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Naturally Graded Leibniz Algebras with Characteristic Sequence $(n-m,m)$

K. K. Masutova*, **B. A. Omirov****, **and A. Kh. Khudoyberdiyev*****

Institute of Mathematics and Information Technologies, Academy of Sciences of Uzbekistan, Tashkent

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Abstract—We consider the classification problem for special classes of nilpotent Leibniz algebras. Namely, we consider "naturally" graded nilpotent n-dimensional Leibniz algebras for which the right multiplication operator (by the generic element) has two Jordan blocks of dimensions m and $n-m$. Earlier, the problem of classifying such algebras was studied for $m < 4$. The present paper continues these studies for the case $m \geq 4$.

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1. INTRODUCTION

The present paper is devoted to the study of Leibniz algebras, which are "noncommutative" generalizations of Lie algebras. The notion of Leibniz algebra was introduced at the beginning of the 1990s by the French mathematician J. L. Loday [1] and was defined by the identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y].$$

Recall that the study of finite-dimensional Lie algebras was reduced to the study of nilpotent algebras in [2], [3]. Methods and approaches related to nilpotent Lie algebras were studied in numerous papers [4]–[6], etc. In this connection, it is natural to apply these results and methods to the study of Leibniz algebras. Since the description of nilpotent Lie algebras is itself a boundless problem, the study of nilpotent Leibniz algebras must be accompanied by imposing additional conditions such as constraints on the index of nilpotency of the algebra, on the characteristic sequence, grading, etc. Note that the classes of null-filiform and filiform Leibniz algebras were studied in [7], [8]. Naturally graded quasifiliform Leibniz algebras were studied in [9], and the case of naturally graded p-filiform Leibniz algebras was considered in [10].

In studying naturally graded quasifiliform Leibniz algebras [9], it was noted that, in contrast to the Lie case, the Leibniz algebras contain a class of n-dimensional algebras whose characteristic sequence is $(n-2,2)$. The subsequent study of naturally graded algebras with characteristic sequence equal to $(n-3,3)$ shows that the class of non-Lie Leibniz algebras in this case is sufficiently wide. In the present paper, we isolate non-Lie Leibniz algebras with characteristic sequence equal to $(n-m,m)$, and provide a description of such algebras. Moreover, we obtain expressions for the changes of the parameters in the multiplication table of such algebras under an isomorphism; these expressions can be used to obtain a complete classification in fixed dimension and a given value of m.

*E-mail: kamilyam81@mail.ru
**E-mail: omirovb@mail.ru
***E-mail: khabror@mail.ru

2. PRELIMINARIES

In this section, we present some necessary definitions and results.

Definition 2.1. An algebra L over a field F is called a *Leibniz algebra* if, for all elements $x, y, z \in L$, the following Leibniz identity holds:

$$[x,[y,z]] = [[x,y],z] - [[x,z],y],$$

where $[\cdot,\cdot]$ is multiplication in L.

For an arbitrary Leibniz algebra L, let us define the lower central series

$$L^1 = L, \quad L^{k+1} = [L^k, L^1], \qquad k \geq 1.$$

Definition 2.2. A Leibniz algebra L is said to be *nilpotent* if there exists an $s \in \mathbb{N}$ such that $L^s = 0$. The minimal number s possessing such a property is called the *index of nilpotency* or the *nilindex* of the algebra L.

Note that the index of nilpotency of an n-dimensional nilpotent Leibniz algebra is at most $n+1$.

Definition 2.3. Let L be a Leibniz algebra of dimension n. The algebra L we said to be *null-filiform* if $\dim L^i = (n+1) - i$, $1 \leq i \leq n+1$.

It is readily seen from the definition that the fact that an algebra L is null-filiform is equivalent to the fact that it has the maximal index of nilpotency.

The following theorem asserts that, in each dimension, up to isomorphism, there exists a unique null-filiform Leibniz algebra.

Theorem 2.4 ([7]). *In any n-dimensional null-filiform Leibniz algebra L, there exists a basis $\{e_1, e_2, \dots, e_n\}$ such that multiplication in the algebra L has the following form*:

$$[e_i, e_1] = e_{i+1}, \qquad 1 \leq i \leq n-1$$

(*the omitted products vanish*).

The set $R(L) = \{x \in L : [y,x] = 0 \text{ for any } y \in L\}$ is called the *right annihilator* of the algebra L.

Let L be an n-dimensional nilpotent Leibniz algebra, and let x be an arbitrary element from the set $L \setminus [L,L]$. For the nilpotent right multiplication operator R_x, we define the decreasing sequence $C(x) = (n_1, n_2, \dots, n_k)$ consisting of the dimensions of the Jordan blocks of the operator R_x. On the set of such sequences, we define the lexicographic order.

Definition 2.5. The sequence $C(L) = \max_{x \in L \setminus L^2} C(x)$ is called the *characteristic sequence* of the algebra L.

Example 1 ([7]). Let L be an n-dimensional Leibniz algebra. L is Abelian if and only if we have $C(L) = (1, 1, \dots, 1)$.

Example 2 ([7]). An n-dimensional Leibniz algebra L is null-filiform if and only if $C(L) = (n)$.

Let us define the notion of a naturally graded algebra.

Let L be a finite-dimensional nilpotent Leibniz algebra. Set

$$\operatorname{gr}(L)_i := L^i / L^{i+1}, \qquad 1 \leq i \leq s-1,$$

where s is the nilindex of the algebra L, and denote

$$\operatorname{gr} L = \operatorname{gr}(L)_1 \oplus \operatorname{gr}(L)_2 \oplus \cdots \oplus \operatorname{gr}(L)_{s-1}.$$

Since $[\operatorname{gr}(L)_i, \operatorname{gr}(L)_j] \subseteq \operatorname{gr}(L)_{i+j}$, we obtain a graded algebra $\operatorname{gr} L$. The grading constructed above will be called the *natural grading*. If a Leibniz algebra G is isomorphic to the algebra $\operatorname{gr} L$, then G is called a *naturally graded Leibniz algebra*.

Definition 2.6. An algebra L is said to be *decomposable* if there exist subalgebras M and N of the algebra L such that $L = M \oplus N$ and $[M,N] = [N,M] = 0$.

3. DESCRIPTION OF NATURALLY GRADED LEIBNIZ ALGEBRAS WITH CHARACTERISTIC SEQUENCE $C(L) = (n-m, m)$, $m \geq 4$

Taking into account results from [9], [10], in what follows, we shall consider n-dimensional naturally graded Leibniz algebras with characteristic sequence $C(L) = (n-m, m)$ for $m \geq 4$.

The definition of the characteristic sequence of a Leibniz algebra implies the existence a basis $\{e_1, e_2, \ldots, e_{n-m}, f_1, \ldots, f_m\}$ such that the matrix of the right multiplication operator R_{e_1} has one of the following two forms:

$$\text{I)} \quad \begin{pmatrix} J_{n-m} & 0 \\ 0 & J_m \end{pmatrix}, \qquad \text{II)} \quad \begin{pmatrix} J_m & 0 \\ 0 & J_{n-m} \end{pmatrix}, \qquad \text{where} \quad n-m \geq m.$$

Definition 3.1. A Leibniz algebra L is called an *algebra of type* I if there exists an element $e_1 \in L \setminus L^2$ such that the right multiplication operator R_{e_1} has a matrix of the form

$$\begin{pmatrix} J_{n-m} & 0 \\ 0 & J_m \end{pmatrix};$$

if R_{e_1} has a matrix of the second form, then L is called an *algebra of type* II.

