [y,z] = \varepsilon x; \\ L_8(\gamma): & [x,z] = y, \ [y,z] = \gamma x + y, \ (\gamma \neq 0); \\ L_9: & [x,z] = x, \ [y,z] = y; \\ L_{10}: & [z,z] = x, \ [x,z] = y; \\ L_{11}: & [z,z] = x, \ [y,z] = x + y, \end{array}$$

where $\varepsilon \in \{1, \eta, p, p\eta\}$ if $p \neq 2$; $\varepsilon \in \{1, 2, 3, 5, 6, 7, 10, 14\}$ if $p = 2$ and $\gamma \in \mathbb{Q}_p$.

Proof. Let $dim R(L) = 2$. Then $R(L) = \langle x, y \rangle$ and multiplication in this case is as follows: $[z,z] = \alpha_1 x + \alpha_2 y, \ [x,z] = \beta_1 x + \beta_2 y, \ [y,z] = \gamma_1 x + \gamma_2 y$ (omitted products are zero).

First, we consider the case of $dim L^2 = 1$, i.e., $[z,z] = \alpha_1 x, \ [x,z] = \alpha_2 x, \ [y,z] = \alpha_3 x$. It should be noted that $\alpha_3 \neq 0$. Indeed, if $\alpha_3 = 0$ then the algebra L is decomposable. Taking $\overline{x} = \alpha_3 x, \overline{y} = y, \overline{z} = z - \dfrac{\alpha_1}{\alpha_3} y$ we obtain $\alpha_1 = 0, \alpha_3 = 1$ and multiplication takes the form: $[z,z] = 0, \ [x,z] = \alpha_2 x, \ [y,z] = x$.

If $\alpha_2 = 0$ then we obtain the algebra L_5. If $\alpha_2 \neq 0$ then substituting $\overline{z} = \dfrac{z}{\alpha_2}$ and $\overline{y} = \alpha_2 y$ we obtain $\alpha_2 = 1$ and therefore, we derive the algebra L_6. Obviously, L_6 is not nilpotent and therefore, it is not isomorphic to the nilpotent algebra L_5.

Now consider the case of $dim L^2 = 2$, i.e., the multiplication has the following form: $[z,z] = \alpha_1 x + \alpha_2 y, \ [x,z] = \beta_1 x + \beta_2 y, \ [y,z] = \gamma_1 x + \gamma_2 y$, where rank $\begin{pmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \\ \gamma_1 & \gamma_2 \end{pmatrix} = 2$.

Case 1. Let $R_{z/R(L)}$ be a non-singular matrix, i.e., $\beta_1\gamma_2 - \gamma_1\beta_2 \neq 0$ and let $\beta_2 \neq 0$. Then, taking $\overline{y} = \beta_1 x + \beta_2 y$ we obtain $\beta_1 = 0$ and $\beta_2 = 1$. Therefore, the multiplication is in the form: $[z,z] = \alpha_1 x + \alpha_2 y, \ [x,z] = y, \ [y,z] = \gamma_1 x + \gamma_2 y$.

Since $R_{z/R(L)}$ is non-degenerate $\gamma_1 \neq 0$. Replacing $\overline{z} = \left(\dfrac{\alpha_1\gamma_2}{\gamma_1} - \alpha_2\right) x - \dfrac{\alpha_1}{\gamma_1} y + z$ we obtain the following two-parametric families of algebras: $[z,z] = 0, \ [x,z] = y, \ [y,z] = \gamma_1 x + \gamma_2 y \ (\gamma_1 \neq 0)$.

Now we check isomorphism of algebras inside this family. In order to do that take two algebras

$$[x,z] = y, \ [y,z] = \gamma_1 x + \gamma_2 y, \ (\gamma_1 \neq 0);$$
$$[\overline{x}, \overline{z}] = \overline{y}, \ [\overline{y}, \overline{z}] = \overline{\gamma_1 x} + \overline{\gamma_2 y}, (\overline{\gamma_1} \neq 0).$$

Consider the general transformation presented in the form of a non-singular matrix

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

Let us consider the following multiplications: $[\overline{x},\overline{z}] = (a_1x + a_2y + a_3z)(c_1x + c_2y + c_3z) = \gamma_1 a_2c_3x + (a_1c_3 + \gamma_2a_2c_3)y$. Since $[\overline{x},\overline{z}] = \overline{y} = b_1x + b_2y + b_3z$ comparing the corresponding coefficients at basic elements we obtain $b_1 = \gamma_1a_2c_3,\ \ b_2 = a_1c_3 + \gamma_2a_2c_3,\ \ b_3 = 0.$

