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# FEW REMARKS ON EVOLUTION ALGEBRAS

ABROR KH. KHUDOYBERDIYEV, BAKHROM A. OMIROV, IZZAT QARALLEH

**ABSTRACT.** In the present paper we study some algebraic properties of evolution algebras. Moreover, we reduce the study of evolution algebras of permutations to two special types of evolution algebras, idempotents and absolute nilpotent elements of the algebra. We study three-dimensional evolution algebras whose each element of evolution basis has infinite period. In addition, for an evolution algebra with some properties we describe its associative enveloping algebra.

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*Key Words and Phrases:* evolution algebra, algebra of permutations, absolute nilpotent element, idempotent, algebra of multiplications, associative enveloping algebra.

## 1. INTRODUCTION.

In 20s and 30s of the last century the new object was introduced to mathematics, which was the product of interactions between Mendelian genetics and mathematics. One of the first scientist who give an algebraic interpretation of the " $\times$ " sign, which indicated sexual reproduction was Serebrowsky.

It is known that there exists an intrinsic and general mathematical structure behind the neutral Wright-Fisher models in population genetics, the reproduction of bacteria involved by bacteriophages, asexual reproduction or generally non-Mendelian inheritance and Markov chains. In [10] a new type of algebras was associated with it – the evolution algebras.

Although an evolution algebra is an abstract system, it gives an insight for the study of non-Mendelian genetics. For instance, an evolution algebra can be applied to the inheritance of organelle genes, one can predict, in particular, all possible mechanisms to establish the homoplasmy of cell populations.

The general genetic algebras developed into a field of independent mathematical interest, because these algebras are in general non-associative and do not belong to any of the well-known classes of non-associative algebras such as Lie algebras, alternative algebras, or Jordan algebras.

Until 1980s, the most comprehensive reference in this area was Wörz-Busekros's book [11]. More recent results, such as genetic evolution in genetic algebras, can be found in Lyubich's book [7]. A good survey is Reed's article [8]. In Tian's book [10] a foundation of the framework of the theory of evolution algebras is established and some applications of evolution algebras in the theory of stochastic processes and genetics are discussed. Recently, Rozikov and Tian [9] studied algebraic structures evolution algebras associated with Gibbs measures defined on some graphs. In [2], [5], [6] derivations, some properties of chain of evolution algebras and dibaricity of evolution algebras were studied. In [1], [3], [4] certain algebraic properties of evolution algebras (like right nilpotency, nilpotency and solvability etc.) in terms of matrix of structural constants have been investigated. In fact, nilpotency, right nilpotency and solvability might be interpreted in a biological way as a various types of vanishing ("deaths") populations.

The present paper is organized as follows: In Section 2 we give some definitions and preliminary results. In Section 3 we reduce the study of arbitrary evolution algebra of permutations into two special evolution algebras. Section 4 is devoted to the description of  $n$ -dimensional associative enveloping algebras of  $n$ -dimensional evolution algebras with some restrictions on *rank* of the matrix  $A$  of structural constants. Moreover, associative enveloping algebras for 2-dimensional evolution algebras are described, as well. In Section 5 we establish some properties of three-dimensional evolution algebras whose each basis element has infinite period.

Throughout the paper we consider finite-dimensional evolution algebras over a field of zero characteristic. Moreover, in the multiplication table of an evolution algebra the omitted products are assumed to be zero.

## 2. PRELIMINARIES.

Let us define the main object of this work - evolution algebra.

**Definition 2.1.** [10] Let  $(E, \cdot)$  be an algebra over a field  $F$ . If it admits a basis  $\{e_1, e_2, \dots\}$  such that

$$e_i \cdot e_j = 0, \quad \text{for } i \neq j, \quad e_i \cdot e_i = \sum_k a_{i,k} e_k, \quad \text{for any } i,$$

then this algebra is called *evolution algebra*.

It is remarkable that this type of algebra depends on evolution basis  $\{e_1, e_2, \dots\}$ .

In the following theorem we present the list (up to isomorphism) of 2-dimensional complex evolution algebras.

**Theorem 2.2.** [4] Any 2-dimensional non abelian complex evolution algebra  $E$  is isomorphic to one of the following pairwise non isomorphic algebras:

- (1)  $\dim E^2 = 1$ 
  - $E_1$  :  $e_1 e_1 = e_1$ ,
  - $E_2$  :  $e_1 e_1 = e_1, \quad e_2 e_2 = e_1$ ,
  - $E_3$  :  $e_1 e_1 = e_1 + e_2, \quad e_2 e_2 = -e_1 - e_2$ ,
  - $E_4$  :  $e_1 e_1 = e_2$ .
- (2)  $\dim E^2 = 2$ 
  - $E_5$  :  $e_1 e_1 = e_1 + a_2 e_2, \quad e_2 e_2 = a_3 e_1 + e_2, \quad 1 - a_2 a_3 \neq 0$ , where  $E_5(a_2, a_3) \cong E'_5(a_3, a_2)$ ,
  - $E_6$  :  $e_1 e_1 = e_2, \quad e_2 e_2 = e_1 + a_4 e_2$ , where for  $a_4 \neq 0$ ,  $E_6(a_4) \cong E_6(a'_4) \Leftrightarrow \frac{a'_4}{a_4} = \cos \frac{2\pi k}{3} + i \sin \frac{2\pi k}{3}$  for some  $k = 0, 1, 2$ .

Further we shall show the role of idempotents and absolute nilpotent elements of an evolution algebra.