Suppose that M and N are null-filiform Leibniz algebras with $\dim M = n-m$ and $\dim N = m$, respectively. Therefore, $C(M) = (n-m)$ and $C(N) = (m)$. It is readily verified that the decomposable algebra $L = M \oplus N$ has the characteristic sequence $C(L) = (n-m, m)$. In the following theorem, it is asserted that the decomposable Leibniz algebras whose characteristic sequence is equal to $C(L) = (n-m, m)$ consist only of the direct sum of two null-filiform Leibniz algebras.

Theorem 3.2. *Let L be a Leibniz algebra with characteristic sequence $C(L) = (n-m, m)$. The algebra L is decomposable if and only if M and N are null-filiform Leibniz algebras with $\dim M = n-m$, $\dim N = m$.*

Proof. *Necessity.* Let L be a decomposable Leibniz algebra whose characteristic sequence is $C(L) = (n-m, m)$, i.e., there exist subalgebras M and N of the algebra L such that $L = M \oplus N$ and $[M, N] = [N, M] = 0$. Then there exists an element $a \in L$ such that $a = x + y$, $x \in M$, $y \in N$, and the matrix of the right multiplication operator R_a has the following form:

$$\begin{pmatrix} J_{n-m} & 0 \\ 0 & J_m \end{pmatrix}.$$

Therefore, there exist bases $\{e_1, e_2, \ldots, e_{n-m}\} \subseteq M$ and $\{f_1, \ldots, f_m\} \subseteq N$ such that

$$ae_i = e_{i+1}, \quad 1 \leq i \leq n-m-1, \qquad af_i = f_{i+1}, \quad 1 \leq i \leq m-1.$$

Hence we have

$$xe_i = e_{i+1}, \quad 1 \leq i \leq n-m-1, \qquad yf_i = f_{i+1}, \quad 1 \leq i \leq m-1.$$

Then the matrix of restriction of the right multiplication operator to M (respectively, to N) has the form (J_{n-m}) (respectively, (J_m)). Thus, it follows from Example 2 that M and N are null-filiform algebras.

Sufficiency. Let M and N be null-filiform Leibniz algebras with $\dim M = n-m$ and $\dim N = m$. Then there exist bases $\{e_1, e_2, \ldots, e_{n-m}\}$ and $\{f_1, \ldots, f_m\}$ in M and N, respectively, such that

$$e_i e_1 = e_{i+1}, \qquad 1 \leq i \leq n-m-1,$$
$$f_i f_1 = f_{i+1}, \qquad 1 \leq i \leq m-1.$$

Take an element $a = e_1 + f_1 \in L = M \oplus N$. Consider

$$R_a(x) = [x, a] = [x, e_1 + f_1] = [x, e_1] + [x, f_1] = R_{e_1}(x) + R_{f_1}(x).$$

Obviously,

$$R_a(e_i) = e_{i+1}, \quad 1 \le i \le n - m - 1, \qquad \text{and} \qquad R_a(f_i) = f_{i+1}, \quad 1 \le i \le m - 1.$$

Then the matrix of the operator R_a has the following form:

$$\begin{pmatrix} J_{n-m} & 0 \\ 0 & J_m \end{pmatrix};$$

therefore, $C(a) = (n - m, m)$.

Suppose that there exists an element $y \in L \setminus L^2$: $C(y) > C(a) = (n - m, m)$. Then $C(y) = (k_1, \dots, k_s)$ satisfies $k_1 > n - m$. Therefore, there exists an element $z \in L$:

$$[\dots[[z, \underbrace{y], y], \dots, y]}_{k_1 \text{ times}} \ne 0.$$

Since the nilindex of L is $n - m$, i.e., $L^{n-m} = \{0\}$, it follows that $k_1 \le n - m$; a contradiction. But if

$$C(y) = (n - m, k_2, \dots, k_s), \qquad \text{where} \quad \sum_{p=2}^{s} k_p = m,$$

then $k_2 > m$; a contradiction. Therefore, $C(L) = (n - m, m)$. □

Theorem 3.2 provides a classification of naturally graded decomposable Leibniz algebras with characteristic sequence equal to $(n - m, m)$, $m \ge 4$. In what follows, we shall consider indecomposable Leibniz algebras and, for convenience, we shall write ab instead of the product $[a, b]$.

Let L be an n-dimensional indecomposable naturally graded Leibniz algebra over the field F. Suppose that $x = (x_1, x_2, \dots, x_n) \in F^n$.

Let us introduce the maps $A_{i,j}, B_{i,j} \colon F^n \to F$ as follows:

$$A_{i,j}(x) = \sum_{l=0}^{j-1} (-1)^l C_{j-1}^l x_{l+i}, \qquad 2 \le i + j \le n,$$
$$B_{i,j}(x) = \sum_{l=0}^{m-i} (-1)^l C_{j-1}^l x_{l+i}, \qquad 2 \le i + j \le n.$$

Here and elsewhere, C_n^m denotes the binomial coefficient $\binom{n}{m}$.

In what follows, we shall need the following lemma.

Lemma 3.3. *For arbitrary $i, j \in \mathbb{N}$, the following equalities are valid*:

$$A_{i,j}(x) - A_{i+1,j}(x) = A_{i,j+1}(x),$$
$$B_{i,j}(x) - B_{i+1,j}(x) = B_{i,j+1}(x).$$

Proof. The proof is carried out by induction, making use of the equality $C_{j-1}^l + C_{j-1}^{l-1} = C_j^l$. □

Theorem 3.4. *Let L be a Leibniz algebra with characteristic sequence $C(L) = (n-m, m)$ of type* I. *Then there exists a basis $\{e_1, e_2, \dots, e_{n-m}, f_1, \dots, f_m\}$ of the algebra L in which the multiplication table has the following form*:

$$\begin{aligned}
&e_i e_1 = e_{i+1}, && 1 \le i \le n-m-1,\\
&f_i e_1 = f_{i+1}, && 1 \le i \le m-1,\\
&e_i f_j = A_{i,j}(\alpha) e_{i+j} + A_{i,j}(\beta) f_{i+j}, && 1 \le i \le m-j,\\
&f_i f_j = A_{i,j}(\gamma) e_{i+j} + A_{i,j}(\delta) f_{i+j}, && 1 \le i \le m-j,\\
&e_i f_j = A_{i,j}(\alpha) e_{i+j}, && m-j+1 \le i \le n-m-j,\\
&f_i f_j = B_{i,j}(\gamma) e_{i+j}, && m-j+1 \le i \le \min\{m, n-m-j\}
\end{aligned} \tag{3.1}$$

(*the other products vanish*).