$[\overline{y},\overline{z}] = (b_1x + b_2y + b_3z)(c_1x + c_2y + c_3z) = \gamma_1b_2c_3x + (b_1c_3 + \gamma_2b_2c_3)y$. Similarly,

$[\overline{y},\overline{z}] = \overline{\gamma_1 x} + \overline{\gamma_2 y} = \overline{\gamma_1}(a_1x + a_2y + a_3z) + \overline{\gamma_2}(b_1x + b_2y + b_3z)$ and

$[\overline{z},\overline{z}] = (c_1x + c_2y + c_3z)(c_1x + c_2y + c_3z) = \gamma_1c_2c_3x + (c_1c_3 + \gamma_2c_2c_3)y$. On the other side, $[\overline{z},\overline{z}] = 0$ and $\gamma_1c_2c_3 = 0,\ \ c_1c_3 + \gamma_2c_2c_3 = 0.$

It is not difficult to see that$\gamma_1 \neq 0$ and $b_3 = 0$ imply $a_3 = 0$ and $c_3 \neq 0$. Hence, $c_1 = 0$, $c_2 = 0$. Thus, we obtain the following relations:

$$\begin{cases} a_3 = b_3 = c_1 = c_2 = 0 \\ b_1 = \gamma_1a_2c_2 \\ b_2 = a_1c_3 + \gamma_2a_2c_3 \\ \gamma_1b_2c_3 = \overline{\gamma_1}a_1 + \overline{\gamma_2}b_1 \\ b_1c_3 + \gamma_2b_2c_3 = \overline{\gamma_1}a_2 + \overline{\gamma_2}b_2. \end{cases} \tag{7}$$

Subtracting from the fifth equation multiplied by a_1 the fourth one multiplied by a_2 we obtain

$$a_1b_1c_3 + \gamma_2a_1b_2c_3 - \gamma_1a_2b_2c_3 = (a_1b_2 - a_2b_1)\overline{\gamma_2}. \tag{8}$$

Then putting the second and third equalities of (7) to (8) we obtain $c_3\gamma_2(a_1^2 - \gamma_1a_2^2 + \gamma_2a_1a_2) = \overline{\gamma_2}(a_1^2 - \gamma_1a_2^2 + \gamma_2a_1a_2)$. Hence, $\overline{\gamma_2} = c_3\gamma_2$.

Case 1.1. Let $\gamma_2 = 0$. Then $\overline{\gamma_2} = 0$ and

$$\begin{cases} b_1 = \gamma_1a_2c_2 \\ b_2 = a_1c_3 \\ \gamma_1b_2c_3 = \overline{\gamma_1}a_1 \\ b_1c_3 = \overline{\gamma_1}a_2. \end{cases}$$

Subtracting from the fourth equation, multiplied by b_1, the third one multiplied by b_2 we obtain $\gamma_1c_3^2(a_1b_2 - a_2b_1) = \overline{\gamma_1}(a_1b_2 - a_2b_1) \Rightarrow \overline{\gamma_1} = \gamma_1c_3^2$.

By Corollaries 2.1 and 2.2 one can represent $\gamma_1 = \varepsilon y_j^2$, $y_j \in Q_p$. Taking $c_3 = \dfrac{1}{y_j}$ we obtain $\overline{\gamma_1} = \varepsilon$. Hence, we derive the algebra L_7.

Case 1.2. Let $\gamma_2 \neq 0$. Then taking $c_3 = \dfrac{1}{\gamma_2}$, we obtain $\gamma_2 = 1$ and

$$\begin{cases} b_1 = \gamma_1a_2c_2 \\ b_2 = a_1c_3 + \gamma_2a_2c_3 \\ \gamma_1b_2c_3 = \overline{\gamma_1}a_1 + b_1 \\ b_1c_3 + \gamma_2b_2c_3 = \overline{\gamma_1}a_2 + b_2. \end{cases}$$

Subtracting from the fourth equation, multiplied by b_1, the third equation multiplied by b_2 we have $c_3(b_1^2 + \gamma_2b_1b_2 - \gamma_1b_2^2) = \overline{\gamma_1}(a_2b_1 - a_1b_2)$. Hence, $c_3(b_1^2 + \gamma_2b_1b_2 - \gamma_1b_2^2) = \dfrac{\overline{\gamma_1}}{\gamma_1c_3}(b_1^2 + \gamma_2b_1b_2 - \gamma_1b_2^2)$; $\overline{\gamma_1} = \gamma_1c_3^2 = \dfrac{\gamma_1}{\gamma_2^2}$. These equalities mean that the parameter γ_1 cannot be cancelled, and we obtain the algebra $L_8(\gamma)$. Moreover, it follows that the family $L_8(\gamma)$ contains pairwise non-isomorphic algebras for different values of the parameter.

