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Local derivations of solvable Lie algebras with Abelian nilradical

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Abstract. In this work we investigate local derivations of solvable Lie algebras. We show that in the class of solvable Lie algebras there exist algebras which admit local derivations which are not ordinary derivation and algebras for which every local derivation is a derivation.

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1 Introduction

The notions of local derivations were first introduced by R.V. Kadison in 1990 [5]. The main problems concerning this notion are to find conditions under which local derivations become derivations and to present examples of algebras with local derivations that are not derivations.

In previous years, the description of local derivations on commutative algebras and algebras of measurable operators are obtained. Investigation of local derivations on Banach algebras, von Neumann algebras, the algebras of measurable operators were initiated in papers [1], [2] and [3].

In [1] by Sh.A.Ayupov and K.K. Kudaybergenov local derivations on finite-dimensional Lie algebras are studied. In particular, it is proved that every local derivation on a finite-dimensional semi-simple Lie algebra over an algebraically closed field of characteristic zero is a derivation. Moreover, examples of finite-dimensional nilpotent Lie algebra which admit local derivations which are not derivations are given.

In this paper we investigate local derivations of solvable Lie algebras. We show that in the class of solvable Lie algebras there exist algebras which admit local derivations which are not ordinary derivation and algebras for which every local derivation is a derivation.

More precisely, local derivations of solvable Lie algebras with abelian nilradical and one-dimensional complementary space are investigated. A necessary and sufficient conditions under which local derivations solvable Lie algebras with abelian nilradical and one-dimensional complementary space become derivations are found.

2 Preliminaries

Definition 2.1. vector space L over a field F with a binary operation $[-,-]$ is called a Lie algebra, if for any $x, y, z \in L$ the following identities are hold:

$[x, x] = 0$ – anticommutative identity,

$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ – Jacobi identity

For an arbitrary Lie algebra L consider the following central lower and derived sequences

$$L^1 = L, \ L^{k+1} = [L^k, L], \quad k \geq 1,$$
$$L^{[1]} = L, \ L^{[k+1]} = [L^{[k]}, L^{[k]}], \quad k \geq 1.$$

Definition 2.2. A Lie algebra L is called solvable (nilpotent) if there exists $m \in \mathbb{N} (s \in \mathbb{N})$, such that $L^{[m]} = 0$ $(L^s = 0)$. The maximal nilpotent ideal of Lie algebra is called nilradical.

Definition 2.3. Linear operator $d : L \to L$ is called a derivation, if

$$d([x, y]) = [d(x), y] + [x, d(y)], \quad \text{for any} \quad x, y \in L.$$

The set of all derivations of a Lie algebra L is a Lie algebra with respect to commutation operation and it is denoted by $Der(L)$

A linear operator Δ is called a local derivation if for any $x \in L$ there exists a derivation $d_x : L \to L$ such that $\Delta(x) = d_x(x)$.

3 Main results

We consider the following solvable Lie algebras [4]:

$$\begin{array}{llll}
L_1: & [e_2, e_1] = e_2; & & \\
L_2(\alpha): & [e_2, e_1] = e_2, & [e_3, e_1] = \alpha e_3; & \\
L_3: & [e_2, e_1] = e_2 + e_3, & [e_3, e_1] = e_1; & \\
L_4(\alpha): & [e_2, e_3] = e_4, & [e_2, e_1] = e_2, & \\
 & [e_3, e_1] = \alpha e_3, & [e_4, e_1] = (1+\alpha) e_4; & \\
L_5: & [e_2, e_3] = e_4, & [e_2, e_1] = e_2 + e_3, & \\
 & [e_3, e_1] = e_3, & [e_4, e_1] = 2e_3; & \\
L_6(\alpha, \beta): & [e_2, e_1] = e_2, & [e_3, e_1] = \alpha e_3, & [e_4, e_1] = \beta e_4; \\
L_7(\alpha): & [e_2, e_1] = e_2 + e_3, & [e_3, e_1] = e_3, & [e_4, e_1] = \alpha e_4; \\
L_8: & [e_2, e_1] = e_2 + e_3, & [e_3, e_1] = e_3 + e_4, & [e_4, e_1] = e_4.
\end{array}$$

