Journal of Algebra and Its Applications
Vol. 20, No. 9 (2021) 2150161 (14 pages)

DOI: 10.1142/S0219498821501619

On Leibniz superalgebras which even part is $\mathfrak{sl}_2$

Kh. A. Muratova*,‡ and A. Kh. Khudoyberdiyev*,†,§

*Institute of Mathematics Academy of Science of Uzbekistan
81, Mirzo Ulug'bek Street, 100170 Tashkent, Uzbekistan

†National University of Uzbekistan
4, University Street, 100174 Tashkent, Uzbekistan
‡xalkulova@gmail.com
§khabror@mail.ru

Received 20 May 2019
Accepted 3 May 2020
Published 15 July 2020

Communicated by V. Futorny

This paper deals with Leibniz superalgebras $L = L_0 \oplus L_1$, whose even part is a simple Lie algebra $\mathfrak{sl}_2$. We describe all such Leibniz superalgebras when odd part is an irreducible Leibniz bi-module on $\mathfrak{sl}_2$. We show that there exist such Leibniz superalgebras with nontrivial odd part only in case of $\dim L_1 = 2$.

Keywords: Leibniz algebra; Leibniz superalgebra; simple Lie algebra; Leibniz representation.

Mathematics Subject Classification 2020: 17A32, 17A36, 17B30

1. Introduction

Extensive investigations in Lie algebras theory have led to the appearance of more general algebraic objects — Mal'cev algebras, binary Lie algebras, Lie superalgebras, Leibniz algebras and others.

During many years, the theory of Lie superalgebras has been actively studied by many mathematicians and physicists. A systematic exposition of basic Lie superalgebras theory can be found in [9]. Many works have been devoted to the study of this topic, but unfortunately most of them do not deal with nilpotent Lie superalgebras. In works [6, 8], the problem of the description of some classes of nilpotent Lie superalgebras have been studied.

Leibniz algebras have been first introduced by Loday in [12] as a nonantisymmetric version of Lie algebras. Leibniz superalgebras are generalizations of Leibniz

‡Corresponding author.

algebras and, on the other hand, they naturally generalize Lie superalgebras. In the description of Leibniz superalgebras structure, the crucial task is to prove the existence of suitable basis (the so-called adapted basis) in which the multiplication of the superalgebra has the most convenient form.

In the work [8], the Lie superalgebras with maximal nilindex were classified. Such superalgebras are two-generated and its nilindex equals to $n+m$ (where n and m are dimensions of even and odd parts, respectively). In fact, there exists unique Lie superalgebra of maximal nilindex. This superalgebra is filiform Lie superalgebra (the characteristic sequence equal to $(n-1, 1 \mid m)$) and some crucial properties of filiform Lie superalgebras are given in [6].

For nilpotent Leibniz superalgebras, it turns to be comparatively easy and was solved in [1]. The distinctive property of such Leibniz superalgebras is that they are single-generated and have the nilindex $n+m+1$. The next step is a description of Leibniz superalgebras with dimensions of even and odd parts, respectively equal to n and m, and with nilindex $n+m$. Such class of Leibniz superalgebras were classified in [2, 4, 5, 7], applying restrictions to the characteristic sequences. Solvable and semi-simple Leibniz superalgerbas are not investigated at this time. The first step of this assertion is to describe Leibniz superalgebras which even part is a semi-simple Lie algebra. Note that the odd part of the superalgebra can be considered as a representation of the even part. Representation or bimodule of a Leibniz algebra L is defined in [13] as a $\mathbb{K}$-module M with two actions — left and right, satisfying compatibility conditions. In [3], it is established that any simple finite-dimensional Leibniz representation is either symmetric, meaning the left and the right actions differ by sign, or antisymmetric, meaning the left action is zero. The classical Weyl's theorem on complete reducibility that claims any finite-dimensional module over a semi-simple Lie algebra is a direct sum of simple modules does not generalize even for the simple Leibniz algebras case.

A Lie algebra can be considered as a Leibniz algebra and one can consider Leibniz representation of a Lie algebra. In [14], the authors describe the indecomposable objects of the category of Leibniz representations of a Lie algebra and as an example, in case the Lie algebra is $\mathfrak{sl}_2$ the indecomposable objects in that category can be described, whereas for $\mathfrak{sl}_n$ $(n > 2)$ they claim that it is of wild type.

Our main focus in this work is to describe Leibniz superalgebras even part is isomorphic to the three-dimensional simple Lie algebra $\mathfrak{sl}_2$. If the multiplication of the odd part is zero, then Leibniz superalgebra is isomorpic to Leibniz algebra, i.e. superalgebra with trivial odd part. We show that there exist such Leibniz superalgebra $L = \mathfrak{sl}_2 \oplus L_1$ with nontrivial odd part only in case of $\dim L_1 = 2$.

Throughout this work, we shall consider spaces, algebras and superalgebras over the field of complex numbers.

2. Preliminaries

In this section, we give necessary definitions and preliminary results.

Definition 2.1. An algebra $(L, [\cdot,\cdot])$ over a field $\mathbb{K}$ is called a *Leibniz algebra* if it is defined by the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y], \quad \text{for all } x,y,z \in L.$$

In fact for Leibniz algebra L the ideal $I = \operatorname{span}\{[x,x] \mid x \in L\}$ coincides with the space spanned by squares of elements of L. Moreover, it is readily to see that this ideal belongs to right annihilator, that is $[L, I] = 0$. Note that the ideal I is the minimal ideal with respect to the property that the quotient algebra L/I is a Lie algebra.