Proof. The condition

$$R_{e_1} = \begin{pmatrix} J_{n-m} & 0 \\ 0 & J_m \end{pmatrix},$$

where $n-m \ge m$, implies that there exists a basis $\{e_1, e_2, \dots, e_{n-m}, f_1, \dots, f_m\}$ such that

$$\begin{aligned}
&e_i e_1 = e_{i+1}, \quad 1 \le i \le n-m-1, \qquad e_{n-m} e_1 = 0,\\
&f_i e_1 = f_{i+1}, \quad 1 \le i \le m-1, \qquad f_m e_1 = 0.
\end{aligned}$$

It is readily verified that

$$\begin{aligned}
&L_i = \langle e_i, f_i \rangle \quad \text{for} \quad 1 \le i \le m,\\
&L_i = \langle e_i \rangle \quad \text{for} \quad m+1 \le i \le n-m,\\
&\langle e_2, \dots, e_{n-m} \rangle \in R(L).
\end{aligned}$$

Consider multiplication in L by the element f_1 on the right. Let

$$\begin{aligned}
&e_i f_1 = \alpha_i e_{i+1} + \beta_i f_{i+1}, \quad 1 \le i \le m-1,\\
&e_i f_1 = \alpha_i e_{i+1}, \quad m \le i \le n-m-1, \qquad e_{n-m} f_1 = 0,\\
&f_i f_1 = \gamma_i e_{i+1} + \delta_i f_{i+1}, \quad 1 \le i \le m-1, \qquad f_m f_1 = \gamma_m e_{m+1}.
\end{aligned}$$

Applying induction on j and the Leibniz identity for the products

$$e_i f_j, \quad 1 \le i \le n-m, \qquad \text{and} \qquad f_i f_j, \quad 1 \le i \le m,$$

for any value of i, we obtain multiplication on the right by the element f_j, $2 \le j \le m$.

For $j = 2$, we consider all possible values of i and obtain the following products:

- $1 \le i \le m-2$,
$$\begin{aligned}
e_i f_2 = e_i(f_1 e_1) = (e_i f_1) e_1 - e_{i+1} f_1 &= \alpha_i e_{i+2} + \beta_i f_{i+2} - (\alpha_{i+1} e_{i+2} + \beta_{i+1} f_{i+2})\\
&= (\alpha_i - \alpha_{i+1}) e_{i+2} + (\beta_i - \beta_{i+1}) f_{i+2};
\end{aligned}$$

- $i = m-1$,
$$\begin{aligned}
e_{m-1} f_2 = e_{m-1}(f_1 e_1) = (e_{m-1} f_1) e_1 - e_m f_1 &= (\alpha_{m-1} e_m + \beta_{m-1} f_m) e_1 - \alpha_m e_{m+1}\\
&= (\alpha_{m-1} - \alpha_m) e_{m+1}.
\end{aligned}$$

- $m \le i \le n-m-2$,
$$\begin{aligned}
e_i f_2 = e_i(f_1 e_1) = (e_i f_1) e_1 - e_{i+1} f_1 &= \alpha_i e_{i+2} - \alpha_{i+1} e_{i+2}\\
&= (\alpha_i - \alpha_{i+1}) e_{i+2};
\end{aligned}$$

- $i = n - m - 1$ or $i = n - m$,

$$e_i f_2 = 0;$$

- $1 \le i \le m - 2$,

$$\begin{aligned} f_i f_2 = f_i(f_1 e_1) = (f_i f_1)e_1 - f_{i+1} f_1 = \gamma_i e_{i+2} + \delta_i f_{i+2} - (\gamma_{i+1} e_{i+2} + \delta_{i+1} f_{i+2}) \\ = (\gamma_i - \gamma_{i+1}) e_{i+2} + (\delta_i - \delta_{i+1}) f_{i+2}. \end{aligned}$$

- $i = m - 1$,

$$\begin{aligned} f_{m-1} f_2 = f_{m-1}(f_1 e_1) = (f_{m-1} f_1) e_1 - f_m f_1 \\ = (\gamma_{m-1} e_m + \delta_{m-1} f_m) e_1 - \gamma_m e_{m+1} = (\gamma_{m-1} - \gamma_m) e_{m+1}. \end{aligned}$$

- $i = m$,

$$f_m f_2 = f_m(f_1 e_1) = (f_m f_1) e_1 - (f_m e_1) f_1 = \gamma_m e_{m+2}.$$

Thus, relations (3.1) hold for $j = 2$.

Suppose that relations (3.1) are valid for $j = q$, and let us prove them for $j = q + 1$.

It follows from the Leibniz identity that

$$e_i f_{q+1} = e_i(f_q e_1) = (e_i f_q) e_1 - e_{i+1} f_q, \qquad f_i f_{q+1} = f_i(f_q e_1) = (f_i f_q) e_1 - f_{i+1} f_q.$$

Using Lemma 3.3, for all i, we obtain

- $1 \le i \le m - q - 1$,

$$e_i f_{q+1} = A_{i,q+1}(\alpha) e_{i+q+1} + A_{i,q+1}(\beta) f_{i+q+1};$$

- $i = m - q$,

$$e_{m-q} f_{q+1} = A_{m-q,q+1}(\alpha) e_{m+1};$$

- $m - q + 1 \le i \le n - m - q - 1$,

$$e_i f_{q+1} = A_{i,q+1}(\alpha) e_{i+q+1};$$

- $i = n - m - q$,

$$e_{n-m-q} f_{q+1} = A_{n-m-q,q}(\alpha) e_{n-m} e_1 = 0;$$

- $n - m - q + 1 \le i \le n - m$,

$$e_i f_{q+1} = 0.$$

Similarly, for products of the form $f_i f_j$, we obtain

- $1 \le i \le m - q - 1$,

$$f_i f_{q+1} = A_{i,q+1}(\gamma) e_{i+q+1} + A_{i,q+1}(\delta) f_{i+q+1};$$

- $i = m-q$,

$$f_{m-q}f_{q+1} = B_{m-q,q+1}(\gamma)e_{m+1};$$

- $m-q+1 \le i \le \min\{n-m-q-1, m\}$,

$$f_i f_{q+1} = B_{i,q+1}(\gamma)e_{i+q+1}.$$

□

Thus, from Theorem 3.4, we obtain the following collection of structure constants defining the multiplication table of the algebra:

$$\alpha_1,\dots,\alpha_{k-1},\alpha_k,\alpha_{k+1},\dots,\alpha_{m-1},\alpha_m,\alpha_{m+1},\dots,\alpha_{n-m-1},\qquad \beta_1,\dots,\beta_{k-1},\beta_k,\beta_{k+1},\dots,\beta_{m-1},$$
$$\gamma_1,\dots,\gamma_{k-1},\gamma_k,\gamma_{k+1},\dots,\gamma_{m-1},\gamma_m,\qquad \delta_1,\dots,\delta_{k-1},\delta_k,\delta_{k+1},\dots,\delta_{m-1}.$$

Let $f_k \notin R(L)$, and let $f_{k+1} \in R(L)$, $1 \le k \le m-1$, where k is a fixed number. Then, for different k, we obtain nonintersecting classes. Indeed, the dimensions of the right annihilators of algebras from these classes will differ.

Let us present the following auxiliary lemmas.

