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The classification of algebras of level two

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ABSTRACT

This paper is devoted to the description of complex finite-dimensional algebras of level two. We obtain the classification of algebras of level two in the varieties of Jordan, Lie and associative algebras.

1. Introduction

Important subjects playing a relevant role in Mathematics and Physics are degenerations, contractions and deformations of algebras.

Degenerations of non-associative algebras were the subject of numerous papers (see for instance [1–4] and references given therein), and their research continues actively [5–7].

The general linear group $GL(V)$ over a field $\mathbb{K}$ acts on the finite-dimensional vector space $V^* \otimes V^* \otimes V$, the space of $\mathbb{K}$-algebra structures, by the change of basis. For two $\mathbb{K}$-algebra structures λ and μ we say that μ is a degeneration of λ if μ lies in the orbit closure of λ with respect to Zariski topology (it is denoted by $\mu \to \lambda$). The orbit closure problem from a geometrical point of view consists of the classification of all degenerations of a certain algebra structure of a fixed dimension. This problem also depends on a complete classification of the corresponding algebra structures. Both problems are highly complicated even in small dimensions.

It is known that closures of orbits in Zariski and standard topologies coincide in the case of an algebraically closed field of characteristic zero and as a particular case usually considered the field $\mathbb{C}$. Therefore, mainly the degenerations of complex objects are investigated.

It is well-known that there are closed relations between associative, Lie and Jordan algebras. In fact, commutator product defined on associative algebra gives us Lie algebra, while symmetrized product gives Jordan algebra. Moreover, any Lie algebra is isomorphic to a subalgebra of a certain commutator algebra. The analogue of this result is not true for Jordan algebras, that is, there are Jordan algebras which cannot be obtained from symmetrized product on associative algebras (such type of algebras are called exceptional Jordan algebras).

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The description of degenerations of dimensions less than five for complex Lie algebras and for nilpotent ones of dimensions less than seven was done in [8,9]. In the case of Jordan algebras we have the description of degenerations up to dimension four [10].

Since any n-dimensional algebra degenerates to the abelian algebra (denoted by $\mathfrak{a}_n$), the lowest edges in degenerations graph end on $\mathfrak{a}_n$. In [11] Gorbatsevich described the nearest-neighbor algebras to $\mathfrak{a}_n$ (algebras of level one) in the degeneration graphs of commutative and skew-symmetric algebras. In the work [12] it was ameliorated and correction of some non-accuracies made in [11]. Namely, a complete list of algebras level one in the variety of finite-dimensional complex algebras is obtained.

In fact, Gorbatsevich studied in [13] a very interesting notion closely related to degeneration: $\lambda \to \mu$ (algebras λ and μ not necessarily have the same dimension) if $\lambda \oplus \mathfrak{a}_k$ degenerates to $\mu \oplus \mathfrak{a}_m$ in the sense considered in this paper for some suitable $k, m \geq 0$. The corresponding first three levels of such type of degenerations are completely classified in [13].

In this paper we study the description of finite-dimensional algebras of level two over the field of complex numbers. More precisely, we obtain the classification of algebras of level two in the varieties of Jordan, Lie and associative algebras.

In the multiplication table of an algebra omitted products are assumed to be zero. Moreover, due to commutatively and anticommutatively of Jordan and Lie algebras, symmetric products for these algebras are also omitted.

2. Preliminaries

In this section we give some basic notions and concepts used through the paper.

Let λ be a n-dimensional algebra. We know that the algebra λ may be considered as an element of the affine variety $Hom(V \otimes V, V)$ via the mapping $\lambda: V \otimes V \to V$ over a field $\mathbb{K}$. The linear reductive group $GL_n(\mathbb{K})$ acts on the variety of n-dimensional algebras Alg_n via change of basis, i.e.,

$$(g * \lambda)(x, y) = g\Big(\lambda\big(g^{-1}(x), g^{-1}(y)\big)\Big), \quad g \in GL_n(\mathbb{K}),\ \lambda \in Alg_n.$$

The orbits $Orb(-)$ under this action are the isomorphism classes of algebras. Note that solvable (respectively, nilpotent) algebras of the same dimension also form an invariant subvariety of the variety of algebras under the mentioned action.

Definition 2.1. An algebra λ is said to degenerate to an algebra μ, if $Orb(\mu)$ lies in the Zariski closure of $Orb(\lambda)$. We denote this by $\lambda \to \mu$.

The degeneration $\lambda \to \mu$ is called *trivial*, if λ is isomorphic to μ. Non-trivial degeneration $\lambda \to \mu$ is called *direct degeneration* if there is no chain of non-trivial degenerations of the form: $\lambda \to \nu \to \mu$.

Definition 2.2. The level of a n-dimensional algebra λ is the maximum length of a chain of direct degenerations, which, of course, ends with the algebra $\mathfrak{a}_n$ (the algebra with zero multiplication).

Here we give the description of the algebras of level one.