Proposition 3.1. *Every local derivation of the algebras $L_1,\ L_2(\alpha),\ L_6(\alpha,\beta)$ is a derivation, and the algebras $L_3,\ L_4,\ L_5,\ L_7,\ L_8$ admits a local derivation which is not a derivation.*

Proof. For the proof of the Proposition we describe the derivations of the algebras $L_1,\ L_2(\alpha),\ L_3,\ L_4,\ L_5,\ L_6(\alpha,\beta),\ L_7,\ L_8$. In particular, the matrix form of the derivations of the algebra L_1 has the form:

$$Der(L_1) = \begin{pmatrix} 0 & \xi_1 \\ 0 & \xi_2 \end{pmatrix}.$$

Let Δ be a local derivation on L_1 and let

$$\Delta(e_1) = \alpha_1 e_1 + \alpha_2 e_2, \quad \Delta(e_2) = \alpha_3 e_1 + \alpha_4 e_2.$$

On the other hand $\Delta(e_1) = d_{e_1}(e_1) = \xi_1 e_2$, which implies $\alpha_1 = 0$. From $\Delta(e_2) = d_{e_2}(e_2) = \xi_2 e_2$, we have $\alpha_3 = 0$. Hence, we get $\Delta(e_1) = \alpha_2 e_2$, $\Delta(e_2) = \alpha_4 e_2$, which derive $\Delta \in Der(L_1)$. Thus, any local derivation on L_1 is a derivation.

Similarly, it is proved that every local derivation of the algebras $L_2(\alpha),\ L_6(\alpha,\beta)$ is a derivation.

Next we consider the algebra L_3. It is not difficult to obtain that the matrix form of the derivation of the algebra L_3 has the form:

$$Der(L_3) = \begin{pmatrix} 0 & \xi_{1,2} & \xi_{1,3} \\ 0 & \xi_{2,2} & \xi_{2,3} \\ 0 & 0 & \xi_{2,2} \end{pmatrix}.$$

Now let us define a linear operator Δ on L_3 by

$$\Delta(e_1) = 0, \quad \Delta(e_2) = 0, \quad \Delta(e_3) = e_3.$$

Evidently, Δ is not a derivation. We show that this operator is a local derivation. For an arbitrary element $x = \eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3$ we have

$$\Delta(x) = \Delta(\eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3) = \eta_3 e_3.$$

Let d_x be a derivation. Consider

$$d_x(x) = d_x(\eta_1 e_1 + \eta_2 e_2 + \eta_3 e_3) = \eta_1 d_x(e_1) + \eta_2 d_x(e_2) + \eta_3 d_x(e_3) =$$

$$\eta_1(\xi_{1,2} e_2 + \xi_{1,3} e_3) + \eta_2(\xi_{2,2} e_2 + \xi_{2,3} e_3) + \eta_3 \xi_{2,2} e_3 =$$

$$(\eta_1\xi_{1,2} + \eta_2\xi_{2,2})e_2 + (\eta_1\xi_{1,3} + \eta_2\xi_{2,3} + \eta_3\xi_{2,2})e_3.$$

It is not difficult to see that for any $\eta_1,\ \eta_2,\ \eta_3$ exists $\xi_{1,2},\ \xi_{1,3},\ \xi_{2,2},\ \xi_{2,3}$ such that

$$\eta_1\xi_{1,2} + \eta_2\xi_{2,2} = 0, \quad \eta_1\xi_{1,3} + \eta_2\xi_{2,3} + \eta_3\xi_{2,2} = \eta_3.$$

Thus, we proved that for any element x there exists a derivation d_x for which $\Delta(x) = d_x(x)$. Hence, Δ is a local derivation.

