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Central extensions of filiform Zinbiel algebras

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ABSTRACT

In this paper we describe central extensions (up to isomorphism) of all complex null-filiform and filiform Zinbiel algebras. It is proven that every non-split central extension of an n-dimensional null-filiform Zinbiel algebra is isomorphic to an $(n+1)$-dimensional null-filiform Zinbiel algebra. Moreover, we obtain all pairwise non isomorphic quasi-filiform Zinbiel algebras.

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1. Introduction

The algebraic classification (up to isomorphism) of an n-dimensional algebras from a certain variety defined by some family of polynomial identities is a classical problem in the theory of non-associative algebras. There are many results related to algebraic classification of small dimensional algebras in the varieties of Jordan, Lie, Leibniz, Zinbiel and many another algebras [1–16]. An algebra $\mathbf{A}$ is called a *Zinbiel algebra* if it satisfies the identity

$$(x \circ y) \circ z = x \circ (y \circ z + z \circ y).$$

Zinbiel algebras were introduced by Loday [17] and studied in [18–29]. Under the Koszul duality, the operad of Zinbiel algebras is dual to the operad of Leibniz algebras. Hence, the tensor product of a Leibniz algebra and a Zinbiel algebra can be given the structure of a Lie algebra. Under the symmetrized product, a Zinbiel algebra becomes an associative and commutative algebra. Zinbiel algebras are also related to Tortkara algebras [22] and Tortkara triple systems [30]. More precisely, every Zinbiel algebra with the commutator multiplication gives a Tortkara algebra (also about Tortkara algebras, see, [9,31,32]), which have recently sprung up in unexpected areas of mathematics [33,34].

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Central extensions play an important role in quantum mechanics: one of the earlier encounters is by means of Wigner's theorem which states that a symmetry of a quantum mechanical system determines an (anti-)unitary transformation of a Hilbert space. Another area of physics where one encounters central extensions is the quantum theory of conserved currents of a Lagrangian. These currents span an algebra which is closely related to so-called affine Kac-Moody algebras, which are universal central extensions of loop algebras. Central extensions are needed in physics, because the symmetry group of a quantized system usually is a central extension of the classical symmetry group, and in the same way the corresponding symmetry Lie algebra of the quantum system is, in general, a central extension of the classical symmetry algebra. Kac-Moody algebras have been conjectured to be symmetry groups of a unified superstring theory. The centrally extended Lie algebras play a dominant role in quantum field theory, particularly in conformal field theory, string theory and in M-theory. In the theory of Lie groups, Lie algebras and their representations, a Lie algebra extension is an enlargement of a given Lie algebra g by another Lie algebra h. Extensions arise in several ways. There is a trivial extension obtained by taking a direct sum of two Lie algebras. Other types are a split extension and a central extension. Extensions may arise naturally, for instance, when forming a Lie algebra from projective group representations. A central extension and an extension by a derivation of a polynomial loop algebra over a finite-dimensional simple Lie algebra gives a Lie algebra which is isomorphic to a non-twisted affine Kac-Moody algebra [35, Chapter 19]. Using the centrally extended loop algebra one may construct a current algebra in two spacetime dimensions. The Virasoro algebra is the universal central extension of the Witt algebra, the Heisenberg algebra is the central extension of a commutative Lie algebra [35, Chapter 18].

The algebraic study of central extensions of Lie and non-Lie algebras has a very long history [36–44]. For example, all central extensions of some filiform Leibniz algebras were classified in [36,43] and all central extensions of filiform associative algebras were classified in [40]. Skjelbred and Sund used central extensions of Lie algebras for a classification of low dimensional nilpotent Lie algebras [42]. After that, the method introduced by Skjelbred and Sund was used to describe all non-Lie central extensions of all 4-dimensional Malcev algebras [39], all non-associative central extensions of 3-dimensional Jordan algebras [38], all anticommutative central extensions of 3-dimensional anticommutative algebras [45]. Note that the method of central extensions is an important tool in the classification of nilpotent algebras. It was used to describe all 4-dimensional nilpotent associative algebras [7], all 4-dimensional nilpotent assosymmetric algebras [46], all 4-dimensional nilpotent bicommutative algebras [47], all 4-dimensional nilpotent Novikov algebras [48], all 4-dimensional commutative algebras [49], all 5-dimensional nilpotent Jordan algebras [10], all 5-dimensional nilpotent restricted Lie algebras [6], all 5-dimensional anticommutative algebras [49], all 6-dimensional nilpotent Lie algebras [5,8], all 6-dimensional nilpotent Malcev algebras [11], all 6-dimensional nilpotent binary Lie algebras[1], all 6-dimensional nilpotent anticommutative $\mathfrak{CD}$-algebras [1], all 6-dimensional nilpotent Tortkara algebras[9,32], and some others.

2. Preliminaries

All algebras and vector spaces in this paper are over $\mathbb{C}$.

2.1. Filiform Zinbiel algebras

An algebra $\mathbf{A}$ is called *Zinbiel algebra* if for any $x, y, z \in \mathbf{A}$ it satisfies the identity

$$(x \circ y) \circ z = x \circ (y \circ z) + x \circ (z \circ y).$$

For an algebra $\mathbf{A}$, we consider the series

$$\mathbf{A}^1 = \mathbf{A}, \quad \mathbf{A}^{i+1} = \sum_{k=1}^{i} \mathbf{A}^k \mathbf{A}^{i+1-k}, \quad i \geq 1.$$

We say that an algebra $\mathbf{A}$ is *nilpotent* if $\mathbf{A}^i = 0$ for some $i \in \mathbb{N}$. The smallest integer satisfying $\mathbf{A}^i = 0$ is called the *nilpotency index* of $\mathbf{A}$.

Definition 2.1: An n-dimensional algebra $\mathbf{A}$ is called null-filiform if $\dim \mathbf{A}^i = (n+1) - i$, $1 \leq i \leq n+1$.

It is easy to see that a Zinbiel algebra has a maximal nilpotency index if and only if it is null-filiform. For a nilpotent Zinbiel algebra, the condition of null-filiformity is equivalent to the condition that the algebra is one-generated.

All null-filiform Zinbiel algebras were described in [50]. Throughout the paper, C_i^j denotes the combinatorial numbers $\binom{i}{j}$.

Theorem 2.2 ([50]): *An arbitrary n-dimensional null-filiform Zinbiel algebra is isomorphic to the algebra* F_n^0 :

$$e_i \circ e_j = C_{i+j-1}^{j} e_{i+j}, \quad 2 \leq i+j \leq n,$$

where omitted products are equal to zero and $\{e_1, e_2, \ldots, e_n\}$ *is a basis of the algebra.*

As an easy corollary from the previous theorem we have the next result.

Theorem 2.3: *Every non-split central extension of* F_n^0 *is isomorphic to* F_{n+1}^0.

Proof: It is easy to see, that every non-split central extension of F_n^0 is a one-generated nilpotent algebra. It follows that every non-split central extension of a null-filiform Zinbiel algebra is a null-filiform Zinbiel algebra. Using the classification of null-filiform algebras (Theorem 2.2) we have the statement of the Theorem. ■

Definition 2.4: An n-dimensional algebra is called filiform if $\dim(\mathbf{A}^i) = n - i$, $2 \leq i \leq n$.