Theorem 2.3 (*[12]*)**.** *A n-dimensional ($n \geq 3$) algebra is algebra of level one if and only if it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{array}{rlll}
p_n^- : & e_1e_i = e_i, & e_ie_1 = -e_i, & 2 \leq i \leq n; \\
n_3^- \oplus \mathfrak{a}_{n-3} : & e_1e_2 = e_3, & e_2e_1 = -e_3; & \\
\lambda_2 \oplus \mathfrak{a}_{n-2} : & e_1e_1 = e_2; & & \\
\nu_n(\alpha) : & e_1e_1 = e_1, & e_1e_i = \alpha e_i, & e_ie_1 = (1-\alpha)e_i, \quad 2 \leq i \leq n,\ \alpha \in \mathbb{C}.
\end{array}$$

Note that algebras $\lambda_2 \oplus \mathfrak{a}_{n-2}$ and $\nu_n\left(\frac{1}{2}\right)$ are Jordan algebras.

It is remarkable that the notion of degeneration considered in [13] is weaker than notions which are used in this paper. For instance, the levels by Gorbatsevich's work of the algebras p_n^- and $\nu_n(\alpha)$ do not equal one, because of $p_n^- \oplus \mathfrak{a}_1 \to n_3^- \oplus \mathfrak{a}_{n-2}$ and $\nu_n(\alpha) \oplus \mathfrak{a}_1 \to \lambda_2 \oplus \mathfrak{a}_{n-1}$.

It is known that any finite-dimensional associative (Jordan) algebra A is decomposed into a semidirect sum of semi-simple subalgebra A_{ss} and nilpotent radical $Rad(A)$. Moreover, an arbitrary finite-dimensional semi-simple associative (Jordan) algebra contains an identity element. Therefore, one can assume that a finite-dimensional associative (Jordan) algebra over a field $\mathbb{K}$ of $char\mathbb{K} = 0$ is either nilpotent or has an idempotent element.

One of the important results of theory of associative algebras related with idempotents is Pierce's decomposition. Let A be an associative algebra which contains an idempotent element e. Then we have decomposition

$$A = A_{1,1} \oplus A_{1,0} \oplus A_{0,1} \oplus A_{0,0}$$

with property $A_{i,j} \cdot A_{k,l} \subseteq \delta_{j,k}A_{i,l}$, where $\delta_{j,k}$ are Kronecker symbols. The subspaces $A_{j,k}$ are called Pierce's components.

Below we present an analogue of Pierce's decomposition for Jordan algebras.

Theorem 2.4 ([14]). *Let e be an idempotent of a Jordan algebra J. Then we have the following decomposition into a direct sum of subspaces*

$$J = P_0 \oplus P_{\frac{1}{2}} \oplus P_1,$$

where $P_i = \{x \in J \mid x \cdot e = ix\},\ i = 0; \frac{1}{2}; 1$ *and the multiplications for the components* P_i *are defined as follows:*

$$P_1^2 \subseteq P_1, \qquad P_1 \cdot P_0 = 0, \qquad P_0^2 \subseteq P_0, \qquad P_0 \cdot P_{\frac{1}{2}} \subseteq P_{\frac{1}{2}}, \qquad P_1 \cdot P_{\frac{1}{2}} \subseteq P_{\frac{1}{2}}, \qquad P_{\frac{1}{2}}^2 \subseteq P_0 \oplus P_1.$$

3. Main result

This section is devoted to the classifications of algebras of level two in the varieties of complex n-dimensional Jordan, Lie and associative algebras.

3.1. Jordan algebras of level two

In this subsection we give the classification of algebras of level two in the variety of complex n-dimensional Jordan algebras.

Theorem 3.1. *A n-dimensional* $(n \geq 3)$ *Jordan algebra is algebra of level two if and only if it is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$\begin{aligned} J_1 = \{e\} \oplus \mathfrak{a}_{n-1} : &\quad e \cdot e = e; \\ J_2 = \{e_1, e_2, e_3, \dots, e_n\} : &\quad e_1 \cdot e_1 = e_1, \quad e_1 \cdot e_i = e_i, \quad 2 \leq i \leq n; \\ J_3 = \{e_1, e_2, e_3\} \oplus \mathfrak{a}_{n-3} : &\quad e_1 \cdot e_2 = e_3. \end{aligned}$$

Proof. Firstly we suppose that semi-simple part of the Jordan algebra J is non-trivial, i.e., $J_{ss} \neq 0$. Thereby, there exists a unit element e of J_{ss} and J admits a basis $\{e, x_1, x_2, \dots, x_p, y_1, y_2, \dots, y_q, z_1, z_2, \dots, z_r\}$ such that

$$P_1 = \{e, x_1, x_2, \dots, x_p\}, \qquad P_0 = \{y_1, y_2, \dots, y_q\}, \qquad P_{\frac{1}{2}} = \{z_1, z_2, \dots, z_r\}.$$

