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The Classification of Filiform Leibniz Superalgebras of Nilindex $n+m$

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Abstract In this work the classification of filiform Leibniz superalgebras of nilindex $n+m$, where n and m ($m \neq 0$) are dimensions of even and odd parts, respectively, is obtained.

Keywords Lie superalgebras, Leibniz superalgebras, nilpotency, characteristic sequence

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1 Introduction

Extensive investigations in Lie algebras theory have led to the appearance of more general algebraic objects — Mal'cev algebras, binary Lie algebras, Lie superalgebras and others.

At the beginning of 90-th of the last century Loday introduced another generalization of Lie algebras — Leibniz algebras [1].

Recall that Leibniz algebras are defined by the Leibniz identity:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

It should be noted, that if a Leibniz algebra satisfies the identity $[x,x]=0$, then Leibniz identity and Jacobi identity coincide. Therefore, Leibniz algebras are "non antisymmetric" generalization of Lie algebras.

In spite of that Leibniz algebras are defined by a single identity they generalize Lie algebras in so natural a way that many properties of Lie algebras remain true also for Leibniz algebras [2–10]. Nevertheless, the complementation of the variety of Lie algebras in the variety of Leibniz algebras forms a Zariski open set (from algebraic geometry it is known that open sets in Zariski topology are "large").

As it was mentioned above the Lie superalgebras are generalizations of Lie algebras and for many years they attract the attention of both the mathematicians and physicists. The systematical exposition of basic Lie superalgebras theory can be found in the monograph [11]. Leibniz superalgebras are generalizations of Leibniz algebras and, on the other hand, they

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naturally generalize Lie superalgebras. In the description of Leibniz superalgebras structure the crucial task is to prove the existence of suitable basis (the so-called adapted basis) in which the multiplication of the superalgebra has the most convenient form.

In contrast to Lie superalgebras for which the problem of description of superalgebras with the maximal nilindex is difficult, for nilpotent Leibniz superalgebras it turns to be comparatively easy and was solved in [3]. The distinctive property of such Leibniz superalgebras is that they are single-generated. The next step — the description of Leibniz superalgebras with dimensions of even and odd parts, respectively equal to n and m, and with nilindex $n+m$ at this moment seems to be too complicated. Therefore, such Leibniz superalgebras can be studied, for example, by applying restrictions on their characteristic sequences. Following this approach in the present paper we investigate Leibniz superalgebras with the characteristic sequence $C(L) = (n-1, 1 \mid m)$. and with nilindex equal to $n+m$. Since such Lie superalgebras with these properties were classified in [12], we shall consider only non Lie superalgebras case.

2 Preliminaries

All over the work we shall consider spaces and algebras over the file of complex numbers.

Definition 2.1 [11] *A Z_2-graded vector space $G = G_0 \oplus G_1$ is called a Lie superalgebra if it is equipped with a product $[-,-]$ which satisfies the following conditions:*

1 $[G_\alpha, G_\beta] \subseteq G_{\alpha+\beta(mod\ 2)}$ *for any* $\alpha, \beta \in Z_2$,

2 $[x,y] = -(-1)^{\alpha\beta}[y,x]$, *for any* $x \in G_\alpha$, $y \in G_\beta$,

3 $(-1)^{\alpha\gamma}[x,[y,z]] + (-1)^{\alpha\beta}[y,[z,x]] + (-1)^{\beta\gamma}[z,[x,y]] = 0$ — *for any* $x \in G_\alpha$, $y \in G_\beta$, $z \in G_\gamma$ *(Jacobi superidentity).*

Definition 2.2 [3] *A Z_2-graded vector space $L = L_0 \oplus L_1$ is called a Leibniz superalgebra if it is equipped with a product $[-,-]$ which satisfies the following conditions:*

1 $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta(mod\ 2)}$ *for any* $\alpha, \beta \in Z_2$,

2 $[x,[y,z]] = [[x,y],z] - (-1)^{\alpha\beta}[[x,z],y]$ —*for any* $x \in L$, $y \in L_\alpha$, $z \in L_\beta$, *(Leibniz superidentity).*

It should be noted that if in a Leibniz superalgebra L the identity

$$[x,y] = -(-1)^{\alpha\beta}[y,x]$$

holds for any $x \in L_\alpha$ and $y \in L_\beta$, then Leibniz superidentity can be easily transformed to Jacobi superidentity.

Thus, Leibniz superalgebras are generalizations of Lie superalgebras. For examples of Leibniz superalgebras we refer to [3].

The set of Leibniz superalgebras with dimensions of the even part (L_0) and the odd part (L_1), respectively equal to n and m, we shall denote by $\mathrm{Leib}_{n,m}$.

Definition 2.3 *Let $V = V_0 \oplus V_1$, $W = W_0 \oplus W_1$ — be Z_2-graded spaces. We say that a linear map $f : V \to W$ has the weight α $(\deg(f) = \alpha)$, if $f(V_\beta) \subseteq W_{\alpha+\beta}$ for any $\beta \in Z_2$.*

Let us define the notion of homomorphism of Leibniz superalgebras.

Definition 2.4 *Let L and L' be Leibniz superalgebras from $Leib_{n,m}$. A linear map $f : L \to L'$ is called a homomorphism of Leibniz superalgebras, if the following conditions are satisfied*:

1 f *preserve gradation, i.e.* $\deg(f) = 0$,

2 $f([x, y]) = [f(x), f(y)]$ *for any* $x, y \in L$.

Moreover, if f is a bijection then it is called isomorphism of Leibniz superalgebras L and L'.

For given Leibniz superalgebra L we define a descending central sequence in the following way:

$$L^1 = L, \quad L^{n+1} = [L^n, L^1].$$

Definition 2.5 *A Leibniz superalgebra L is called nilpotent, if there exists $s \in \mathbb{N}$ such that $L^s = 0$. The minimal number s with this property is called index of nilpotency (nilindex) of the superalgebra L.*

Definition 2.6 *The set $R(L) = \{z \in L \mid [L, z] = 0\}$ is called the right annihilator of a superalgebra L.*

Using Leibniz superidentity it is not difficult to see that $R(L)$ is an ideal of the superalgebra L. Moreover, elements of the form $[a, b] + (-1)^{ab}[b, a]$ belong to $R(L)$.

Let L be an arbitrary Leibniz algebra or superalgebra of dimension n and let $\{e_1, \dots, e_n\}$ be a basis of the algebra L. Then the multiplication on L is defined by the products of the basic elements, namely, $[e_i, e_j] = \sum_{k=1}^{n} \gamma_{ij}^k e_k$, where γ_{ij}^k are the structural constants. Thus the problem of classification of algebras or superalgebras can be reduced to the problem of finding a description of the structural constants up to a non-degenerate basis transformation.

The following theorem from [3] describes nilpotent Leibniz superalgebras with maximal nilindex.

Theorem 2.1 *Let L be a Leibniz superalgebra of the variety $Leib_{n,m}$ with nilindex equal to $n+m+1$. Then L is isomorphic to the one of the following two non isomorphic superalgebras*:

$$[e_i, e_1] = e_{i+1},\ 1 \le i \le n-1; \qquad \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \le i \le n+m-1, \\ [e_i, e_2] = 2e_{i+2}, & 1 \le i \le n+m-2 \end{cases}$$

(*omitted products are equal to zero*).

Remark 2.1 From Theorem 2.1 we have that if the odd part L_1 of the superalgebra L in the theorem 2.1 is non trivial, then either $m = n$ or $m = n+1$.

Let $L = L_0 \oplus L_1$ be a nilpotent Leibniz superalgebra. For an arbitrary element $x \in L_0$, the operator of right multiplication R_x is nilpotent endomorphism of the space L_i, where $i \in \{0, 1\}$. Denote by $C_i(x)$ ($i \in \{0, 1\}$) the descending sequence of the dimensions of Jordan blocks of the operator R_x. Consider the lexicographical order on the set $C_i(L_0)$.

The following notion in the case of Lie algebras was introduced in [13].

Definition 2.7 *A sequence*

$$C(L) = \left(\max_{x \in L_0 \setminus [L_0, L_0]} C_0(x) \,\middle|\, \max_{\widetilde{x} \in L_0 \setminus [L_0, L_0]} C_1(\widetilde{x}) \right)$$

is said to be the characteristic sequence of the Leibniz superalgebra L.

Similarly to [14, corollary 3.0.1] it can be proved that the characteristic sequence is invariant under isomorphisms.

Let us introduce the analog of filiform Leibniz algebras in the case of Leibniz superalgebras.

Definition 2.8 *A Leibniz superalgebra L is said to be of filiform, if $C(L) = (n-1, 1 \mid m)$. Denote by $F_{n,m}$ the set of filiform Leibniz superalgebras.*

For non-Lie filiform Leibniz superalgebras the existence of adapted basis is given in the following theorem, which follows from the results of [4], [15].