Similarly, it is not difficult to show that there exists a local derivation of the algebras $L_4,\ L_5,\ L_7,\ L_8$ which is not a derivation.

Let L be a solvable Lie algebra with nilradical N and let $dimN = n$, $dimL = n + 1$. Then we take a basis $\{x, e_1, e_2, \dots, e_n\}$ of L such that $\{e_1, e_2, \dots, e_n\}$ basis of N.

It is known that the operator of right multiplication $ad_x : N \to N$, $ad_x(y) = [y, x], \forall y \in N$ is a non nilpotent operator [6]. Moreover, if nilradical N is an abelian ideal, then solvable algebra L is characterized by the operator ad_x, i.e., two solvable algebras with abelian nilradical N are isomorphic if and only if the corresponding operators of right multiplication have the same Jordan forms.

For example, algebras $L_2(\alpha)$ and L_3 are three-dimensional solvable Lie algebras with two-dimensional abelian nilradical. Operators ad_{e_1} on algebras $L_2(\alpha)$ and L_3 respectively have the form:

$$\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is obvious that the operator ad_{e_1} on the algebra $L_2(\alpha)$ is a diagonalized, and on the algebra L_3 it is not a diagonalized.

According to Proposition 1 it is known that every local derivation on $L_2(\alpha)$ ia a derivation, and algebra L_3 has a local derivation which is not a derivation.

In the following theorem we give a necessary and sufficient condition that a solvable Lie algebra with abelian nilradical and one-dimensional complementary space has local derivation which is not a derivation.

Theorem 3.2. *Let L be a solvable Lie algebra with abelian nilradical N. Let $dimL = dimN + 1$ and $x \in L \setminus N$. Every local derivation on L is a derivation if and only if ad_x is a diagonalized operator.*

Proof. Let $\{x, e_1, e_2, \ldots, e_n\}$ be a basis of L such that $\{e_1, e_2, \ldots e_n\}$ is a basis of N and let ad_x diagonalized operator on N. Then the Jordan form of the operator ad_x has the form:

$$ad_x = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

Consequently,

$$[e_i, x] = \lambda_i e_i, \quad 1 \leq i \leq n.$$

Case 1. Let $\lambda_i \neq \lambda_j$ for $i \neq j$. Let $d \in Der(L)$, then

$$\begin{cases} d(x) = \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n, \\ d(e_i) = \alpha_{i,1} e_1 + \alpha_{i,2} e_2 + \cdots + \alpha_{i,n} e_n, \ 1 \leq i \leq n. \end{cases}$$

From the property of derivation we have

$$d([e_i, x]) = [d(e_i), x] + [e_i, d(x)] =$$

$$= [\alpha_{i,1} e_1 + \alpha_{i,2} e_2 + \cdots + \alpha_{i,n} e_n, x] + [e_i, \beta_1 e_1 + \beta_2 e_2 + \cdots + \beta_n e_n] =$$

$$= \alpha_{i,1} \lambda_1 e_1 + \alpha_{i,2} \lambda_2 e_2 + \cdots + \alpha_{i,n} \lambda_n e_n.$$

On the other hand,

$$d([e_i, x]) = \lambda_i d(e_i) = \lambda_i (\alpha_{i,1} e_1 + \alpha_{i,2} e_2 + \cdots + \alpha_{i,n} e_n).$$

Comparing the coefficients at the basis elements we obtain

$$\alpha_{i,j}(\lambda_i - \lambda_j) = 0, \quad 1 \leq j \leq n.$$

Since $\lambda_i \neq \lambda_j$, we get $\alpha_{i,j} = 0$ for $i \neq j$.