Lemma 3.5 ([11]). *For an arbitrary polynomial P of degree less than n, the following equality holds*:

$$\sum_{i=0}^{n}(-1)^i C_n^i P(i) = 0.$$

Lemma 3.6 ([11]). *For arbitrary $a, n \in \mathbb{N}$, the following identity holds*:

$$\sum_{k=0}^{n}(-1)^k C_a^k C_{a+(n-k)-1}^{n-k} = 0.$$

Theorem 3.7. *Let L be a Leibniz algebra of type* I. *Let $f_k \notin R(L)$, $f_{k+1} \in R(L)$, $1 \le k \le m-1$. Then the following relations hold*:

$$\begin{gathered}\beta_1 = -1, \qquad \beta_i = 0, \quad 2 \le i \le k-1,\\ \beta_{k+t} = C_{k+t-1}^{k-1}\beta + (-1)^k C_{k+t-2}^{k-1}, \quad 1 \le t \le m-k-1,\end{gathered} \tag{3.2}$$

where $\beta = \beta_k$ is a fixed number.

Proof. Let $f_k \notin R(L)$, and let $f_{k+1} \in R(L)$, $1 \le k \le m-1$. Obviously,

$$f_i e_1 + e_1 f_i \in R(L), \qquad 1 \le i \le k-1.$$

Since

$$f_i e_1 + e_1 f_i = f_{i+1} + A_{1,i}(\alpha)e_{i+1} + A_{1,i}(\beta)f_{i+1}$$

and $\langle e_2,\dots,e_{n-m}\rangle \in R(L)$, we obtain $1 + A_{1,i}(\beta) = 0$.

For $i = 1$, we have $1 + A_{1,1}(\beta) = 1 + \beta_1 = 0$ and, therefore, $\beta_1 = -1$.

For $i > 1$, using the equality

$$1 + A_{1,i}(\beta) = 1 + \beta_1 + \sum_{l=1}^{i-1}(-1)^l C_{i-1}^l \beta_{l+1}$$

we obtain

$$\sum_{l=1}^{i-1}(-1)^l C_{i-1}^l \beta_{l+1} = 0, \qquad i > 2.$$

This yields

$$\beta_i = (-1)^i \sum_{l=1}^{i-2} (-1)^l C_{i-1}^l \beta_{l+1}, \qquad i > 2.$$

Using the resulting relation, we shall show by induction on i that $\beta_i = 0$ for all $2 \le i \le k-1$. Indeed, for $i = 2$, we have $\beta_2 = 0$ and, for $i = 3$, the equality $-2\beta_2 + \beta_3 = 0$ implies that $\beta_3 = 0$.

Suppose that $\beta_i = 0$ for $i = q < k-1$. Then, using

$$\sum_{l=1}^{q} (-1)^l C_q^l \beta_{l+1} = \sum_{l=1}^{q-1} (-1)^l C_q^l \beta_{l+1} + (-1)^q \beta_{q+1} = 0,$$

we obtain $\beta_{q+1} = 0$.

Thus, we have obtained $\beta_1 = -1$, $\beta_2 = 0, \dots, \beta_{k-1} = 0$.

Now consider the products $e_i f_{k+1}$, $1 \le i \le m-k-1$.

The condition $f_{k+1} \in R(L)$ implies that

$$e_1 f_{k+1} = 0 \qquad \text{and} \qquad A_{1,k+1}(\alpha) e_{k+2} + A_{1,k+1}(\beta) f_{k+2} = 0.$$

Therefore,

$$\begin{cases} A_{1,k+1}(\alpha) = 0, \\ A_{1,k+1}(\beta) = 0. \end{cases}$$

Since

$$A_{1,k+1}(\beta) = \sum_{l=0}^{k} (-1)^l C_k^l \beta_{l+1} = \beta_1 + 0 + \dots + 0 + (-1)^{k-1} k \beta_k + (-1)^k \beta_{k+1},$$

we have $\beta_{k+1} = k\beta + (-1)^k$.

Similarly, considering products of the form $e_i f_{k+1}$, $2 \le i \le m-k-1$, we have $A_{i,k+1}(\beta) = 0$. Let us prove the following dependence by induction:

$$\beta_{k+t} = C_{k+t-1}^{k-1} \beta + (-1)^k C_{k+t-2}^{k-1}, \qquad 1 \le t \le m-k-1.$$

The base of the induction was obtained earlier. It follows from the condition

$$A_{i,k+1}(\beta) = \sum_{l=0}^{k} (-1)^l C_k^l \beta_{l+i}$$

that

$$\beta_{k+i} = (-1)^{k+1} \sum_{l=0}^{k-1} (-1)^l C_k^l \beta_{l+i}, \qquad 2 \le i \le m-k-1.$$

Suppose that the equality

$$\beta_{k+i} = C_{k+i-1}^{k-1} \beta + (-1)^k C_{k+i-2}^{k-1}$$

is valid for all $1 \le i \le q-1$ $(2 \le q \le m-k-1)$; let us prove it for $i = q$. Using the equality $A_{i,k+1}(\beta) = 0$, we obtain

$$\sum_{l=0}^{k} (-1)^l C_k^l \beta_{l+q} = (-1)^{k-q} C_k^{k-q} \beta + \sum_{l=1-q}^{-1} (-1)^{l+k} C_k^{l+k} \beta_{k+(l+q)} + (-1)^k \beta_{k+q} = 0.$$

Using Lemma 3.6 in the following equalities:

$$\beta_{k+q} = (-1)^{k+1} \left[(-1)^{k-q} C_k^{k-q} \beta + (-1)^k \sum_{l=1-q}^{-1} (-1)^l C_k^{l+k} (C_{k+l+q-1}^{k-1} \beta + (-1)^k C_{k+l+q-2}^{k-1}) \right]$$

$$= (-1)^{1-q}C_k^{k-q}\beta - \sum_{l=1-q}^{-1}(-1)^l(C_k^{k+l}C_{k+l+q-1}^{k-1}\beta + (-1)^k C_k^{k+l}C_{k+l+q-2}^{k-1})$$

$$= (-1)^{1-q}C_k^{k-q}\beta - \sum_{l=1-q}^{-1}(-1)^l C_k^{k+l}C_{k+l+q-1}^{k-1}\beta - (-1)^k\sum_{l=1-q}^{-1}(-1)^l C_k^{k+l}C_{k+l+q-2}^{k-1}$$

$$= -\beta\sum_{l=-q}^{-1}(-1)^l C_k^{k+l}C_{k+l+q-1}^{k-1} - (-1)^k\sum_{l=1-q}^{-1}(-1)^l C_k^{k+l}C_{k+l+q-2}^{k-1}$$

$$= -\beta\sum_{l=1}^{q}(-1)^l C_k^{k-l}C_{k-l+q-1}^{k-1} - (-1)^k\sum_{l=1}^{q-1}(-1)^l C_k^{k-l}C_{k-l+q-2}^{k-1}$$

$$= \beta\, C_{k+q-1}^{k-1} - \beta\sum_{l=0}^{q}(-1)^l C_k^{l}C_{k-l+q-1}^{k-1} + (-1)^k C_{k+q-2}^{k-1} - (-1)^k\sum_{l=0}^{q-1}(-1)^l C_k^{l}C_{k-l+q-2}^{k-1}$$

$$= \beta C_{k+q-1}^{k-1} + (-1)^k C_{k+q-2}^{k-1},$$

we find that

$$\beta_{k+q} = \beta C_{k+q-1}^{k-1} + (-1)^k C_{k+q-2}^{k-1}.$$

Therefore, for $i = q$ formula (3.2) is also valid. □

Theorem 3.7 shows that the constants $\beta_{k+1}, \dots, \beta_{m-1}$ are expressed in terms of β_k. In the subsequent theorems, we show that the constants α_i, γ_i, δ_i, $i \ge k+1$, can also be linearly expressed in terms of α_i, γ_i, δ_i, $1 \le i \le k$.