Theorem 2.2 *Let L be an arbitrary non-Lie filiform Leibniz superalgebra. Then there exists a basis $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$ of the superalgebra L, in which the multiplication satisfies one of the following three conditions:*

$$\text{a)}\quad \begin{aligned}
&[x_1, x_1] = x_3,\\
&[x_i, x_1] = x_{i+1}, \quad 2 \le i \le n-1,\\
&[y_j, x] = y_{j+1}, \quad 1 \le j \le m-1 \text{ and for some } x \in L_0 \setminus [L_0, L_0],\\
&[x_1, x_2] = \alpha_4 x_4 + \alpha_5 x_5 + \cdots + \alpha_{n-1} x_{n-1} + \theta x_n,\\
&[x_j, x_2] = \alpha_4 x_{j+2} + \alpha_5 x_{j+3} + \cdots + \alpha_{n+2-j} x_n, \quad 2 \le j \le n-2,
\end{aligned}$$

where the omitted products in L_0 are equal to zero;

$$\text{b)}\quad \begin{aligned}
&[x_1, x_1] = x_3,\\
&[x_i, x_1] = x_{i+1}, \quad 3 \le i \le n-1,\\
&[y_j, x] = y_{j+1}, \quad 1 \le j \le m-1 \text{ and for some } x \in L_0 \setminus [L_0, L_0],\\
&[x_1, x_2] = \beta_4 x_4 + \beta_5 x_5 + \cdots + \beta_n x_n,\\
&[x_2, x_2] = \gamma x_n,\\
&[x_j, x_2] = \beta_4 x_{j+2} + \beta_5 x_{j+3} + \cdots + \beta_{n+2-j} x_n, \quad 3 \le j \le n-2,
\end{aligned}$$

where the omitted products in L_0 are equal to zero;

$$\text{c)}\quad \begin{aligned}
&[x_i, x_1] = -[x_1, x_i] = x_{i+1}, \quad 3 \le i \le n-1,\\
&[x_1, x_1] = x_n, \quad [x_2, x_1] = x_3, \quad [x_1, x_2] = -x_3 + \alpha x_n, \quad [x_2, x_2] = \beta x_n\\
&[x_i, x_j] = -[x_j, x_i] \in \mathrm{lin}\langle x_{i+j}, x_{i+j+1}, \ldots, x_n\rangle, \quad 2 \le i \le n-2, \quad 3 \le j \le n-i,\\
&[y_j, x] = y_{j+1}, \quad 1 \le j \le m-1 \text{ and for some } x \in L_0 \setminus [L_0, L_0].
\end{aligned}$$

3 The Classification of Filiform Leibniz Superalgebras of the Nilindex $n+m$ $(m \neq 0)$

Let a superalgebra $L \in F_{n,m}$ have the nilindex equal to $n+m$, and let $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$ be basis of L.

From the restriction on the nilpotency index it is not difficult to see that the superalgebra L has two generators, the first generator belongs to L_0 and the second one belongs to L_1. Indeed, from Theorem 2.2 we have that $y_2, y_3, \ldots, y_m$ are not generators. Therefore, the case when both generators lie in L_1 is impossible and the case when both generators lie in odd part is degenerated, since $m = 0$ (i.e. we have filiform Leibniz algebra).

Lemma 3.1 *In the three cases of multiplication in Theorem* 2.2 *instead of the element x one can choose the element x_1.*

Proof Without loss of generality we can assume that x has the form: $x = A_1x_1 + A_2x_2$, where $|A_1| + |A_2| \neq 0$.

Let us investigate the first class of Theorem 2.2 and consider the three cases.

Case 1 Let $A_1(A_1 + A_2) \neq 0$. Then applying the following change of basis:

$$x'_1 = A_1x_1 + A_2x_2, \quad x'_2 = (A_1 + A_2)x_2 + A_2(\theta - \alpha_n)x_{n-1},$$
$$x'_i = [x'_{i-1}, x'_1], \quad 3 \leq i \leq n, \quad y'_j = y_j, \quad 1 \leq j \leq m,$$

we obtain that the first three multiplications in class a) do not change.

Case 2 Let $A_1 = 0$. Let us take change of basis in the form:

$$x'_1 = x_1 + aA_2x_2, \text{ where } a \neq 0 \text{ and } 1 + aA_2 \neq 0, \ x'_2 = (1 + A_2)x_2 + aA_2(\theta - \alpha_n)x_{n-1},$$
$$x'_i = [x'_{i-1}, x'_1], \ 3 \leq i \leq n, \quad y'_1 = y_1, \quad y'_j = [y'_{j-1}, x'_1], \ 2 \leq j \leq m.$$

If we take sufficiently great value of the parameter a then we obtain that the first three multiplications in the class a) also do not change.

Case 3 Let $A_1 \neq 0$ and $A_1 = -A_2$. Then taking the following change of basis:

$$x'_1 = A_1x_1 - A_1x_2 + ax_2, \quad x'_2 = ax_2 + (a - A_1)(\theta - \alpha_n)x_{n-1}, \quad a \neq 0,$$
$$x'_i = [x'_{i-1}, x'_1], \ 3 \leq i \leq n, \quad y'_1 = y_1, \quad [y'_j, x'_1] = y'_{j+1}, \ 1 \leq j \leq m - 1,$$

it is not difficult to check that for sufficiently small value of the parameter a the first three multiplications in the class a) are preserved.

Thus, we have shown that in the first case of Theorem 2.2 instead of x one can choose x_1, i.e. the equality: $[y_j, x_1] = y_{j+1}$, $1 \leq j \leq m - 1$.

In the classes b) and c) for the same reason as in the first class and by using elementary changes of basis obtained in [10], [16], one can show that the element can be replaced by the x_1.

Note, that in the case $m = 1$ we easily obtain contradiction to the condition $m \neq 0$ (because we have Lie algebra).

3.1 The case of $F_{n,m}$ $(n = 2)$

In case $F_{2,2}$ we have the following result

Theorem 3.1 *Let L be an arbitrary Leibniz superalgebra from $F_{2,2}$ with nilindex equal to* 4*. Then it is isomorphic to one of the following two non-isomorphic superalgebras*:

$$L_1 = \begin{cases} [y_1, x_1] = y_2, \\ [x_1, y_1] = \frac{1}{2}y_2, \\ [x_2, y_1] = \frac{1}{2}y_2, \\ [y_1, x_2] = y_2, \\ [y_1, y_1] = x_2, \end{cases} \qquad L_2 = \begin{cases} [y_1, x_1] = y_2, \\ [x_2, y_1] = \frac{1}{2}y_2, \\ [x_2, y_1] = \frac{1}{2}y_2, \\ [y_1, x_2] = y_2, \\ [y_1, y_1] = x_2, \end{cases}$$

(*omitted products are equal to zero*).

Proof Let $\{x_1, x_2, y_1, y_2\}$ be a basis of the superalgebra L. Evidently, Leibniz algebra L_0 is abelian.

Suppose that the element x_1 is the generator. Then $L^2 = \{x_2, y_2\}$.

Let $[x_1, y_1] = \alpha y_2$.

If we suppose that $L^3 = \{x_2\}$, then $[y_2, y_1] = cx_2$ where $c \neq 0$.

If $\alpha \neq -1$, then sum $[y_1, x_1] + [x_1, y_1] = (1+\alpha)y_2$ lies in $R(L) \Rightarrow$ therefore, $y_2 \in R(L)$. And since $y_2 \in R(L)$, then $[y_1, y_2] = 0$.

On the other hand, $[y_1, y_2] = [y_1, [y_1, x_1]] = [[y_1, y_1], x_1] - [[y_1, x_1], y_1] = -cx_2$. Hence, $c = 0$ and we obtain contradiction to the assumption $c \neq 0$.

If $\alpha = -1$, then $0 = [x_1, [y_1, y_1]] = 2[[x_1, y_1], y_1] = -2[y_2, y_1] = -2cx_2 \Rightarrow c = 0$ and again we obtain a contradiction, i.e. $L^3 \neq \{x_2\}$.

Thus, $L^3 = \{y_2\}$, so $[x_2, y_1] = \beta y_2$, where $\beta \neq 0$ and $[y_1, y_1] = \gamma x_2$, where $\gamma \neq 0$ (since $x_2 \in L^2 \setminus L^3$). Without loss of generality, we can suppose that is equal to 1.

From Leibniz superidentity we obtain the following multiplication in the superalgebra L :

$$L(\alpha, \beta) = \begin{cases} [y_1, x_1] = y_2, \\ [x_1, y_1] = \alpha y_2, \\ [x_2, y_1] = \beta y_2, \ \beta \neq 0, \\ [y_1, x_2] = 2\beta y_2, \\ [y_1, y_1] = x_2 \end{cases}$$

(omitted products are equal to zero).

Let us suppose the opposite case, i.e. the element x_1 is not a generator. Then without loss of generality we can suppose that x_2 is a generator. Hence, $L^2 = \{x_1, y_2\}$, $L^3 = \{y_2\}$ and $[y_1, y_1] = \tau x_1$, where $\tau \neq 0$. Using Leibniz superidentity and appropriate change of basis we obtain the following multiplication in L :

$$L(\delta, \theta) = \begin{cases} [y_1, x_1] = y_2, \\ [x_1, y_1] = \frac{1}{2} y_2, \\ [x_2, y_1] = \delta y_2, \\ [y_1, x_2] = \theta y_2, \\ [y_1, y_1] = x_1 \end{cases}$$

(omitted products are equal to zero).