**Definition 2.3.** An element  $x$  of an evolution algebra  $E$  is called *idempotent*, if  $xx = x$ . An element  $y$  of an evolution algebra  $E$  is called *absolute nilpotent* if  $yy = 0$ .

Consider a complex evolution algebra  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  with a basis  $\{e_1, e_2, \dots, e_n\}$  and the table of multiplications given by

$$\begin{cases} e_i \cdot e_i = a_i e_{\pi(i)}, & 1 \leq i \leq n, \\ e_i \cdot e_j = 0, & i \neq j, \end{cases}$$

where  $\pi$  is an element of the group of permutations  $S_n$ .

An evolution algebra  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  is said to be *evolution algebra of permutations*.

In what follows, by a *cycle permutation* we mean a permutation in which a part of symbols  $\{l_1, l_2, \dots, l_t\} \subseteq \{1, 2, \dots, n\}$  are cyclic permuted and the rest ones are stationary, i.e.,  $l_1 \rightarrow l_2 \rightarrow \dots \rightarrow l_t \rightarrow l_1$ , and we denote  $\pi = (l_1, l_2, \dots, l_t)$ .

It is known that any permutation is up to order uniquely decomposed into product of independent cycles.

For permutations of the form  $\pi = (l_1, l_2, \dots, l_r)(m_1, m_2, \dots, m_s) \dots (p_1, p_2, \dots, p_t)$  it is known the following result.

**Proposition 2.4.** Two permutations are conjugated in  $S_n$  if and only if the corresponding sets  $\{r, s, \dots, t\}$  are coincided.

**Definition 2.5.** [10] An evolution algebra  $E$  with a table of multiplications

$$e_i \cdot e_i = \sum_k a_{i,k} e_k, \quad a_i e_j = 0, \quad i \neq j$$

is called *Markov evolution algebra* if  $\sum_k a_{i,k} = 1$ .

For a given element  $x$  of an evolution algebra  $E$  we consider the right multiplication operator  $R_x : E \rightarrow E$  defined by  $R_x(y) = yx$ ,  $y \in E$ .

Note that operators of right and left multiplications are coincided, since evolution algebras are commutative.

For an evolution algebra  $E$ , by  $M(E)$  we denote an *associative enveloping algebra* which is generated by the set  $R(E) = \{R_x \mid x \in E\}$ . It is clear that  $M(E)$  is a subalgebra of  $\text{End}(E)$ .

For an element  $x \in E$  we define plenary powers as follows:

$$x^{[1]} = x, \quad x^{[k+1]} = x^{[k]} \cdot x^{[k]}, \quad k \geq 1.$$

**Definition 2.6.** Let  $e_j$  be a generator of an evolution algebra  $E$ , the period  $d$  of  $e_j$  is defined to be the greatest common divisor of the set  $\{\log_2 m \mid e_j < e_j^{[m]}\}$ . That is

$$d = \text{g.c.d.}\{\log_2 m \mid e_j < e_j^{[m]}\}.$$

### 3. EVOLUTION ALGEBRA OF PERMUTATIONS

Let us first present two important examples of evolution algebra of permutations.

**Example 3.1.** Consider the following evolution algebra:

$$E_n : \begin{cases} e_i \cdot e_i = e_{i+1}, & 1 \leq i \leq n-1, \\ e_n \cdot e_n = e_1, \\ e_i \cdot e_j = 0, & i \neq j. \end{cases}$$

Evidently, the algebra  $E_n$  is evolution algebra of permutations of the form  $E_{n,\pi}(1, 1, \dots, 1)$ , with  $\pi = (1, 2, 3, \dots, n)$ .

**Example 3.2.** Evolution algebra defined as follows:

$$EN_n : \begin{cases} e_i \cdot e_i = e_{i+1}, & 1 \leq i \leq n-1, \\ e_n \cdot e_n = 0, \\ e_i \cdot e_j = 0, & i \neq j, \end{cases}$$

is the algebra of permutations of the form  $E_{n,\pi}(1, 1, \dots, 1, 0)$  with  $\pi = (1, 2, 3, \dots, n)$ .

Note that  $E_n, EN_n$  are single-generated simple and nilpotent evolution algebras, respectively. Moreover, algebras  $E_1, EN_1$  define one-dimensional evolution algebras, whose basis elements are idempotent and absolute nilpotent elements, respectively.

Now we shall consider some properties of evolution algebra of permutations.

**Proposition 3.3.** Let  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  be an evolution algebra of permutations with the following conditions:

- (i)  $a_i \neq 0$  for all  $i$  ( $1 \leq i \leq n$ ),
- (ii)  $\pi = \pi_1 \circ \pi_2 \circ \dots \circ \pi_r$ , where  $\pi_1 = (l_1, l_2, \dots, l_{k_1})$ ,  $\pi_2 = (m_1, m_2, \dots, m_{k_2})$ ,  $\dots$ ,  $\pi_r = (p_1, p_2, \dots, p_{k_r})$  are independent cycles and  $k_1 + k_2 + \dots + k_r = n$ .