The assertion of Theorem 2.4 provides the table of multiplication in this basis:

$$J : \begin{cases} e \cdot x_i = x_i, & x_i \cdot x_j = \alpha_{i,j} e + \sum\limits_{k=1}^{p} \beta_{i,j}^k x_k, & x_i \cdot z_j = \sum\limits_{k=1}^{r} \delta_{i,j}^k z_k, \\ & y_i \cdot y_j = \sum\limits_{k=1}^{q} \gamma_{i,j}^k y_k, & y_i \cdot z_j = \sum\limits_{k=1}^{r} \nu_{i,j}^k z_k, \\ e \cdot z_i = \frac{1}{2} z_i, & z_i \cdot z_j = \lambda_{i,j} e + \sum\limits_{k=1}^{p} \mu_{i,j}^k x_k + \sum\limits_{k=1}^{q} \theta_{i,j}^k y_k. & \end{cases}$$

It is easy to see that condition $p = q = 0$ implies the multiplication

$$e \cdot e = e, \qquad e \cdot z_i = \frac{1}{2} z_i, \qquad z_i \cdot z_j = \lambda_{i,j} e.$$

From Jordan identity we get $\lambda_{i,j} = 0$ and the algebra $\nu_n(\frac{1}{2})$ is obtained. However, this algebra is an algebra of level one.

Therefore, we assume that $(p, q) \neq (0, 0)$. Taking the degeneration

$$g_t : g_t(e) = e, \qquad g_t(x_i) = t^{-1} x_i, \qquad g_t(y_j) = t^{-1} y_j, \qquad g_t(z_k) = t^{-1} z_k,$$

we easily obtain that any Jordan algebra J with condition of non-triviality of semi-simple part is degenerated to the following algebra

$$e \cdot e = e, \qquad e \cdot x_i = x_i, \qquad e \cdot y_j = 0, \qquad e \cdot z_k = \frac{1}{2} z_k, \quad 1 \leq i \leq p,\ 1 \leq j \leq q,\ 1 \leq k \leq r.$$

- If $p = r = 0$, then we obtain the algebra J_1.
- If $q = r = 0$, then we get the algebra J_2.

- If two of the $p,\ q,\ r$ are non-zero, then denoting by $e_1 = e$ and elements $\{x_i, y_j, z_k\}$ by elements $e_i,\ 2 \le i \le n$, we rewrite the table of multiplication as follows:

$$J(\zeta_i):\ e_1 \cdot e_1 = e_1, \qquad e_1 \cdot e_i = \zeta_i e_i, \quad 2 \le i \le n,$$

where $\zeta_i \in \{0; \frac{1}{2}; 1\}$ and there exist i, j such that $\zeta_i \ne \zeta_j$. Without loss of generality, one can suppose $\zeta_2 \ne \zeta_3$. Taking the degeneration g_t defined as

$$g_t^{-1}: \begin{cases} g_t^{-1}(e_1) = te_1, & g_t^{-1}(e_2) = e_2 + e_3, \\ g_t^{-1}(e_3) = t(\zeta_2 e_2 + \zeta_3 e_3), & g_t^{-1}(e_i) = e_i, \qquad 4 \le i \le n, \end{cases}$$

we obtain that the algebra $J(\zeta_i)$ degenerates to J_3.

Now we consider the case of $J_{ss} = 0$, i.e., the Jordan algebra J is nilpotent.

Case 1. Let $dim J^2 \ \ge\ 2$. Then the algebra J admits a basis $\{x_1, x_2, \dots, x_n\}$ such that $\{x_1, x_2, \dots, x_k\} \ \in\ J \setminus J^2$ and $x_{k+1}, x_{k+2} \in J^2$. Moreover, one can assume $x_1 \in J \setminus J^2$ and $x_1 \cdot x_1 = x_{k+1} \in J^2 \setminus J^3$.

Case 1.1. Let $dim(J^2/J^3) \ge 2$. Then $x_{k+2} \in J^2 \setminus J^3$ and we can suppose $x_1 \cdot x_2 = x_{k+2}$.

Indeed, if there exists some i such that $x_1 \cdot x_i \notin span\langle x_{k+1}\rangle$, then without loss of generality, we can suppose $i = 2$ and derive $x_1 \cdot x_2 = x_{k+2}$.

Let now $x_1 \cdot x_i \in span\langle x_{k+1}\rangle$ for any i. We set $x_1 \cdot x_i = \alpha_i x_{k+1},\ 2 \le i \le k$. The condition $x_{k+2} \in J^2 \setminus J^3$ implies the existence of $j,\ 2 \le j \le k$ such that $x_j \cdot x_j = x_{k+2}$. Without loss of generality, one can assume $j = 2$. Hence, we obtain the products

$$x_1 \cdot x_1 = x_{k+1}, \qquad x_1 \cdot x_2 = \alpha_2 x_{k+1}, \qquad x_2 \cdot x_2 = x_{k+2}.$$