Taking a transformation of the following form:

$$\begin{aligned} x_1' &= (\theta + 1)x_1 - x_2, \\ x_2' &= x_1, \\ y_1' &= y_1, \\ y_2' &= y_2, \end{aligned}$$

we obtain the embedding of the family $L(\delta, \theta)$ into the family $L(\alpha, \beta)$.

Let us consider the isomorphism problem inside the family $L(\alpha,\beta)$, where

$$L(\alpha,\beta)=\begin{cases}[y_1,x_1]=y_2,\\ [x_1,y_1]=\alpha y_2,\\ [x_2,y_1]=\beta y_2,\ \beta\neq 0,\\ [y_1,x_2]=2\beta y_2,\\ [y_1,y_1]=x_2.\end{cases}$$

Take the general change of the generators in the form:

$$x_1'=a_1x_1+a_2x_2,\quad y_1'=b_1y_1+b_2y_2,\qquad \text{where } a_1b_1(a_1+2a_2\beta)\neq 0.$$

Then expressing the multiplication in the new basis $\{x_1',x_2',y_1',y_2'\}$ via the basic elements $\{x_1,x_2,y_1,y_2\}$ and comparing coefficients of basic elements, we obtain

$$x_2'=b_1^2x_2,\quad y_2'=b_1(a_1+2a_2\beta)y_2,$$
$$\alpha'=\frac{a_1\alpha+a_2\beta}{a_1+2a_2\beta},\quad \beta'=\frac{b_1^2\beta}{a_1+2a_2\beta}.$$

Besides, it is not difficult to see that the following equality holds

$$\left(\alpha'-\frac{1}{2}\right)=\frac{a_1}{a_1+2a_2\beta}\left(\alpha-\frac{1}{2}\right).$$

Putting $b_1=\sqrt{\frac{a_1+2a_2\beta}{\beta}}$, we have $\beta'=1$. If $\alpha\neq\frac{1}{2}$, then taking $a_2=-\frac{\alpha a_1}{\beta}$, we obtain $\alpha'=0$. If $\alpha=\frac{1}{2}$, then $\alpha'=\frac{1}{2}$.

Thus, we have obtained superalgebras $L(0,1)$ and $L\left(\frac{1}{2},1\right)$, which are not isomorphic.

In the case when a superalgebra from $F_{2,m}$ $(m\geq 3)$ has nilindex $m+2$, the following result is true.

Lemma 3.2 *Let L be a Leibniz superalgebra from $F_{2,m}$ $(m\geq 3)$ and suppose that it has nilindex equal to $m+2$. Then L is Lie superalgebra.*

Proof Let $\{x_1,x_2,y_1,y_2,\dots,y_m\}$ be a basis of superalgebra L. Put

$$[y_i,y_1]=\beta_ix_2,\ 1\leq i\leq m,\quad [x_j,y_1]=\sum_{i=2}^{m}\alpha_{j,i}y_i,\ j=1,2.$$

Let us prove that x_2 can not be a generator. Suppose the opposite, i.e. x_2 is a generator. Then from Theorem 2.2 and conditions $[L^k,L]=L^{k+1}$, $[L^i,L^j]\subseteq L^{i+j}$ we have

$$L^2=\{x_1,y_2,\dots,y_m\},\quad L^3=\{y_2,\dots,y_m\},$$
$$L^4=L^5=\{y_3,y_4,\dots,y_m\}\ (\text{since } y_3=[y_2,x_1] \text{ and } y_2\in L^3,\ x_1\in L^2),$$

i.e. we obtain a contradiction to the nilpotency of the superalgebra L. Therefore, we can take the element x_1 as a generator. Then $L^2=\{x_2,y_2,\dots,y_m\}$.

Let s be a natural number such that $x_2\in L^s\setminus L^{s+1}$.

Suppose that $s=2$. Then

$$L^i=\{y_{i-1},y_i,\dots,y_m\},\quad 3\leq i\leq m+1,\quad \alpha_{2,2}\neq 0,\qquad [y_1,y_1]=x_2.$$

Consider the product

$$[x_2,y_2]=[x_2,[y_1,x_1]]=[[x_2,y_1],x_1]-[[x_2,x_1],y_1]=[\alpha_{2,2}y_2+\alpha_{2,3}y_3+\dots++\alpha_{2,m}y_m,x_1]$$

$$= \alpha_{2,2}y_3 + \alpha_{2,3}y_4 + \cdots + \alpha_{2,m-1}y_m.$$

Evidently, $[x_2, y_2] \in L^4 \setminus L^5$. On the other hand $[x_2, y_2] \in L^5$ (since $x_2 \in L^2$ and $y_2 \in L^3$), i.e. we obtain a contradiction to the condition $s = 2$.

Suppose that $3 \leq s \leq m$. Then $L^i = \{x_2, y_i, \dots, y_m\}$ for $2 \leq i \leq s$, and $L^i = \{y_{i-1}, \dots, y_m\}$ for $s+1 \leq i \leq m+1$. Note that $\alpha_{2,j} = 0$ for $2 \leq j \leq s-1$, $\alpha_{2,s} \neq 0$ and $\beta_{s-1} \neq 0$.

Consider the product

$$\beta_{s-1}[x_2, y_1] = [[y_{s-1}, y_1], y_1] = 2[y_{s-1}, [y_1, y_1] = 2\beta_1[y_{s-1}, x_2].$$

Since $\alpha_{2,s} \neq 0$ and $x_2 \in L^s$, $y_{s-1} \in L^{s-1}$, then $0 \neq \beta_1[y_{s-1}, x_2] \in L^{2s-1}$.

On the other hand, $[x_2, y_1] = \alpha_{2,s}y_s + \alpha_{2,s+1}y_{s+1} + \cdots + \alpha_{2,m}y_m \in L^{s+1} \setminus L^{s+2}$ and since $2s - 1 > s + 1$, then we obtain a contradiction to the condition $3 \leq s \leq m$.

Thus, $s = m+1$. Therefore $L^i = \{x_2, y_i, \dots, y_m\}$, $2 \leq i \leq m$, $L^{m+1} = \{x_2\}$ and $\beta_m \neq 0$.

Applying Leibniz superidentity we have:

$$\begin{cases} [y_i, y_j] = (-1)^{j-1}\beta_{i+j+1}x_2, & 1 \leq i+j \leq m+1, \\ [y_i, y_j] = 0, & m+2 \leq i+j \leq 2m, \\ [x_1, y_i] = \alpha_{1,2}y_{i+1} + \alpha_{1,3}y_{i+2} + \cdots + \alpha_{1,m+i-1}y_m, & 1 \leq i \leq m-1. \end{cases}$$

Suppose that the sum $[y_1, x_1] + [x_1, y_1] = (1 + \alpha_{1,2})y_2 + \alpha_{1,3}y_3 + \cdots + \alpha_{1,m}y_m$, which lies in $R(L)$ is different from zero. Then multiplying sufficiently times from the right side by the element x_1, we obtain $y_m \in R(L)$. Hence, $[y_1, y_m] = (-1)^{m-1}\beta_m x_2 = 0$, i.e. $\beta_m = 0$ and so, we have contradiction to $\beta_m \neq 0$. Thus, $[x_1, y_i] = -y_{i+1}$. Furthermore, from

$$0 = [x_1, [y_i, y_i]] = 2[[x_1, y_i], y_i] = -2[y_{i+1}, y_i] = 2(-1)^{i-1}\beta_{2i}x_2 \;\Rightarrow \beta_{2i} = 0,\; 1 \leq i \leq \left[\frac{m}{2}\right],$$

we have that m is odd and $[y_i, y_j] = [y_j, y_i]$.

Thus, we have proved the identity:

$$[a, b] = -(-1)^{\alpha\beta}[b, a], \quad \text{for any } a \in L_\alpha,\; b \in L_\beta.$$

The following theorem contains a classification of Lie superalgebras from the set $\mathrm{Leib}_{n,m}$ with nilindex equal to $n + m$.

Theorem 3.2 [12] *Let G be a Lie superalgebra from $F_{2,m}$ with nilindex equal to $m+2$. Then $n = 2$, m is odd and there exists a basis $\{x_1, x_2, y_1, y_2, \dots, y_m\}$ of the superalgebra G such that its multiplication w.r.t. this basis has the following form*:

$$[y_i, x_1] = y_{i+1}, \quad 1 \leq i \leq m-1,$$
$$[y_{m+1}, y_i] = (-1)^{i+1}x_2, \quad 1 \leq i \leq \frac{m+1}{2}$$

(*omitted products are zero*).

Remark 3.1 It is easy to see that the superalgebra from Theorem 3.2 belongs to $F_{2,m}$.

3.2 The Case of $F_{n,m}$ $(n \geq 3)$

Put $[x_i, y_1] = \sum_{j=2}^{m} \alpha_{i,j}y_j$, $1 \leq i \leq n$.