Case 2. Let $\beta_2 = 0$. Then taking $\overline{x} = \dfrac{x}{\beta_1}$ we obtain $\beta_1 = 1$ and the table of multiplication becomes as follows: $[z, z] = \alpha_1 x + \alpha_2 y,\ \ [x, z] = x,\ \ [y, z] = \gamma_1 x + \gamma_2 y.$

Similar to the case 1, from non-singularity of the operator $R_{z/R(L)}$ it follows that $\gamma_2 \neq 0$.

Substituting $\overline{z} = (\dfrac{\alpha_2\gamma_1}{\gamma_2} - \alpha_1)x - \dfrac{\alpha_2}{\gamma_2}y + z$ we obtain $\alpha_1 = \alpha_2 = 0$. So we derive the following parametric family of algebras: $[z, z] = 0,\ \ [x, z] = x,\ \ [y, z] = \gamma_1 x + \gamma_2 y,\ \ (\gamma_2 \neq 0).$

Substituting $\overline{x} = ax + by, \overline{z} = z$ we obtain $[\overline{x}, \overline{z}] = [ax + by, z] = ax + b(\gamma_1 x + \gamma_2 y) = (a + \gamma_1 b)x + b\gamma_2 y$.

If $(\gamma_1, \gamma_2) \neq (0, 1)$ then we can choose $\overline{y} = (a + \gamma_1 b)x + b\gamma_2^2 y$, i.e., $\begin{vmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{a} + \gamma_1\mathrm{b} & \mathrm{b}\gamma_2 \end{vmatrix} = ab(\gamma_2 - 1) - b^2\gamma_1 \neq 0$ and we have $[\overline{x}, \overline{z}] = \overline{y}$. Hence, checking multiplication $[\overline{y}, \overline{z}] = (a + \gamma_1 b)x + b\gamma_2(\gamma_1 x + \gamma_2 y) = (a + \gamma_1 b + b\gamma_1\gamma_2)x + b\gamma_2^2 y$, it is not difficult to see that $\begin{vmatrix} \mathrm{a} + \gamma_1\mathrm{b} & \mathrm{b}\gamma_2 \\ \mathrm{a} + \gamma_1\gamma_2\mathrm{b} + \gamma_1\mathrm{b} & \mathrm{b}\gamma_2^2 \end{vmatrix} \neq 0$, i.e., $[\overline{y}, \overline{z}] = \overline{\gamma_1 x} + \overline{\gamma_2 y}, \overline{\gamma_1} \neq 0$.

This means that if $(\gamma_1, \gamma_2) \neq (0, 1)$ then it can be reduced to the case 1. If $\gamma_1 = 0$ and $\gamma_2 = 1$ then we obtain an algebra: $L_9 : \quad [x, z] = x, \quad [y, z] = y.$

Now we check isomorphism between the following algebras:

$$\begin{aligned} &L_7(\alpha): \ [x, z] = y, \quad [y, z] = \alpha x,\ (\alpha \neq 0); \\ &L_9: \qquad [x, z] = x, \quad [y, z] = y. \end{aligned}$$

Applying the standard methods we obtain the following relations:

$$\begin{cases} a_3 = b_3 = 0 \\ \alpha a_1 = b_1 c_1 \\ \alpha a_2 = b_2 c_2 \\ b_1 = a_1 c_3 \\ b_2 = a_2 c_3. \end{cases}$$

Now, if we subtract from the fifth equation multiplied by a_1 the fourth one multiplied by a_2 we obtain $a_1 b_2 - a_2 b_1 = 0$. This equality implies singularity of the matrix, which is a contradiction. Hence, algebras $L_7(\alpha)$ and L_9 are non-isomorphic.