Taking the change of basis

$$x_1' = x_1 + Ax_2, \qquad x_2' = x_2, \qquad x_{k+1}' = (1 + 2A\alpha_2)x_{k+1} + A^2 x_{k+2}, \qquad x_{k+2}' = \alpha_2 x_{k+1} + Ax_{k+2},$$

where $A(1 + A\alpha_2) \ne 0$, we derive

$$x_1' \cdot x_1' = x_{k+1}', \qquad x_1' \cdot x_2' = x_{k+2}'.$$

Therefore, in this subcase we have shown that there exists a basis $\{x_1, x_2, \dots, x_{k+1}, x_{k+2} \dots, x_n\}$ such that

$$x_1 \cdot x_1 = x_{k+1}, \qquad x_1 \cdot x_2 = x_{k+2}, \qquad x_2 \cdot x_2 = \gamma_{k+1}x_{k+1} + \gamma_{k+2}x_{k+2} + \dots + \gamma_n x_n.$$

Taking the degeneration

$$g_t: \begin{cases} g_t(x_1) = t^{-2}x_1, & g_t(x_2) = t^{-2}x_2, \\ g_t(x_{k+2}) = t^{-4}x_{k+2}, & g_t(x_i) = t^{-3}x_i, \quad i \ne k+2,\ 3 \le i \le n, \end{cases}$$

we obtain that the algebra J degenerates to the algebra with the following table of multiplication:

$$x_1 \cdot x_2 = x_{k+2}, \qquad x_2 \cdot x_2 = \gamma_{k+2}x_{k+2}.$$

Obviously, this algebra is isomorphic to the algebra J_3 (by the basis transformation $x_2' := x_2 - \gamma_{k+2}x_1$ and $x_i' := x_i$ for $i \ne 2$).

Case 1.2. Let $dim(J^2/J^3) = 1$. Then $x_{k+2} \in J^3$. If there exist $i,\ j$ such that $x_1 \cdot x_i \notin span\langle x_{k+1}\rangle$ or $x_i \cdot x_j \notin span\langle x_{k+1}\rangle$, then similarly to Case 1.1 we conclude that the algebra J degenerates to the algebra J_3. Now we consider the case of $x_1 \cdot x_i,\ x_i \cdot x_j \in span\langle x_{k+1}\rangle$.

We set

$$x_1 \cdot x_i = \alpha_{1,i}x_{k+1}, \qquad x_i \cdot x_j \in \alpha_{i,j}x_{k+1}, \quad 2 \le i, j \le k.$$

Due to $x_{k+2} \in J^3$, we get the existence of some i_0 $(1 \le i_0 \le k+1)$ such that $x_{i_0} \cdot x_{k+1} = x_{k+2}$. Without loss of generality, we can assume $i_0 = 1$. Indeed, if $x_1 \cdot x_{k+1} = 0$, then taking the change

$$x_1' = x_1 + Ax_{i_0}, \qquad x_{k+1}' = (1 + 2A\alpha_{1,i_0} + A^2\alpha_{i_0,i_0})x_{k+1}, \qquad x_{k+2}' = (1 + 2A\alpha_{1,i_0} + A^2\alpha_{i_0,i_0})Ax_{i_0}x_{k+1},$$

we obtain

$$x_1 \cdot x_1 = x_{k+1}, \qquad x_1 \cdot x_{k+1} = x_{k+2}, \qquad x_{k+1} \cdot x_{k+1} = \gamma_{k+2}x_{k+2} + \dots + \gamma_n x_n.$$

Taking the degeneration

$$g_t: \begin{cases} g_t(x_1) = t^{-2}x_1, & g_t(x_i) = t^{-3}x_i, \qquad 2 \le i \le n,\ i \ne k+1; k+2, \\ g_t(x_{k+1}) = t^{-2}x_{k+1}, & g_t(x_{k+2}) = t^{-4}x_{k+2}, \end{cases}$$

we conclude that the algebra J degenerates to the algebra with the following table of multiplication:

$$x_1 \cdot x_{k+1} = x_{k+2}, \qquad x_{k+1} \cdot x_{k+1} = \gamma_{k+2}x_{k+2},$$

which is isomorphic to J_3.

Case 2. Let $dim J^2 = 1$. Then $J^3 = 0$ and either J has a three-dimensional indecomposable subalgebra $\widetilde{J}$ with conditions $dim\widetilde{J}^2 = 1$, $\widetilde{J}^3 = 0$ or J is isomorphic to the algebra $\lambda_2 \oplus \mathbb{C}^{n-2}$. Taking into account that J is not isomorphic to $\lambda_2 \oplus \mathbb{C}^{n-2}$ and that any three-dimensional indecomposable Jordan algebra satisfying the above conditions is isomorphic to the algebra: $x_1 \cdot x_2 = x_3$ (in notation of [10] this algebra is T_4), we conclude that the Jordan algebra J admits a basis $\{x_1, x_2, \dots, x_n\}$ such that the table of multiplication in this basis is as follows:

$$x_1 \cdot x_2 = x_n, \qquad x_1 \cdot x_i = \alpha_i x_n, \qquad x_2 \cdot x_i = \beta_i x_n, \qquad x_j \cdot x_i = \gamma_{i,j} x_n, \quad 2 \le i, j \le n.$$

Taking the following degeneration

$$g_t: \quad g_t(x_1) = x_1, \qquad g_t(x_2) = x_2, \qquad g_t(x_n) = x_n, \qquad g_t(x_i) = t^{-1}x_i, \quad 3 \le i \le n-1,$$

we obtain that the algebra J degenerates to J_3.