All filiform Zinbiel algebras were classified in [50].

Theorem 2.5: *An arbitary n-dimensional* $(n \geq 5)$ *filiform Zinbiel algebra is isomorphic to one of the following pairwise non-isomorphic algebras*:

$$F_n^1 : e_i \circ e_j = C_{i+j-1}^{j} e_{i+j}, \quad 2 \leq i+j \leq n-1;$$

$$F_n^2 : e_i \circ e_j = C_{i+j-1}^{j} e_{i+j}, \quad 2 \leq i+j \leq n-1, \quad e_n \circ e_1 = e_{n-1};$$

$$F_n^3 : e_i \circ e_j = C_{i+j-1}^{j} e_{i+j}, \quad 2 \leq i+j \leq n-1, \quad e_n \circ e_n = e_{n-1}.$$

2.2. Basic definitions and methods

Throughout this paper, we are using the notations and methods well written in [38,39] and adapted for the Zinbiel case with some modifications. From now, we will give only some important definitions.

Let $(\mathbf{A}, \circ)$ be a Zinbiel algebra and $\mathbb{V}$ a vector space. Then the $\mathbb{C}$-linear space $\mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ is defined as the set of all bilinear maps $\theta \colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{V}$, such that

$$\theta(x \circ y, z) = \theta(x, y \circ z + z \circ y).$$

Its elements will be called *cocycles*. For a linear map f from $\mathbf{A}$ to $\mathbb{V}$, if we write $\delta f \colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{V}$ by $\delta f(x, y) = f(x \circ y)$, then $\delta f \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$. We define $\mathrm{B}^2(\mathbf{A}, \mathbb{V}) = \{\theta = \delta f \ : f \in \mathrm{Hom}(\mathbf{A}, \mathbb{V})\}$. One can easily check that $\mathrm{B}^2(\mathbf{A}, \mathbb{V})$ is a linear subspace of $\mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ whose elements are called *coboundaries*. We define the *second cohomology space* $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$ as the quotient space $\mathrm{Z}^2(\mathbf{A}, \mathbb{V}) \big/ \mathrm{B}^2(\mathbf{A}, \mathbb{V})$.

Let $\mathrm{Aut}(\mathbf{A})$ be the automorphism group of the Zinbiel algebra $\mathbf{A}$ and let $\phi \in \mathrm{Aut}(\mathbf{A})$. For $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ define $\phi\theta(x, y) = \theta(\phi(x), \phi(y))$. Then $\phi\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$. So, $\mathrm{Aut}(\mathbf{A})$ acts on $\mathrm{Z}^2(\mathbf{A}, \mathbb{V})$. It is easy to verify that $\mathrm{B}^2(\mathbf{A}, \mathbb{V})$ is invariant under the action of $\mathrm{Aut}(\mathbf{A})$ and so we have that $\mathrm{Aut}(\mathbf{A})$ acts on $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$.

Let $\mathbf{A}$ be a Zinbiel algebra of dimension $m < n$, and $\mathbb{V}$ be a $\mathbb{C}$-vector space of dimension $n{-}m$. For any $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ define on the linear space $\mathbf{A}_\theta := \mathbf{A} \oplus \mathbb{V}$ the bilinear product '$[-,-]_{\mathbf{A}_\theta}$' by $[x + x', y + y']_{\mathbf{A}_\theta} = x \circ y + \theta(x, y)$ for all $x, y \in \mathbf{A}, x', y' \in \mathbb{V}$. The algebra $\mathbf{A}_\theta$ is a Zinbiel algebra which is called an *$(n - m)$-dimensional central extension* of $\mathbf{A}$ by $\mathbb{V}$. Indeed, we have, in a straightforward way, that $\mathbf{A}_\theta$ is a Zinbiel algebra if and only if $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$.

We also call the set $\mathrm{Ann}(\theta) = \{x \in \mathbf{A} : \theta(x, \mathbf{A}) + \theta(\mathbf{A}, x) = 0\}$ the *annihilator* of θ. We recall that the *annihilator* of an algebra $\mathbf{A}$ is defined as the ideal $\mathrm{Ann}(\mathbf{A}) = \{x \in \mathbf{A} : x \circ \mathbf{A} + \mathbf{A} \circ x = 0\}$ and observe that $\mathrm{Ann}(\mathbf{A}_\theta) = (\mathrm{Ann}(\theta) \cap \mathrm{Ann}(\mathbf{A})) \oplus \mathbb{V}$.

We have the next key result:

Lemma 2.6: *Let $\mathbf{A}$ be an n-dimensional Zinbiel algebra such that $\dim(\mathrm{Ann}(\mathbf{A})) = m \neq 0$. Then there exists, up to isomorphism, a unique $(n - m)$-dimensional Zinbiel algebra $\mathbf{A}'$ and a bilinear map $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ with $\mathrm{Ann}(\mathbf{A}) \cap \mathrm{Ann}(\theta) = 0$, where $\mathbb{V}$ is a vector space of dimension m, such that $\mathbf{A} \cong \mathbf{A}'_\theta$ and $\mathbf{A}/\mathrm{Ann}(\mathbf{A}) \cong \mathbf{A}'$.*

However, in order to solve the isomorphism problem we need to study the action of $\mathrm{Aut}(\mathbf{A})$ on $\mathrm{H}^2(\mathbf{A}, \mathbb{V})$. To do that, let us fix $e_1, \ldots, e_s$ a basis of $\mathbb{V}$, and $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$. Then θ can be uniquely written as $\theta(x, y) = \sum_{i=1}^{s} \theta_i(x, y) e_i$, where $\theta_i \in \mathrm{Z}^2(\mathbf{A}, \mathbb{C})$. Moreover, $\mathrm{Ann}(\theta) = \mathrm{Ann}(\theta_1) \cap \mathrm{Ann}(\theta_2) \cap \ldots \cap \mathrm{Ann}(\theta_s)$. Further, $\theta \in \mathrm{B}^2(\mathbf{A}, \mathbb{V})$ if and only if all $\theta_i \in \mathrm{B}^2(\mathbf{A}, \mathbb{C})$.

Definition 2.7: Let $\mathbf{A}$ be an algebra and I be a subspace of $\mathrm{Ann}(\mathbf{A})$. If $\mathbf{A} = \mathbf{A}_0 \oplus I$ then I is called an *annihilator component* of $\mathbf{A}$.

Definition 2.8: A central extension of an algebra $\mathbf{A}$ without annihilator component is called a non-split central extension.

It is not difficult to prove (see [39, Lemma 13]), that given a Zinbiel algebra $\mathbf{A}_\theta$, if we write as above $\theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)e_i \in \mathrm{Z}^2(\mathbf{A}, \mathbb{V})$ and we have $\mathrm{Ann}(\theta) \cap \mathrm{Ann}(\mathbf{A}) = 0$, then $\mathbf{A}_\theta$ has an annihilator component if and only if $[\theta_1], [\theta_2], \ldots, [\theta_s]$ are linearly dependent in $\mathrm{H}^2(\mathbf{A}, \mathbb{C})$.