Definition 2.2. Let L be a Leibniz algebra, M be a $\mathbb{K}$ vector space and bilinear maps $[-,-] : L \times M \to M$ and $[-,-] : M \times L \to M$ satisfy the following three axioms:

$$\begin{aligned} [m,[x,y]] &= [[m,x],y] - [[m,y],x], \\ [x,[m,y]] &= [[x,m],y] - [[x,y],m], \\ [x,[y,m]] &= [[x,y],m] - [[x,m],y]. \end{aligned} \tag{2.1}$$

Then M is called a *representation* of the Leibniz algebra L or an L-*bimodule*.

Definition 2.3. A $\mathbb{Z}_2$-graded vector space $L = L_0 \oplus L_1$ is called a Leibniz superalgebra if it is equipped with a product $[-,-]$ which satisfies the following conditions:

$$[x,[y,z]] = [[x,y],z] - (-1)^{\alpha\beta}[[x,z],y] - \text{Leibniz superidentity}$$

for all $x \in L, y \in L_\alpha, z \in L_\beta$.

The vector spaces L_0 and L_1 are said to be the even and odd parts of the superalgebra L, respectively. It is obvious that L_0 is a Leibniz algebra and L_1 is a representation of L_0. Note that if in Leibniz superalgebra L the identity

$$[x,y] = -(-1)^{\alpha\beta}[y,x]$$

holds for any $x \in L_\alpha$ and $y \in L_\beta$, then the Leibniz superidentity can be transformed into the Jacobi superidentity. Thus, Leibniz superalgebras are a generalization of Lie superalgebras and Leibniz algebras.

Definition 2.4. The set

$$\mathcal{R}(L) = \{z \in L \mid [L,z] = 0\}$$

is called the right annihilator of a superalgebra L.

Using the Leibniz superidentity, it is easy to see that $\mathcal{R}(L)$ is an ideal of the superalgebra L. Moreover, the elements of the form $[a,b] + (-1)^{\alpha\beta}[b,a]$, $(a \in L_\alpha, b \in L_\beta)$ belong to $\mathcal{R}(L)$.

Definition 2.5. An L-bimodule is called *simple* or *irreducible*, if it does not admit nontrivial L-subbimodules. An L-bimodule is called *indecomposable*, if it is not a direct sum of its L-subbimodules. An L-bimodule M is called *completely reducible* if for any L-subbimodule N, there exists a complementing L-subbimodule N' such that $M = N \oplus N'$.

In [3], it is proved that a finite-dimensional simple L-bimodule is either symmetric or antisymmetric for any finite-dimensional Leibniz algebra L.

Lemma 2.6 ([3]). *Let L be a finite-dimensional Leibniz algebra, and let M be a finite-dimensional simple L-bimodule. Then either $[L, M] = 0$ or $[x, m] = -[m, x]$ for all $x \in L$ and $m \in M$.*

An L-bimodule with trivial left actions is called *symmetric*. If the left action is the negative of the right action, then it is called *antisymmetric*.

In particular, in case of L is isomorphic to the three-dimensional simple Lie algebra $\mathfrak{sl}_2$, we have that there exist a basis $\{x_0, x_1, \ldots, x_n\}$ of M such that one of the table of multiplication holds:

$$N_1 : \begin{cases} [x_i, h] = (n-2i)x_i, & [h, x_i] = -(n-2i)x_i, \\ [x_i, f] = x_{i+1}, & [f, x_i] = -x_{i+1}, \\ [x_i, e] = -i(n-i+1)x_{i-1}, & [e, x_i] = i(n-i+1)x_{i-1}. \end{cases} \tag{2.2}$$

$$N_2 : \begin{cases} [x_i, h] = (n-2i)x_i, & [h, x_i] = 0, \\ [x_i, f] = x_{i+1}, & [f, x_i] = 0, \\ [x_i, e] = -i(n-i+1)x_{i-1}, & [e, x_i] = 0. \end{cases} \tag{2.3}$$

In [10], it is studied indecomposable Leibniz $\mathfrak{sl}_2$-bimodules with condition that as a Lie algebra module is split into a direct sum of two simple $\mathfrak{sl}_2$-submodules.

Theorem 2.7 ([10]). *An $\mathfrak{sl}_2$-module $M = X \oplus Y$, where X and Y are simple $\mathfrak{sl}_2$-modules is indecomposable as a Leibniz $\mathfrak{sl}_2$-bimodule if and only if $\dim X - \dim Y = 2$. Moreover, upto $\mathfrak{sl}_2$-bimodule isomorphism there are only two indecomposable $\mathfrak{sl}_2$-bimodules, which in basis $\{x_0, x_1, \ldots, x_n, y_0, \ldots, y_{n-2}\}$ have the following brackets:*

$$M_1 : \begin{array}{ll} [x_i, h] = (n-2i)x_i, & [h, x_i] = -(n-2i)x_i - 2iy_{i-1}, \\ [x_i, f] = x_{i+1}, & [f, x_i] = -x_{i+1} + y_i, \\ [x_i, e] = -i(n-i+1)x_{i-1}, & [e, x_i] = i(n-i+1)x_{i-1} + i(i-1)y_{i-2}, \\ [y_j, h] = (n-2-2j)y_j, & [h, y_j] = 0, \\ [y_j, f] = y_{j+1}, & [f, y_j] = 0, \\ [y_j, e] = -j(n-j-1)y_{j-1}, & [e, y_j] = 0. \end{array}$$

$$M_2 : \begin{array}{ll} [x_i, h] = (n-2i)x_i, & [h, x_i] = 0, \\ [x_i, f] = x_{i+1}, & [f, x_i] = 0, \\ [x_i, e] = -i(n-i+1)x_{i-1}, & [e, x_i] = 0, \\ [y_j, h] = (n-2-2j)y_j, & [h, y_j] = 2(n-j-1)x_{j+1} - (n-2j-2)y_j, \\ [y_j, f] = y_{j+1}, & [f, y_j] = x_{j+2} - y_{i+1}, \\ [y_j, e] = -j(n-j-1)y_{j-1}, & [e, y_j] = (n-j-1)((n-j)x_j + jy_{j-1}). \end{array}$$

Following theorem generalize of Theorem 2.7, in case of M is a direct sum of k simple $\mathfrak{sl}_2$-submodules.