In order to complete the proof of the theorem we need to establish that the algebras J_1, J_2 and J_3 do not degenerate to each other. For this purpose we shall apply invariant argumentations.

Due to nilpotency of J_3 we have $J_1, J_2 \notin \overline{Orb(J_3)}$. Computing of dimensions of the spaces of derivations we get

$$dim(Der(J_1)) = n^2 - 2n + 1, \qquad dim(Der(J_2)) = n^2 - 2n + 1, \qquad dim(Der(J_3)) = n^2 - 3n + 4.$$

Since $dim(Der(J_1)) = dim(Der(J_2)) \ge dim(Der(J_3))$ we obtain that $J_2, J_3 \notin \overline{Orb(J_1)}$ and $J_1, J_3 \notin \overline{Orb(J_2)}$. $\square$

Remark 3.2. Note that in the variety of 2-dimensional Jordan algebras, the algebras of level two are J_1 and J_2.

3.2. Lie algebras of level two

In this subsection we will describe algebras of level two in the varieties of complex n-dimensional Lie and associative algebras.

We denote by $Lie_n(\mathbb{C})$ the variety of n-dimensional complex Lie algebras.

Thanks to work [8] we have the lists of algebras of level two in the varieties $Lie_3(\mathbb{C})$ and $Lie_4(\mathbb{C})$. Namely, we can state the next proposition.

Proposition 3.3. *Algebras of level two of the variety $Lie_3(\mathbb{C})$ up to isomorphism are the following:*

$$\begin{aligned} r_2 \oplus \mathfrak{a}_1: &\quad [e_1, e_2] = e_2, \\ r_3(\alpha): &\quad [e_1, e_2] = e_2, \quad [e_1, e_3] = \alpha e_3, \quad |\alpha| < 1, \ or\ \alpha = \pm 1. \end{aligned}$$

Algebras of level two of the variety $Lie_4(\mathbb{C})$ up to isomorphism are the following:

$$\begin{array}{rlll} n_4: & [e_1, e_2] = e_3, & [e_1, e_3] = e_4, & \\ r_2 \oplus \mathfrak{a}_2: & [e_1, e_2] = e_2, & & \\ r_3(1) \oplus \mathfrak{a}_1: & [e_1, e_2] = e_2, & [e_1, e_3] = e_3, & \\ \mathfrak{g}_{4,1}(\alpha): & [e_1, e_2] = \alpha e_2, & [e_1, e_3] = e_3, & [e_1, e_4] = e_4, \quad \alpha \ne 1, \ \alpha \in \mathbb{C}^*, \\ \mathfrak{g}_{4,2}: & [e_1, e_2] = e_2 + e_3, & [e_1, e_3] = e_3, & [e_1, e_4] = e_4. \end{array}$$

We consider Lie algebras

$$\begin{array}{rlll} n_{5,1} \oplus \mathfrak{a}_{n-5}: & [e_1, e_3] = e_5, & [e_2, e_4] = e_5, & \\ n_{5,2} \oplus \mathfrak{a}_{n-5}: & [e_1, e_2] = e_4, & [e_1, e_3] = e_5, & \\ r_2 \oplus \mathfrak{a}_{n-2}: & [e_1, e_2] = e_2, & & \\ \mathfrak{g}_{n,1}(\alpha): & [e_1, e_2] = \alpha e_2, & [e_1, e_i] = e_i, & 3 \le i \le n, \ \alpha \ne 1, \ \alpha \in \mathbb{C}^* \\ \mathfrak{g}_{n,2}: & [e_1, e_2] = e_2 + e_3, & [e_1, e_i] = e_i, & 3 \le i \le n. \end{array}$$

Further we shall need the following lemma.

Lemma 3.4.

$$\begin{aligned} dim(Der(n_{5,1} \oplus \mathfrak{a}_{n-5})) &= n^2 - 5n + 15, & dim(ab(n_{5,1} \oplus \mathfrak{a}_{n-5})) &= n - 2, \\ dim(Der(n_{5,2} \oplus \mathfrak{a}_{n-5})) &= n^2 - 5n + 13, & dim(ab(n_{5,2} \oplus \mathfrak{a}_{n-5})) &= n - 1, \\ dim(Der(r_2 \oplus \mathfrak{a}_{n-2})) &= n^2 - 3n + 4, & dim(ab(r_2 \oplus \mathfrak{a}_{n-2})) &= n - 1, \\ dim(Der(\mathfrak{g}_{n,1}(\alpha))) &= n^2 - 3n + 4, & dim(ab(\mathfrak{g}_{n,1}(\alpha))) &= n - 1, \\ dim(Der(\mathfrak{g}_{n,2})) &= n^2 - 3n + 4, & dim(ab(\mathfrak{g}_{n,2})) &= n - 1, \end{aligned}$$

where $ab(G)$ is a maximal abelian ideal of G.