If x_1 is a generating basic element, then put

$$[y_i, y_1] = \sum_{j=2}^{n} \beta_{i,j}x_j, \quad 1 \leq i \leq m,$$

In this notation we have

Lemma 3.3 *The following equality holds*:

$$[y_i, y_j] = \sum_{k=0}^{\min\{i+j-1,m\}-i} (-1)^k C_{j-1}^k \sum_{t=2}^{n-j+k+1} \beta_{i+k,t} x_{t+j-k-1}, \tag{3.1}$$

where $1 \le j \le m$, $1 \le i \le m$.

Proof The proof is found by induction on j at any value of i.

Lemma 3.4 *Let L be a Leibniz superalgebra from $F_{n,m}$ $(n \ge 3)$ with nilindex equal to $n+m$. Then x_2 can be supposed to be a generator.*

Proof Obviously, we can take either x_1 or x_2 as a generator of the superalgebra L which lies in L_0. Suppose that x_1 is a generator. Let s be such a number that $x_2 \in L^s \setminus L^{s+1}$. Suppose $s = 2$. Then $L^2 = \{x_2, x_3, \ldots, x_n, y_2, \ldots, y_m\}$, $L^3 = \{x_3, \ldots, x_n, y_2, \ldots, y_m\}$ and $\beta_{1,2} \ne 0$.

From Leibniz superidentity we have

$$\begin{aligned}[x_{i+1}, y_1] &= [[x_i, x_1], y_1] = [x_i, [x_1, y_1]] + [[x_i, y_1], x_1] \\ &= \left[x_i, \sum_{j=2}^{m} \alpha_{i,j} y_j\right] + \left[\sum_{j=2}^{m} \alpha_{i,j} y_j, x_1\right] = \sum_{j=3}^{m} A_{i+1,j} y_j, \quad 2 \le i \le n-1.\end{aligned}$$

Hence, $y_2 \in L^3 \setminus L^4$, i.e. $L^4 = \{x_3, \ldots, x_n, y_3, y_4, \ldots, y_m\}$. Therefore there exists $t \ge 2$ such that $\beta_{t,3} \ne 0$ and $\alpha_{2,2} \ne 0$.

Consider the products

$$[x_2, [y_1, y_1]] = 2[[x_2, y_1], y_1] = 2[\alpha_{2,2} y_2 + \alpha_{2,3} y_3 + \ldots + \alpha_{2,m} y_m, y_1], \tag{3.2}$$

$$[x_2, [y_1, y_1]] = [x_2, \beta_{1,2} x_2 + \beta_{1,3} x_3 + \ldots + \beta_{1,n} x_n]. \tag{3.3}$$

Evidently, coefficient of the element x_3 in (3.3) is equal to zero. Therefore, if $\beta_{2,3} \ne 0$, then from (3.2) we have the existence of $t_0 > 2$ such that $\beta_{t_0,3} \ne 0$. Thus, $x_3 \in L^5$ and $L^5 = \{x_3, \ldots, x_n, y_4, \ldots, y_m\}$.

Consider the products

$$[y_1, [y_1, y_1]] = 2[[y_1, y_1], y_1] = 2\left[\sum_{i=2}^{n} \beta_{1,i} x_i, y_1\right] = 2\sum_{i=2}^{n} \beta_{1,i} [x_i, y_1], \tag{3.4}$$

$$[y_1, [y_1, y_1]] = \left[y_1, \sum_{i=2}^{n} \beta_{1,i} x_i\right] = \beta_{1,2}[y_1, x_2] + \left[y_1, \sum_{i=3}^{n} \beta_{1,i} x_i\right]. \tag{3.5}$$

Since $\alpha_{2,2} \ne 0$ and $L^5 = \{x_3, \ldots, x_n, y_4, \ldots, y_m\}$, then from (3.4)–(3.5) it follows that in the decomposition of $[y_1, x_2]$ the coefficient at the basic element y_2 is equal to $2\alpha_{2,2}$.

Using Leibniz superidentity and Theorem 2.2, we have that in the decomposition of $[y_2, x_2]$ the coefficient at basic element y_3 is equal to $2\alpha_{2,2}$. Since $y_2 \in L^3$ and $x_2 \in L^2$, then $[y_2, x_2] \in L^5$ and, hence, y_3 also lies in L_5, i.e. we obtain a contradiction which implies that $s \ge 3$.

Suppose that $s \le m$. Then $\beta_{s-1,2} \ne 0$, $L^i = \{x_2, x_3, \ldots, x_n, y_i, y_{i+1}, \ldots, y_m\}$, $2 \le i \le s$ and $L^{s+1} = \{x_3, x_4, \ldots, x_n, y_s, \ldots, y_m\}$.

From the equality (3.1) we have expression

$$[y_1, y_s] = \sum_{k=0}^{s-1} (-1)^k C_{s-1}^k \sum_{t=2}^{n-s+k+1} \beta_{1_k,t} x_{t+s-k-1},$$

in which the coefficient $\beta_{s-1,2}$ occurs. Taking into account the equality $[y_s, y_1] = C_{s-1}^{s-1} \sum_{t=2}^{n} \beta_{s,t} x_t$, we conclude that $x_3 \in \operatorname{lin}\langle [y_1, y_s], [y_s, y_1], x_4, x_5, \dots, x_n \rangle$. Therefore, $L^{s+2} = \{x_3, x_4, \dots, x_n, y_{s+1}, \dots, y_{m-1}, y_m\}$, $y_s \in L^{s+1} \setminus L^{s+2}$ and $\alpha_{2,s} \neq 0$.

Consider the products

$$[y_{s-1}, [y_1, y_1]] = 2[[y_{s-1}, y_1], y_1] = 2\left[\sum_{i=2}^{n} \beta_{s-1,i} x_i, y_1\right] = 2\beta_{s-1,2}\alpha_{2,s} y_s + \sum_{k \geq s+1} d_k y_k. \quad (3.6)$$

$$[y_{s-1}, [y_1, y_1]] = \left[y_{s-1}, \sum_{i=2}^{n} \beta_{1,i} x_i\right] = \sum_{i=2}^{n} \beta_{1,i} [y_{s-1}, x_i]. \quad (3.7)$$

Since $L^{s-1} = \{x_2, x_3, \dots, x_n, y_{s-1}, y_s, \dots, y_m\}$ and $L^s = \{x_2, x_3, \dots, x_n, y_s, y_{s+1}, \dots, y_m\}$, then from (3.6)–(3.7) we have $y_s \in L^{2s-1}$. Since for $s \geq 3$, the inequality $2s - 1 > s + 1$ holds, we obtain a contradiction to the condition $y_s \in L^{s+1} \setminus L^{s+2}$, from which it follows that the case $3 \leq s \leq m$ is impossible.

It is easy to see that $s = m + 1$ and

$$\begin{aligned}
&L^i = \{x_2, x_3, \dots, x_n, y_i, y_{i+1}, \dots, y_m\}, \quad 2 \leq i \leq m,\\
&L^{m+i-1} = \{x_i, x_{i+1}, \dots, x_n\}, \quad 2 \leq i \leq n,\\
&L^{m+n} = \{0\}.
\end{aligned}$$

For basic elements of L^i we have

$$\begin{aligned}
&[x_i, y_j] = [y_j, x_i] = 0, \quad 2 \leq i \leq n, \quad 1 \leq j \leq m,\\
&[y_m, y_1] = \beta_{m,2} x_2 + \beta_{m,3} x_3 + \dots + \beta_{m,n} x_n, \quad \text{where } \beta_{m,2} \neq 0.
\end{aligned}$$

The element $[y_1, x_1] + [x_1, y_1] = (1 + \alpha_{1,2}) y_2 + \alpha_{1,3} y_3 + \dots + \alpha_{1,m} y_m$ belongs to $R(L)$.

Case 1 Assume that $[y_1, x_1] + [x_1, y_1] = 0$.

Then consider the products

$$\begin{aligned}
[x_1, [y_m, y_1]] &= [[x_1, y_m], y_1] + [[x_1, y_1], y_m] = -[y_2, y_m]]\\
&= -\sum_{k=0}^{m-2} C_{m-1}^{k} \sum_{t=2}^{n-m+k+1} \beta_{2+k,t} x_{t+m-k-1}.
\end{aligned}$$

$$[x_1, [y_m, y_1]] = \left[x_1, \sum_{i=2}^{n} \beta_{m,i} x_i\right] = [x_1, x_i].$$

Comparing coefficients of the basic elements in products in the classes a) and b) of Theorem 2.2, we obtain $\sum_{i=2}^{n} \beta_{m,i} [x_1, x_i] \in \operatorname{lin}\langle x_4, x_5, \dots, x_n \rangle$ and, hence, $\beta_{m,2} = 0$.

Thus, we obtain a contradiction to $\beta_{m,2} \neq 0$.

If the multiplication in L belongs to the class b) of Theorem 2.2, then $\sum_{i=2}^{n} \beta_{m,i} [x_1, x_i] = -\sum_{i=2}^{n-1} \beta_{m,i} x_{i+1}$. Comparing coefficients at the element x_3 in the above products we obtain $\beta_{m,2} = (-1)^m \beta_{m,2}$. For $m \geq 3$ we have $\beta_{m,2} = 0$, i.e. in this case we also obtain a contradiction. For $m = 2$, consider the product

$$[x_1, [y_1, y_1]] = 2[[x_1, y_1], y_1] = -2[y_2, y_1] = -2 \sum_{i=2}^{n} \beta_{2,i} x_i.$$

On the other hand, $[x_1, [y_1, y_1]] = [x_1, \sum_{i=2}^{n} \beta_{1,i} x_i] = -\sum_{i=2}^{n-1} \beta_{1,i} x_{i+1}$, hence, $\beta_{2,2} = 0$.