Let us present the following auxiliary lemma.

Lemma 3.8. *For arbitrary $i, k, l \in \mathbb{N}$, the following equality holds*:

$$C_{k+i}^l - \sum_{p=0}^{i-1}(-1)^p C_{k+i}^{k+p}C_{k+p}^{l}C_{k-l+p-1}^{k-l-1} = (-1)^i C_{k+i}^{l}C_{k-l+i-1}^{k-l-1}. \tag{3.3}$$

Proof. The proof is carried out by induction, making use of Lemma 3.6. □

Theorem 3.9. *Let L be a Leibniz algebra of type* I. *Let $f_k \notin R(L)$, $f_{k+1} \in R(L)$, $1 \le k \le m-1$. Then*

$$\alpha_{k+t} = (-1)^{k+1}\sum_{l=0}^{k-1}(-1)^l C_{k+t-1}^{l}C_{k-l+t-2}^{k-l-1}\alpha_{l+1}, \qquad 1 \le t \le m-k-1. \tag{3.4}$$

Proof. It follows from the assumptions of the theorem that $e_1 f_{k+t} = 0$ for $1 \le t \le m-k-1$. Then

$$A_{1,k+t}(\alpha) = \sum_{l=0}^{k+t-1}(-1)^l C_{k+t-1}^{l}\alpha_{l+1} = 0,$$

whence

$$\alpha_{k+t} = (-1)^{k+t}\sum_{l=0}^{k+t-2}(-1)^l C_{k+t-1}^{l}\alpha_{l+1}, \qquad 1 \le t \le m-k-1.$$

Using the resulting relations, let us prove equalities (3.4) by induction.

For $t = 1$, we obviously have

$$\alpha_{k+1} = (-1)^{k+1} \sum_{l=0}^{k-1} (-1)^l C_k^l \alpha_{l+1}.$$

For $t \le q < m - k - 1$, suppose that relations (3.4) hold; let us prove them for $t = q + 1$.

In view of the relations

$$\alpha_{k+t} = (-1)^{k+t} \sum_{l=0}^{k+t-2} (-1)^l C_{k+t-1}^l \alpha_{l+1},$$
$$\alpha_{k+t} = (-1)^{k+1} \sum_{l=0}^{k-1} (-1)^l C_{k+t-1}^l C_{k-l+t-2}^{k-l-1} \alpha_{l+1}, \qquad t \le q,$$

and equality (3.3) from the chain of equalities

$$\begin{aligned}
\alpha_{k+q+1} &= (-1)^{k+q+1} \sum_{l=0}^{k+q-1} (-1)^l C_{k+q}^l \alpha_{l+1} \\
&= (-1)^{k+q+1} \biggl(\sum_{l=0}^{k-1} (-1)^l C_{k+q}^l \alpha_{l+1} + \sum_{l=k}^{k+q-1} (-1)^l C_{k+q}^l \alpha_{l+1} \biggr) \\
&= (-1)^{k+q+1} \biggl(\sum_{l=0}^{k-1} (-1)^l C_{k+q}^l \alpha_{l+1} + \sum_{l=0}^{q-1} (-1)^{l+k} C_{k+q}^{l+k} \alpha_{l+k+1} \biggr) \\
&= (-1)^{k+q+1} \sum_{l=0}^{k-1} (-1)^l C_{k+q}^l \alpha_{l+1} \\
&\qquad + (-1)^{k+q+1} \sum_{l=0}^{q-1} (-1)^{l+k} C_{k+q}^{k+l} \biggl((-1)^{k+1} \sum_{p=0}^{k-1} (-1)^p C_{k+l}^p C_{k-p+l-1}^{k-p-1} \alpha_{p+1} \biggr) \\
&= (-1)^{k+q+1} \biggl[\sum_{l=0}^{k-1} (-1)^l C_{k+q}^l \alpha_{l+1} - \sum_{p=0}^{q-1} (-1)^p C_{k+q}^{k+p} \sum_{l=0}^{k-1} (-1)^l C_{k+p}^l C_{k-l+p-1}^{k-l-1} \alpha_{l+1} \biggr] \\
&= (-1)^{k+q+1} \sum_{l=0}^{k-1} (-1)^l \biggl[C_{k+q}^l - \sum_{p=0}^{q-1} (-1)^p C_{k+q}^{k+p} C_{k+p}^l C_{k-l+p-1}^{k-l-1} \biggr] \alpha_{l+1} \\
&= (-1)^{k+1} \sum_{l=0}^{k-1} (-1)^l C_{k+q}^l C_{k-l+q-1}^{k-l-1} \alpha_{l+1}
\end{aligned}$$

we find that, for $t = q + 1$, relations (3.4) hold; thus, Theorem 3.9 is proved. □

Remark 3.10. Using Theorem 3.9, we have obtained the dependence of the structure constants $\alpha_{k+1}, \dots, \alpha_{m-1}$ in terms of $\alpha_1, \dots, \alpha_k$. In the case $n - m > m$, the parameters $\alpha_m, \dots, \alpha_{n-m-1}$ are found from the equalities $e_i f_{k+t} = 0$ for $1 \le i \le n - 2m$, $1 \le t \le m - k$. Thus, relations (3.4) are extended to the case $m - k \le t \le n - m - k - 1$, i.e.,

$$\alpha_{k+t} = (-1)^{k+1} \sum_{l=0}^{k-1} (-1)^l C_{k+t-1}^l C_{k-l+t-2}^{k-l-1} \alpha_{l+1}, \qquad 1 \le t \le n - m - k - 1.$$

Taking identities $f_1 f_{k+t} = 0$, $1 \le t \le m - k$, into account, we can prove the following theorem just as we proved Theorem 3.9.

Theorem 3.11. *Let L be a Leibniz algebra of type* I. *Let $f_k \notin R(L)$, and let $f_{k+1} \in R(L)$, $1 \le k \le m-1$. Then*

$$\gamma_{k+t} = (-1)^{k+1} \sum_{l=0}^{k-1} (-1)^l C_{k+t-1}^{l} C_{k-l+t-2}^{k-l-1} \gamma_{l+1}, \qquad 1 \le t \le m-k,$$

$$\delta_{k+t} = (-1)^{k+1} \sum_{l=0}^{k-1} (-1)^l C_{k+t-1}^{l} C_{k-l+t-2}^{k-l-1} \delta_{l+1}, \qquad 1 \le t \le m-k-1.$$

Remark 3.12. Using Theorem 3.11, we obtain the dependence δ_{k+t} in terms of the constants $\delta_1, \delta_2, \dots, \delta_k$. Taking into account the condition $f_i f_1 + f_1 f_i \in R(L)$, $1 \le i \le m$, it is easy to see that all the odd δ_i are linearly expressed in terms of the even ones, i.e., the number of free parameters decreases twofold.

Let L be a naturally graded Leibniz algebra with characteristic sequence $C(L) = (n-m, m)$ of type II.