Then

$$E_{n,\pi}(a_1, a_2, \dots, a_n) \cong E_{k_1,\pi_1}(b_1, b_2, \dots, b_{k_1}) \oplus E_{k_2,\pi_2}(c_1, c_2, \dots, c_{k_2}) \oplus \dots \oplus E_{k_r,\pi_r}(d_1, d_2, \dots, d_{k_r}).$$

*Proof.* The isomorphism is provided by the following change of basis:

$$e_{i,1} = e_{l_i}, \quad 1 \leq i \leq k_1, \quad e_{i,2} = e_{m_i}, \quad 1 \leq i \leq k_2, \quad \dots, \quad e_{i,r} = e_{p_i}, \quad 1 \leq i \leq k_r.$$

Thus, we have the evolution algebra  $E_{k_s,\pi_s}(*, *, \dots, *)$  with the basis  $e_{i,s}$ ,  $1 \leq i \leq k_s$ ,  $1 \leq s \leq r$  and

$$E_{n,\pi}(a_1, a_2, \dots, a_n) \cong E_{k_1,\pi_1}(b_1, b_2, \dots, b_{k_1}) \oplus E_{k_2,\pi_2}(c_1, c_2, \dots, c_{k_2}) \oplus \dots \oplus E_{k_r,\pi_r}(d_1, d_2, \dots, d_{k_r})$$

for some non-zero values of  $b_i, c_j, \dots, d_s$ . □

In the following proposition we specify more details on the terms of direct sum in the statement of Proposition 3.3.

**Proposition 3.4.** Any evolution algebra of permutation  $E_{n,\tau}(a_1, a_2, \dots, a_n)$  with  $\tau = (l_1, l_2, \dots, l_n)$  and condition  $a_i \neq 0$  for all  $i$  ( $1 \leq i \leq n$ ) is isomorphic to the algebra  $E_{n,\pi}(a_1, a_{\pi(1)}, \dots, a_{\pi^{n-1}(1)})$  with  $\pi = (1, 2, \dots, n)$ .

*Proof.* The isomorphism is established by basis permutation:

$$e'_1 = e_1, \quad e'_i = e_{\pi^{i-1}(1)}, \quad 2 \leq i \leq n.$$

□

**Theorem 3.5.** Any evolution algebra of permutation  $E_{n,\tau}(a_1, a_2, \dots, a_n)$  with  $\tau = (l_1, l_2, \dots, l_n)$  and condition  $a_i \neq 0$  for all  $i$  ( $1 \leq i \leq n$ ) is isomorphic to the algebra  $E_n$ .

*Proof.* Taking into account Proposition 3.4 it is sufficient to establish isomorphism between evolution algebra  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  with  $\pi = (1, 2, \dots, n)$  and non zero values of  $a_i$ ,  $1 \leq i \leq n$  and evolution algebra  $E_n$ .

The application of the following scaling of basis:

$$e'_i = A_i e_i, \quad 1 \leq i \leq n \quad \text{with } A_i = \sqrt[2^n - 1]{\frac{1}{a_i^{2^{n-1}} a_{i+1}^{2^{n-2}} \dots a_n^{2^{i-1}} a_1^{2^{i-2}} a_2^{2^{i-3}} \dots a_{i-1}}}},$$

deduces products

$$\begin{cases} e'_i \cdot e'_i = e'_{i+1}, & 1 \leq i \leq n-1, \\ e'_n \cdot e'_n = e'_1, \\ e'_i \cdot e'_j = 0, & i \neq j. \end{cases}$$

□

Now we consider the case of  $a_i = 0$  for some  $i \in \{1, 2, \dots, n\}$ .

**Proposition 3.6.** *Any evolution algebra of permutation  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  with  $\pi = (l_1, l_2, \dots, l_n)$  and condition  $a_i = 0$  for some  $i \in \{1, 2, \dots, n\}$  is isomorphic to the algebra  $EN_{k_1} \oplus EN_{k_2} \oplus \dots \oplus EN_{k_r}$ .*

*Proof.* Similarly to the proof of Proposition 3.4 taking the change

$$e'_1 = e_1, \quad e'_i = e_{\pi^{i-1}(1)}, \quad 2 \leq i \leq n,$$

we can suppose  $\pi = (1, 2, \dots, n)$ .

Let  $a_{i_1} = a_{i_2} = \dots = a_{i_r} = 0$  for  $i_1 < i_2 < \dots < i_r$  and the rest are non-zero.

If  $i_r = n$ , then similarly as above we can assume that all  $a_i = 1$  for  $1 \leq i \leq n-1$  and hence,  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  with  $\pi = (1, 2, \dots, n)$  is isomorphic to the algebra  $E_n$ .

If  $i_r < n$ , then taking the following change of basis:

$$\begin{aligned} e_s^1 &= e_{i_r+s}, \quad 1 \leq s \leq n - i_r, \\ e_{n-i_r+s}^1 &= e_s, \quad 1 \leq s \leq i_1, \\ e_s^2 &= e_{i_1+s}, \quad 1 \leq s \leq i_2 - i_1, \\ &\dots \quad \dots \quad \dots \\ e_s^r &= e_{i_{r-1}+s}, \quad 1 \leq s \leq i_r - i_{r-1}. \end{aligned}$$

we obtain that  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  with  $\pi = (1, 2, \dots, n)$  is isomorphic to the algebra  $EN_{k_1}(a_1, \dots, a_{k_1-1}) \oplus EN_{k_2}(b_1, \dots, b_{k_2-1}) \oplus \dots \oplus EN_{k_r}(c_1, \dots, c_{k_r-1})$ , where each of the algebra  $EN_{k_s}(*, *, \dots, *)$ ,  $1 \leq s \leq r$  has the model of the following evolution algebra:

$$EN_k(a_1, \dots, a_{k-1}) : \quad \begin{cases} e_i \cdot e_i = a_i e_{i+1}, & 1 \leq i \leq k-1, \quad a_i \neq 0, \\ e_i \cdot e_j = 0, & i \neq j. \end{cases}$$

Taking the basis transformation in the algebra  $EN_k(a_1, \dots, a_{k-1})$ :

$$e'_1 = e_1, \quad e'_2 = a_1 e_2, \quad e'_3 = a_1^2 a_2 e_3, \quad \dots, \quad e'_k = a_1^{2^{k-2}} a_2^{2^{k-3}} \dots a_{k-1} e_k$$

we have that  $EN_k(a_1, \dots, a_{k-1})$  is isomorphic to the algebra  $EN_k$ , which complete the proof of proposition. □

Below we establish the isomorphism of evolution algebras of permutations with given conjugated permutations.