Now we check isomorphism between the following algebras:

$$\begin{aligned} &L_8(\gamma): \ [x, z] = y, \quad [y, z] = \gamma x + y,\ (\gamma \neq 0); \\ &L_9: \qquad [x, z] = x, \quad [y, z] = y. \end{aligned}$$

To this end let us make the general basis transformation in algebra $L_9 : \overline{x} = a_1 x + a_2 y + a_3 z, \quad \overline{y} = b_1 x + b_2 y + b_3 z, \quad \overline{z} = c_1 x + c_2 y + c_3 z$, and suppose that the base $\overline{x}, \overline{y}, \overline{z}$ satisfies to multiplication in

$L_8(\gamma)$, where $\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ is a non-singular matrix. Analogously, we obtain the following relations:

$$\begin{cases} a_3 = b_3 = 0 \\ \gamma a_1 + b_1 = b_1 c_c \\ \gamma a_2 + b_2 = b_2 c_3 \\ b_1 = a_1 c_3 \\ b_2 = a_2 c_3. \end{cases}$$

Subtracting from the fifth equation multiplied by a_1 the fourth one multiplied by a_2 yields $a_1 b_2 - a_2 b_1 = 0$. The obtained equality implies singularity of the matrix, which is a contradiction. Therefore, the algebras $L_8(\gamma)$ and L_9 are non-isomorphic.

Case 3. Let $R_{z/R(L)}$ be a singular matrix. Then multiplication is as follows: $[z, z] = \alpha_1 x + \alpha_2 y,\ [x, z] = k_1(\beta_1 x + \beta_2 y),\ [y, z] = k_2(\beta_1 x + \beta_2 y)$.

Since x and y are symmetric, one can assume $k_1 \neq 0$. Then taking $\overline{y} = \dfrac{k_2}{k_1}x - y$ we obtain $k_2 = 0$. Therefore, the multiplication becomes $[z, z] = \alpha_1 x + \alpha_2 y,\ [x, z] = \beta_1 x + \beta_2 y$.

If $\alpha_1 \neq 0$, then replacing $\overline{x} = \alpha_1 x + \alpha_2 y$ we obtain multiplication $[z, z] = x,\ [x, z] = \beta_1 x + \beta_2 y$.

Since $\beta_2 \neq 0$ taking $\overline{y} = \beta_2 y$ we obtain the algebra $[z, z] = x,\ [x, z] = \beta_1 x + y$.

But if in this algebra $\beta_1 = 0$, we obtain the following algebra: $L_{10}:\ [z, z] = x,\ [x, z] = y$.

And in case $\beta_1 \neq 0$, taking $\overline{x} = \dfrac{x}{\beta_1^2},\ \overline{y} = \dfrac{y}{\beta_1^3},\ \overline{z} = \dfrac{z}{\beta_1}$ we obtain the algebra $L_{11}:\quad [z, z] = x,\quad [x, z] = x + y$.

Obviously, these two obtained algebras are non-isomorphic:

$$[z, z] = x, \quad [x, z] = y \text{ (nilpotent)};$$
$$[z, z] = x, \quad [y, z] = x + y \text{ (not nilpotent)}.$$

If $\alpha_1 = 0$ then $\alpha_2 \neq 0,\ \beta_1 \neq 0$ and replacing $\overline{y} = \dfrac{\alpha_2}{\beta_1^2}y,\ \overline{z} = \dfrac{z}{\beta_1}$ we obtain the algebra $[z, z] = y, \qquad [x, z] = x + \beta_2 y$.

It is not difficult to check that substituting $\overline{x} = x + (1 + \beta_2)y, \overline{y} = -y, \overline{z} = x + z$ in this algebra we derive the algebra L_{11}, i.e., $[\overline{z}, \overline{z}] = \overline{x},\ [\overline{x}, \overline{z}] = \overline{x} + \overline{y}$. In fact, $[\overline{z}, \overline{z}] = [x + z, x + z] = x + (1 + \beta_2)y = \overline{x},\quad [\overline{x}, \overline{z}] = [x + (1 + \beta_2)y, x + z] = x + \beta_2 y = x + (1 + \beta_2)y - y = \overline{x} + \overline{y}$. □

Comparing the classifications of non-Lie Leibniz algebras over the fields of complex and p-adic numbers we conclude that the lists coincide except two algebras, L_4 and L_7. Namely, in the list of classification over complex numbers in these algebras parameter ε is equal to 1.

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