Theorem 3.13. *Let L be a Leibniz algebra with characteristic sequence $C(L) = (n-m, m)$ of type* II. *Then, in L, there exists a basis $\{e_1, e_2, \dots, e_m, f_1, \dots, f_{n-m}\}$ such that multiplication in the algebra is of the following form*:

$$\begin{aligned}
&e_i e_1 = e_{i+1}, && 1 \le i \le m-1,\\
&f_i e_1 = f_{i+1}, && 1 \le i \le n-m-1,\\
&e_i f_j = A_{i,j}(\alpha) e_{i+j} + A_{i,j}(\beta) f_{i+j}, && 1 \le i \le m-j,\\
&f_i f_j = A_{i,j}(\gamma) e_{i+j} + A_{i,j}(\delta) f_{i+j}, && 1 \le i \le m-j,\\
&e_i f_j = B_{i,j}(\beta) f_{i+j}, && m-j+1 \le i \le \min\{m, n-m-j\},\\
&f_i f_j = A_{i,j}(\delta) f_{i+j}, && m-j+1 \le i \le n-m-j
\end{aligned} \tag{3.5}$$

(*the other products vanish*).

Proof. The proof is similar to that of Theorem 3.4. □

Thus, we have obtained the following collection of structure constants defining the algebra:

$$\alpha_1, \dots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \dots, \alpha_{m-1}, \qquad \beta_1, \dots, \beta_{k-1}, \beta_k, \beta_{k+1}, \dots, \beta_{m-1}, \beta_m,$$
$$\gamma_1, \dots, \gamma_{k-1}, \gamma_k, \gamma_{k+1}, \dots, \gamma_{m-1}, \qquad \delta_1, \dots, \delta_{k-1}, \delta_k, \delta_{k+1}, \dots, \delta_{m-1}, \delta_m, \delta_{m+1}, \dots, \delta_{n-m-1}.$$

Theorem 3.14. *Let L be a Leibniz algebra of type* II. *Let $f_k \notin R(L)$, and let $f_{k+1} \in R(L)$, where $1 \le k \le m-1$. Hence*

$$\beta_1 = -1, \qquad \beta_i = 0, \quad 2 \le i \le k-1,$$
$$\beta_{k+t} = C_{k+t-1}^{k-1} \beta + (-1)^k C_{k+t-2}^{k-1}, \quad 1 \le t \le m-k,$$

where $\beta = \beta_k$ is a fixed number.

Proof. The proof is similar to that of Theorem 3.7. □

Using the same arguments as for type I, we obtain the following theorem.

Theorem 3.15. *Let L be a Leibniz algebra of type* II. *Let $f_k \notin R(L)$, and let $f_{k+1} \in R(L)$, where $1 \le k \le m-1$. Then*

$$\alpha_{k+t} = (-1)^{k+1} \sum_{l=0}^{k-1} (-1)^l C_{k+t-1}^{l} C_{k-l+t-2}^{k-l-1} \alpha_{l+1}, \qquad 1 \le t \le m-k-1,$$

$$\gamma_{k+t} = (-1)^{k+1} \sum_{l=0}^{k-1} (-1)^l C_{k+t-1}^{l} C_{k-l+t-2}^{k-l-1} \gamma_{l+1}, \qquad 1 \le t \le m-k-1,$$

$$\delta_{k+t} = (-1)^{k+1}\sum_{l=0}^{k-1}(-1)^l C_{k+t-1}^{l} C_{k-l+t-2}^{k-l-1}\delta_{l+1}, \qquad 1 \le t \le m-k.$$

Proof. The proof is similar to that of Theorem 3.9. □

Remark 3.16. If $m \le k \le n-m-1$, all the fixed constants α_i, γ_i, $1 \le i \le m-1$, and β_i, $1 \le i \le m$, remain as parameters. Therefore, it suffices to find the dependence only for δ_i, $k+1 \le i \le n-m-1$, in terms of δ_i, $1 \le i \le k$. In Theorem 3.14, the constants δ_{k+t}, $1 \le t \le m-k$, are linearly expressed in terms of $\delta_1, \dots, \delta_k$ for $1 \le k \le m-1$. We can extend the relation for δ_{k+t} from Theorem 3.14 in the interval $1 \le k \le n-m-1$. Thus, we obtain

$$\delta_{k+t} = (-1)^{k+1}\sum_{l=0}^{k-1}(-1)^l C_{k+t-1}^{l} C_{k-l+t-2}^{k-l-1}\delta_{l+1}, \qquad 1 \le t \le n-m-k-1.$$

Moreover, just as in Remark 3.12, we find that the odd constants δ_i are expressed in terms of even.

4. ON TRANSFORMATIONS OF NATURALLY GRADED LEIBNIZ ALGEBRAS WITH CHARACTERISTIC SEQUENCE $C(L) = (n-m, m)$, $m \ge 4$

.

Let L be an n-dimensional naturally graded Leibniz algebra whose characteristic sequence is $C(L) = (n-m, m)$, $m \ge 4$, of type I, and let $\{e_1, e_2, \dots, e_{n-m}, f_1, \dots, f_m\}$ be a basis in L. Then, from Theorem 3.4, we find that multiplication in L is defined by the equalities (3.1). Thus, the classification problem can be reduced to the problem of finding the structure constants α_i, β_i, γ_i, and δ_i for $1 \le i \le k$.

Statement 4.1. *Let L be a Leibniz algebra of type* I, *let $f_k \notin R(L)$, and let $f_{k+1} \in R(L)$, where $1 \le k \le m-1$. Then $e'_i = A_i e_i + B_i f_i$, $f'_i = C_i e_i + D_i f_i$, $2 \le i \le k$, where*

$$\begin{gathered} A_2 = A_1^2 + A_1 B_1 \alpha_1 + B_1^2 \gamma_1, \qquad B_2 = B_1^2 \delta_1, \\ C_2 = A_1 C_1 + B_1 C_1 \alpha_1 + B_1 D_1 \gamma_1, \qquad D_2 = A_1 D_1 - B_1 C_1 + B_1 D_1 \delta_1, \\ A_i = A_{i-1}(A_1 + B_1\alpha_{i-1}) + B_{i-1} B_1 \gamma_{i-1}, \\ B_i = B_{i-1}(A_1 + B_1\delta_{i-1}) = B_1 \prod_{l=1}^{i-2}(A_1 + B_1\delta_{l+1}), \\ C_i = C_{i-1}(A_1 + B_1\alpha_{i-1}) + D_{i-1} B_1 \gamma_{i-1}, \\ D_i = D_{i-1}(A_1 + B_1\delta_{i-1}) = D_1 \prod_{l=1}^{i-2}(A_1 + B_1\delta_{l+1}). \end{gathered} \tag{4.1}$$

Proof. Consider the general transformation of the basic elements. It is well known that, for naturally graded Leibniz algebras, it suffices to consider the transformation

$$e'_1 = A_1 e_1 + B_1 f_1, \qquad f'_1 = C_1 e_1 + D_1 f_1.$$

The proof of the statement is concluded by using the products $e'_i e'_1 = e'_{i+1}$ and $f'_i f'_1 = f'_{i+1}$. □

In the general transformation of the basis elements, the new constants α'_i, β'_i, γ'_i, δ'_i appear; they must be expressed via the initial constants α_i, β_i, γ_i, δ_i.