Case 2 Now assume that $[y_1, x_1] + [x_1, y_1] \neq 0$. Then by argumentations as in Lemma 3.2 we have $y_m \in R(L)$. Therefore,

$$0 = [y_2, y_m] = \sum_{k=0}^{m-2} (-1)^k C_{m-1}^k \sum_{t=2}^{n-m+k+1} \beta_{2+k,t} x_{t+m-k-1} \ \Rightarrow\ \beta_{m,2} = 0$$

and again we obtain a contradiction to $\beta_{m,2} \neq 0$.

Thus, x_1 can not be a generator and so as a generator we can choose x_2. □

Now let x_2 be a generator. Then put

$$[y_1, x_2] = \sum_{i=2}^{m} a_i y_i, \quad [y_i, y_1] = \sum_{j=1, j\neq 2}^{m} \beta_{i,j} x_j.$$

Lemma 3.5 *Let L be a Leibniz superalgebra, which belongs to the class* b) *of Theorem* 2.2. *Then its nilindex is less than $n + m$.*

Proof Suppose the opposite, i.e. nilindex of superalgebra L is equal to $n + m$.

Let $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m\}$ be the basis of L.

We'll show that $[y_1, y_i] + [y_i, y_1] \in \{x_4, x_5, \dots, x_n\}$ for any $2 \leq i \leq m$.

Consider the product

$$\begin{aligned}[y_1, y_2] &= [y_1, [y_1, x_1]] = [[y_1, y_1], x_1] - [[y_1, x_1], y_1] \\ &= \left[\sum_{j=1, j\neq 2}^{m} \beta_{1,j} x_j, x_1\right] - [y_2, y_1] \ \Rightarrow\ [y_1, y_2] + [y_2, y_1] = \sum_{j=3}^{m} \beta_{1,j} x_{j+1} \in \{x_4, x_5, \dots, x_n\}.\end{aligned}$$

Similarly to the proof of Lemma 3.4 we conclude that $y_2 \in L^3 \setminus L^4$. Therefore, $L^2 = \{x_1, x_3, \dots, x_n, y_2, \dots, y_m\}$, $L^3 = \{x_3, \dots, x_n, y_2, \dots, y_m\}$, $L^4 = \{x_3, \dots, x_n, y_3, \dots, y_m\}$, $L^5 = \{x_4, \dots, x_n, y_3, \dots, y_m\}$ and $\beta_{1,1} \neq 0$. Hence

$$\begin{aligned}[[y_1, y_1], x_2] &= [y_1, [y_1, x_2]] + [[y_1, x_2], y_1] = \left[y_1, \sum_{i=2}^{m} a_i y_i\right] + \left[\sum_{i=2}^{m} a_i y_i, y_1\right] \\ &= \sum_{i=2}^{m} a_i([y_1, y_i] + [y_i, y_1]) \in \{x_4, x_5, \dots, x_n\}.\end{aligned}$$

On the other hand,

$$[[y_1, y_1], x_2] = \left[\sum_{j=1, j\neq 2}^{m} \beta_{1,j} x_j, x_2\right] = -\beta_{1,1} x_3 + \sum_{j=3}^{n} \beta_{1,j} [x_i, x_2].$$

Since $[x_i, x_2] \in \{x_4, x_5, \dots, x_n\}$ for any $3 \leq i \leq n$, then $\beta_{1,1} = 0$. We have a contradiction to the condition $x_1 \in L^2$, and therefore $L^2 = \{x_3, \dots, x_n, y_2, \dots, y_m\}$ and $\dim L/L^2 = 3$. Thus, the superalgebra L has nilindex less than $n + m$.

Lemma 3.6 *Let L be a Leibniz superalgebra which belongs to $F_{n,m}$ with nilindex equal to $n + m$. Then either $m = n$ or $m = n - 1$ and in the latter case in the class* a) *there exists a basis $\{x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m\}$ of the superalgebra L such that its multiplication has the following form*:

$$[x_1, x_1] = x_3,$$

$$[x_i, x_1] = x_{i+1}, \quad 2 \leq i \leq n - 1,$$

$$
\begin{aligned}
&[y_j, x_1] = y_{j+1}, \quad 1 \le j \le n-2,\\
&[x_1, y_1] = y_2,\\
&[x_i, y_1] = y_i, \quad 2 \le i \le n-1,\\
&[y_1, y_1] = x_1,\\
&[y_j, y_1] = x_{j+1}, \quad 2 \le j \le n-1,\\
&[x_1, x_2] = \alpha_4 x_4 + \alpha_5 x_5 + \cdots + \alpha_{n-1} x_{n-1} + \theta x_n,\\
&[x_j, x_2] = \alpha_4 x_{j+2} + \alpha_5 x_{j+3} + \cdots + \alpha_{n+2-j} x_n,\\
&[y_1, x_2] = \alpha_4 y_3 + \alpha_5 y_4 + \cdots + \alpha_{n-1} y_{n-2} + \theta y_{n-1},\\
&[y_j, x_2] = \alpha_4 y_{j+2} + \alpha_5 y_{j+3} + \cdots + \alpha_{n+1-j} y_{n-1}, \quad 2 \le j \le n-3,
\end{aligned}
$$

where omitted products are equal to zero.

Proof Since x_2 and y_1 are generators of L, it follows that $\beta_{1,1} \neq 0$. Without loss of generality we can assume that $[y_1, y_1] = x_1$. Considering the subsuperalgebra $\langle y_1 \rangle$ of the superalgebra L, then from multiplication rules in the classes a) and b) we obtain

$$\langle y_1 \rangle = \{x_1, x_3, x_4, \ldots, x_n, y_1, y_2, \ldots, y_m\}.$$

The subsuperalgebra $\langle y_1 \rangle$ is sigle-generated and from Theorem 2.1 we have that either $m = n-1$, or $m = n$ and the multiplication in $\langle y_1 \rangle$ has the following form:

$$
\begin{aligned}
&[x_1, x_1] = x_3,\\
&[x_i, x_1] = x_{i+1}, \quad 2 \le i \le n-1,\\
&[y_j, x_1] = y_{j+1}, \quad 1 \le j \le m-1,\\
&[x_1, y_1] = \frac{1}{2} y_2,\\
&[x_i, y_1] = \frac{1}{2} y_i, \quad 3 \le i \le m,\\
&[y_1, y_1] = x_1,\\
&[y_j, y_1] = x_{j+1}, \quad 2 \le j \le n-1
\end{aligned}
$$

(omitted products are equal to zero).

Suppose now that $m = n-1$ and L belongs to the class a) of Theorem 2.2. Then using the above multiplication rules we obtain

$$
\begin{aligned}
&L^2 = \{x_1, x_3, \ldots, x_n, y_2, \ldots, y_{n-1}\},\\
&L^{2k-1} = \{x_{k+1}, x_{k+2}, \ldots, x_n, y_k, y_{k+1}, \ldots, y_{n-1}\}, \quad 2 \le k \le n-1,\\
&L^{2k} = \{x_{k+1}, x_{k+2}, \ldots, x_n, y_{k+1}, y_{k+2}, \ldots, y_{n-1}\}, \quad 2 \le k \le n-2,\\
&L^{2(n-1)} = \{x_n\},\\
&L^{2n-1} = 0.
\end{aligned}
$$

In the superalgebra L consider multiplication on the right side by the element x_2.

Since $[y_1, x_2] = a_2 y_2 + a_3 y_3 + \cdots + a_{n-1} y_{n-1}$ then

$$[y_i, x_2] = [[y_{i-1}, x_1], x_2] = [y_{i-1}, [x_1, x_2]] + [[y_{i-1}, x_2], x_1] = \sum_{k=2}^{n-i} a_k y_{i+k-1}, \ 2 \le i \le m.$$

Furthermore, $[x_2, y_1] = b_2y_2 + b_3y_3 + \cdots + b_{n-1}y_{n-1}$. Then by Leibniz superidentity we have

$$[x_2, [y_1, y_1] = 2[[x_2, y_1], y_1] = 2[b_2y_2 + b_3y_3 + \cdots + b_{n-1}y_{n-1}, y_1]$$
$$= 2b_2x_3 + 2b_3x_4 + \cdots + 2b_{n-1}x_n.$$

On the other hand, $[x_2, [y_1, y_1] = [x_2, x_1] = x_3$. If we compare the coefficients at the basic elements, we obtain the following restrictions:

$$b_2 = \frac{1}{2}, \quad b_3 = b_4 = \cdots = b_{n-1} = 0.$$

Therefore, $[x_2, y_1] = \frac{1}{2}y_2$.