**Theorem 3.7.** *If permutations  $\pi_1, \pi_2 \in S_n$  are conjugated, then evolution algebras  $E_{n,\pi_1}(a_1, \dots, a_n)$  and  $E_{n,\pi_2}(a_1, \dots, a_n)$  are isomorphic.*

*Proof.* Let  $\pi_1, \pi_2 \in S_n$  are conjugated, then there exists  $g \in S_n$  such that  $g\pi_1 = \pi_2 g$ . The map  $f : E_{n,\pi_1} \rightarrow E_{n,\pi_2}$  defined by  $f(e_i) = e_{g(i)}$  is isomorphism. Indeed,

$$a_i e_{g\pi_1(i)} = f(a_i e_{\pi_1(i)}) = f(e_i \cdot e_i) = f(e_i) \cdot f(e_i) = a_i e_{g(i)} * e_{g(i)} = a_i e_{\pi_2 g(i)}.$$

□

Thus, for an algebra  $E_{n,\pi}(a_1, \dots, a_n)$  we can always assume that  $\pi = (1, 2, \dots, n)$  and table of multiplication is

$$\begin{cases} e_i \cdot e_i = a_i e_{i+1}, & 1 \leq i \leq n-1, \\ e_n \cdot e_n = a_n e_1, \\ e_i \cdot e_j = 0, & i \neq j, \end{cases}$$

where  $a_i \in \{0; 1\}$ .

**Proposition 3.8.** *An arbitrary evolution algebra  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  with  $\pi = (1, 2, \dots, n)$  is isomorphic to the algebra  $E_n$  or the direct sum of evolution algebras  $EN_{k_1} \oplus EN_{k_2} \oplus \dots \oplus EN_{k_r}$ .*

*Proof.* If all  $a_i \neq 0$ , then due to Theorem 3.5 we have that algebra  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  is isomorphic to the  $E_n$ .

Let  $a_k = 0$  for some  $k$  and  $a_i \neq 0$  for  $i \neq k$ . Taking the basis transformation in the following form:

$$\begin{aligned} e'_1 &= A_1 e_{k+1}, \quad e'_2 = A_2 e_{k+2}, \quad \dots, \quad e'_{n-k} = A_{n-k} e_n, \\ e'_{n-k+1} &= A_{n-k+1} e_1, \quad e'_{n-k+2} = A_{n-k+2} e_2, \quad \dots, \quad e'_n = A_n e_k, \end{aligned}$$

where

$$\begin{aligned} A_1 &= 1, \quad A_2 = a_{k+1}, \quad A_3 = a_{k+1}^2 a_{k+2}, \quad \dots, \quad A_{n-k+1} = a_{k+1}^{2^{n-k-1}} a_{k+2}^{2^{n-k-2}} \dots a_n, \\ A_{n-k+2} &= a_{k+1}^{2^{n-k}} a_{k+2}^{2^{n-k-1}} \dots a_n^2 a_1, \quad \dots, \quad A_n = a_{k+1}^{2^{n-2}} a_{k+2}^{2^{n-3}} \dots a_n^{2^{k-1}} a_1^{2^{k-2}} a_2^{2^{k-3}} \dots a_{k-1}, \end{aligned}$$

we derive isomorphism between algebra  $E_{n,\pi}(a_1, \dots, a_{k-1}, 0, a_{k+1}, \dots, a_n)$  and algebra  $EN_n$ .

Applying similar arguments, we can establish that for  $(r+1)$ -times of parameters  $a_i$  are equal to zero an algebra  $E_n(a_1, a_2, \dots, a_n)$  is isomorphic to

$$EN_{k_1} \oplus EN_{k_2} \oplus \dots \oplus EN_{k_r}.$$

□

We resume the above results in the main theorem of this section.

**Theorem 3.9.** *An arbitrary evolution algebra of permutations  $E_{n,\pi}(a_1, a_2, \dots, a_n)$  is isomorphic to a direct sum of algebras  $E_{p_1}, E_{p_2}, \dots, E_{p_s}, EN_{k_1}, EN_{k_2}, \dots, EN_{k_r}$ , i.e.,*

$$E_{n,\pi}(a_1, a_2, \dots, a_n) \cong E_{p_1} \oplus E_{p_2} \oplus \dots \oplus E_{p_s} \oplus EN_{k_1} \oplus EN_{k_2} \oplus \dots \oplus EN_{k_r}.$$

In the description of evolution algebras of permutations from above theorem we get the importance of algebras  $E_n, EN_n$ , idempotents and absolute nilpotent elements.