Let $\mathbb{V}$ be a finite-dimensional vector space. The *Grassmannian* $G_k(\mathbb{V})$ is the set of all k-dimensional linear subspaces of $\mathbb{V}$. Let $G_s(\mathrm{H}^2(\mathbf{A}, \mathbb{C}))$ be the Grassmannian of subspaces of dimension s in $\mathrm{H}^2(\mathbf{A}, \mathbb{C})$. There is a natural action of $\mathrm{Aut}(\mathbf{A})$ on $G_s(\mathrm{H}^2(\mathbf{A}, \mathbb{C}))$. Let $\phi \in \mathrm{Aut}(\mathbf{A})$. For $W = \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in G_s(\mathrm{H}^2(\mathbf{A}, \mathbb{C}))$ define $\phi W = \langle [\phi\theta_1], [\phi\theta_2], \ldots, [\phi\theta_s] \rangle$. Then $\phi W \in G_s(\mathrm{H}^2(\mathbf{A}, \mathbb{C}))$. We denote the orbit of $W \in G_s(\mathrm{H}^2(\mathbf{A}, \mathbb{C}))$ under the action of $\mathrm{Aut}(\mathbf{A})$ by $\mathrm{Orb}(W)$. Since given

$$W_1 = \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle, \quad W_2 = \langle [\vartheta_1], [\vartheta_2], \ldots, [\vartheta_s] \rangle \in G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right),$$

we easily have that in case $W_1 = W_2$, then $\bigcap_{i=1}^{s} \mathrm{Ann}(\theta_i) \cap \mathrm{Ann}(\mathbf{A}) = \bigcap_{i=1}^{s} \mathrm{Ann}(\vartheta_i) \cap \mathrm{Ann}(\mathbf{A})$, and so we can introduce the set

$$T_s(\mathbf{A}) = \left\{ W = \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in G_s\left(\mathrm{H}^2(\mathbf{A}, \mathbb{C})\right) : \bigcap_{i=1}^{s} \mathrm{Ann}(\theta_i) \cap \mathrm{Ann}(\mathbf{A}) = 0 \right\},$$

which is stable under the action of $\mathrm{Aut}(\mathbf{A})$.

Now, let $\mathbb{V}$ be an s-dimensional linear space and let us denote by $E(\mathbf{A}, \mathbb{V})$ the set of all *non-split s-dimensional central extensions* of $\mathbf{A}$ by $\mathbb{V}$. We can write

$$E(\mathbf{A}, \mathbb{V}) = \left\{ \mathbf{A}_\theta : \theta\left(x,y\right) = \sum_{i=1}^{s} \theta_i\left(x,y\right) e_i \text{ and } \langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle \in T_s(\mathbf{A}) \right\}.$$

We also have the next result, which can be proved as in [39, Lemma 17].

Lemma 2.9: *Let $\mathbf{A}_\theta, \mathbf{A}_\vartheta \in E(\mathbf{A}, \mathbb{V})$. Suppose that $\theta(x,y) = \sum_{i=1}^{s} \theta_i(x,y)e_i$ and $\vartheta(x,y) = \sum_{i=1}^{s} \vartheta_i(x,y)e_i$. Then the Zinbiel algebras $\mathbf{A}_\theta$ and $\mathbf{A}_\vartheta$ are isomorphic if and only if*

$$\mathrm{Orb}\,\langle [\theta_1], [\theta_2], \ldots, [\theta_s] \rangle = \mathrm{Orb}\,\langle [\vartheta_1], [\vartheta_2], \ldots, [\vartheta_s] \rangle.$$

From here, there exists a one-to-one correspondence between the set of $\mathrm{Aut}(\mathbf{A})$-orbits on $T_s(\mathbf{A})$ and the set of isomorphism classes of $E(\mathbf{A}, \mathbb{V})$. Consequently we have a procedure that allows us, given the Zinbiel algebra $\mathbf{A}'$ of dimension $n-s$, to construct all non-split central extensions of $\mathbf{A}'$. This procedure would be:

2.3. Procedure

(1) For a given Zinbiel algebra $\mathbf{A}'$ of dimension $n-s$, determine $\mathrm{H}^2(\mathbf{A}', \mathbb{C})$, $\mathrm{Ann}(\mathbf{A}')$ and $\mathrm{Aut}(\mathbf{A}')$.
(2) Determine the set of $\mathrm{Aut}(\mathbf{A}')$-orbits on $T_s(\mathbf{A}')$.
(3) For each orbit, construct the Zinbiel algebra corresponding to a representative of it.

Finally, let us introduce some of notation. Let $\mathbf{A}$ be a Zinbiel algebra with a basis $e_1, e_2, \ldots, e_n$. Then by $\Delta_{i,j}$ we will denote the bilinear form $\Delta_{i,j} \colon \mathbf{A} \times \mathbf{A} \longrightarrow \mathbb{C}$ with

$\Delta_{i,j}(e_l, e_m) = \delta_{il}\delta_{jm}$. Then the set $\{\Delta_{i,j} : 1 \le i, j \le n\}$ is a basis for the linear space of the bilinear forms on $\mathbf{A}$. Then every $\theta \in \mathrm{Z}^2(\mathbf{A}, \mathbb{C})$ can be uniquely written as $\theta = \sum_{1\le i,j\le n} c_{ij}\Delta_{i,j}$, where $c_{ij} \in \mathbb{C}$.

3. Central extension of filiform Zinbiel algebras

Proposition 3.1: *Let F_n^1, F_n^2 and F_n^3 be n-dimensional filiform Zinbiel algebras defined in Theorem 2.5. Then:*

- *A basis of $\mathrm{Z}^2(F_n^k, \mathbb{C})$ is formed by the following cocycles*

$$\mathrm{Z}^2(F_n^1, \mathbb{C}) = \left\langle \Delta_{1,1}, \Delta_{1,n}, \Delta_{n,1}, \Delta_{n,n}, \sum_{i=1}^{s-1} C_{s-1}^{i-1}\Delta_{i,s-i};\ 3 \le s \le n \right\rangle,$$

$$\mathrm{Z}^2(F_n^k, \mathbb{C}) = \left\langle \Delta_{1,1}, \Delta_{1,n}, \Delta_{n,1}, \Delta_{n,n}, \sum_{i=1}^{s-1} C_{s-1}^{i-1}\Delta_{i,s-i};\ 3 \le s \le n-1 \right\rangle, \quad k = 2, 3.$$

- *A basis of $\mathrm{B}^2(F_n^k, \mathbb{C})$ is formed by the following coboundaries*

$$\mathrm{B}^2(F_n^1, \mathbb{C}) = \left\langle \Delta_{1,1}, \sum_{i=1}^{s-1} C_{s-1}^{i-1}\Delta_{i,s-i}, 3 \le s \le n-1 \right\rangle,$$

$$\mathrm{B}^2(F_n^2, \mathbb{C}) = \left\langle \Delta_{1,1}, \sum_{i=1}^{s-1} C_{s-1}^{i-1}\Delta_{i,s-i}, 3 \le s \le n-2, \sum_{i=1}^{n-2} C_{n-2}^{i-1}\Delta_{i,n-1-i} + \Delta_{n,1} \right\rangle,$$

$$\mathrm{B}^2(F_n^3, \mathbb{C}) = \left\langle \Delta_{1,1}, \sum_{i=1}^{s-1} C_{s-1}^{i-1}\Delta_{i,s-i}, 3 \le s \le n-2, \sum_{i=1}^{n-2} C_{n-2}^{i-1}\Delta_{i,n-1-i} + \Delta_{n,n} \right\rangle.$$