Theorem 2.8 ([11]). *Let M be an $\mathfrak{sl}_2$-bimodule and as a right $\mathfrak{sl}_2$-module let it decompose as $M = V(n_1) \oplus V(n_2) \oplus \cdots \oplus V(n_k)$, where $V(n_i)$ are simple $\mathfrak{sl}_2$-modules with base $\{v_0^i, \dots, v_{n_i}^i\}$, $1 \le i \le k$ and $n_1 \ge n_2 \ge \cdots \ge n_k$. Then M is an indecomposable Leibniz $\mathfrak{sl}_2$-bimodule only if $n_i - n_{i+1} = 2$ for all $1 \le i \le k-1$. Moreover, up to $\mathfrak{sl}_2$-bimodule isomorphism there are exactly two indecomposable $\mathfrak{sl}_2$-bimodules*:

$$
M_3: \begin{array}{ll}
[v_i^{2p-1}, h] = (n-4p+4-2i)v_i^{2p-1}, & [h, v_i^{2p-1}] = 0,\\
[v_i^{2p-1}, f] = v_{i+1}^{2p-1}, & [f, v_i^{2p-1}] = 0,\\
[v_i^{2p-1}, e] = -i(n-4p+5-i)v_{i-1}^{2p-1}, & [e, v_i^{2p-1}] = 0,\\
[v_i^{2p}, h] = (n-4p+2-2i)v_i^{2p}, & [h, v_i^{2p}] = 2(n-2p-i+3)v_{i+1}^{2p-1}\\
& \qquad - (n-2p-2i+2)v_{i+1}^{2p}\\
& \qquad - 2iv_{i-1}^{2p+1},\\
[v_i^{2p}, f] = v_{i+1}^{2p}, & [f, v_i^{2p}] = v_{i+2}^{2p-1} - v_{i+1}^{2p} + v_i^{2p+1},\\
[v_i^{2p}, e] = -i(n-4p+1-i)v_{i-1}^{2p}, & [e, v_i^{2p}] = (n-2p-i+3)\\
& \qquad \times ((n-2p-i+4)v_i^{2p-1}\\
& \qquad + iv_{i-1}^{2p}) + i(i-1)v_{i-2}^{2p+1},\\
[v_i^{2p+1}, h] = (n-4p+1-i)v_i^{2p+1}, & [h, v_i^{2p+1}] = 0,\\
[v_i^{2p+1}, f] = v_{i+1}^{2p+1}, & [f, v_i^{2p+1}] = 0,\\
[v_i^{2p+1}, e] = -i(n-4p-1-i)v_{i-1}^{2p+1}, & [e, v_i^{2p+1}] = 0.
\end{array}
$$

for all $1 \le p \le \frac{k}{2}$,

$$
M_4: \begin{array}{ll}
[v_i^{1}, h] = (n-2i)v_i^{1}, & [h, v_i^{1}] = -(n-2i)v_i^{1} - 2iv_{i-1}^{2},\\
[v_i^{1}, f] = v_{i+1}^{1}, & [f, v_i^{1}] = -v_{i+1}^{1} + v_i^{2},\\
[v_i^{1}, e] = -i(n-i+1)v_{i-1}^{1}, & [e, v_i^{1}] = i(n-i+1)v_{i-1}^{1} + i(i-1)v_{i-2}^{2},\\
[v_i^{2p}, h] = (n-4p+2-2i)v_i^{2p}, & [h, v_i^{2p}] = 0,\\
[v_i^{2p}, f] = v_{i+1}^{2p}, & [f, v_i^{2p}] = 0,\\
[v_i^{2p}, e] = -i(n-4p+1-i)v_{i-1}^{2p}, & [e, v_i^{2p}] = 0,\\
[v_i^{2p+1}, h] = (n-4p-2i)v_i^{2p+1}, & [h, v_i^{2p+1}] = (n-4p-i+1)v_{i+1}^{2p}\\
& \qquad - (n-4p-2i)v_{i+1}^{2p+1}\\
& \qquad - 2iv_{i-1}^{2p+2},\\
[v_i^{2p+1}, f] = v_{i+1}^{2p+1}, & [f, v_i^{2p+1}] = v_{i+2}^{2p} - v_{i+1}^{2p+1} + v_i^{2p+2},\\
[v_i^{2p+1}, e] = -i(n-4p-1-i)v_{i-1}^{2p+1}, & [e, v_i^{2p+1}] = (n-4p-i+1)\\
& \qquad \times ((n-4p-i+2)v_i^{2p}\\
& \qquad + iv_{i-1}^{2p+1}) + i(i-1)v_{i-2}^{2p+2},
\end{array}
$$

for all $1 \le p \le \frac{k-1}{2}$, *where* $n = n_1$.

3. Main Part

In this section, we describe Leibniz superalgebras which even part is $\mathfrak{sl}_2$.

