Matematika Instituti Byulleteni
2020,№5, 20-24 b.

Bulletin of the Institute of Mathematics
2020,№5, pp.20-24

Бюллетень Института математики
2020,№5, стр.20-24

Maximal torus of quasi-filiform Leibniz algebras

Khudoyberdiyev A. Kh. [1] **Shermatova Z. Kh.** [2]

Kvazi-filiform Leybnits algebralarining maksimal tori
Ushbu ishda Li algebralarining maksimal torusi uchun olingan ba'zi natijalarni Leybnits algebralari uchun kengaytiramiz. Matritsa usuli yordamida kvazi-filiform Leybnits algebralarining maksimal torini aniqlaymiz.
Kalit so'zlar: Leybnits algebrasi; nilpotent ideal; nilradical; differensialash; maksimal tor.

Максимальный тор квази-филиформных алгебр Лейбница
Мы обобщаем некоторые результаты, полученные для максимального тора алгебр Ли. к случаю алгебр Лейбница. В этой статье, мы явно определяем максимальный тор квази-филиформных алгебр Лейбница с помощью матричного метода.
Ключевые слова: Алгебра Лейбница; нильпотентный идеал; нильрадикал; дифференцирование; максимальный тор.

MSC 2010: 17A32, 17A36, 17B30, 17B56.
Keywords: Leibniz algebra; nilpotent ideal; nilradical; derivation; maximal torus.

Introduction

A Lie algebra is called complete if its center is zero, and all its derivations are inner. The definition of complete Lie algebras was given by N. Jacobson in 1962 [4]. The first important result of complete Lie algebras first appeared in 1951, in the context of Schenkman's theory of subinvariant Lie algebras [3]. In last years, different authors have concentrated on classifications and structural properties of complete Lie algebras.

In 1996 solvable complete Lie algebras were studied by D. J. Meng and.L. S. Zhu. In 2002 Y. C. Gao and D. J. Meng first have given a necessary and sufficient condition for some solvable Lie algebras with l-step nilpotent radicals to be complete and a method to construct non-solvable complete Lie algebras.

The complete nilpotent Lie algebra is one of the interesting results have been obtained. In 2001 J. R. Gomez, A. Jimenez-Merchan and J. Reyes have given the classification of quasi-filiform Lie algebras of maximal length. In 2016, M. Wu has explicitly determined the maximal torus of a class of quasi-filiform Lie algebras B_n and C_n, then proved that they were complete.

In this work we extend some results obtained for maximal torus of Lie algebras to the case of Leibniz algebras. By using a matrix method, we explicitly determine the maximal torus of a class of quasi-filiform Leibniz algebras.

Throughout the paper we denote by $\mathbb{F}$ a field of characteristic zero and by L a finite dimensional Leibniz algebra over $\mathbb{F}$.

[1] National University of Uzbekistan, 4, University str., 100174, Tashkent, Uzbekistan. E-mail: abror.khudoyberdiyev@mathinst.uz
[2] V.I.Romanovskiy Institute of Mathematics of Uzbekistan Academy of Sciences, Tashkent, Uzbekistan. E-mail: z.shermatova@mathinst.uz

Preliminaries

Definition 1. A vector space with bilinear bracket $(L, [-,-])$ over a field $\mathbf{F}$ is called a Leibniz algebra if for any $x, y, z \in L$ the so-called Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

holds.

Definition 2. A linear map $d\colon L \to L$ of a Leibniz algebra $(L, [-,-])$ is said to be a derivation if for all $x, y \in L$, the following condition holds:

$$d([x,y]) = [d(x),y] + [x,d(y)]$$

The set of all derivations of L (denoted by $Der(L)$) forms a Lie algebra with respect to the commutator.

Definition 3. For a given Leibniz algebra $(L, [-,-])$ the sequence of two-sided ideals defined recursively as follows:

$$L^1 = L,\ L^{k+1} = [L^k, L],\ k \geq 1$$

is said to be the lower central series of L.

Definition 4. A Leibniz algebra L is said to be nilpotent, if there exists $n \in \mathbb{N}$ such that $L^n = 0$.