- *A basis of $\mathrm{H}^2(F_n^k, \mathbb{C})$ is formed by the following cocycles*

$$\mathrm{H}^2(F_n^1, \mathbb{C}) = \left\langle [\Delta_{1,n}], [\Delta_{n,1}], [\Delta_{n,n}], \left[\sum_{i=1}^{n-1} C_{n-1}^{i-1}\Delta_{i,n-i}\right] \right\rangle,$$

$$\mathrm{H}^2(F_n^k, \mathbb{C}) = \langle [\Delta_{1,n}], [\Delta_{n,1}], [\Delta_{n,n}] \rangle, \quad k = 2, 3.$$

Proof: The proof follows directly from the definition of a cocycle. ∎

Proposition 3.2: *Let $\phi_k^n \in \mathrm{Aut}(F_n^k)$. Then*

$$\phi_1^n = \begin{pmatrix} a_{1,1} & 0 & 0 & \dots & 0 & 0 \\ a_{2,1} & a_{1,1}^2 & 0 & \dots & 0 & 0 \\ a_{3,1} & * & a_{1,1}^3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & * & * & & a_{1,1}^{n-1} & a_{n-1,n} \\ a_{n,1} & 0 & 0 & \dots & 0 & a_{n,n} \end{pmatrix},$$

$$\phi_2^n = \begin{pmatrix} a_{1,1} & 0 & 0 & \dots & 0 & 0 \\ a_{2,1} & a_{1,1}^2 & 0 & \dots & 0 & 0 \\ a_{3,1} & * & a_{1,1}^3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & * & * & & a_{1,1}^{n-1} & a_{n-1,n} \\ a_{n,1} & 0 & 0 & \dots & 0 & a_{1,1}^{n-2} \end{pmatrix},$$

$$\phi_3^n = \begin{pmatrix} a_{1,1} & 0 & 0 & \dots & 0 & 0 \\ a_{2,1} & a_{1,1}^2 & 0 & \dots & 0 & 0 \\ a_{3,1} & * & a_{1,1}^3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & * & * & & a_{1,1}^{n-1} & a_{n-1,n} \\ a_{n,1} & 0 & 0 & \dots & 0 & a_{1,1}^{(n-1)/2} \end{pmatrix}$$

3.1. Central extensions of F_n^1

Let us denote

$$\nabla_1 = [\Delta_{1,n}], \quad \nabla_2 = [\Delta_{n,1}], \quad \nabla_3 = [\Delta_{n,n}], \quad \nabla_4 = \left[\sum_{j=1}^{n-1} C_{n-1}^{j-1} \Delta_{j,n-j}\right]$$

and $x = a_{1,1}, y = a_{n,n}, z = a_{n-1,n}, w = a_{n,1}$. Since

$$\begin{pmatrix} * & \dots & * & C_{n-1}^0 \alpha_4' & \alpha_1' \\ * & \dots & C_{n-1}^1 \alpha_4' & 0 & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots \\ C_{n-1}^{n-2} \alpha_4' & \dots & 0 & 0 & 0 \\ \alpha_2' & \dots & 0 & 0 & \alpha_3' \end{pmatrix}$$

$$= (\phi_1^n)^T \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & C_{n-1}^0 \alpha_4 & \alpha_1 \\ 0 & 0 & 0 & \cdots & C_{n-1}^1 \alpha_4 & 0 & 0 \\ 0 & 0 & 0 & \vdots & \iddots & 0 & 0 \\ \vdots & \vdots & \iddots & C_{n-1}^{n-1-i} \alpha_4 & \iddots & \vdots & \vdots \\ \vdots & 0 & \iddots & 0 & \vdots & \vdots & \vdots \\ 0 & C_{n-1}^{n-3} \alpha_4 & 0 & \vdots & \vdots & \vdots & \vdots \\ C_{n-1}^{n-2} \alpha_4 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \alpha_2 & 0 & 0 & \cdots & 0 & 0 & \alpha_3 \end{pmatrix} \phi_1^n,$$

for any $\theta = \alpha_1 \nabla_1 + \alpha_2 \nabla_2 + \alpha_3 \nabla_3 + \alpha_4 \nabla_4$, we have the action of the automorphism group on the subspace $\langle \theta \rangle$ as

$$\Big\langle (\alpha_1 xy + \alpha_3 yw + \alpha_4 xz) \nabla_1 + (\alpha_2 xy + \alpha_3 yw + (n-1) \alpha_4 xz) \nabla_2 + \alpha_3 y^2 \nabla_3 + \alpha_4 x^n \nabla_4 \Big\rangle.$$

3.1.1. 1-dimensional central extensions of F_n^1

Let us consider the following cases:

(1) if $\alpha_1 \neq 0, \alpha_2 = \alpha_3 = \alpha_4 = 0$, then by choosing $x = 1, y = 1/\alpha_1$, we have the representative $\langle \nabla_1 \rangle$.
(2) if $\alpha_2 \neq 0, \alpha_3 = \alpha_4 = 0$, then by choosing $x = 1, y = 1/\alpha_2, \alpha = \alpha_1/\alpha_2$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 \rangle$.
(3) if $\alpha_1 = \alpha_2, \alpha_3 \neq 0, \alpha_4 = 0$, then by choosing $y = 1/\sqrt{\alpha_3}, w = -\alpha_2/\alpha_3, x = 1$, we have the representative $\langle \nabla_3 \rangle$.
(4) if $\alpha_1 \neq \alpha_2, \alpha_3 \neq 0, \alpha_4 = 0$, then by choosing $x = \frac{\sqrt{\alpha_3}}{\alpha_1 - \alpha_2}, y = \frac{1}{\sqrt{\alpha_3}}, w = \frac{\alpha_2}{\sqrt{\alpha_3}(\alpha_2 - \alpha_1)}$, we have the representative $\langle \nabla_1 + \nabla_3 \rangle$.
(5) if $(n-1)\alpha_1 = \alpha_2, \alpha_3 = 0, \alpha_4 \neq 0$, then by choosing $x = 1/\sqrt[n]{\alpha_4}, y = 1, z = -\alpha_1/\alpha_4$, we have the representative $\langle \nabla_4 \rangle$.
(6) if $(n-1)\alpha_1 \neq \alpha_2, \alpha_3 = 0, \alpha_4 \neq 0$, then by choosing $x = 1/\sqrt[n]{\alpha_4}, y = \frac{\sqrt[n]{\alpha_4}}{\alpha_2 - (n-1)\alpha_1}$, $z = -\frac{\sqrt[n]{\alpha_4}}{\alpha_2 - (n-1)\alpha_1}$, we have the representative $\langle \nabla_2 + \nabla_4 \rangle$.
(7) if $\alpha_3 \neq 0, \alpha_4 \neq 0$, then by choosing $x = 1/\sqrt[n]{\alpha_4}, y = 1/\sqrt{\alpha_3}, z = \frac{\alpha_1 - \alpha_2}{(n-2)\sqrt{\alpha_3}\alpha_4}$, $w = \frac{\alpha_2 - (n-1)\alpha_1}{(n-2)\sqrt[n]{\alpha_4}\alpha_3}$, we have the representative $\langle \nabla_3 + \nabla_4 \rangle$.