In the following theorem we present a complete list of algebras of level two in the variety $Lie_n(\mathbb{C})$, $n \ge 5$.

Theorem 3.5. *An arbitrary n $(n \geq 5)$-dimensional Lie algebra of level two is isomorphic to one of the following pairwise non-isomorphic algebras:*

$$n_{5,1} \oplus \mathfrak{a}_{n-5}, \qquad n_{5,2} \oplus \mathfrak{a}_{n-5}, \qquad r_2 \oplus \mathfrak{a}_{n-2} \qquad g_{n,1}(\alpha),\ g_{n,2}.$$

Proof. I. Firstly, we consider where G is a nilpotent algebra. We distinguish the following cases.

Case 1. Let $dim G^2 = 1$. Then G is isomorphic to either Heisenberg algebra $H_{n=2k+1}$ or $H_{2k+1} \oplus \mathfrak{a}_{n-2k-1}$. Thus, there exists a basis $\{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k, z, p_1, \ldots, p_{n-2k-1}\}$ of G such that $[x_i, y_i] = z,\ 1 \leq i \leq k$.

Clearly, $k \geq 2$ because otherwise G is an algebra of level one. Taking the degeneration

$$g_t : \begin{cases} g_t(x_1) = x_1, \quad g_t(x_2) = x_2, \quad g_t(x_i) = t^{-1}x_i, \quad 3 \leq i \leq k, \\ g_t(y_1) = y_1, \quad g_t(y_2) = y_2, \quad g_t(y_i) = t^{-1}y_i, \quad 3 \leq i \leq k, \\ g_t(z) = z, \end{cases}$$

we obtain that the algebra G degenerates to $n_{5,1} \oplus \mathfrak{a}_{n-5}$.

Case 2. Let $dim G^2 \geq 2$. We suppose that $\{x_1, x_2, \ldots, x_k\}$ are generator basis elements of G. Then, without loss of generality, we can assume $[x_1, x_2] = x_{k+1}$.

Below, we show that it may always be assumed

$$[x_1, x_2] = x_4, \qquad [x_1, x_3] = x_5.$$

- Let there exist i_0 such that $[x_1, x_{i_0}] \notin span\langle x_{k+1}\rangle$. Then taking
$$x_1' = x_1, \qquad x_2' = x_2, \qquad x_3' = x_{i_0}, \qquad x_4' = x_{k+1}, \qquad x_5' = [x_1, x_{i_0}]$$
we obtain $[x_1', x_2'] = x_4',\ [x_1', x_3'] = x_5'$.
- Let $[x_1, x_i] \in span\langle x_{k+1}\rangle$ for all $3 \leq i \leq k$ and there exists some i_0 such that $[x_2, x_{i_0}] \notin span\langle x_{k+1}\rangle$. Due to symmetricity of x_1 and x_2, similarly to the previous case we can choose a basis $\{x_1', x_2', \ldots, x_n'\}$ with condition $[x_1', x_2'] = x_4',\ [x_1', x_3'] = x_5'$.
- Let $[x_1, x_i],\ [x_2, x_i] \in span\langle x_{k+1}\rangle$ for all $3 \leq i \leq k$. We set $[x_1, x_i] = \alpha_i x_{k+1}$ and $[x_2, x_i] = \beta_i x_{k+1}$. Let x_{i_0} and x_{j_0} be generators of G such that $[x_{i_0}, x_{j_0}] \notin span\langle x_{k+1}\rangle$. Since $dim G^2 \geq 2$ one can assume $[x_{i_0}, x_{j_0}] = x_{k+2}$.
Putting
$$x_1' = x_1 + Ax_{i_0}, \qquad x_2' = x_2, \qquad x_3' = x_{j_0}, \qquad x_4' = (1 - A\beta_{i_0})x_{k+1}, \qquad x_5' = Ax_{k+2} + \alpha_{i_0}x_{k+1}$$
with $A(1 - A\beta_{i_0}) \neq 0$, we deduce $[x_1', x_2'] = x_4',\ [x_1', x_3'] = x_5'$.
- Let $[x_i, x_j] \in span\langle x_{k+1}\rangle$ for all $1 \leq i, j \leq k$. Then for some i_0 we have $[x_{i_0}, x_{k+1}] \neq 0$. Without loss of generality, one can assume $[x_1, x_{k+1}] = x_{k+2}$.
  - If $k \geq 3$, then setting
  $$x_1' = x_1, \qquad x_2' = x_2, \qquad x_3' = x_3 + x_{k+1}, \qquad x_4' = x_{k+1}, \qquad x_5' = x_{k+2} + \alpha_{1,3}x_{k+1},$$
  we obtain $[x_1', x_2'] = x_4',\ [x_1', x_3'] = x_5'$.
  - If $k = 2$, then we have $[x_1, x_2] = x_3,\ [x_1, x_3] = x_4$. It is not difficult to obtain that $[x_1, x_4] = x_5$ or $[x_2, x_3] = x_5$ (because of $n \geq 5$). Indeed, taking
  $$x_1' = x_1, \qquad x_2' = x_2, \qquad x_3' = x_4, \qquad x_4' = x_3, \qquad x_5' = x_5$$
  in the case of $[x_1, x_4] = x_5$ and
  $$x_1' = -x_3, \qquad x_2' = x_1, \qquad x_3' = x_4, \qquad x_4' = x_2, \qquad x_5' = x_5$$
  in the case of $[x_2, x_3] = x_5$, we derive the products $[x_1, x_2] = x_4,\ [x_1, x_3] = x_5$.