Consider the following equalities:

$$[x_2, y_2] = [x_2, [y_1, x_1]] = [[x_2, y_1], x_1] - [[x_2, x_1], y_1]$$
$$= \left[\frac{1}{2}y_2, x_1\right] - [x_3, y_1] = \frac{1}{2}y_3 - \frac{1}{2}y_3 = 0.$$

Thus, we obtain the multiplication rules in the Leibniz superalgebra L.

Consider the product

$$[y_1, [y_1, x_2]] = [[y_1, y_1], x_2]] - [[y_1, x_2], y_1] = [x_1, x_2] - [a_2y_2 + a_3y_3 + \cdots + a_{n-1}y_{n-1}, y_1].$$

On the other hand, $[y_1, [y_1, x_2]] = 0$, i.e. equality $4x_4 + 5x_5 + \cdots + \alpha_{n-1}x_{n-1} + \theta x_n = a_2x_3 + a_3x_4 + \cdots + a_{n-1}x_n$ holds. From this we have the following relations:

$$\theta = a_{n-1}, \quad a_2 = 0, \quad a_i = \alpha_{i+1}, \quad 3 \le i \le n-2.$$

Denote the superalgebras from Lemma 3.6 by $L(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta)$.

Proposition 3.1 *Two superalgebras $L(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta)$ and $L'(\alpha'_4, \alpha'_5, \ldots, \alpha'_n, \theta')$ are isomorphic if and only if there exists $a \in C$ which satisfies the following conditions*:

$$\begin{cases} \alpha_j = a^{2(j-3)}\alpha'_j, & 4 \le j \le n, \\ \theta = a^{2n-6}\theta'. \end{cases}$$

Proof Take the following general change of generating elements of the superalgebra $L(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta)$:

$$y'_1 = \sum_{i=1}^{n-1} c_iy_i, \quad x'_2 = \sum_{i=1}^{n} d_ix_i, \quad \text{where } c_1d_2 \neq 0.$$

Then the expression for the element x'_1 in old basis has the form:

$$x'_1 = [y'_1, y'_1] = c_1^2x_1 + c_1\sum_{i=2}^{n-1} c_ix_{i+1}.$$

Expressions for x'_{i+1} for $2 \le i \le n-1$ and y'_{j+1} for $1 \le j \le n-2$ have the following form:

$$\begin{aligned} x'_{i+1} &= c_i^{2i-1}\sum_{k=2}^{n-i} c_kx_{k+i}, \quad 2 \le i \le n-1, \\ y'_{j+1} &= c_i^{2j}\sum_{k=1}^{n-j} c_ky_{k+j}, \quad 1 \le j \le n-2. \end{aligned} \tag{3.8}$$

Since $x'_3 = [x'_2, x'_1] = [\sum_{i=1}^{n} d_ix_i, c_1^2x_1 + c_1\sum_{i=2}^{n-1} c_ix_{i+1}] = c_1^2(d_1x_3 + d_2x_3 + d_3x_4 + \cdots + d_{n-1}x_n)$, then by comparing the coefficients at the basic elements in (3.8) for $i = 3$ we obtain restrictions:

$$\begin{cases} d_1 + d_2 = c_1^2, \\ d_i = c_1 c_{i-1}, \ 3 \le i \le n-1. \end{cases} \tag{3.9}$$

The verification of the rest of products gives either an identity or the relations (3.9).

Applying (3.9), we obtain

$$\begin{aligned}[][x_1', x_2'] &= \Big[c_1^2 x_1 + c_1 \sum_{i=2}^{n-1} c_i x_{i+1}, d_1 x_1 + d_2 x_2 + c_1 c_2 x_3 + \cdots + c_1 c_{n-2} x_n + d_n x_n\Big] \\ &= c_1^2 d_1 x_3 + c_1 d_1 \sum_{i=2}^{n-2} c_i x_{i+2} + c_1 d_2 \sum_{i=3}^{n-3} [x_i, x_2] \\ &= c_1 d_1 \sum_{i=2}^{n-2} c_i x_{i+2} + c_1^2 d_2 (\alpha_4 x_4 + \alpha_5 x_5 + \cdots + \alpha_{n-1} x_n - 1 + \theta x_n) \\ &\quad + c_1 d_2 \sum_{i=3}^{n-3} \sum_{j=4}^{n+1-i} \alpha_j x_{j+i-2}. \end{aligned}$$

On the other side,

$$\begin{aligned}[][x_1', x_2'] &= \alpha_4' x_4' + \alpha_5' x_5' + \cdots + \alpha_{n-1}' x_{n-1}' + \theta' x_n' \\ &= \alpha_4' \Big(c_1^5 \sum_{k=1}^{n-3} c_k x_{k+3}\Big) + \alpha_5' \Big(c_1^7 \sum_{k=1}^{n-4} c_k x_{k+4}\Big) + \cdots \\ &\quad + \alpha_{n-1}' \Big(c_1^{2n-5} \sum_{k=1}^{2} c_k x_{k+n-2}\Big) + \theta' c_1^{2(n-1)} x_n. \end{aligned}$$

If we compare coefficients at the basic elements, we obtain the following restrictions:

$$\begin{cases} d_1 = 0, \\ c_1 d_2 \sum\limits_{i=1}^{k} c_{k+1-i} \alpha_{i+3} = c_1^5 \sum\limits_{i=1}^{k} c_1^{2(i-1)} c_{k+1-i} \alpha_{i+3}', \ 1 \le k \le n-4, \\ c_1 d_2 \Big(\sum\limits_{i=1}^{n-4} c_{n-2-i} \alpha_{i+3} + c_1 \theta\Big) = c_1^5 \Big(\sum\limits_{i=1}^{n-4} c_1^{2(i-1)} c_{n-2-i} \alpha_{i+3}' + c_1^{2n-7} \theta'\Big). \end{cases} \tag{3.10}$$

From (3.9) and (3.10) we have $d_2 = c_1^2$ and $\begin{cases} \alpha_j = c_1^{2(j-3)} \alpha_j', \ 4 \le j \le n-1, \\ \theta = c_1^{2(n-6)} \theta'. \end{cases}$

In a similar way we have

$$\begin{aligned}[][x_2', x_2'] &= \Big[c_1 \sum_{i=1}^{n-1} c_i x_{i+1}, c_1 \sum_{i=1}^{n-1} c_i x_{i+1}\Big] = c_1^3 \sum_{i=1}^{n-3} c_i [x_{i+1}, x_2] \\ &= c_1^3 \sum_{i=1}^{n-3} c_i (\alpha_4 x_{i+3} + \alpha_5 x_{i+4} + \cdots + \alpha_{n-i} x_{n-1} + \alpha_{n+1-i} x_n), \end{aligned}$$

and on the other side,

$$[x_2', x_2'] = \alpha_4' x_4' + \alpha_5' x_5' + \cdots + \alpha_{n-1}' x_{n-1}' + \alpha_n' x_n'.$$

Therefore,

$$\alpha_n = c_1^{2n-6} \alpha_n'.$$

Lemma 3.7 *Let L be a Leibniz superalgebra from the family* a) *and $m = n$. Then there exists a basis $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$ of the superalgebra L such that its multiplication has the following form*:

$$
\begin{aligned}
&[x_1, x_1] = x_3, \\
&[x_i, x_1] = x_{i+1}, \quad 2 \le i \le n-1, \\
&[y_j, x_1] = y_j + 1, \quad 1 \le j \le n-1, \\
&[x_1, y_1] = \frac{1}{2} y_2, \\
&[x_i, y_1] = \frac{1}{2} y_i, \quad 2 \le i \le n, \\
&[y_1, y_1] = x_1, \\
&[y_j, y_1] = x_{j+1}, \quad 2 \le j \le n-1, \\
&[x_1, x_2] = \alpha_4 x_4 + \alpha_5 x_5 + \cdots + \alpha_{n-1} x_{n-1} + \theta x_n, \\
&[x_j, x_2] = \alpha_4 x_{j+2} + \alpha_5 x_{j+3} + \cdots + \alpha_{n+2-j} x_n, \\
&[y_1, x_2] = \alpha_4 y_3 + \alpha_5 y_4 + \cdots + \alpha_{n-1} y_{n-2} + \theta y_{n-1} + \tau y_n, \\
&[y_2, x_2] = \alpha_4 y_4 + \alpha_5 y_4 + \cdots + \alpha_{n-1} y_{n-1} + \theta y_n, \\
&[y_j, x_2] = \alpha_4 y_{j+2} + \alpha_5 y_{j+3} + \cdots + \alpha_{n+2-j} y_n, \quad 3 \le j \le n-2,
\end{aligned}
$$

where omitted products are equal to zero.

Proof The proof of this lemma is similar to the proof of Lemma 3.6.

Denote of superalgebra from Lemma 3.7 by $M(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta, \tau)$.

Then we have the following criterion of isomorphism.

Proposition 3.2 *Two superalgebras $M(\alpha_4, \alpha_5, \ldots, \alpha_n, \theta, \tau)$ and $M(\alpha'_4, \alpha'_5, \ldots, \alpha'_n, \theta', \tau')$ are isomorphic if and only if there exists $a \in C$ such that*

$$
\begin{cases}
\alpha_j = a^{2(j-3)} \alpha'_j, \quad 4 \le j \le n, \\
\theta = a^{2n-6} \theta', \\
\tau = a^{2n-4} \tau'.
\end{cases}
$$

Proof The proof of this proposition is similar to the proof of Proposition 3.1.