Theorem 4.2. *Let L be a Leibniz algebra of type* I, *let $f_k \notin R(L)$, and let $f_{k+1} \in R(L)$, where $1 \le k \le m-1$. Then, in the general transformation of the basis, the parameters α_i, β_i, γ_i, δ_i, $1 \le i \le k$, take the form*

$$\alpha'_i = \frac{(A_i C_1 + D_1(\alpha_i A_i + \gamma_i B_i))D_{i+1} - B_i C_{i+1}(C_1 + \delta_i D_1)}{A_{i+1} D_{i+1} - B_{i+1} C_{i+1}}, \qquad 1 \le i \le k-1,$$

$$\beta_i' = \frac{A_{i+1}B_i(C_1+\delta_i D_1) - (A_iC_1 + D_1(\alpha_i A_i + \gamma_i B_i))B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \qquad 1 \le i \le k-1,$$
$$\gamma_i' = \frac{(C_iC_1 + D_1(\alpha_i C_i + \gamma_i D_i))D_{i+1} - D_iC_{i+1}(C_1+\delta_i D_1)}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \qquad 1 \le i \le k-1,$$
$$\delta_i' = \frac{A_{i+1}D_i(C_1+\delta_i D_1) - (C_iC_1 + D_1(\alpha_i C_i + \gamma_i D_i))B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \qquad 1 \le i \le k-1,$$
$$\alpha_k' = \frac{(A_kC_1 + D_1(\alpha_k A_k + \gamma_k B_k))(D_{k+1} + \beta_k B_1 C_k) - (B_kC_1 + D_1(\beta_k A_k + \delta_k B_k))C_{k+1}}{A_{k+1}D_{k+1} - B_{k+1}C_{k+1} + \beta_k B_1(A_{k+1}C_k - A_kC_{k+1})},$$
$$\beta_k' = \frac{A_{k+1}(C_1C_k + D_1(\beta_k A_k + \delta_k B_k)) - (B_{k+1} + \beta_k A_k B_1)(A_kC_1 + D_1(\alpha_k A_k + \gamma_k B_k))}{A_{k+1}D_{k+1} - B_{k+1}C_{k+1} + \beta_k B_1(A_{k+1}C_k - A_kC_{k+1})},$$
$$\gamma_k' = \frac{(C_1C_k + D_1(\alpha_k C_k + \gamma_k D_k))(D_{k+1} + \beta_k B_1 C_k) - (C_1D_k + D_1(\beta_k C_k + \delta_k D_k))C_{k+1}}{A_{k+1}D_{k+1} - B_{k+1}C_{k+1} + \beta_k B_1(A_{k+1}C_k - A_kC_{k+1})},$$
$$\delta_k' = \frac{A_{k+1}(C_1D_k + D_1(\beta_k C_k + \delta_k D_k)) - (B_{k+1} + \beta_k A_k B_1)(C_kC_1 + D_1(\alpha_k C_k + \gamma_k D_k))}{A_{k+1}D_{k+1} - B_{k+1}C_{k+1} + \beta_k B_1(A_{k+1}C_k - A_kC_{k+1})},$$

where A_i, B_i, C_i, D_i satisfy relations (4.1).

Proof. It follows from the assumptions of the theorem that

$$e_i = A_ie_i + B_if_i, \qquad 1 \le i \le k,$$
$$f_i = C_ie_i + D_if_i, \qquad 1 \le i \le k,$$

where the A_i, B_i, C_i, D_i are defined by identities (4.1).

For $1 \le i \le k-1$, we have

$$e_i'f_1' = (A_ie_i + B_if_i)(C_1e_1 + D_1f_1)$$
$$= (A_iC_1 + \alpha_iA_iD_1 + \gamma_iB_iD_1)e_{i+1} + (B_iC_1 + \beta_iA_iD_1 + \delta_iB_iD_1)f_{i+1}.$$

On the other hand,

$$e_i'f_1' = \alpha_i'e_{i+1}' + \beta_i'f_{i+1}' = \alpha_i'(A_{i+1}e_{i+1} + B_{i+1}f_{i+1}) + \beta_i'(C_{i+1}e_{i+1} + D_{i+1}f_{i+1}).$$

Thus, we obtain the system of equalities

$$A_{i+1}\alpha_i' + C_{i+1}\beta_i' = A_iC_1 + \alpha_iA_iD_1 + \gamma_iB_iD_1,$$
$$B_{i+1}\alpha_i' + D_{i+1}\beta_i' = B_iC_1 + \beta_iA_iD_1 + \delta_iB_iD_1;$$

this yields

$$\alpha_i' = \frac{(A_iC_1 + D_1(\alpha_iA_i + \gamma_iB_i))D_{i+1} - (B_iC_1 + D_1(\beta_iA_i + \delta_iB_i))C_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}},$$
$$\beta_i' = \frac{(B_iC_1 + D_1(\beta_iA_i + \delta_iB_i))A_{i+1} - (A_iC_1 + D_1(\alpha_iA_i + \gamma_iB_i))B_{i+1}}{A_{i+1}Di + 1 - B_{i+1}C_{i+1}}.$$

Similarly, from the equalities

$$f_i'f_1' = (C_ie_i + D_if_i)(C_1e_1 + D_1f_1)$$
$$= (C_iC_1 + \alpha_iC_iD_1 + \gamma_iD_iD_1)e_{i+1} + (D_iC_1 + \beta_iC_iD_1 + \delta_iD_iD_1)f_{i+1},$$
$$f_i'f_1' = \gamma_i'e_{i+1}' + \delta_i'f_{i+1}' = \gamma_i'(A_{i+1}e_{i+1} + B_{i+1}f_{i+1}) + \delta_i'(C_{i+1}e_{i+1} + D_{i+1}f_{i+1}),$$

we obtain

$$A_{i+1}\gamma_i' + C_{i+1}\delta_i' = C_iC_1 + \alpha_iC_iD_1 + \gamma_iD_iD_1,$$
$$B_{i+1}\gamma_i' + D_{i+1}\delta_i' = D_iC_1 + \beta_iC_iD_1 + \delta_iD_iD_1,$$

$$\gamma_i' = \frac{(C_iC_1 + D_1(\alpha_iC_i + \gamma_iD_i))D_{i+1} - (D_iC_1 + D_1(\beta_iC_i + \delta_iD_i))C_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}},$$
$$\delta_i' = \frac{(D_iC_1 + D_1(\beta_iC_i + \delta_iD_i))A_{i+1} - (C_iC_1 + D_1(\alpha_iC_i + \gamma_iD_i))B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}.$$

For $i = k$, the following relations are also valid:

$$e_k' = A_ke_k + B_kf_k,$$
$$f_k' = C_ke_k + D_kf_k.$$

Using the Leibniz identity for e_{k+1}', f_{k+1}', we find the constants α_k', β_k', γ_k', δ_k':

$$e_{k+1}' = e_k'e_1' = A_{k+1}e_{k+1} + (B_{k+1} + \beta_kA_kB_1)f_{k+1},$$
$$f_{k+1}' = f_k'e_1' = C_{k+1}e_{k+1} + (D_{k+1} + \beta_kB_1C_k)f_{k+1}.$$

Consider

$$e_k'f_1' = (A_kC_1 + \alpha_kA_kD_1 + \gamma_kB_kD_1)e_{k+1} + (B_kC_1 + \beta_kA_kD_1 + \delta_kB_kD_1)f_{k+1}.$$