Thus, there exists a basis $\{x_1, x_2, x_3, \ldots, x_n\}$ of G with the products

$$[x_1, x_2] = x_4, \qquad [x_1, x_3] = x_5.$$

Note that G degenerates to the algebra with multiplication:

$$[x_1, x_2] = x_4, \qquad [x_1, x_3] = x_5, \qquad [x_2, x_3] = \gamma_4 x_4 + \gamma_4 x_5$$

via the following degeneration:

$$g_t : \begin{cases} g_t(x_1) = t^{-2}x_1, \quad g_t(x_2) = t^{-2}x_2, \quad g_t(x_3) = t^{-2}x_3, \\ g_t(x_4) = t^{-4}x_4, \quad g_t(x_5) = t^{-4}x_5, \quad g_t(x_i) = t^{-3}x_i, \quad 6 \leq i \leq n. \end{cases}$$

From the change of basis $x_2' = x_2 - \gamma_5 x_1, x_3' = x_3 + \gamma_4 x_1$, we obtain that this algebra is isomorphic to $n_{5,2} \oplus \mathfrak{a}_{n-5}$.

II. Let G be a solvable Lie algebra with nilradical N. Since the nilradical N degenerates to the abelian algebra, we conclude that any solvable Lie algebra degenerates to the solvable algebra with abelian nilradical. Therefore, one can assume that G is a solvable Lie algebra with abelian nilradical.

Moreover, if $codim N \geq 2$, then choosing a basis $\{x_1, x_2, x_3, \ldots, x_n\}$ such that $\{x_1, x_2, \ldots, x_k\}$ is a basis of the complementary space to N and taking the degeneration

$$g_t(x_1) = x_1, \qquad g_t(x_2) = t^{-1}x_2, \ldots, \qquad g_t(x_k) = t^{-1}x_k, \qquad g_t(x_{k+1}) = x_{k+1}, \ldots, g_t(x_n) = x_n,$$

we obtain that G degenerates to a solvable Lie algebra with nilradical of codimension equal to 1.

Therefore, we assume that the algebra G admits a basis $\{x_1, x_2, \dots, x_n\}$ with nilradical $N = \{x_2, x_3, \dots, x_n\}$ and the restriction of the operator $ad(x_1)$ on N has the Jordan normal form $ad(x_1)_{|N} = (J_{k_1}, J_{k_2}, \dots J_{k_s})$.

It is easy to see that, if the operator $ad(x_1)_{|N}$ is a scalar matrix, that is, $ad(x_1)_{|N}$ has a unique eigenvalue and $k_i = 1$ for all i $(1 \le i \le s)$, then G is an algebra of level one (namely, $G \cong p_n^-$).

Let the operator $ad(x_1)_{|N}$ have a unique eigenvalue, but there exists a Jordan block of order greater than one. One can assume $k_1 \ge 2$. Taking the degeneration

$$g_t : \begin{cases} g_t(x_1) = x_1, & g_t(x_2) = t^{2-k_1}x_2, \\ g_t(x_i) = t^{i-1-k_1}x_i, & 3 \le i \le k_1 + 1, \\ g_t(x_{k_1+\cdots+k_{j-1}+i}) = t^{i-1-k_j}x_{k_1+\cdots+k_{j-1}+i}, & 2 \le j \le s, \qquad 2 \le i \le k_j + 1, \end{cases}$$

we obtain that G degenerates to the algebra $g_{n,2}$.

Let the operator $ad(x_1)_{|N}$ have different eigenvalues. Taking the following degeneration:

$$g_t : g_t(x_1) = x_1, \qquad g_t(x_{k_1+\cdots+k_{j-1}+i}) = t^{i-1-k_j}x_{k_1+\cdots+k_{j-1}+i}, \quad 1 \le j \le s,\ 2 \le i \le k_j + 1,$$

we conclude that the algebra G degenerates to an algebra of the family:

$$g_{n,1}(\alpha_3, \dots \alpha_n) : \quad [x_1, x_2] = x_2, \quad [x_1, x_i] = \alpha_i x_i, \quad 3 \le i \le n,\ (\alpha_3, \dots \alpha_n) \ne (1, \dots, 1).$$

Note that $g_{n,1}(0, 0, \dots, 0)$ is isomorphic to the algebra $r_2 \oplus \mathfrak{a}_{n-2}$ and the algebras $g_{n,1}(1, \dots, 1, \alpha_j, 1, \dots, 1)$ with $\alpha_j \ne 1$ and $g_{n,1}(\alpha, \alpha, \dots \alpha)$ with $\alpha \ne 1$ are isomorphic to $g_{n,1}(\alpha)$, via change of the basis $x_1' = \frac{1}{\alpha}x_1, x_i' = x_i, 2 \le i \le n$.