Definition 5. A Leibniz algebra L is called quasi-filiform if $L^{n-2} \neq 0$ and $L^{n-1} = 0$, where $n = \dim L$.

A Leibniz algebra L is called $\mathbb{Z}$-graded if $L = \oplus_{i\in\mathbb{Z}} V_i$, where $[V_i, V_j] \subseteq V_{i+j}$ for any $i, j \in \mathbb{Z}$ with a finite number of non-null spaces V_i.

A gradation $L = V_{k_1} \oplus \cdots \oplus V_{k_t}$ of a Leibniz algebra L is called *connected gradation* if $V_{k_i} \neq 0$ for any i $(1 \leq i \leq t)$ and the number $l(\oplus L) := l(V_{k_1} \oplus \cdots \oplus V_{k_t}) = k_t - k_1 + 1$ is called *the length of the gradation.*

Definition 6. A Leibniz algebra L is called to be *of maximum length* if $max\{l(\oplus L)$ such that $L = V_{k_1} \oplus \cdots \oplus V_{k_t}$ is a connected gradation$\} = \dim(L)$.

In the following theorem we give the classification of quasi-filiform non-Lie Leibniz algebras of maximum length given in [1], [2].

Theorem 1. *An arbitrary n-dimensional quasi-filiform non-Lie Leibniz algebra of maximum length is isomorphic to one algebra of the following pairwise non-isomorphic algebras of the families:*

$$M^{1,\delta}: \begin{cases} [e_1,e_1] = e_n, & [e_{n-1},e_1] = e_2, \\ [e_i,e_1] = e_{i+1}, & 2 \leq i \leq n-3, \\ [e_{n-1},e_{n-1}] = \delta e_4, & \delta \in \{0,1\}, \\ [e_i,e_{n-1}] = \delta e_{i+3}, & 2 \leq i \leq n-5, \end{cases} \qquad M^{2,\lambda}: \begin{cases} [e_i,e_1] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_{n-1},e_1] = e_n, & \\ [e_1,e_{n-1}] = \lambda e_n, & \lambda \in \mathbb{C}, \end{cases}$$

$$M^{3,\alpha}: \begin{cases} [e_1,e_1] = e_2, & \\ [e_i,e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [e_1,e_i] = -e_{i+1}, & 3 \leq i \leq n-1, \\ [e_3,e_3] = \alpha e_6, & \alpha = 0,\ if\ n > 6, \\ & \alpha \in \{0,1\},\ if\ n = 6, \end{cases} \qquad M^4: \begin{cases} [e_i,e_1] = e_{i+1}, & 1 \leq i \leq n-3, \\ [e_1,e_{n-1}] = e_n, & \end{cases}$$

where $\{e_1, e_2, \ldots, e_n\}$ is a basis of the algebra.

Definition 7. Let N be a Lie algebra. A maximal torus on N is a maximal abelian subalgebra of derivation algebra $Der(N)$, which consists of semisimple linear transformations.

Lemma 1. [5] *Let H_1 and H_2 be two maximal torus on N, then there exist $\theta \in AutN$, such that $H_2 = \theta H_1 \theta^{-1}$.*

As all maximal torus on N are mutually conjugated, so the dimension of a maximal torus on N is an invariant of N called the rank of N (denoted by $rank(N)$). A nilpotent Lie algebra is called maximal rank nilpotent Lie algebra if $rank(N) = dimN/[N;N]$.

If H is a maximal torus on a nilpotent Lie algebra N, define the bracket in $H \dotplus N$, by $[h_1 + n_1, h_2 + n_2] = h_1(n_2) - h_2(n_1) + [n_1, n_2]$, where $h_i \in H$, $n_i \in N$, $i = 1, 2$, then $H \dotplus N$ is a solvable Lie algebra.

Main results

Now we calculate a maximal abelian subalgebra of derivation algebra $Der(L)$, which consists of semisimple linear transformations of L Leibniz algebras similar to the case of Lie algebras.