Lemma 3.1. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra, such that L_0 is a semi-simple Lie algebra. Then $[x, y] = [y, x]$ for any $x, y \in L_1$.*

Proof. Note that for any $x, y \in L_1$ an element $[x, y] - [y, x]$ belongs to the right annihilator of L. Consequently, $[x, y]-[y, x]$ belongs to the center of the semi-simple Lie algebra L_0. Since the center of semi-simple Lie algebra is zero, we have that $[x, y] = [y, x]$. □

Lemma 3.2. *Let $L = L_0 \oplus L_1$ be a Leibniz superalgebra, such that L_0 is a semi-simple Lie algebra and $[L_0, L_1] = 0$, then $[L_1, L_1] = 0$.*

Proof. By the condition of the lemma, we have that $[x, y] = 0$ for any $x \in L_0$ and $y \in L_1$. Considering the Leibniz superidentity for the elements $x \in L_0$ and $y, z \in L_1$, we have

$$[x, [y, z]] = [[x, y], z] + [[x, z], y] = 0,$$

which follows that $[y, z] \in \mathcal{R}(L_0)$. Since $\mathcal{R}(L_0) = 0$, we derive that $[y, z] = 0$ for any $y, z \in L_1$. □

Let $L = \mathfrak{sl}_2 \oplus L_1$ be a Leibniz superalgebra, i.e. even part is isomorphic to $\mathfrak{sl}_2$. If L_1 is a simple $\mathfrak{sl}_2$-bimodule, then by Lemma 2.6, we have that either $[L_1, \mathfrak{sl}_2] = 0$ or $[x, y] = -[y, x]$ for all $x \in \mathfrak{sl}_2$, $y \in L_1$.

In case of $[L_1, \mathfrak{sl}_2] = 0$, from Lemma 3.2, we derive that $[L_1, L_1] = 0$ and L is isomorphic to the Leibniz algebra with the following multiplication [15]:

$$\begin{cases} [e, h] = 2e, & [h, f] = 2f, \quad [e, f] = h, \\ [h, e] = -2e, & [f, h] = -2f, \quad [f, e] = -h, \\ [x_i, h] = (n - 2i)x_i, & [x_i, f] = x_{i+1}, \quad [x_i, e] = -i(n - i + 1)x_{i-1}, \quad 0 \leq i \leq n. \end{cases}$$

Therefore, it is sufficient to consider the case when $[x, y] = -[y, x]$ for all $x \in \mathfrak{sl}_2$, $y \in L_1$. In the following proposition, we describe such Leibniz superalgebras in case of $\dim L_1 = 2$.

Proposition 3.3. *Let $L = \mathfrak{sl}_2 \oplus L_1$ be a Leibniz superalgebra, such that L_1 is a simple bimodule. Let $\dim L_1 = 2$ and $[x, y] = -[y, x]$ for all $x \in \mathfrak{sl}_2$, $y \in L_1$, then L*

is isomorphic one of the following two Leibniz superalgebras:

$$S_1:\begin{cases}[e,h]=2e, & [h,f]=2f, & [e,f]=h, & \\ [h,e]=-2e, & [f,h]=-2f, & [f,e]=-h, & \\ [x_0,h]=x_0, & [x_1,h]=-x_1, & [x_0,f]=x_1, & [x_1,e]=-x_0, \\ [h,x_0]=-x_0, & [h,x_1]=x_1, & [f,x_0]=-x_1, & [e,x_1]=x_0.\end{cases}$$

$$S_2:\begin{cases}[e,h]=2e, & [h,f]=2f, & [e,f]=h, & \\ [h,e]=-2e, & [f,h]=-2f, & [f,e]=-h, & \\ [x_0,h]=x_0, & [x_1,h]=-x_1, & [x_0,f]=x_1, & [x_1,e]=-x_0, \\ [h,x_0]=-x_0, & [h,x_1]=x_1, & [f,x_0]=-x_1, & [e,x_1]=x_0, \\ [x_0,x_0]=2e, & [x_1,x_1]=2f, & [x_0,x_1]=h, & [x_1,x_0]=h.\end{cases}$$

Proof. From Lemma 2.6 and (2.2), we have the following products:

$$\begin{cases}[[e,h]=2e, & [h,f]=2f, & [e,f]=h, & \\ [h,e]=-2e, & [f,h]=-2f, & [f,e]=-h, & \\ [x_0,h]=x_0, & [x_1,h]=-x_1, & [x_0,f]=x_1, & [x_1,e]=-x_0, \\ [h,x_0]=-x_0, & [h,x_1]=x_1, & [f,x_0]=-x_1, & [e,x_1]=x_0.\end{cases}$$

Put

$$\begin{cases}[x_0,x_0]=a_{0,0}e+b_{0,0}f+c_{0,0}h, \\ [x_0,x_1]=a_{0,1}e+b_{0,1}f+c_{0,1}h, \\ [x_1,x_1]=a_{1,1}e+b_{1,1}f+c_{1,1}h.\end{cases}$$

Consider following Leibniz superidentities:

- $[e,[x_0,x_0]]=2[[e,x_0],x_0]=0.$

  On the other hand: $[e,[x_0,x_0]]=[e,a_{0,0}e+b_{0,0}f+c_{0,0}h]=b_{0,0}h+2c_{0,0}e$, which implies

$$b_{0,0}=c_{0,0}=0.$$

- $[e,[x_1,x_1]]=2[[e,x_1],x_1]=2[x_0,x_1]=2a_{0,1}e+2b_{0,1}f+2c_{0,1}h,$

  On the other hand: $[e,[x_1,x_1]]=[e,a_{1,1}e+b_{1,1}f+c_{1,1}h]=b_{1,1}h+2c_{1,1}e$, which derive

$$b_{0,1}=0, \quad c_{1,1}=a_{0,1}, \quad b_{1,1}=2c_{0,1}.$$

- $[h,[x_1,x_1]]=2[[h,x_1],x_1]=2[x_1,x_1]=2a_{1,1}e+2b_{1,1}f+2c_{1,1}h,$

  On the other hand: $[h,[x_1,x_1]]=[h,a_{1,1}e+b_{1,1}f+c_{1,1}h]=-2a_{1,1}e+2b_{1,1}f$, which implies

$$a_{1,1}=0, \quad c_{1,1}=0$$

- $[f, [x_0, x_0]] = 2[[f, x_0], x_0] = -2[x_1, x_0] = -2c_{0,1}h$.