#### 4. ASSOCIATIVE ENVELOPING ALGEBRAS OF SOME EVOLUTION ALGEBRAS

For a complex two-dimensional evolution algebra  $E$  of the list of Theorem 2.2 we describe its associative enveloping algebra  $M(E)$ :

$$\begin{aligned} M(E_1) &= \text{alg} \left\langle R_{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle \cong xx = x; \\ M(E_2) &= \text{alg} \left\langle R_{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, R_{e_2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle \cong xx = x, yx = x; \\ M(E_3) &= \text{alg} \left\langle R_{e_1} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, R_{e_2} = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix} \right\rangle \cong xx = x, xy = -x, yx = y, yy = -y; \\ M(E_4) &= \text{alg} \left\langle R_{e_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle \cong xx = 0; \\ M(E_5(0, 0)) &= \text{alg} \left\langle R_{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, R_{e_2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle \cong xx = x, yy = y; \\ M(E_5(a_2, a_3))(a_2 = 0 \text{ or } a_3 = 0) &\cong \left\{ P \in M_2(\mathbb{C}) \mid P = \begin{pmatrix} b_1 & b_2 \\ 0 & b_3 \end{pmatrix} \right\}; \end{aligned}$$



The condition  $\dim M(E) = n$  implies  $t_i \neq 0$  for any  $i$  ( $1 \leq i \leq n$ ) and basis of  $M(E)$  can be chosen as  $\{R_{e_1}, R_{e_2}, \dots, R_{e_n}\}$ , where  $R_{e_i} = t_i \sum_{k=1}^n a_k e_{i,k}$ . Therefore, the table of multiplication of the algebra  $M(E)$  has the form:

$$R_{e_i} R_{e_j} = t_j a_j R_{e_i}, \quad 1 \leq i, j \leq n.$$

By appropriate shifting of basis elements, without loss of generality, one can assume  $a_i \neq 0$  for  $1 \leq i \leq s$ ,  $s \leq n$  and  $a_j = 0$  for  $s+1 \leq j \leq n$ . The scaling the basis elements  $e_i$ ,  $1 \leq i \leq s$  reduces our study to the case of  $t_i a_i = 1$ . Thus, we obtain the tables of multiplication of associative enveloping algebras  $M^s$ .  $\square$

Below we present the description of  $n$ -dimensional associative enveloping algebras  $M(E)$  for evolution algebra  $E$  with  $\text{rank} A = n$ .

**Proposition 4.3.** *Let  $\text{rank} A = n$ . Then associative enveloping algebra  $M(E)$  is isomorphic to the algebra*

$$M_1 : \quad x_i x_i = x_i, \quad 1 \leq i \leq n.$$

*Proof.* Taking into account in the equalities

$$R_{e_i} R_{e_j} = a_{i,j} \sum_{k=1}^n a_{j,k} e_{i,k} = \beta_1 R_{e_1} + \beta_2 R_{e_2} + \dots + \beta_n R_{e_n}$$

that  $\{R_{e_1}, R_{e_2}, \dots, R_{e_n}\}$  are linear independent (they are forms a basis of  $M(E)$ ), we conclude  $\beta_k = 0$  for  $k \neq i$ .

Therefore,

$$a_{i,j} \sum_{k=1}^n a_{j,k} e_{i,k} = \beta_i R_{e_i} = \beta_i \sum_{k=1}^n a_{i,k} e_{i,k}. \quad (4.2)$$

From (4.2) we derive

$$a_{i,j} a_{j,k} = \beta_i a_{i,k}, \quad 1 \leq k \leq n. \quad (4.3)$$

The condition  $\text{rank} A = n$ , implies  $\text{rank} \begin{pmatrix} a_{i,1} & a_{i,2} & \dots & a_{i,n} \\ a_{j,1} & a_{j,2} & \dots & a_{j,n} \end{pmatrix} = 2$  for any  $i \neq j$ .

Using the arbitrariness  $k$  in the equality (4.3) we obtain

$$a_{i,j} = 0, \quad i \neq j \quad \Rightarrow \quad a_{i,i} \neq 0.$$

Therefore,

$$R_{e_i} R_{e_i} = a_{i,i} R_{e_i}, \quad 1 \leq i \leq n.$$

By scaling the basis elements, we can suppose  $a_{i,i} = 1$  and the algebra  $M_1$  is obtained.  $\square$

The list of  $n$ -dimensional algebras  $M(E)$  for an evolution algebra, satisfying the condition  $\text{rank} A = n - 1$ , is presented in the following theorem.

**Theorem 4.4.** *Let  $\text{rank} A = n - 1$ . Then associative enveloping algebra  $M(E)$  is isomorphic to one of the following algebras:*

$$M_2 : \quad x_i x_i = x_i, \quad 1 \leq i \leq n, \quad x_1 x_n = x_1, \quad x_n x_1 = x_n,$$

$$M_3 : \quad x_i x_i = x_i, \quad 1 \leq i \leq n - 1, \quad x_n x_1 = x_n,$$

$$M_4 : \quad x_i x_i = x_i, \quad 1 \leq i \leq n - 1, \quad x_1 x_2 = x_n, \quad x_1 x_n = x_n, \quad x_n x_2 = x_n.$$

*Proof.* Without loss of generality, one can assume

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n} \\ \sum_{i=1}^{n-1} \alpha_i a_{i,1} & \sum_{i=1}^{n-1} \alpha_i a_{i,2} & \dots & \sum_{i=1}^{n-1} \alpha_i a_{i,n} \end{pmatrix}.$$