Let σ be a linear transformation of a Leibniz algebra L, then $\sigma \in Der(L)$ if and only if with respect to a basis $\{e_1, e_2, \ldots, e_n\}$ of L, σ satisfies for any $1 \leq i, j \leq n$,

$$\sigma[e_i, e_j] = [\sigma e_i, e_j] + [e_i, \sigma e_j].$$

Lemma 2. *Let σ be a linear transformation of a Leibniz algebra L and $A = diag(a_1, a_2, \ldots, a_n)$ be the matrix of σ with respect to a basis $\{e_1, e_2, \ldots, e_n\}$ of L. Suppose that*

$$[e_i, e_j] = \sum_{k=1}^{n} b_{ij}^k e_k.$$

Then $\sigma \in Der(L)$ if only if for any $1 \leq i, j \leq n$,

$$(a_i + a_j) b_{ij}^k = a_k b_{ij}^k, \qquad 1 \leq k \leq n.$$

Lemma 3. *Let $M^{1,0}$ be the n-dimensional quasi-filiform Leibniz algebra of maximum length defined in Theorem 1. Suppose that ϕ is a semisimple linear transformation of $M^{1,0}$ with respect to the given basis $\{e_1, e_2, \ldots, e_n\}$. Then ϕ is a derivation of $M^{1,0}$ if and only if ϕ satisfies*

$$\phi(e_1) = a_1 e_1, \quad \phi(e_2) = (a_1 + a_{n-1}) e_2, \quad \phi(e_3) = (2a_1 + a_{n-1}) e_3, \ldots,$$

$$\phi(e_{n-2}) = ((n-3)a_1 + a_{n-1}) e_{n-2}, \quad \phi(e_{n-1}) = a_{n-1} e_{n-1}, \quad \phi(e_n) = 2a_1 e_n,$$

where $a_1, a_{n-1} \in \boldsymbol{F}$.

Proof. Let ϕ be a semisimple linear transformation of $M^{1,0}$, we may assume that

$$\phi(e_1) = a_1 e_1, \ \phi(e_2) = a_2 e_2, \ \ldots, \ \phi(e_{n-1}) = a_{n-1} e_{n-1}, \ \phi(e_n) = a_n e_n.$$

Then by Lemma 2, we have that $\phi \in Der(M^{1,0})$ if and only if

$$\begin{cases} a_1 + a_1 = a_n, \\ a_i + a_1 = a_{i+1}, \quad 2 \leq i \leq n-3, \\ a_{n-1} + a_1 = a_2. \end{cases}$$

Solving the system of linear equations, we have $\phi \in Der(M^{1,0})$ if and only if

$$\phi(e_1) = a_1 e_1, \quad \phi(e_2) = (a_1 + a_{n-1}) e_2, \quad \phi(e_3) = (2a_1 + a_{n-1}) e_3, \ldots,$$

$$\phi(e_{n-2}) = ((n-3)a_1 + a_{n-1}) e_{n-2}, \quad \phi(e_{n-1}) = a_{n-1} e_{n-1}, \quad \phi(e_n) = 2a_1 e_n,$$

Hence the conclusion holds.

Lemma 3 is proved. □

Theorem 2. *Let $M^{1,0}$ be the n-dimensional quasi-filiform Leibniz algebra of maximum length defined in Theorem 1. Then the two linear transformations*

$$\phi_1(e_1) = e_1, \quad \phi_1(e_2) = e_2, \quad \phi_1(e_3) = 2e_3, \quad \phi_1(e_4) = 3e_4, \ldots,$$

$$\phi_1(e_{n-2}) = (n-3) e_{n-2}, \quad \phi_1(e_{n-1}) = 0, \quad \phi_1(e_n) = 2e_n.$$

$$\phi_2(e_1) = 0, \quad \phi_2(e_2) = e_2, \quad \phi_2(e_3) = e_3, \quad \phi_2(e_4) = e_4, \ldots,$$

$$\phi_2(e_{n-2}) = e_{n-2}, \quad \phi_2(e_{n-1}) = e_{n-1}, \quad \phi_2(e_n) = 0.$$

generate a maximal torus on $M^{1,0}$.