It is easy to verify that all previous orbits are different, and so we obtain

$$
\begin{aligned}
T_1(F_n^1) = {} & \mathrm{Orb}\langle \nabla_1 \rangle \cup \mathrm{Orb}\langle \alpha\nabla_1 + \nabla_2 \rangle \cup \mathrm{Orb}\langle \nabla_3 \rangle \cup \mathrm{Orb}\langle \nabla_1 + \nabla_3 \rangle \\
& \cup \mathrm{Orb}\langle \nabla_4 \rangle \cup \mathrm{Orb}\langle \nabla_2 + \nabla_4 \rangle \cup \mathrm{Orb}\langle \nabla_3 + \nabla_4 \rangle.
\end{aligned}
$$

3.1.2. 2-dimensional central extensions of F_n^1

We may assume that a 2-dimensional subspace is generated by

$$
\begin{aligned}
\theta_1 &= \alpha_1\nabla_1 + \alpha_2\nabla_2 + \alpha_3\nabla_3 + \alpha_4\nabla_4, \\
\theta_2 &= \beta_1\nabla_1 + \beta_2\nabla_2 + \beta_3\nabla_3.
\end{aligned}
$$

Then we have the six following cases:

(1) if $\alpha_4 \neq 0, \beta_3 \neq 0$, then we can suppose that $\alpha_3 = 0$. Now
  (a) for $(n-1)\alpha_1 \neq \alpha_2, \beta_1 \neq \beta_2$, by choosing $x = (\frac{(\alpha_2 - (n-1)\alpha_1)(\beta_2 - \beta_1)}{\alpha_4})^{1/(n-2)}, y = \frac{\beta_2 - \beta_1}{\beta_3}x$, $z = \frac{\alpha_1(\beta_1 - \beta_2)}{\alpha_4\beta_3}x$, $w = -\beta_1 x/\beta_3$, we have the representative $\langle \nabla_2 + \nabla_3, \nabla_2 + \nabla_4 \rangle$.
  (b) for $(n-1)\alpha_1 \neq \alpha_2, \beta_1 = \beta_2$, by choosing $x = (\frac{\alpha_2 - (n-1)\alpha_1}{\alpha_4\sqrt{\beta_3}})^{1/(n-1)}, y = 1/\sqrt{\beta_3}$, $z = -\frac{\alpha_1}{\alpha_4\sqrt{\beta_3}}, w = -\beta_1 x/\beta_3$, we have the representative $\langle \nabla_3, \nabla_2 + \nabla_4 \rangle$.
  (c) for $(n-1)\alpha_1 = \alpha_2, \beta_1 \neq \beta_2$, by choosing $x = 1/\sqrt[n]{\alpha_4}$, $y = \frac{\beta_2 - \beta_1}{\beta_3}x$, $z = \frac{\alpha_1(\beta_1 - \beta_2)}{\alpha_4\beta_3}x, w = -\beta_1 x/\beta_3$, we have the representative $\langle \nabla_2 + \nabla_3, \nabla_4 \rangle$.

  (d) for $(n-1)\alpha_1 = \alpha_2, \beta_1 = \beta_2$, by choosing $x = 1/\sqrt[n]{\alpha_4}$, $y = 1/\sqrt{\beta_3}$, $z = -\alpha_1 y/\alpha_4$, $w = -\beta_1 x/\beta_3$, we have the representative $\langle \nabla_3, \nabla_4 \rangle$.

(2) if $\alpha_4 \neq 0, \beta_3 = 0, \beta_2 \neq 0$, then we can suppose that $\alpha_2 = 0$. Now
  (a) for $\alpha_3 \neq 0$, by choosing $x = 1/\sqrt[n]{\alpha_4}, y = 1/\sqrt{\alpha_3}, z = \frac{\alpha_1}{(n-2)\sqrt{\alpha_3}\alpha_4}$, $w = -\frac{(n-1)\alpha_1}{(n-2)\sqrt[n]{\alpha_4}\alpha_3}$, and $\alpha = \beta_1/\beta_2$ we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2, \nabla_3 + \nabla_4 \rangle$.
  (b) for $\alpha_3 = 0, (n-1)\beta_1 \neq \beta_2$, then by choosing $x = 1/\sqrt[n]{\alpha_4}, y = 1$, $z = -\frac{\alpha_1\beta_2}{\alpha_4((n-1)\beta_1 - \beta_2)}$ and $\alpha = \beta_1/\beta_2$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2, \nabla_4 \rangle_{\alpha \neq \frac{1}{n-1}}$.
  (c) for $\alpha_3 = 0, (n-1)\beta_1 = \beta_2$ and $\alpha_1 = 0$, by choosing $x = 1/\sqrt[n]{\alpha_4}, y = 1, z = 0$, we have the representative $\langle \frac{1}{n-1}\nabla_1 + \nabla_2, \nabla_4 \rangle$.
  (d) for $\alpha_3 = 0, (n-1)\beta_1 = \beta_2$ and $\alpha_1 \neq 0$, by choosing $x = 1/\sqrt[n]{\alpha_4}, y = -\frac{\sqrt[n]{\alpha_4}}{(n-1)\alpha_1}$, $z = \frac{\sqrt[n]{\alpha_4}}{(n-1)\alpha_4}$, we have the representative $\langle \frac{1}{n-1}\nabla_1 + \nabla_2, \nabla_2 + \nabla_4 \rangle$.

(3) if $\alpha_4 \neq 0, \beta_3 = \beta_2 = 0, \beta_1 \neq 0$, then
  (a) for $\alpha_3 \neq 0$, by choosing $x = 1/\sqrt[n]{\alpha_4}, y = 1/\sqrt{\alpha_3}, z = \frac{\alpha_1 - \alpha_2}{(n-2)\sqrt{\alpha_3}\alpha_4}$, $w = \frac{\alpha_2 - (n-1)\alpha_1}{(n-2)\sqrt[n]{\alpha_4}\alpha_3}$, we have the representative $\langle \nabla_1, \nabla_3 + \nabla_4 \rangle$.
  (b) for $\alpha_3 = 0$, after a linear combination of θ_1 and θ_2 we can suppose that $(n-1)\alpha_1 = \alpha_2$, by choosing $x = 1/\sqrt[n]{\alpha_4}, y = 1, z = -\alpha_1/\alpha_4$, we have the representative $\langle \nabla_1, \nabla_4 \rangle$.