Therefore, the derivations of the algebra L have the following form:

$$Der(L) = \begin{pmatrix} 0 & \beta_1 & \beta_2 & \ldots & \beta_n \\ 0 & \alpha_{1,1} & 0 & \ldots & 0 \\ 0 & 0 & \alpha_{2,2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \alpha_{n,n} \end{pmatrix}$$

Let Δ be a local derivation on L and let

$$\begin{cases} \Delta(x) = \xi x + \xi_1 e_1 + \xi_2 e_2 + \cdots + \xi_n e_n, \\ \Delta(e_i) = \zeta_i x + \zeta_{i,1} e_1 + \zeta_{i,2} e_2 + \cdots + \zeta_{i,n} e_n, \ 1 \le i \le n. \end{cases}$$

From the equalities $\Delta(x) = d_x(x)$ and $\Delta(e_i) = d_{e_i}(e_i)$ for $1 \le i \le n$, it is easy to obtain, that

$$\xi = \zeta_i = \zeta_{j,k} = 0, \quad 1 \le i, j, k \le n, \ j \ne k.$$

Consequently, we have $\Delta \in Der(L)$, i.e., any local derivation on L is a derivation.

Case 2. Let $\lambda_i = \lambda_j$ for some i and j. Without loss of generality, we can assume

$$\lambda_1 = \cdots = \lambda_s, \ \lambda_{s+1} = \cdots = \lambda_{s+p}, \quad \ldots \quad \lambda_{n-q} = \cdots = \lambda_n.$$

Similarly, to the case 1, using the property of derivation we describe all derivation of the algebra L and odtain that the general matrix form of $Der(L)$ has the form

$$\begin{pmatrix}
0 & \beta_1 & ... & \beta_s & \beta_{s+1} & ... & \beta_{s+p} & ... & \beta_{n-q} & ... & \beta_n \\
0 & \alpha_{1,1} & ... & \alpha_{1,s} & 0 & ... & 0 & ... & 0 & ... & 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
0 & \alpha_{s,1} & ... & \alpha_{s,s} & 0 & ... & 0 & ... & 0 & ... & 0 \\
0 & 0 & ... & 0 & \alpha_{s+1,s+1} & ... & \alpha_{s+1,s+p} & ... & 0 & ... & 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
0 & 0 & ... & 0 & \alpha_{s+p,s+1} & ... & \alpha_{s+p,s+p} & ... & 0 & ... & 0 \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
0 & 0 & ... & 0 & 0 & ... & 0 & ... & \alpha_{n-q,n-q} & ... & \alpha_{n-q,n} \\
... & ... & ... & ... & ... & ... & ... & ... & ... & ... & ... \\
0 & 0 & ... & 0 & 0 & ... & 0 & ... & \alpha_{n,n-q} & ... & \alpha_{n,n}
\end{pmatrix}$$

It should be noted that all parameters of the previous matrix are free. Thus, for any local derivation Δ on L, considering $\Delta(x)$ and $\Delta(e_i)$ for $1 \le i \le n$, analogously to the case 1, it is not difficult to show that Δ is a derivation. Therefore, any local derivation on L is a derivation.

Now let ad_x be a not diagonalized operator on N. Then in this Jordan form

$$ad_x = \begin{pmatrix} J_1 & 0 & \ldots & 0 \\ 0 & J_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & ... & J_s \end{pmatrix}$$

there exists a Jordan block with order $k(k \geq 2)$. Without loss of generality one can suppose J_1 has order $k \geq 2$. Then the table of multiplication of the algebra L on the basis $\{x, e_1, e_2, \dots e_k, e_{k+1}, \dots e_n\}$ has the following form:

$$\begin{cases} e_1 x = \lambda_1 e_1 + e_2, \\ e_2 x = \lambda_1 e_2 + e_3, \\ \dots\dots\dots\dots \\ e_k x = \lambda_1 e_k, \\ e_i x = \lambda_i e_i + \mu_i e_{i+1}, \ k+1 \leq i \leq n, \end{cases}$$

where $\mu_i = 0; 1$.