**Case 1.** Let  $R_{e_n} = \sum_{k=1}^n \left( \sum_{i=1}^{n-1} \alpha_i a_{i,k} \right) e_{n,k} \neq 0$ . Then  $\{R_{e_1}, R_{e_1}, \dots, R_{e_{n-1}}, R_{e_n}\}$  is a basis of  $M(E)$ . Similarly as in the proof of Proposition 4.3, from the condition

$$\text{rank} \begin{pmatrix} a_{i,1} & a_{i,2} & \dots & a_{i,n} \\ a_{j,1} & a_{j,2} & \dots & a_{j,n} \end{pmatrix} = 2,$$

for any  $1 \leq i, j \leq n-1$ ,  $i \neq j$ , we deduce

$$a_{i,j} = 0, \quad 1 \leq i, j \leq n-1, \quad i \neq j.$$

Consequently,

$$R_{e_i} = a_{i,i} e_{i,i} + a_{i,n} e_{i,n}, \quad 1 \leq i \leq n-1$$

and

$$R_{e_n} = \sum_{k=1}^{n-1} \alpha_k a_{k,k} e_{n,k} + \left( \sum_{s=1}^{n-1} \alpha_s a_{s,n} \right) e_{n,n}.$$

If  $\alpha_k a_{k,k} \neq 0$  for some  $k$  ( $1 \leq k \leq n-1$ ), then without loss of generality we can suppose  $\alpha_1 a_{1,1} \neq 0$ . Consider

$$R_{e_n} R_{e_1} = \alpha_1 a_{1,1}^2 e_{n,1} + \alpha_1 a_{1,1} a_{1,n} e_{n,n}.$$

Since the product  $R_{e_n} R_{e_i}$  should be expressed by  $R_{e_n}$ , we conclude  $\alpha_k a_{k,k} = \alpha_k a_{k,n} = 0$ ,  $2 \leq k \leq n-1$ , which yield

$$\alpha_k = 0, \quad 2 \leq k \leq n-1.$$

If  $\alpha_k a_{k,k} = 0$  for any  $k$  ( $1 \leq k \leq n-1$ ), then from the condition  $R_{e_n} \neq 0$  we have the existence some  $k_0$  such that  $\alpha_{k_0} \neq 0$ . Hence,  $a_{k_0,k_0} = 0$ . Since  $\text{rank} A = n-1$ , then  $a_{k,k} \neq 0$  for any  $k \neq k_0$ , consequently,  $\alpha_k = 0$  for any  $k \neq k_0$ . Without loss of generality, one can assume  $k_0 = 1$ .

Thus, in Case 1 we obtain

$$\alpha_k = 0, \quad 2 \leq k \leq n-1.$$

Therefore,

$$R_{e_n} R_{e_i} = 0, \quad 2 \leq i \leq n-1, \quad R_{e_1} R_{e_n} = a_{1,n} R_{e_1}.$$

Consider

$$R_{e_i} R_{e_n} = (a_{i,i} e_{i,i} + a_{i,n} e_{i,n}) (\alpha_1 a_{1,1} e_{n,1} + \alpha_1 a_{1,n} e_{n,n}) = a_{i,n} (\alpha_1 a_{1,1} e_{i,1} + \alpha_1 a_{i,n} e_{i,n}).$$

Since  $R_{e_i} R_{e_n}$  should belong to  $\langle R_{e_i} \rangle$ , we get  $a_{i,n} = 0$ ,  $2 \leq i \leq n-1$ .

Thus, the table of multiplication of the algebra  $M(E)$  has the following form:

$$R_{e_i} R_{e_i} = a_{i,i} R_{e_i}, \quad 1 \leq i \leq n-1,$$

$$R_{e_1} R_{e_n} = a_{1,n} R_{e_1}, \quad R_{e_n} R_{e_1} = a_{1,1} R_{e_n}, \quad R_{e_n} R_{e_n} = a_{1,n} R_{e_n},$$

where  $(a_{1,1}, a_{1,n}) \neq (0, 0)$  and  $a_{i,i} \neq 0$ ,  $2 \leq i \leq n$ .

Considering the possible cases:  $a_{1,1} a_{1,n} \neq 0$  and  $a_{1,1} a_{1,n} = 0$ , one finds the algebras  $M_2$ ,  $M_3$ .

**Case 2.** Let  $R_{e_n} = 0$ . Since a non-zero products of the form  $R_{e_i} R_{e_j} = a_{i,j} \sum_{k=1}^n a_{j,k} e_{i,k}$  are linear independent, we obtain the existence of a unique non-zero coefficient  $a_{i_0,j_0}$ ,  $1 \leq i_0, j_0 \leq n-1$ ,  $i_0 \neq j_0$ . Without loss of generality, we can suppose  $i_0 = 1, j_0 = 2$ , i.e.  $a_{1,2} \neq 0$ .

Therefore,

$$a_{i,j} = 0, \quad 1 \leq i, j \leq n-1, \quad i \neq j, \quad (i, j) \neq (1, 2).$$

$$a_{i,i} \neq 0, \quad 1 \leq i \leq n-1,$$

Putting  $x_i = R_{e_i}$ ,  $1 \leq i \leq n-1$  and  $x_n = a_{2,2} e_{1,2} + a_{2,n} e_{1,n}$ , we obtain the table of multiplications of the algebra  $M(E)$  in the form:

$$x_i x_i = a_{i,i} x_i, \quad 1 \leq i \leq n-1, \quad x_1 x_2 = a_{1,2} x_n, \quad x_1 x_n = a_{1,1} x_n, \quad x_n x_2 = a_{2,2} x_n.$$

Taking the change

$$x'_1 = \frac{1}{a_{1,1}} x_1, \quad 1 \leq i \leq n-1, \quad x'_n = \frac{a_{1,2}}{a_{1,1} a_{2,2}} x_n$$

we get the algebra  $M_4$ . □

## 5. THREE-DIMENSIONAL EVOLUTION ALGEBRAS WHOSE GENERATORS HAVE INFINITE PERIOD.

In this section we study three-dimensional evolution algebras whose generators have infinite period. Let  $E$  be a three-dimensional evolution algebra  $E$  with table of multiplications:

$$e_1 \cdot e_1 = a_1 e_1 + a_2 e_2 + a_3 e_3, \quad e_2 \cdot e_2 = b_1 e_1 + b_2 e_2 + b_3 e_3, \quad e_3 \cdot e_3 = c_1 e_1 + c_2 e_2 + c_3 e_3. \quad (5.1)$$

Since the periods of generators are infinite, we get  $a_1 = b_2 = c_3 = 0$ .