(4) if $\alpha_4 = 0, \alpha_3 \neq 0, \beta_2 \neq 0$, then
  (a) for $\beta_1 \neq \beta_2$, after a linear combination of θ_1 and θ_2 we can suppose that $\alpha_1 = \alpha_2$, by choosing $y = 1/\sqrt{\alpha_3}, w = -\alpha_2/\alpha_3, x = 1$ and $\alpha = \beta_1/\beta_2$ we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2, \nabla_3 \rangle_{\alpha \neq 1}$.
  (b) for $\beta_1 = \beta_2, \alpha_1 = \alpha_2$, after a linear combination of θ_1 and θ_2 we have the representative $\langle \nabla_1 + \nabla_2, \nabla_3 \rangle$.
  (c) for $\beta_1 = \beta_2, \alpha_1 \neq \alpha_2$, by choosing $x = \frac{\sqrt{\alpha_3}}{\alpha_1 - \alpha_2}, y = 1/\sqrt{\alpha_3}, w = \frac{\alpha_2}{\sqrt{\alpha_3}(\alpha_2 - \alpha_1)}$, we have the representative $\langle \nabla_1 + \nabla_2, \nabla_1 + \nabla_3 \rangle$.

(5) if $\alpha_4 = 0, \alpha_3 \neq 0, \beta_2 = 0, \beta_1 \neq 0$, then after a linear combination of θ_1 and θ_2 we can suppose that $\alpha_1 = \alpha_2$, by choosing $y = 1/\sqrt{\alpha_3}, w = -\alpha_2/\alpha_3, x = 1$ and $\alpha = \beta_1/\beta_2$ we have the representative $\langle \nabla_1, \nabla_3 \rangle$.

(6) if $\alpha_3 = \alpha_4 = 0, \beta_3 = 0$, then we have the representative $\langle \nabla_1, \nabla_2 \rangle$.

It is easy to verify that all previous orbits are different, and so we obtain

$$
\begin{aligned}
T_2(F_n^1) = {} & \mathrm{Orb}\langle \nabla_1, \nabla_2 \rangle \cup \mathrm{Orb}\langle \nabla_1, \nabla_3 \rangle \cup \mathrm{Orb}\langle \nabla_1, \nabla_3 + \nabla_4 \rangle \cup \mathrm{Orb}\langle \nabla_1, \nabla_4 \rangle \\
& \cup \mathrm{Orb}\left\langle \frac{1}{n-1}\nabla_1 + \nabla_2, \nabla_2 + \nabla_4 \right\rangle \cup \mathrm{Orb}\langle \nabla_1 + \nabla_2, \nabla_1 + \nabla_3 \rangle \\
& \cup \mathrm{Orb}\langle \alpha\nabla_1 + \nabla_2, \nabla_3 \rangle \cup \mathrm{Orb}\langle \alpha\nabla_1 + \nabla_2, \nabla_3 + \nabla_4 \rangle \\
& \cup \mathrm{Orb}\langle \alpha\nabla_1 + \nabla_2, \nabla_4 \rangle \cup \mathrm{Orb}\langle \nabla_2 + \nabla_3, \nabla_2 + \nabla_4 \rangle \\
& \cup \mathrm{Orb}\langle \nabla_2 + \nabla_3, \nabla_4 \rangle \cup \mathrm{Orb}\langle \nabla_3, \nabla_2 + \nabla_4 \rangle \cup \mathrm{Orb}\langle \nabla_3, \nabla_4 \rangle.
\end{aligned}
$$

3.1.3. 3-dimensional central extensions of F_n^1

We may assume that a 3-dimensional subspace is generated by

$$\theta_1 = \alpha_1\nabla_1 + \alpha_2\nabla_2 + \alpha_3\nabla_3 + \alpha_4\nabla_4,$$
$$\theta_2 = \beta_1\nabla_1 + \beta_2\nabla_2 + \beta_3\nabla_3,$$
$$\theta_3 = \gamma_1\nabla_1 + \gamma_2\nabla_2.$$

Then we have the following cases:

(1) if $\alpha_4 \neq 0, \beta_3 \neq 0, \gamma_2 \neq 0$, then we can suppose that $\alpha_2 = 0, \alpha_3 = 0, \beta_2 = 0$ and
  (a) for $\gamma_1 \neq \gamma_2$, $(n-1)\gamma_1 \neq \gamma_2$, then by choosing $x = 1/\sqrt[n]{\alpha_4}$, $y = 1/\sqrt{\beta_3}$, $z = \frac{\alpha_1\gamma_2 y}{\alpha_4((n-1)\gamma_1-\gamma_2)}$, $w = \frac{\beta_1\gamma_2 x}{\alpha_4(\gamma_1-\gamma_2)}$, we have the family of representatives $\langle\alpha\nabla_1 + \nabla_2, \nabla_3, \nabla_4\rangle_{\alpha\notin\{1,\frac{1}{n-1}\}}$.
  (b) for $\gamma_1 = \gamma_2$, then
    (i) for $\beta_1 \neq 0$, by choosing $x = 1/\sqrt[n]{\alpha_4}$, $y = \frac{\beta_1 x}{\beta_3}$, $z = \frac{\alpha_1 y}{(n-2)\alpha_4}$, $w = 0$, we have the representative $\langle\nabla_1 + \nabla_2, \nabla_1 + \nabla_3, \nabla_4\rangle$.
    (ii) for $\beta_1 = 0$, by choosing $x = 1/\sqrt[n]{\alpha_4}$, $y = 1/\sqrt{\beta_3}$, $z = \frac{\alpha_1 y}{(n-2)\alpha_4}$, $w = 0$, we have the representative $\langle\nabla_1 + \nabla_2, \nabla_3, \nabla_4\rangle$.
  (c) for $(n-1)\gamma_1 = \gamma_2$, then
    (i) for $\alpha_1 \neq 0$, by choosing $y = 1/\sqrt{\beta_3}$, $z = -\frac{\alpha_1 y}{\alpha_4}$, $x = \sqrt[n-1]{(n-1)z}$, $w = -\frac{(n-1)\beta_1 x}{(n-2)\beta_3}$, we have the representative $\langle\frac{1}{n-1}\nabla_1 + \nabla_2, \nabla_3, \nabla_2 + \nabla_4\rangle$.
    (ii) for $\alpha_1 = 0$, by choosing $x = 1/\sqrt[n]{\alpha_4}$, $y = 1/\sqrt{\beta_3}$, $z = 0$, $w = -\frac{(n-1)\beta_1 x}{(n-2)\beta_3}$, we have the representative $\langle\frac{1}{n-1}\nabla_1 + \nabla_2, \nabla_3, \nabla_4\rangle$.

(2) if $\alpha_4 \neq 0, \beta_3 \neq 0, \gamma_2 = 0, \gamma_1 \neq 0$, then we can suppose that $\alpha_3 = 0$ and after a linear combination of $\theta_1, \theta_2, \theta_3$ we can suppose that $(n-1)\alpha_1 = \alpha_2, \beta_1 = \beta_2$. By choosing $x = 1/\sqrt[n]{\alpha_4}, y = 1/\sqrt{\beta_3}, z = -\alpha_1 y/\alpha_4, w = -\beta_1 x/\beta_3$, we have the representative $\langle\nabla_1, \nabla_3, \nabla_4\rangle$.