  On the other hand: $[f, [x_0, x_0]] = [f, a_{0,0}e] = -a_{0,0}h$, which derive

$$2c_{0,1} = a_{0,0}.$$

The rest Leibniz superidentities give us the same restrictions. Thus, we have the remaining multiplications:

$$\begin{cases} [x_0, x_0] = 2c_{0,1}e, & [x_1, x_1] = 2c_{0,1}f, \\ [x_0, x_1] = c_{0,1}h, & [x_1, x_0] = c_{0,1}h. \end{cases}$$

In case of $c_{0,1} = 0$, we have superalgebra S_1 and if $c_{0,1} \neq 0$, then taking the change

$$x_0' = \frac{1}{\sqrt{c_{0,1}}}x_0, \quad x_1' = \frac{1}{\sqrt{c_{0,1}}}x_1,$$

we obtain the Leibniz superalgebra S_2. □

It should be noticed that in the superalgebra S_1, the multiplication $[L_1, L_1]$ is zero, thus, S_1 is a Leibniz algebra.

In the following proposition, we investigate the case when $\dim L_1 \geq 3$.

Proposition 3.4. *Let* $L = \mathfrak{sl}_2 \oplus L_1$ *be a Leibniz superalgebra, such that* L_1 *is a simple bimodule. Let* $\dim L_1 \geq 3$ *and* $[x, y] = -[y, x]$ *for all* $x \in \mathfrak{sl}_2$, $y \in L_1$, *then* $[L_1, L_1] = 0$.

Proof. By the condition of proposition, we have that there exist a basis $\{e, f, h, x_{0,1}, \dots, x_n\}$ of L such that

$$\begin{cases} [e, h] = 2e, & [h, f] = 2f, & [e, f] = h, \\ [h, e] = -2e, & [f, h] = -2f, & [f, e] = -h, \\ [x_k, h] = (n-2k)x_k, & [h, x_k] = (2k-n)x_k, & \\ [x_k, f] = x_{k+1}, & [f, x_k] = -x_{k+1}, & \\ [x_k, e] = -k(n+1-k)x_{k-1}, & [e, x_k] = k(n+1-k)x_{k-1}. & \end{cases}$$

Put

$$[x_i, x_j] = a_{i,j}e + b_{i,j}f + c_{i,j}h, \quad 0 \leq i \leq j \leq n.$$

Now, we consider Leibniz superidentity for the elements $[h, [x_i, x_j]]$, $[f, [x_i, x_j]]$ and $[e, [x_i, x_j]]$.

$$\begin{aligned} [h, [x_i, x_j]] &= [[h, x_i], x_j] + [[h, x_j], x_i] = (2i-n)[x_i, x_j] + (2j-n)[x_j, x_i] \\ &= 2(i+j-n)(a_{i,j}e + b_{i,j}f + c_{i,j}h) \\ &= 2(i+j-n)a_{i,j}e + 2(i+j-n)b_{i,j}f + 2(i+j-n)c_{i,j}h, \end{aligned}$$

where $0 \leq i \leq j \leq n$.

$$[f,[x_i,x_j]] = [[f,x_i],x_j] + [[f,x_j],x_i] = -[x_{i+1},x_j] - [x_{j+1},x_i]$$
$$= -(a_{i+1,j} + a_{i,j+1})e - (b_{i+1,j} + b_{i,j+1})f - (c_{i+1,j} + c_{i,j+1})h,$$

where $0 \leq i \leq j \leq n-1$.

$$\begin{aligned}[e,[x_i,x_j]] &= [[e,x_i],x_j] + [[e,x_j],x_i] \\ &= i(n+1-i)[x_{i-1},x_j] + j(n+1-j)[x_{j-1},x_i] \\ &= (i(n+1-i)a_{i-1,j} + j(n+1-j)a_{i,j-1})e \\ &\quad + (i(n+1-i)b_{i-1,j} + j(n+1-j)b_{i,j-1})f \\ &\quad + (i(n+1-i)c_{i-1,j} + j(n+1-j)c_{i,j-1})h,\end{aligned}$$

where $1 \leq i \leq j \leq n$.