Consider  $e_i^{[3]}$  and  $e_i^{[4]}$  for  $1 \leq i \leq 3$

$$\begin{cases} e_1^{[3]} = (a_2^2 b_1 + a_3^2 c_1) e_1 + a_2^2 c_2 e_2 + a_2^2 b_3 e_3, \\ e_2^{[3]} = b_3^2 c_1 e_1 + (b_1^2 a_2 + b_3^2 c_2) e_2 + b_1^2 a_3 e_3, \\ e_3^{[3]} = c_2^2 b_1 e_1 + c_1^2 a_2 e_2 + (c_1^2 a_3 + c_2^2 b_3) e_3, \end{cases}$$

$$\begin{cases} e_1^{[4]} = (a_3^4 c_2^2 b_1 + a_2^4 b_3^2 c_1) e_1 + a_2^4 b_3^2 c_2 e_2 + a_3^4 c_2^2 b_3 e_3, \\ e_2^{[4]} = b_1^4 a_3^2 c_1 e_1 + (b_3^4 c_1^2 a_2 + b_1^4 a_3^2 c_2) e_2 + b_3^4 c_1^2 a_3 e_3, \\ e_3^{[4]} = c_1^4 a_2^2 b_1 e_1 + c_2^4 b_1^2 a_2 e_2 + (c_2^4 b_1^2 a_3 + c_1^4 a_2^2 b_3) e_3. \end{cases}$$

Taking account the condition on periods of generators, we derive

$$a_2^2 b_1 + a_3^2 c_1 = 0, \quad b_1^2 a_2 + b_3^2 c_2 = 0, \quad c_1^2 a_3 + c_2^2 b_3 = 0, \quad (5.2)$$

$$a_3^4 c_2^2 b_1 + a_2^4 b_3^2 c_1 = 0, \quad b_3^4 c_1^2 a_2 + b_1^4 a_3^2 c_2 = 0, \quad c_2^4 b_1^2 a_3 + c_1^4 a_2^2 b_3 = 0. \quad (5.3)$$

**Theorem 5.1.** *Let  $E$  be a three-dimensional evolution algebra with the table of multiplication (5.1). Let any basis element has an infinite period and  $a_2 a_3 b_1 b_3 c_1 c_2 = 0$ . Then  $E$  is isomorphic to the following evolution algebra:*

$$E^1 : e_1 \cdot e_1 = a_2 e_2 + a_3 e_3, \quad e_2 \cdot e_2 = b_3 e_3.$$

*Proof.* Let  $a_2 a_3 b_1 b_3 c_1 c_2 = 0$ , then, without loss of generality, we can assume  $b_1 = 0$ . The equalities (5.2) and (5.3) imply

$$a_3 c_1 = 0, \quad b_3 c_2 = 0, \quad a_2 b_3 c_1 = 0. \quad (5.4)$$

**Case 1.** Let  $a_3 = b_3 = 0$ . Then we obtain products

$$e_1 \cdot e_1 = a_2 e_2, \quad e_3 \cdot e_3 = c_1 e_1 + c_2 e_2.$$

Taking the change  $e'_1 = e_3, e'_2 = e_1, e'_3 = e_2$ , we derive that this algebra is isomorphic to the algebra  $E^1$ .

**Case 2.** Let  $a_3 = 0$  and  $b_3 \neq 0$ . Then from (5.4) we have  $c_2 = a_2 c_1 = 0$ .

If  $a_2 \neq 0$ , then  $c_1 = 0$  and we obtain evolution algebra with multiplications:

$$e_1 \cdot e_1 = a_2 e_2, \quad e_2 \cdot e_2 = b_3 e_3.$$

If  $a_2 = 0$ , then the table of multiplications of the algebra  $E$  is as follows:

$$e_2 \cdot e_2 = b_3 e_3, \quad e_3 \cdot e_3 = c_1 e_1.$$

Putting  $e'_1 = e_2, e'_2 = e_3, e'_3 = e_1$ , we derive that this algebra is isomorphic to the algebra  $E^1$ .

**Case 3.** Let  $a_3 \neq 0$ . Then restrictions (5.4) imply  $c_1 = b_3 c_2 = 0$ .

If  $b_3 \neq 0$ , then  $c_2 = 0$  and the algebra  $E^1$  is obtained.