(3) if $\alpha_4 \neq 0, \beta_3 = 0, \beta_2 \neq 0, \gamma_2 = 0, \gamma_1 \neq 0$, then and after a linear combination of $\theta_1, \theta_2, \theta_3$ we can suppose that $\alpha_1 = \alpha_2 = \beta_1 = 0$. Now
  (a) for $\alpha_3 \neq 0$, by choosing $y = 1/\sqrt{\alpha_3}, x = 1/\sqrt[n]{\alpha_4}$ we have the representative $\langle\nabla_1, \nabla_2, \nabla_3 + \nabla_4\rangle$.
  (b) for $\alpha_3 = 0$, we have the representative $\langle\nabla_1, \nabla_2, \nabla_4\rangle$.

(4) if $\alpha_4 = 0, \beta_3 = 0, \gamma_2 = 0$, then we have the representative $\langle\nabla_1, \nabla_2, \nabla_3\rangle$.

It is easy to verify that all previous orbits are different, and so we obtain

$$\begin{aligned} T_3(F_n^1) = {} & \mathrm{Orb}\langle\nabla_1, \nabla_2, \nabla_3\rangle \cup \mathrm{Orb}\langle\nabla_1, \nabla_2, \nabla_3 + \nabla_4\rangle \cup \mathrm{Orb}\langle\nabla_1, \nabla_2, \nabla_4\rangle \\ & \cup \mathrm{Orb}\langle\nabla_1 + \nabla_2, \nabla_1 + \nabla_3, \nabla_4\rangle \cup \mathrm{Orb}\left\langle\frac{1}{n-1}\nabla_1 + \nabla_2, \nabla_3, \nabla_2 + \nabla_4\right\rangle \\ & \cup \mathrm{Orb}\langle\alpha\nabla_1 + \nabla_2, \nabla_3, \nabla_4\rangle \cup \mathrm{Orb}\langle\nabla_1, \nabla_3, \nabla_4\rangle. \end{aligned}$$

3.1.4. 4-dimensional central extensions of F_n^1

There is only one 4-dimensional non-split central extension of the algebra F_n^1. It is defined by $\langle\nabla_1, \nabla_2, \nabla_3, \nabla_4\rangle$.

3.1.5. Non-split central extensions of F_n^1

So we have the next theorem

Theorem 3.3: *An arbitrary non-split central extension of the algebra F_n^1 is isomorphic to one of the following pairwise non-isomorphic algebras*

- *one-dimensional central extensions:*

$$\mu_1^{n+1},\ \mu_2^{n+1}(\alpha),\ \mu_3^{n+1},\ \mu_4^{n+1},\ F_{n+1}^1,\ F_{n+1}^2,\ F_{n+1}^3$$

- *two-dimensional central extensions:*

$$\mu_5^{n+2},\ \mu_6^{n+2},\ \mu_7^{n+2},\ \mu_1^{n+2},\ \mu_8^{n+2},\ \mu_9^{n+2},\ \mu_{10}^{n+2}(\alpha),\ \mu_{11}^{n+2}(\alpha),\ \mu_2^{n+2}(\alpha),$$
$$\mu_{12}^{n+2},\ \mu_4^{n+2},\ \mu_{13}^{n+2},\ \mu_3^{n+2}$$

- *three-dimensional central extensions:*

$$\mu_{14}^{n+3},\ \mu_{15}^{n+3},\ \mu_5^{n+3},\ \mu_9^{n+3},\ \mu_{16}^{n+3},\ \mu_{10}^{n+3}(\alpha),\ \mu_6^{n+3}$$

- *four-dimensional central extensions:*

$$\mu_{14}^{n+4}$$

with $\alpha \in \mathbb{C}$.

3.2. Central extensions of F_n^2

Let us denote

$$\nabla_1 = [\Delta_{1,n}], \quad \nabla_2 = [\Delta_{n,1}], \quad \nabla_3 = [\Delta_{n,n}]$$

and $x = a_{1,1}, w = a_{n,1}$. Let $\theta = \alpha_1\nabla_1 + \alpha_2\nabla_2 + \alpha_3\nabla_3$. Then by

$$\begin{pmatrix} * & \dots & 0 & 0 & \alpha_1' \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ \alpha_2' & \dots & 0 & 0 & \alpha_3' \end{pmatrix} = (\phi_2^n)^T \begin{pmatrix} 0 & \dots & 0 & 0 & \alpha_1 \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ \alpha_2 & \dots & 0 & 0 & \alpha_3 \end{pmatrix} \phi_2^n,$$

we have the action of the automorphism group on the subspace $\langle\theta\rangle$ as

$$\left\langle x^{n-2}(x\alpha_1 + w\alpha_3)\nabla_1 + x^{n-2}(x\alpha_2 + w\alpha_3)\nabla_2 + x^{2n-4}\alpha_3\nabla_3 \right\rangle.$$

3.2.1. 1-dimensional central extensions of F_n^2

Let us consider the following cases:

(1) if $\alpha_3 = 0$, then
 (a) for $\alpha_2 = 0, \alpha_1 \neq 0$, we have the representative $\langle\nabla_1\rangle$.

(b) for $\alpha_2 \neq 0$, by choosing $x = \alpha_2^{-1/(n-1)}$ and $\alpha = \alpha_1/\alpha_2$, we have the family of representatives $\langle \alpha\nabla_1 + \nabla_2 \rangle$.

(2) if $\alpha_3 \neq 0$, then
   (a) for $\alpha_1 \neq \alpha_2$, by choosing $x = (\frac{\alpha_2-\alpha_1}{\alpha_3})^{1/(n-3)}$, $w = -\frac{x\alpha_1}{\alpha_3}$ we have the representative $\langle \nabla_2 + \nabla_3 \rangle$.
   (b) for $\alpha_1 = \alpha_2$, by choosing $w = -\frac{x\alpha_1}{\alpha_3}$ we have the representative $\langle \nabla_3 \rangle$.

It is easy to verify that all previous orbits are different, and so we obtain

$$T_1(F_n^2) = \mathrm{Orb}\langle \nabla_1 \rangle \cup \mathrm{Orb}\langle \alpha\nabla_1 + \nabla_2 \rangle \cup \mathrm{Orb}\langle \nabla_2 + \nabla_3 \rangle \cup \mathrm{Orb}\langle \nabla_3 \rangle.$$

3.2.2. 2-dimensional central extensions of F_n^2

We may assume that a 2-dimensional subspace is generated by

$$\theta_1 = \alpha_1\nabla_1 + \alpha_2\nabla_2 + \alpha_3\nabla_3,$$
$$\theta_2 = \beta_1\nabla_1 + \beta_2\nabla_2.$$

We consider the following cases:

(1) if $\alpha_3 \neq 0$ and $\beta_1 \neq \beta_2$, then after a linear combination of θ_1 and θ_2 we can suppose that $\alpha_1 = \alpha_2$. Now,
   (a) for $\beta_2 \neq 0$, by choosing $x = \beta_2^{-1/(n-1)}$, $w = -\frac{x\alpha_1}{\alpha_3}$ and $\alpha = \beta_1/\beta_2$ we have the family of respresentatives $\langle \alpha\nabla_1 + \nabla_2, \nabla_3 \rangle_{\alpha\neq 1}$.
   (b) for $\beta_2 = 0$, by choosing $x = \beta_1^{-1/(n-1)}$, $w = -\frac{x\alpha_1}{\alpha_3}$, we have the respresentative $\langle \nabla_1, \nabla_3 \rangle$.