Consider now the class b) of Theorem 2.2.

Lemma 3.8 *Let L be a superalgebra from class* b). *Then in case when $m = n-1$, there exists a basis $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$ of the superalgebra L such that its multiplication has the following form*:

$$
\begin{aligned}
&[x_1, x_1] = x_3, \\
&[x_i, x_1] = x_{i+1}, \quad 3 \le i \le n-1, \\
&[y_j, x_1] = y_j + 1, \quad 1 \le j \le n-2, \\
&[x_1, x_2] = \beta_4 x_4 + \beta_5 x_5 + \ldots + \beta_n x_n, \\
&[x_2, x_2] = \gamma x_n, \\
&[x_j, x_2] = \beta_4 x_{j+2} + \beta_5 x_{j+3} + \cdots + \beta_{n+2-j} x_n, \ 3 \le j \le n-2,
\end{aligned}
$$

$$
\begin{aligned}
&[y_1, y_1] = x_1,\\
&[y_j, y_1] = x_{j+1}, \quad 2 \le j \le n-1,\\
&[x_1, y_1] = \frac{1}{2} y_2,\\
&[x_i, y_1] = \frac{1}{2} y_i,\ 3 \le i \le n-1,\\
&[y_j, x_2] = \beta_4 y_{j+2} + \beta_5 y_{j+3} + \cdots + \beta_{n+1-j} y_{n-1},\ 1 \le j \le n-3,
\end{aligned}
$$

where omitted products are equal to zero.

Proof The proof of this lemma is similar to the proof of Lemma 3.6.

Denote the superalgebra from Lemma 3.8 by $G(\beta_4, \beta_5, \ldots, \beta_n, \gamma)$.

Proposition 3.3 *Two superalgebras $G(\beta_4, \beta_5, \ldots, \beta_n, \gamma)$ and $G(\beta'_4, \beta'_5, \ldots, \beta'_n, \gamma')$ are isomorphic if and only if there exist $a, b \in C$ such that*

$$
\begin{cases}
\beta b_j = a^{2(j-2)} \beta'_j,\ 4 \le j \le n,\\
b^2 \gamma = a^{2(n-1)} \gamma'.
\end{cases}
$$

Proof Let us take the general change of generators in the superalgebra $G(\beta_4, \beta_5, \ldots, \beta_n, \gamma)$:

$$
y'_1 = \sum_{i=1}^{n-1} c_i y_i, \quad x'_2 = \sum_{i=1}^{n} d_i x_i, \quad \text{where } c_1 d_2 \neq 0.
$$

Then for new basic elements x'_i and y'_j we have the following expressions:

$$
\begin{aligned}
x'_1 &= [y'_1, y'_1] = c_1^2 x_1 + c_1 \sum_{i=2}^{n-1} c_i x_{i+1},\\
x'_{i+1} &= c_1^{2i-1} \sum_{k=1}^{n-i} c_k x_{k+i}, \quad 2 \le i \le n-1,\\
y'_{j+1} &= c_1^{2j} \sum_{k=1}^{n-j-1} c_k y_{k+j}, \quad 1 \le j \le n-2.
\end{aligned}
$$

We have

$$
[x'_2, x'_1] = \left[\sum_{i=1}^{n} d_i x_i, c_1^2 x_1 + c_1 \sum_{i=2}^{n-1} c_i x_{i+1} \right] = c_1^2 \Big(d_1 x_3 + \sum_{i=3}^{n-1} d_i x_{i+1} \Big).
$$

On the other hand $[x'_2, x'_1] = 0$. Comparing coefficients at the basic elements we obtain

$$
d_1 = d_3 = d_4 = \cdots = d_{n-1} = 0,
$$

i.e. $x'_2 = d_2 x_2 + d_n x_n$.

Furthermore, we have

$$
\begin{aligned}
[x'_1, x'_2] &= \left[c_1^2 x_1 + c_1 \sum_{i=2}^{n-1} c_i x_{i+1}, d_2 x_2 + d_n x_n \right] = c_1^2 d_2 [x_1, x_2] + c_1 d_2 \sum_{i=2}^{n-3} c_i [x_{i+1}, x_2]\\
&= c_1^2 d_2 (\beta_4 x_4 + \beta_5 x_5 + \ldots + \beta_n x_n) + c_1 d_2 \sum_{i=2}^{n-3} c_i \left(\beta_4 x_{i+3} + \beta_5 x_{i+4} + \cdots + \beta_{n-i+1} x_n \right).
\end{aligned}
$$

On the other hand,

$$
[x'_1, x'_2] = \beta'_4 x'_4 + \beta'_5 x'_5 + \cdots + \beta'_n x'_n
$$

$$= \beta'_4 c_1^5 \sum_{k=1}^{n-3} c_k x_{k+3} + \beta'_5 c_1^7 \sum_{k=1}^{n-4} c_k x_{k+4} + \cdots + \beta'_n c_1^{2(n-1)} x_n.$$

Again, comparing coefficients at the basic elements we obtain the following restrictions:

$$\beta'_j c_1^{2(j-2)} = d_2 \beta_j, \quad 4 \le j \le n. \tag{3.11}$$

Consider the product

$$[x'_2, x'_2] = [d_2 x_2 + d_n x_n, d_2 x_2 + d_n x_n] = d_2^2 \gamma x_n.$$

On the other hand, we have $[x'_2, x'_2] = \gamma' x'_n = c_1^{2(n-1)} x_n$, i.e. $\gamma' c_1^{2(n-1)} = d_2^2 \gamma$.

The verification of the rest of products gives either an identity or the above restrictions. □

Consider now the case when the superalgebra belongs to the class b) and $m = n$.

Lemma 3.9 *Let L be a superalgebras from class* b) *and $m = n$. Then there exists a basis $\{x_1, x_2, , x_n, y_1, y_2, \ldots, y_m\}$ of the superalgebra L such that its multiplication has the form:*

$$\begin{aligned}
&[x_1, x_1] = x_3,\\
&[x_i, x_1] = x_{i+1}, \quad 3 \le i \le n-1,\\
&[y_j, x_1] = y_j + 1, \quad 1 \le j \le n-2,\\
&[x_1, x_2] = \beta_4 x_4 + \beta_5 x_5 + \cdots + \beta_n x_n,\\
&[x_2, x_2] = \gamma x_n,\\
&[x_j, x_2] = \beta_4 x_{j+2} + \beta_5 x_{j+3} + \cdots + \beta_{n+2-j} x_n, \ 3 \le j \le n-2,\\
&[y_1, y_1] = x_1,\\
&[y_j, y_1] = x_{j+1}, \quad 2 \le j \le n-1,\\
&[x_1, y_1] = \frac{1}{2} y_2,\\
&[x_i, y_1] = \frac{1}{2} y_i, \ 3 \le i \le n-1,\\
&[y_1, x_2] = \beta_4 y_3 + \beta_5 y_4 + \cdots + \beta_n y_{n-1} + \delta y_n,\\
&[y_j, x_2] = \beta_4 y_{j+2} + \beta_5 y_{j+3} + \cdots + \beta_{n+2-j} y_n, \ 2 \le j \le n-2,
\end{aligned}$$

(*omitted products are equal to zero*).

Proof The proof of this lemma is similar to the proof of Lemma 3.8.

Denote a superalgebra from Lemma 3.9 by $H(\beta_4, \beta_5, \ldots, \beta_n, \delta, \gamma)$.

Proposition 3.4 *Two superalgebras $H(\beta_4, \beta_5, \ldots, \beta_n, \delta, \gamma)$ and $H(\beta'_4, \beta'_5, \ldots, \beta'_n, \delta', \gamma')$ are isomorphic if and only if there exist $a, b \in C$ such that*

$$\begin{cases} b\beta_j = a^{2(j-2)} \beta'_j, \ 4 \le jl4n,\\ b\delta = a^{2(n-1)} \delta',\\ b^2 \gamma = a^{2(n-1)} \gamma'. \end{cases}$$

Proof Similar to that of Proposition 3.3.