On the other hand,

$$[h,[x_i,x_j]] = [h, a_{i,j}e + b_{i,j}f + c_{i,j}h] = -2a_{i,j}e + 2b_{i,j}f,$$
$$[f,[x_i,x_j]] = [f, a_{i,j}e + b_{i,j}f + c_{i,j}h] = -a_{i,j}h - 2c_{i,j}f,$$
$$[e,[x_i,x_j]] = [e, a_{i,j}e + b_{i,j}f + c_{i,j}h] = b_{i,j}h + 2c_{i,j}e.$$

Comparing the coefficients at the basis elements, we obtain following restrictions:

$$\begin{cases} (i+j+1-n)a_{i,j} = 0, \\ (i+j-1-n)b_{i,j} = 0, \\ (i+j-n)c_{i,j} = 0. \end{cases} \tag{3.1}$$

$$\begin{cases} a_{i+1,j} + a_{j+1,i} = 0, \\ b_{i+1,j} + b_{j+1,i} = 2c_{i,j}, \\ c_{i+1,j} + c_{j+1,i} = a_{i,j}. \end{cases} \tag{3.2}$$

$$\begin{cases} i(n+1-i)a_{i-1,j} + j(n+1-j)a_{i,j-1} = 2c_{i,j}, \\ i(n+1-i)b_{i-1,j} + j(n+1-j)b_{i,j-1} = 0, \\ i(n+1-i)c_{i-1,j} + j(n+1-j)c_{i,j-1} = b_{i,j}. \end{cases} \tag{3.3}$$

It is obvious that from (3.1), we have

$$\begin{cases} a_{i,j} = 0, \quad i+j \neq n-1, \\ b_{i,j} = 0, \quad i+j \neq n+1, \\ c_{i,j} = 0, \quad i+j \neq n. \end{cases} \tag{3.4}$$

From the first equality of (3.2), we derive:

$$a_{i,n-i-1} = (-1)^i a_{0,n-1}, \quad 0 \le i \le n-1. \tag{3.5}$$

If n is even, then we have $a_{n-1,0} = -a_{0,n-1}$. On the other hand, $a_{n-1,0} = a_{0,n-1}$. Thus, we get $a_{0,n-1} = 0$, which implies $a_{i,j} = 0,\ 0 \le i \le j \le n$.

If n is odd, then in case of $n = 3$ from the Leibniz superidentity $[x_1, [x_1, x_1]] - 2[[x_1, x_1], x_1] = 0$, we get $a_{1,1} = 0$. If $n \ge 4$, then considering Leibniz superidentity $[x_{n-2}, [x_1, x_1]] - 2[[x_{n-2}, x_1], x_1] = 0$, we obtain $a_{n-2,0} = 0$. According to the equality (3.5), it follows $a_{i,j} = 0, 0 \le i \le j \le n$.

Since $a_{i,j} = 0$, then from the equality (3.3), we have

$$b_{i,j} = 0, \quad c_{i,j} = 0, \quad 0 \le i \le j \le n.$$

Therefore, we get $[L_1, L_1] = 0$. □

Now, we consider the Leibniz superalgebra $L = \mathfrak{sl}_2 \oplus L_1$, such that L_1 is a split module into direct sum of the two simple $\mathfrak{sl}_2$-modules. Then by the Theorem 2.7, we obtain that L_1 as a $\mathfrak{sl}_2$-bimodule isomorphic to M_1 or M_2.

Proposition 3.5. *Let $L = \mathfrak{sl}_2 \oplus L_1$ be a Leibniz superalgebra such that L_1 is a bimodule isomorphic to M_1. Then $[L_1, L_1] = 0$.*

Proof. By the condition of the proposition, we have that there exist a basis $\{e, f, h, x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_{n-2}\}$ of the superalgebra L such that the following products hold:

$$\begin{cases} [e,h] = 2e, & [h,f] = 2f, & [e,f] = h, \\ [h,e] = -2e, & [f,h] = -2f, & [f,e] = -h, \\ [x_i,h] = (n-2i)x_i, & [h,x_i] = -(n-2i)x_i - 2iy_{i-1}, & \\ [x_i,f] = x_{i+1}, & [f,x_i] = -x_{i+1} + y_i, & \\ [x_i,e] = -i(n-i+1)x_{i-1}, & [e,x_i] = i(n-i+1)x_{i-1} & \\ & \qquad + i(i-1)y_{i-2}, & \\ [y_j,h] = (n-2-2j)y_j, & [h,y_j] = 0, & \\ [y_j,f] = y_{j+1}, & [f,y_j] = 0, & \\ [y_j,e] = -j(n-j-1)y_{j-1}, & [e,y_j] = 0. & \end{cases}$$

Since

$$[y_0, h] + [h, y_0] = (n-2)y_0, \quad [y_j, f] + [f, y_j] = y_{j+1}, \quad 0 \le j \le n-3,$$

we have that $y_j \in \mathcal{R}(L)$ for $0 \le j \le n-2$. Thus, we have $[y_i, y_j] = [x_i, y_j] = 0$ for any values i and j.

Put

$$[x_i, x_j] = a_{i,j}e + b_{i,j}f + c_{i,j}h, \quad 0 \le i \le j \le n.$$

Considering Leibniz superidentities for the elements $[h, [x_i, x_j]]$, $[f, [x_i, x_j]]$ and $[e, [x_i, x_j]]$, we obtain following restrictions:

$$\begin{cases} (i + j + 1 - n)a_{i,j} = 0, \\ (i + j - 1 - n)b_{i,j} = 0, \\ (i + j - n)c_{i,j} = 0. \end{cases} \tag{3.6}$$

$$\begin{cases} a_{i+1,j} + a_{j+1,i} = 0, \\ b_{i+1,j} + b_{j+1,i} = 2c_{i,j}, \\ c_{i+1,j} + c_{j+1,i} = a_{i,j}. \end{cases} \tag{3.7}$$

$$\begin{cases} i(n + 1 - i)a_{i-1,j} + j(n + 1 - j)a_{i,j-1} = 2c_{i,j}, \\ i(n + 1 - i)b_{i-1,j} + j(n + 1 - j)b_{i,j-1} = 0, \\ i(n + 1 - i)c_{i-1,j} + j(n + 1 - j)c_{i,j-1} = b_{i,j}. \end{cases} \tag{3.8}$$