On the other hand,

$$e_k'f_1' = \alpha_k'e_{k+1}' + \beta_k'f_{k+1}' = \alpha_k'(A_{k+1}e_{k+1} + (B_{k+1} + \beta_kA_kB_1)f_{k+1}) + \beta_k'(C_{k+1}e_{k+1} + (D_{k+1} + \beta_kB_1C_k)f_{k+1}).$$

Therefore,

$$A_{k+1}\alpha_k' + C_{k+1}\beta_k' = A_kC_1 + \alpha_kA_kD_1 + \gamma_kB_kD_1,$$
$$(B_{k+1} + \beta_kA_kB_1)\alpha_k' + (D_{k+1} + \beta_kB_1C_k)\beta_k' = B_kC_1 + \beta_kA_kD_1 + \delta_kB_kD_1,$$

whence

$$\alpha_k' = \frac{(A_kC_1 + D_1(\alpha_kA_k + \gamma_kB_k))(D_{k+1} + \beta_kB_1C_k) - (B_kC_1 + D_1(\beta_kA_k + \delta_kB_k))C_{k+1}}{A_{k+1}(D_{k+1} + \beta_kB_1C_k) - C_{k+1}(B_{k+1} + \beta_kA_kB_1)},$$
$$\beta_k' = \frac{A_{k+1}(B_kC_1 + D_1(\beta_kA_k + \delta_kB_k)) - (B_{k+1} + \beta_kA_kB_1)(A_kC_1 + D_1(\alpha_kA_k + \gamma_kB_k))}{A_{k+1}(D_{k+1} + \beta_kB_1C_k) - C_{k+1}(B_{k+1} + \beta_kA_kB_1)}.$$

Similarly, from the product $f_k'f_1'$, we find the following identities:

$$\gamma_k' = \frac{(C_1C_k + D_1(\alpha_kC_k + \gamma_kD_k))(D_{k+1} + \beta_kB_1C_k) - (C_1D_k + D_1(\beta_kC_k + \delta_kD_k))C_{k+1}}{A_{k+1}(D_{k+1} + \beta_kB_1C_k) - C_{k+1}(B_{k+1} + \beta_kA_kB_1)},$$
$$\delta_k' = \frac{A_{k+1}(C_1D_k + D_1(\beta_kC_k + \delta_kD_k)) - (B_{k+1} + \beta_kA_kB_1)(C_kC_1 + D_1(\alpha_kC_k + \gamma_kD_k))}{A_{k+1}(D_{k+1} + \beta_kB_1C_k) - C_{k+1}(B_{k+1} + \beta_kA_kB_1)}.$$

□

Using the same arguments for Leibniz algebras of type II as for type I, we obtain the following results.

Statement 4.3. *Let L be a Leibniz algebra of type* II. *Let $f_k \notin R(L)$, and let $f_{k+1} \in R(L)$, where $m \le k \le n-m-1$. Then, in the general transformation of the basis, we find the dependence*

$$e_i' = A_ie_i + B_if_i, \qquad 1 \le i \le m,$$
$$f_i' = C_ie_i + D_if_i, \qquad 1 \le i \le m,$$
$$f_{m+1}' = (\beta_mB_1C_m + D_{m+1})f_{m+1},$$
$$f_i' = \left[\beta_mB_1C_m\prod_{l=m+1}^{i-1}(A_1 + B_1\delta_l) + D_i\right]f_i, \qquad m+2 \le i \le n-m,$$

where A_i, B_i, C_i, D_i are defined by equalities (4.1).

Theorem 4.4. *Let L be a Leibniz algebra of type* II. *Let $f_k \notin R(L)$, and let $f_{k+1} \in R(L)$, where $m \le k \le n-m-1$. Then, for $k = m$, we have*

$$\alpha'_i = \frac{(A_iC_1 + D_1(\alpha_iA_i + \gamma_iB_i))D_{i+1} - B_iC_{i+1}(C_1 + \delta_iD_1)}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \qquad 1 \le i \le m-1,$$

$$\beta'_i = \frac{A_{i+1}B_i(C_1 + \delta_iD_1) - (A_iC_1 + D_1(\alpha_iA_i + \gamma_iB_i))B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \qquad 1 \le i \le m-1,$$

$$\gamma'_i = \frac{(C_iC_1 + D_1(\alpha_iC_i + \gamma_iD_i))D_{i+1} - D_iC_{i+1}(C_1 + \delta_iD_1)}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \qquad 1 \le i \le m-1,$$

$$\delta'_i = \frac{A_{i+1}D_i(C_1 + \delta_iD_1) - (C_iC_1 + D_1(\alpha_iC_i + \gamma_iD_i))B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \qquad 1 \le i \le m-1,$$

$$\beta'_m = \frac{\beta A_mD_1 + B_m(C_1 + \delta_mD_1)}{\beta B_1C_m + D_{m+1}}, \qquad \delta'_m = \frac{\beta C_mD_1 + D_m(C_1 + \delta_mD_1)}{\beta B_1C_m + D_{m+1}}.$$

For $m+1 \le k \le n-m-1$, we have

$$\alpha'_i = \frac{(A_iC_1 + D_1(\alpha_iA_i + \gamma_iB_i))D_{i+1} - B_iC_{i+1}(C_1 + \delta_iD_1)}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \qquad 1 \le i \le m-1,$$

$$\beta'_i = \frac{A_{i+1}B_i(C_1 + \delta_iD_1) - (A_iC_1 + D_1(\alpha_iA_i + \gamma_iB_i))B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \qquad 1 \le i \le m-1,$$

$$\gamma'_i = \frac{(C_iC_1 + D_1(\alpha_iC_i + \gamma_iD_i))D_{i+1} - D_iC_{i+1}(C_1 + \delta_iD_1)}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \qquad 1 \le i \le m-1,$$

$$\delta'_i = \frac{A_{i+1}D_i(C_1 + \delta_iD_1) - (C_iC_1 + D_1(\alpha_iC_i + \gamma_iD_i))B_{i+1}}{A_{i+1}D_{i+1} - B_{i+1}C_{i+1}}, \qquad 1 \le i \le m-1,$$

$$\beta'_m = \frac{B_m(C_1 + \delta_mD_1)}{D_{m+1}}, \qquad \delta'_m = \frac{D_m(C_1 + \delta_mD_1)}{D_{m+1}},$$

$$\delta'_i = \frac{D_i(C_1 + D_1\delta_i)}{D_{i+1}}, \qquad m+1 \le i \le n-m-1.$$

5. CONCLUSIONS

It should be noted that the solution of the classification problem consists of the following stages: the description of the algebras satisfying the given conditions, i.e., the determination of the multiplication table of the algebras with the least number of parameters; the determination of the relations determining the change of the parameters in the new basis (in the general transformation of the basis); the study of the given relations for the parameters and the determination of pairwise of nonisomorphic algebras with given conditions. The results presented in this paper carry out the first two stages of the classification of naturally graded Leibniz algebras with characteristic sequence equal to $(n-m, m)$. The available classifications for $m = 2$ and $m = 3$ ([9], [12]) show that, in the general case (i.e., for any values of m), the classification is boundless. However, for fixed values of n and m, it can be obtained with the help of the results of the present paper.

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