Consider the following operators which act on k-dimensional vectors:

$$V_{j,k}^m(\alpha_1, \alpha_2, \ldots, \alpha_k) = (0, 0, \ldots, \overset{j-1}{0}, 1, S_{m,j}^{j+1} \alpha_{j+1}, S_{m,j}^{j+2} \alpha_{j+2}, \ldots S_{m,j}^{k-1} \alpha_{k-1}, S_{m,j}^k \alpha_k),$$
$$V_{k+1,k}^m(\alpha_1, \alpha_2, \ldots, \alpha_k) = (0, 0, \ldots, 0),$$

$$W^m_{s,k}(0,0,\ldots,\overset{j-1}{0},\overset{j}{1},S^{j+1}_{m,j}\alpha_{s+j+1},S^{j+2}_{m,j}\alpha_{s+j+2},\ldots,S^{k}_{m,j}\alpha_k,\gamma)$$
$$=(0,0,\ldots,\overset{j}{1},0,\ldots,\overset{s+j}{1},S^{s+1}_{m,s}\alpha_{s+j+1},S^{s+2}_{m,s}\alpha_{s+j+2},\ldots,S^{k-j}_{m,s}\alpha_k,S^{k+6-2j}_{m,s}\gamma),$$
$$W^m_{k+1-j,k}(0,0,\ldots,\overset{j-1}{0},\overset{j}{1},S^{j+1}_{m,j}\alpha_{s+j+1},S^{j+2}_{m,j}\alpha_{s+j+2},\ldots,S^{k}_{m,j}\alpha_k,\gamma)=(0,0,\ldots,\overset{j}{1},0,\ldots,1),$$
$$W^m_{k+2-j,k}(0,0,\ldots,\overset{j-1}{0},\overset{j}{1},S^{j+1}_{m,j}\alpha_{s+j+1},S^{j+2}_{m,j}\alpha_{s+j+2},\ldots,S^{k}_{m,j}\alpha_k,\gamma)=(0,0,\ldots,\overset{j}{1},0,\ldots,0),$$

where $k\in N$, $1\le j\le k$, $1\le s\le k-j$, $S_{m,t}=\cos\frac{2\pi m}{t}+\mathrm{i}\sin\frac{2\pi m}{t}$ $(m=0,1,\ldots,t-1)$.

Theorem 3.3 *Let L be a filiform non-Lie Leibniz superalgebra with nilindex equal to $n+m$. Then for $m=n-1$ it is isomorphic to one of the following pairwise non-isomorphic superalgebras:*

$$L\left(V_{j,n-3}\left(\alpha_4,\alpha_5,\ldots,\alpha_n\right),S^{n-3}_{m,j}\theta\right),\quad 1\le j\le n-3,$$
$$L(0,0,\ldots,0,1),$$
$$L(0,0,\ldots,0),$$
$$G\left(W_{s,n-2}\left(V_{j,n-3}\left(\beta_4,\beta_5,\ldots,\beta_n\right),\gamma\right)\right),\quad 1\le j\le n-3,\ 1\le s\le n-j,$$
$$G(0,0,\ldots,0,1),$$
$$G(0,0,\ldots,0).$$

Proof First we consider the family $L(\alpha_4,\alpha_5,\ldots,\alpha_n,\theta)$. From Proposition 3.1 we have the following conditions

$$\begin{cases}\alpha_j=a^{2(j-3)}\alpha'_j,\ 4\le j\le n,\\ \theta=a^{2n-6}\theta'.\end{cases}$$

Case 1 Let $\alpha_i=0$ for any $4\le i\le n$. Then if $\theta=0$ we obtain the superalgebra $L(0,0,\ldots,0)$. If $\theta\neq0$, then taking $a=\sqrt[2n-6]{\theta}$, we obtain $\theta'=1$, i.e. the superalgebra $L(0,0,\ldots,1)$.

Case 2 Let $\alpha_i=0$ for any i, for certain t $(4\le t\le n)$ and $\alpha_t\neq0$. Then putting $a^{2(t-3)}=\alpha_t$

$$\left(\text{i.e } a_1^{-2}=\sqrt[t-3]{\left|\frac{1}{\alpha_t}\right|}\left(\cos\frac{\varphi}{t-3}+\mathrm{i}\sin\frac{\varphi}{t-3}\right)\left(\cos\frac{2\pi m}{t-3}+\mathrm{i}\sin\frac{2\pi m}{t-3}\right)\right.,$$

where $\varphi=\arg\left(\frac{1}{\alpha_t}\right)$, $m=0,1,\ldots,t-4$), we obtain

$$\alpha'_t=1,\quad a'_{t+j}=S^{t+j-3}_{m,t-3}\mu_{t+j}\ \text{ for } 1\le j\le n-t,\ \theta'=S^{n-3}_{m,t-3}\nu,$$

where $\mu_{t+j}=\left|\frac{1}{\alpha_t}\right|^{\frac{t+j-3}{t-3}}\left(\cos\frac{\varphi(t+j-3)}{t-3}+\mathrm{i}\sin\frac{\varphi(t+j-3)}{t-3}\right)\alpha_{t+j}$,

$$\nu=\left|\frac{1}{\alpha_t}\right|^{\frac{n-3}{t-3}}\left(\cos\frac{\varphi(n-3)}{t-3}+\mathrm{i}\sin\frac{\varphi(n-3)}{t-3}\right).$$

Thus, in this case we obtain the following family of superalgebras

$$L\left(V_{j,n-3}\left(\alpha_4,\alpha_5,\ldots,\alpha_n\right),S^{n-3}_{m,j}\theta\right),\quad 1\le j\le n-3.$$

Consider the family of superalgebras $G(\beta_4,\beta_5,\ldots,\beta_n,\gamma)$. From Proposition 3.3 we have the following conditions:

$$\begin{cases}\beta b_j=a^{2(j-2)}\beta'_j,\ 4\le j\le n,\\ b^2\gamma=a^{2(n-1)}\gamma'.\end{cases}$$

Case 1 Let $\beta_i = 0$ for any $i < t$, where $4 \le t \le n$ and $\beta_t \ne 0$. Then putting $b = \frac{a^{2(t-2)}}{\beta_t}$, we obtain

$$\beta'_t = 1, \quad \beta'_{t+j}a^{2j} = \frac{\beta_{t+j}}{\beta_t}\ (1 \le j \le n-t), \quad \gamma' = a_1^{2(2t-n-3)}\frac{\gamma}{\beta_t^2}.$$

Case 1.1 Let $\beta_{t+j} = 0$ for any $j < q$, where $1 \le q \le n-t$ $\beta_{t+q} \ne 0$. Then taking

$$a^{-2} = \sqrt[l]{\left|\frac{\beta_t}{\beta_{t+1}}\right|}\left(\cos\frac{\varphi}{q} + \mathrm{i}\sin\frac{\varphi}{q}\right), \text{ where } \varphi = \arg\left(\frac{\beta_t}{\beta_{t+q}}\right), \ m = 0, 1, \ldots, q-1,$$

we obtain

$$\beta'_{t+q+j} = S_{m,q}^{q+j}\lambda_{t+q+j}\ (1 \le j \le n-t-1) \quad \gamma' = S_{m,q}^{n+3-2j}\gamma,$$

where $\lambda_{t+q+j} = \left|\frac{\beta_t}{\beta_{t+q}}\right|^{\frac{q+j}{q}}\left(\cos\frac{\varphi(q+j)}{q} + \mathrm{i}\sin\frac{\varphi(q+j)}{q}\right)\beta_{t+q+j}$,

$$\varepsilon = \left|\frac{\beta_t}{\beta_{t+q}}\right|^{\frac{n+3-2j}{q}}\left(\cos\frac{\varphi(n+3-2j)}{q} + \mathrm{i}\sin\frac{\varphi(n+3-2j)}{q}\right)\frac{\gamma}{\beta_j^2}.$$

Thus, in this case we obtain the following family of superalgebras:

$$G\left(W_{s,n-2}\left(V_{j,n-3}\left(\beta_4, \beta_5, \ldots, \beta_n\right), \gamma\right)\right), \ 1 \le j \le n-4, \ 1 \le s \le n-3-j.$$

Case 1.2 Let $\beta_{t+q} = 0$ for any $q \in \{1, 2, \ldots, n-t\}$. Then we have $\gamma' = a^{2(2t-n-3)}\frac{\gamma}{\beta_t^2}$, i.e. we obtain two superalgebras

$$G(0, 0, \ldots, \overset{t}{1}, 0, \ldots, 1), \quad G(0, 0, \ldots, \overset{t}{1}, 0, \ldots, 0).$$

Case 2 Let $\beta_t = 0$ for any $t \in \{4, 5, \ldots, n\}$. Then taking into account that $\gamma' a^{2(n-1)} = b^2\gamma$, we obtain the superalgebras

$$G(0, 0, \ldots, 1), \qquad G(0, 0, \ldots, 0).$$

In a similar way one can prove the following theorem.

Theorem 3.4 *Let L be a filiform non-Lie Leibniz superalgebra with nilindex equal to $n+m$. Then for $m = n$ it is isomorphic to one of the following pairwise non-isomorphic superalgebras*:

$$M\left(V_{j,n-2}\left(\alpha_4, \alpha_5, \ldots, \alpha_n\right), S_{m,j}^{n-3}\theta\right), \quad 1 \le j \le n-2,$$
$$M(0, 0, \ldots, 0, 1),$$
$$M(0, 0, \ldots, 0),$$
$$H\left(W_{s,n-1}\left(V_{j,n-2}\left(\beta_4, \beta_5, \ldots, \beta_n\right), \gamma\right)\right), \quad 1 \le j \le n-2, \ 1 \le s \le n+1-j,$$
$$H(0, 0, \ldots, 0, 1),$$
$$H(0, 0, \ldots, 0).$$

Proof Similar to that of Theorem 3.3.

Thus, summarizing the results of Theorems 3.1–3.4 we obtain the classification of all filiform Leibniz superalgebras with nilindex equal to $n+m$, where $m \ne 0$.

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