Proof. Let $a_1 = 1$, $a_{n-1} = 3$. By Lemma 3, we know that the transformation

$$\tau(e_1) = e_1, \ \tau(e_2) = 4e_2, \ \tau(e_3) = 5e_2, \ \ldots, \ \tau(e_{n-2}) = n e_{n-2}, \ \tau(e_{n-1}) = 3e_{n-1}, \ \tau(e_n) = 2e_n$$

is a derivation of $M^{1,0}$. Let H be a maximal torus on $M^{1,0}$ such that $\tau \in H$. $\forall h \in H$ and suppose the matrix of h with respect to the given basis is $M_h = (h_{ij})_{n \times n}$. Since any maximal torus is abelian, we have

$$[\tau, h] = 0,$$

which means $[M_h, diag(1,4,5,\dots,n,3,2)] = 0$. Since the diagonal entries of the matrix $diag(1,4,5,\dots,n,3,2)$ are different to each others, we have M_h is a diagonal matrix. Similarly as the proof of Lemma 3, we have $h \in Der(M^{1,0})$ if and only if

$$h(e_1) = a_1e_1, \quad h(e_2) = (a_1 + a_{n-1})e_2, \quad h(e_3) = (2a_1 + a_{n-1})e_3, \dots,$$

$$h(e_{n-2}) = ((n-3)a_1 + a_{n-1})e_{n-2}, \quad h(e_{n-1}) = a_{n-1}e_{n-1}, \quad h(e_n) = 2a_1e_n,$$

where $a_1, a_{n-1} \in \mathbf{F}$. Therefore $h = a_1\phi_1 + a_{n-1}\phi_2$. Hence H is generated by ϕ_1 and ϕ_2.

Theorem 2 is proved. □

The following statements are proved similarly.

Lemma 4. *Let $M^{2,\lambda}$ be the n-dimensional quasi-filiform Leibniz algebra of maximum length defined in Theorem 1. Suppose that ϕ is a semisimple linear transformation of $M^{2,\lambda}$ with respect to the given basis $\{e_1, e_2, \dots, e_n\}$. Then ϕ is a derivation of $M^{2,\lambda}$ if and only if ϕ satisfies*

$$\phi(e_1) = a_1e_1, \quad \phi(e_2) = 2a_1e_2, \quad \phi(e_3) = 3a_1e_3, \quad \phi(e_4) = 4a_1e_4, \dots,$$

$$\phi(e_{n-2}) = (n-2)a_1e_{n-2}, \quad \phi(e_{n-1}) = a_{n-1}e_{n-1}, \quad \phi(e_n) = (a_1 + a_{n-1})e_n,$$

where $a_1, a_{n-1} \in \boldsymbol{F}$.

Theorem 3. *Let $M^{2,\lambda}$ be the n-dimensional quasi-filiform Leibniz algebra of maximum length defined in Theorem 1. Then the two semisimple linear transformations*

$$\phi_1(e_1) = e_1, \quad \phi_1(e_2) = 2e_2, \quad \phi_1(e_3) = 3e_3, \quad \phi_1(e_4) = 4e_4, \dots,$$

$$\phi_1(e_{n-2}) = (n-2)e_{n-2}, \quad \phi_1(e_{n-1}) = 0, \quad \phi_1(e_n) = e_n,$$

$$\phi_2(e_1) = 0, \quad \phi_2(e_2) = 0, \quad \phi_2(e_3) = 0, \quad \phi_2(e_4) = 0, \dots,$$

$$\phi_2(e_{n-2}) = 0, \quad \phi_2(e_{n-1}) = e_{n-1}, \quad \phi_2(e_n) = e_n$$

generate a maximal torus on $M^{2,\lambda}$.