Analogously, to the proof of Proposition 3.4 from (3.6)–(3.8), we conclude that $a_{i,j} = b_{i,j} = c_{i,j} = 0$. Therefore, $[x_i, x_j] = 0$ for any $0 \le i \le j \le n$, which implies $[L_1, L_1] = 0$. □

Proposition 3.6. *Let $L = \mathfrak{sl}_2 \oplus L_1$ be a Leibniz superalgebra such that L_1 is a bimodule isomorphic to M_2. Then $[L_1, L_1] = 0$.*

Proof. According to [10, Theorem 2.7], there exist a basis $\{e, f, h, x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_{n-2}\}$ of the superalgebra L such that the following products are hold:

$$M_2 : \begin{cases} [e, h] = 2e, & [h, f] = 2f, & [e, f] = h, \\ [h, e] = -2e, & [f, h] = -2f, & [f, e] = -h, \\ [x_i, h] = (n - 2i)x_i, & [h, x_i] = 0, & \\ [x_i, f] = x_{i+1}, & [f, x_i] = 0, & \\ [x_i, e] = -i(n - i + 1)x_{i-1}, & [e, x_i] = 0, & \\ [y_j, h] = (n - 2 - 2j)y_j, & [h, y_j] = 2(n - j - 1)x_{j+1} & \\ & \qquad - (n - 2j - 2)y_j, & \\ [y_j, f] = y_{j+1}, & [f, y_j] = x_{j+2} - y_{j+1}, & \\ [y_j, e] = -j(n - j - 1)y_{j-1}, & [e, y_j] = (n - j - 1) & \\ & \qquad \times ((n - j)x_j + jy_{j-1}). & \end{cases}$$

Since

$$[x_0, h] + [h, x_0] = nx_0,$$
$$[x_i, f] + [f, x_i] = x_{i+1}, \quad 0 \le i \le n-1,$$

we have that $x_i \in \mathcal{R}(L)$ for $0 \le i \le n$. Thus, we have $[x_j, x_i] = [y_j, x_i] = 0$ for any values i and j.

Put

$$[y_i, y_j] = a_{i,j}e + b_{i,j}f + c_{i,j}h, \quad 0 \le i \le j \le n-2.$$

Analogously, to the proof of Proposition 3.4 considering Leibniz superidentities for the elements $[h, [y_i, y_j]]$, $[f, [y_i, y_j]]$ and $[e, [y_i, y_j]]$, we obtain following restrictions:

$$\begin{cases} (i+j+3-n)a_{i,j} = 0, \\ (i+j+1-n)b_{i,j} = 0, \\ (i+j+2-n)c_{i,j} = 0. \end{cases} \tag{3.9}$$

$$\begin{cases} a_{i+1,j} + a_{j+1,i} = 0, \\ b_{i+1,j} + b_{j+1,i} = 2c_{i,j}, \\ c_{i+1,j} + c_{j+1,i} = a_{i,j}. \end{cases} \tag{3.10}$$

$$\begin{cases} i(n-1-i)a_{i-1,j} + j(n-1-j)a_{i,j-1} = 2c_{i,j}, \\ i(n-1-i)b_{i-1,j} + j(n-1-j)b_{i,j-1} = 0, \\ i(n-1-i)c_{i-1,j} + j(n-1-j)c_{i,j-1} = b_{i,j}. \end{cases} \tag{3.11}$$

From Eqs. (3.9)–(3.11), similarly of the proof of Proposition 3.4, we conclude that $a_{i,j} = 0, b_{i,j} = 0, c_{i,j} = 0$. Therefore, we obtain $[L_1, L_1] = 0$. □

In the next two propositions, we consider Leibniz superalgebra $L = \mathfrak{sl}_2 \oplus L_1$, such that L_1 is a split $\mathfrak{sl}_2$-modules as $V_1 \oplus V_2 \oplus \cdots \oplus V_k$. By Theorem 2.8, we obtain that L_1 is a bimodule isomorphic to M_3 or M_4.

Proposition 3.7. *Let $L = \mathfrak{sl}_2 \oplus L_1$ be a Leibniz superalgebra such that L_1 is a bimodule isomorphic to M_3. Then $[L_1, L_1] = 0$.*

Proof. According to [11, Theorem 2.8], there exist a basis $\{e, f, h, v_0^j, v_1^j, \ldots, v_{n-2k+2}^j\}, 1 \le j \le k$ of the superalgebra L such that the following products are holds:

$$\begin{cases} [e,h]=2e, & [h,f]=2f, \quad [e,f]=h, \\ [h,e]=-2e, & [f,h]=-2f, \quad [f,e]=-h, \\ [v_i^{2p-1},h]=(n-4p+4-2i)v_i^{2p-1}, & [h,v_i^{2p-1}]=0, \\ [v_i^{2p-1},f]=v_{i+1}^{2p-1}, & [f,v_i^{2p-1}]=0, \\ [v_i^{2p-1},e]=-i(n-4p+5-i)v_{i-1}^{2p-1}, & [e,v_i^{2p-1}]=0, \\ [v_i^{2p},h]=(n-4p+2-2i)v_i^{2p}, & [h,v_i^{2p}]=2(n-2p-i+3)v_{i+1}^{2p-1} \\ & \qquad -(n-2p-2i+2)v_{i+1}^{2p} \\ & \qquad -2iv_{i-1}^{2p+1}, \\ [v_i^{2p},f]=v_{i+1}^{2p}, & [f,v_i^{2p}]=v_{i+2}^{2p-1}-v_{i+1}^{2p}+v_i^{2p+1}, \\ [v_i^{2p},e]=-i(n-4p+1-i)v_{i-1}^{2p}, & [e,v_i^{2p}]=(n-2p-i+3)((n-2p \\ & \qquad -i+4)v_i^{2p-1}+iv_{i-1}^{2p}) \\ & \qquad +i(i-1)v_{i-2}^{2p+1}, \\ [v_i^{2p+1},h]=(n-4p+1-i)v_i^{2p+1}, & [h,v_i^{2p+1}]=0, \\ [v_i^{2p+1},f]=v_{i+1}^{2p+1}, & [f,v_i^{2p+1}]=0, \\ [v_i^{2p+1},e]=-i(n-4p-1-i)v_{i-1}^{2p+1}, & [e,v_i^{2p+1}]=0. \end{cases}$$