By the direct verification of the property of derivation we obtain that the general form of the matrix of $Der(L)$ has the form:

$$\begin{pmatrix} 0 & \beta_1 & \beta_2 & ... & \beta_{k-1} & \beta_k & \beta_{k+1} & ... & \beta_n \\ 0 & \alpha_{1,1} & \alpha_{1,2} & \dots & \alpha_{1,k-1} & \alpha_{1,k} & 0 & ... & 0 \\ 0 & 0 & \alpha_{1,1} & ... & \alpha_{1,k-2} & \alpha_{1,k-1} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & \\ 0 & 0 & 0 & ... & \alpha_{1,1} & \alpha_{1,2} & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 & \alpha_{1,1} & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 & 0 & H_2 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 & 0 & 0 & ... & H_s \end{pmatrix}$$

where H_i are block matrices with the same order of Jordan blocks J_i.

Consider the linear operator $\Delta : L \to L$ by

$$\Delta(x) = 0, \quad \Delta(e_i) = e_i, \quad 1 \leq i \leq k-1,$$

$$\Delta(e_k) = 2e_k, \quad \Delta(e_i) = 0, \quad k+1 \leq i \leq n.$$

It is easy to see that Δ is not a derivation, but it is local derivation.

Indeed, for any element $y = \gamma x + \eta_1 e_1 + \eta_2 e_2 + \dots + \eta_n e_n \in L$ we consider

$$\Delta(y) = \Delta(\gamma x + \eta_1 e_1 + \eta_2 e_2 + \dots + \eta_n e_n) = \eta_1 e_1 + \eta_2 e_2 + \dots + \eta_{k-1} e_{k-1} + 2\eta_k e_k.$$

Consider the derivation d_y such that

$$d_y(x) = 0, \qquad d_y(e_i) = 0, \quad k+1 \leq i \leq n,$$

$$d_y(e_i) = \alpha_{1,1} e_i + \alpha_{1,2} e_{i+1} + \dots + \alpha_{1,k-i+1} e_k, \quad 1 \leq i \leq k.$$

Then

$$d_y(y) = d_y(\gamma x + \eta_1 e_1 + \eta_2 e_2 + \dots + \eta_n e_n) =$$

$$= \eta_1 \alpha_{1,1} e_1 + (\eta_2 \alpha_{1,1} + \eta_1 \alpha_{1,2}) e_2 + \dots + (\eta_k \alpha_{1,1} + \eta_{k-1} \alpha_{1,2} + \dots + \eta_1 \alpha_{1,k}) e_k.$$

Suppose that $\Delta(y) = d_y(y)$. Then we get

$$\begin{cases} \eta_1 = \eta_1 \alpha_{1,1}, \\ \eta_2 = \eta_2 \alpha_{1,1} + \eta_1 \alpha_{1,2}, \\ \dots\dots\dots \\ \eta_{k-1} = \eta_{k-1} \alpha_{1,1} + \eta_{k-2} \alpha_{1,2} + \dots + \eta_1 \alpha_{1,k-1}, \\ 2\eta_k = \eta_k \alpha_{1,1} + \eta_{k-1} \alpha_{1,2} + \dots + \eta_1 \alpha_{1,k}. \end{cases}$$

Note that this system has a solution with respect to $\alpha_{i,j}$ for any parameters η_i. Indeed,

- if $\eta_1 \neq 0$, then

$$\alpha_{1,1} = 1, \ \alpha_{1,2} = \dots = \alpha_{1,k-1} = 0, \ \alpha_{1,k} = \frac{\eta_k}{\eta_1},$$

- if $\eta_1 = \dots = \eta_{s-1} = 0$ и $\eta_s \neq 0, \ 2 \leq s \leq k-1$, then

$$\alpha_{1,1} = 1, \ \alpha_{1,2} = \dots = \alpha_{1,k-s} = 0, \ \alpha_{1,k-s+1} = \frac{\eta_k}{\eta_s},$$

- if $\eta_1 = \dots = \eta_{k-1} = 0$ и $\eta_k \neq 0$ then we have $\alpha_{1,1} = 2$.

Thus, we proved that Δ is a local derivation.

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