For the remaining cases of parameters α_i we can assume that $\alpha_3 \ne 1$ and $\alpha_4 \ne \alpha_5$.

Making the basis transformation

$$e_1 = x_1, \qquad e_2 = x_2 + x_3, \qquad e_3 = x_2 + \alpha_3 e_3, \qquad e_4 = x_4 + x_5, \qquad e_5 = \alpha_4 x_4 + \alpha_5 x_5, \qquad e_i = x_i, \quad 6 \le i \le n,$$

we get the multiplication

$$\begin{aligned} &[e_1, e_2] = e_3, \quad [e_1, e_3] = -\alpha_3 e_2 + (1 + \alpha_3)e_3, \\ &[e_1, e_4] = e_5, \quad [e_1, e_5] = -\alpha_4\alpha_5 e_4 + (\alpha_4 + \alpha_5)e_5, \quad [e_1, e_i] = e_i, \quad 6 \le i \le n. \end{aligned}$$

Similarly to the nilpotent case, G degenerates to the algebra $n_{5,2} \oplus \mathfrak{a}_{n-5}$ via the degeneration

$$g_t : \begin{cases} g_t(x_1) = t^{-1}x_1, & g_t(x_2) = t^{-1}x_2, & g_t(x_3) = t^{-2}x_3, \\ g_t(x_4) = t^{-1}x_4, & g_t(x_4) = t^{-2}x_5, & g_t(x_i) = x_i, \qquad 6 \le i \le n. \end{cases}$$

III. Let us suppose that G has not-trivial semi-simple part. Due to Levi's decomposition we have $G = (S_1 \oplus \cdots \oplus S_k) \dot{+} R$, where S_i are simple Lie ideals and R is solvable radical. From the classical theory of Lie algebras [15] we know that any simple Lie algebra S has root decomposition with respect to regular element x. Namely we have

$$S = S_0 \oplus S_\alpha \oplus S_{-\alpha} \oplus S_\beta \oplus S_{-\beta} \oplus \cdots \oplus S_\gamma \oplus S_{-\gamma}, \quad x \in S_0.$$

Let $\{x_1, x_2, \dots, x_n\}$ be a basis such that $x_1 = x$, $x_2 \in S_\alpha$ and $x_3 \in S_{-\alpha}$ with $\alpha \ne 0$. Then $[x_1, x_2] = \alpha x_2$ and $[x_1, x_3] = -\alpha x_3$. By scaling of basis elements we can assume that $\alpha = 1$.

Taking the degeneration

$$g_t(x_1) = x_1, \qquad g_t(x_i) = t^{-1}x_i, \quad 2 \le i \le n,$$

we obtain that G is degenerated to the following algebra:

$$[x_1, x_2] = x_2, \qquad [x_1, x_3] = -x_3, \qquad [x_1, x_i] \in lin\langle x_2, x_3, \dots, x_n\rangle.$$

Obviously, this solvable algebra is not an algebra of level one (from Case **II**).

Hence, any Lie algebra G with non-trivial semi-simple part has not level two.

Thus, we have proved that any Lie algebra, which is not level one, degenerates to one of the algebras:

$$n_{5,1} \oplus \mathfrak{a}_{n-5}, \qquad n_{5,2} \oplus \mathfrak{a}_{n-5}, \qquad r_2 \oplus \mathfrak{a}_{n-2}, \qquad g_{n,1}(\alpha), \quad g_{n,2}.$$

Taking into account that the property $\lambda \to \mu$ implies $dimDer(\lambda) < dimDer(\mu)$ and $dimab(\lambda) < dimab(\mu)$ and Lemma 3.4 we conclude that these algebras do not degenerate to each other. □

Applying similar techniques as in the proof of Theorems 3.1 and 3.5 we obtain the list of n-dimensional associative algebras of level two.

Theorem 3.6. *Any n-dimensional associative algebra of level two is isomorphic to one of the following algebras:*

$$\begin{array}{rllll} A_1: & e\cdot e=e; & & & \\ A_2: & e_1\cdot e_1=e_1, & e_1\cdot e_i=e_i, & e_i\cdot e_1=e_i, & 2\le i\le n; \\ A_3: & e_1\cdot e_1=e_1, & e_1\cdot e_i=e_i, & & 2\le i\le n; \\ A_4: & e_1\cdot e_1=e_1, & e_i\cdot e_1=e_i, & & 2\le i\le n; \\ A_5(\alpha): & e_2\cdot e_1=e_3, & e_1\cdot e_2=\alpha e_3, & & \alpha\in\{|z|<1\}\cup\{|z|=1, Im(z)\ge 0\}; \\ A_6: & e_1\cdot e_1=e_3, & e_2\cdot e_1=e_3, & e_1\cdot e_2=-e_3. & \end{array}$$

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