Since

$$\begin{aligned} [v_i^{2p-1},e]+[e,v_i^{2p-1}]&=-i(n-4p+5-i)v_{i-1}^{2p-1}, \\ [v_i^{2p-1},f]+[f,v_i^{2p-1}]&=v_{i+1}^{2p-1}, \\ [v_i^{2p-1},h]+[h,v_i^{2p-1}]&=-i(n-4p+5-i)v_{i-1}^{2p-1}, \end{aligned}$$

we have that $v_i^{2p-1}\in\mathcal{R}(L)$, $1\le p\le[\frac{k+1}{2}]$, $0\le i\le n-4p+4$. Then according to Lemma 3.1, we have

$$[v_i^{2p-1},v_j^q]=[v_j^q,v_i^{2p-1}]=0, \quad 1\le q\le k,\ \ 0\le j\le n-2p+2.$$

Put

$$[v_i^{2p},v_j^{2q}]=a_{i,j}^{p,q}e+b_{i,j}^{p,q}f+c_{i,j}^{p,q}h, \quad 1\le p\le q\le k,$$
$$0\le i\le n-4p+2,\ \ 0\le j\le n-4q+2.$$

Considering Leibniz superidentities for the elements $[v_0^{2p-1},[v_i^{2p},v_j^{2q}]]$ and $[v_1^{2p-1},[v_i^{2p},v_j^{2q}]]$, we have $[V_{2p},V_{2q}]=0$. Which implies $[L_1,L_1]=0$. □

Proposition 3.8. *Let* $L=\mathfrak{sl}_2\oplus L_1$ *be a Leibniz superalgebra such that* L_1 *is a bimodule isomorphic to* M_4. *Then* $[L_1,L_1]=0$.

Proof. The proof of this proposition is proved similarly to the previous one. □

References

[1] S. Albeverio, Sh. A. Ayupov and B. A. Omirov, On nilpotent and simple Leibniz algebras, *Comm. Algebra* **33**(1) (2005) 159–172.

[2] Sh. A. Ayupov, A. Kh. Khudoyberdiyev and B. A. Omirov, The classification of filiform Leibniz superalgebras of nilindex $n + m$, *Acta Math. Sinica* (*Engl. Ser.*) **25**(1) (2009) 171–190.

[3] D. Barnes, Some Theorems on Leibniz algebras, *Comm. Algebra* **39** (2011) 2463–2472.

[4] L. M. Camacho, J. R. Gómez, R. M. Navarro and B. A. Omirov, Classification of some nilpotent class of Leibniz superalgebras, *Acta Math. Sinica* (*Engl. Ser.*) **26**(5) (2010) 799–816.

[5] L. M. Camacho, J. R. Gómez, B. A. Omirov and A. Kh. Khudoyberdiyev, Complex nilpotent Leibniz superalgebras with nilindex equal to dimension, *Comm. Algebra* **41**(7) (2013) 2720–2735.

[6] M. Gilg, Super-algèbres de Lie nilpotentes, PhD thesis, University of Haute Alsace (2000) 126 p.

[7] J. R. Gómez, B. A. Omirov and A. Kh. Khudoyberdiyev, The classification of Leibniz superalgebras of nilindex $n + m$ $(m \neq 0)$, *J. Algebra* **324**(10) (2010) 2786–2803.

[8] J. R. Gómez, Yu. Khakimdjanov and R. M. Navarro, Some problems concerning to nilpotent Lie superalgebras, *J. Geom. Phys.* **51**(4) (2004) 473–486.

[9] V. G. Kac, Lie superalgebras, *Adv. Math.* **26**(1) (1977) 8–96.

[10] T. Kurbanbaev and R. Turdibaev, Some Leibniz bimodules $\mathfrak{sl}_2$, *J. Algebra Appl.*, **19**(4) (2020) 2050064.

[11] T. Kurbanbaev and R. Turdibaev, On irreducible finite-dimensional Leibniz bimodules of $\mathfrak{sl}_2$, arXiv:1904.13137v1 (2019) 9 p.

[12] J.-L. Loday, Une version non commutative des algèbres de Lie: Les algèbres de Leibniz, *Enseign. Math.* (*2*) **39**(3–4) (1993) 269–293.

[13] J.-L. Loday and T. Pirashvili, Universal enveloping algebras of Leibniz algebras and (co)homology, *Math. Ann.* **296** (1993) 139–158.

[14] J.-L. Loday and T. Pirashvili, Leibniz representations of Lie algebras, *J. Algebra* **181**(2) (1996) 414–425.

[15] I. S. Rakhimov, B. A. Omirov and R. M. Turdibaev, On description of Leibniz algebras corresponding to $\mathfrak{sl}_2$, *Algebr. Represent. Theory* **16**(5) (2013) 1507–1519.