If  $b_3 = 0$ , then by taking basis transformation as follows:

$$e'_1 = e_1, e'_2 = e_3, e'_3 = e_2$$

we get  $E^1$ . □

**Theorem 5.2.** *Let  $E$  be a three-dimensional evolution algebra with the table of multiplication (5.1) and let  $a_2 a_3 b_1 b_3 c_1 c_2 \neq 0$ . Then period of each basis elements of the algebra  $E$  is infinite if and only if equations (5.2) hold true.*

*Proof.* It is sufficient to proof the part *Only if*. From equalities (5.2) we get

$$a_2 = -\frac{b_3^2 c_2}{b_1^2}, \quad a_3 = -\frac{c_2^2 b_3}{c_1^2}.$$

Putting this restrictions to the equality  $a_2^2 b_1 + a_3^2 c_1 = 0$  we obtain

$$0 = a_2^2 b_1 + a_3^2 c_1 = \frac{b_3^2 c_2^2 (b_3^2 c_1^3 + b_1^3 c_2^2)}{b_1^3 c_1^3},$$

which implies

$$b_3^2 c_1^3 + b_1^3 c_2^2 = 0. \quad (5.5)$$

Similarly, we derive

$$b_1 = -\frac{a_3^2 c_1}{a_2^2}, \quad b_3 = -\frac{c_1^2 a_3}{c_2^2} \Rightarrow a_3^2 c_2^3 + a_2^3 c_1^2 = 0, \quad (5.6)$$

$$c_1 = -\frac{a_2^2 b_1}{a_3^2}, \quad c_2 = -\frac{b_1^2 a_2}{b_3^2} \Rightarrow a_2^2 b_3^3 + a_3^3 b_1^2 = 0. \quad (5.7)$$

Applying induction we will prove the following:

$$e_1^{[k]} = A_{k,2}e_2 + A_{k,3}e_3, \quad e_2^{[k]} = B_{k,1}e_1 + B_{k,3}e_3, \quad e_3^{[k]} = C_{k,1}e_1 + C_{k,2}e_2, \quad k \geq 3 \quad (5.8)$$

with recurrence expressions

$$\begin{aligned} A_{k,2} &= A_{k-1,3}^2 c_2, & A_{k,3} &= A_{k-1,2}^2 b_3, & A_{k-1,2}^2 b_1 + A_{k-1,3}^2 c_1 &= 0, \\ B_{k,1} &= B_{k-1,3}^2 c_1, & B_{k,3} &= B_{k-1,1}^2 a_3, & B_{k-1,1}^2 a_2 + B_{k-1,3}^2 c_2 &= 0, \\ C_{k,1} &= C_{k-1,2}^2 b_1, & C_{k,2} &= C_{k-1,1}^2 a_2, & C_{k-1,1}^2 a_3 + C_{k-1,3}^2 b_3 &= 0, \end{aligned} \quad (5.9)$$

where  $A_{2,2} = a_2$ ,  $A_{2,3} = a_3$ ,  $B_{2,1} = b_1$ ,  $B_{2,3} = b_3$ ,  $C_{2,1} = c_1$ ,  $C_{2,2} = c_2$ .

In fact, the correctness of expressions (5.8) is equivalent to that each basis element of evolution algebra  $E$  has infinite period.

From decompositions of  $e_i^{[3]}$ ,  $1 \leq i \leq 3$  it is easy to see the correctness of (5.9) for  $k = 3$ , i.e.

$$\begin{aligned} A_{3,2} &= a_3^2 c_2 = A_{2,3}^2 c_2, & A_{3,3} &= a_2^2 b_3 = A_{2,2}^2 b_3, & A_{2,2}^2 b_1 + A_{2,3}^2 c_1 &= a_2^2 b_1 + a_3^2 c_1 = 0, \\ B_{3,1} &= b_3^2 c_1 = B_{2,3}^2 c_1, & B_{3,3} &= b_1^2 a_3 = B_{2,1}^2 a_3, & B_{2,1}^2 a_2 + B_{2,3}^2 c_2 &= b_1^2 a_2 + b_3^2 c_2 = 0, \\ C_{3,1} &= c_2^2 b_1 = C_{2,2}^2 b_1, & C_{3,2} &= c_1^2 a_2 = C_{2,1}^2 a_2, & C_{2,1}^2 a_3 + C_{2,2}^2 b_3 &= c_1^2 a_3 + c_2^2 b_3 = 0. \end{aligned}$$

Suppose that (5.8) and (5.9) are true for  $k$ . We will prove it for  $k + 1$ .

The chain of equalities

$$e_1^{[k+1]} = e_1^{[k]} \cdot e_1^{[k]} = (A_{k,2}e_2 + A_{k,3}e_3) \cdot (A_{k,2}e_2 + A_{k,3}e_3) = (A_{k,2}^2 b_1 + A_{k,3}^2 c_1)e_1 + A_{k,3}^2 c_2 e_2 + A_{k,2}^2 b_3 e_3$$

deduces

$$A_{k+1,2} = A_{k,3}^2 c_2, \quad A_{k+1,3} = A_{k,2}^2 b_3.$$

Applying induction assumption, that is,

$$A_{k,2} = A_{k-1,3}^2 c_2, \quad A_{k,3} = A_{k-1,2}^2 b_3, \quad A_{k-1,2}^2 b_1 + A_{k-1,3}^2 c_1 = 0,$$

we get

$$A_{k,2}^2 b_1 + A_{k,3}^2 c_1 = A_{k-1,3}^4 c_2^2 b_1 + A_{k-1,2}^4 b_3^2 c_1 = \frac{A_{k-1,2}^4 b_1^2}{c_1^2} c_2^2 b_1 + A_{k-1,2}^4 b_3^2 c_1 = \frac{A_{k-1,2}^4}{c_1^2} (b_1^3 c_2^2 + b_3^2 c_1^3)$$

The equality (5.5) implies  $A_{k,2}^2 b_1 + A_{k,3}^2 c_1 = 0$ .

Similarly, one finds

$$B_{k,1}^2 a_2 + B_{k,3}^2 c_2 = 0, \quad C_{k,1}^2 a_3 + C_{k,2}^2 b_3 = 0.$$

□

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