Lemma 5. *Let $M^{3,0}$ be the n-dimensional quasi-filiform Leibniz algebra of maximum length $l(L) = n \geq 6$ defined in Theorem 1. Suppose that ϕ is a semisimple linear transformation of $M^{3,0}$ with respect to the given basis $\{e_1, e_2, \dots, e_n\}$. Then ϕ is a derivation of $M^{3,0}$ if and only if ϕ satisfies*

$$\phi(e_1) = a_1e_1, \quad \phi(e_2) = 2a_1e_2, \quad \phi(e_3) = a_3e_3, \ \phi(e_4) = (a_1 + a_3)e_4,$$

$$\phi(e_5) = (2a_1 + a_3)e_5, \dots, \phi(e_n) = ((n-3)a_1 + a_3)e_n,$$

where $a_1, a_3 \in \boldsymbol{F}$.

Theorem 4. *Let $M^{3,0}$ be the n-dimensional quasi-filiform Leibniz algebra of maximum length $l(L) = n \geq 6$ defined in Theorem 1. Then the two semisimple linear transformations*

$$\phi_1(e_1) = e_1, \quad \phi_1(e_2) = 2e_2, \quad \phi_1(e_3) = 0,$$

$$\phi_1(e_4) = e_4, \quad \phi_1(e_5) = 2e_5, \dots, \phi_1(e_n) = (n-3)e_n.$$

$$\phi_2(e_1) = 0, \quad \phi_2(e_2) = 0, \quad \phi_2(e_3) = e_3, \quad \phi_2(e_4) = e_4,$$

$$\phi_2(e_5) = e_5, \dots, \phi_2(e_n) = e_n.$$

generate a maximal torus on $M^{3,0}$.

Lemma 6. *Let M^4 be the n-dimensional quasi-filiform Leibniz algebra of maximum length defined in Theorem 1. Suppose that ϕ is a semisimple linear transformation of M^4 with respect to the given basis $\{e_1, e_2, \dots, e_n\}$. Then ϕ is a derivation of $M^{3,0}$ if and only if ϕ satisfies*

$$\phi(e_1) = a_1e_1, \quad \phi(e_2) = 2a_1e_2, \quad \phi(e_3) = 3a_1e_3, \quad \phi(e_4) = 4a_1e_4, \dots,$$

$$\phi(e_{n-2}) = (n-2)a_1e_{n-2}, \quad \phi(e_{n-1}) = a_{n-1}e_{n-1}, \quad \phi(e_n) = (a_1 + a_{n-1})e_n,$$

where $a_1, a_{n-1} \in \boldsymbol{F}$.

Theorem 5. *Let M^4 be the n-dimensional quasi-filiform Leibniz algebra of maximum length defined in Theorem 1. Then the two semisimple linear transformations*

$$\phi_1(e_1) = e_1, \quad \phi_1(e_2) = 2e_2, \quad \phi_1(e_3) = 3e_3,$$

$$\phi_1(e_4) = 4e_4, \dots, \ \phi_1(e_{n-2}) = (n-2)e_{n-2}, \ \phi_1(e_{n-1}) = 0, \ \phi_1(e_n) = e_n.$$

$$\phi_2(e_1) = 0, \quad \phi_2(e_2) = 0, \quad \phi_2(e_3) = 0,$$

$$\phi_2(e_4) = 0, \dots, \phi_2(e_{n-2}) = 0, \ \phi_2(e_{n-1}) = e_{n-1}, \ \phi_2(e_n) = e_n.$$

generate a maximal torus on M^4.

References

1. Cabezas J. M. , Camacho L. M. , Rodríguez I. M. On filiform and 2-filiform Leibniz algebras of maximum length. *Journal of Lie Theory*, 18, 2008, pp. 335-350.

2. Camacho L. M. , Cañete E. M. , Gómez J. R. , Omirov B. A. Quasi-filiform Leibniz algebras of maximum length. *Sib. Math. J.*, 52(5), 2011, pp. 840-853.

3. Schenkman E. A theory of subinvariant Lie algebras. *Amer. Journal of Math.*, 73, 1951, pp. 453-474.

4. Jacobson N. Lie Algebras. *Wiley Interscience, New York,* 1962.

5. Mostow G. D. Fully reducible subgroup of algebaic group. *Amer. Journal of Math.*, 78, 1956, pp. 200-221.

Received: 04/10/2020

Accepted: 30/10/2020