(2) if $\alpha_3 \neq 0$ and $\beta_1 = \beta_2$, then
   (a) for $\alpha_1 \neq \alpha_2$, by choosing $x = (\frac{\alpha_1-\alpha_2}{\alpha_3})^{1/(n-1)}$, $w = -\frac{x\alpha_2}{\alpha_3}$ we have the representative $\langle \nabla_1 + \nabla_2, \nabla_1 + \nabla_3 \rangle$.
   (b) for $\alpha_1 = \alpha_2$, after a linear combination of θ_1 and θ_2 we have the representative $\langle \nabla_1 + \nabla_2, \nabla_3 \rangle$.

(3) if $\alpha_3 = 0$, then we have the representative $\langle \nabla_1, \nabla_2 \rangle$.

It is easy to verify that all previous orbits are different, and so we obtain

$$T_2(F_n^2) = \mathrm{Orb}\langle \nabla_1, \nabla_2 \rangle \cup \mathrm{Orb}\langle \nabla_1, \nabla_3 \rangle \cup \mathrm{Orb}\langle \nabla_1 + \nabla_2, \nabla_1 + \nabla_3 \rangle \cup \mathrm{Orb}\langle \alpha\nabla_1 + \nabla_2, \nabla_3 \rangle.$$

3.2.3. 3-dimensional central extensions of F_n^2

There is only one 3-dimensional non-split central extension of the algebra F_n^2. It is defined by $\langle \nabla_1, \nabla_2, \nabla_3 \rangle$.

3.2.4. Non-split central extensions of F_n^2

So we have the next result.

Theorem 3.4: *An arbitrary non-split central extension of the algebra F_n^2 is isomorphic to one of the following pairwise non-isomorphic algebras*

- *one-dimensional central extensions*:

$$\mu_1^{n+1},\ \mu_2^{n+1}(\alpha) \quad \textit{with}\ \alpha \neq \frac{1}{n-3},\ \mu_8^{n+1},\ \mu_{12}^{n+1},\ \mu_{16}^{n+1}$$

- *two-dimensional central extensions*:

$$\mu_5^{n+2},\ \mu_6^{n+2},\ \mu_9^{n+2},\ \mu_{10}^{n+2}(\alpha) \quad \textit{with}\ \alpha \neq \frac{1}{n-4},\ \mu_{16}^{n+2}$$

- *three-dimensional central extensions*:

$$\mu_{14}^{n+3}$$

with $\alpha \in \mathbb{C}$.

3.3. Central extensions of F_n^3

Let us denote

$$\nabla_1 = [\Delta_{1,n}], \quad \nabla_2 = [\Delta_{n,1}], \quad \nabla_3 = [\Delta_{n,n}]$$

and $x = a_{1,1}, w = a_{n,1}$. Let $\theta = \alpha_1\nabla_1 + \alpha_2\nabla_2 + \alpha_3\nabla_3$. Then by

$$\begin{pmatrix} * & \dots & 0 & 0 & \alpha_1' \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ \alpha_2' & \dots & 0 & 0 & \alpha_3' \end{pmatrix} = (\phi_3^n)^T \begin{pmatrix} 0 & \dots & 0 & 0 & \alpha_1 \\ 0 & \dots & 0 & 0 & 0 \\ \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & 0 \\ \alpha_2 & \dots & 0 & 0 & \alpha_3 \end{pmatrix} \phi_3^n,$$

we have the action of the automorphism group on the subspace $\langle\theta\rangle$ as

$$\left\langle x^{(n-1)/2}(x\alpha_1 + w\alpha_3)\nabla_1 + x^{(n-1)/2}(x\alpha_2 + w\alpha_3)\nabla_2 + x^{n-1}\alpha_3\nabla_3 \right\rangle.$$

3.3.1. 1-dimensional central extensions of F_n^3

Let us consider the following cases:

(1) if $\alpha_3 = 0$, then
   (a) for $\alpha_2 = 0, \alpha_1 \neq 0$, by choosing $x = \alpha_1^{-2/(n+1)}$, we have the representative $\langle\nabla_1\rangle$.
   (b) for $\alpha_2 \neq 0$, by choosing $x = \alpha_2^{-2/(n+1)}$ and $\alpha = \alpha_1/\alpha_2$ we have the family of representatives $\langle\alpha\nabla_1 + \nabla_2\rangle$.
(2) if $\alpha_3 \neq 0$, then
   (a) for $\alpha_2 \neq \alpha_1$, by choosing $x = (\frac{\alpha_2-\alpha_1}{\alpha_3})^{2/(n-3)}, w = -\frac{x\alpha_1}{\alpha_3}$ we have the representative $\langle\nabla_2 + \nabla_3\rangle$.
   (b) for $\alpha_2 = \alpha_1$, by choosing $w = -\frac{x\alpha_1}{\alpha_3}$ we have the representative $\langle\nabla_3\rangle$.

It is easy to verify that all previous orbits are different, and so we obtain

$$T_1(F_n^3) = \mathrm{Orb}\langle\nabla_1\rangle \cup \mathrm{Orb}\langle\alpha\nabla_1 + \nabla_2\rangle \cup \mathrm{Orb}\langle\nabla_2 + \nabla_3\rangle \cup \mathrm{Orb}\langle\nabla_3\rangle.$$

3.3.2. 2-dimensional central extensions of F_n^3

We may assume that a 2-dimensional subspace is generated by

$$\theta_1 = \alpha_1\nabla_1 + \alpha_2\nabla_2 + \alpha_3\nabla_3,$$
$$\theta_2 = \beta_1\nabla_1 + \beta_2\nabla_2.$$

We consider the following cases:

(1) if $\alpha_3 \neq 0$ and $\beta_1 \neq \beta_2$, then after a linear combination of θ_1 and θ_2 we can suppose that $\alpha_1 = \alpha_2$. Now,
  (a) for $\beta_2 \neq 0$, by choosing $x = \beta_2^{-2/(n+1)}, w = -\frac{x\alpha_1}{\alpha_3}$ and $\alpha = \beta_1/\beta_2$ we have the family of respresentatives $\langle \alpha\nabla_1 + \nabla_2, \nabla_3\rangle_{\alpha\neq 1}$.
  (b) for $\beta_2 = 0$, by choosing $x = \beta_1^{-2/(n+1)}, w = -\frac{x\alpha_1}{\alpha_3}$, we have the respresentative $\langle \nabla_1, \nabla_3\rangle$.

(2) if $\alpha_3 \neq 0$ and $\beta_1 = \beta_2$, then
  (a) for $\alpha_1 \neq \alpha_2$, by choosing $x = (\frac{\alpha_1-\alpha_2}{\alpha_3})^{2/(n-3)}, w = -\frac{x\alpha_2}{\alpha_3}$ we have the representative $\langle \nabla_1 + \nabla_2, \nabla_1 + \nabla_3\rangle$.
  (b) for $\alpha_1 = \alpha_2$, after a linear combination of θ_1 and θ_2 we have the representative $\langle \nabla_1 + \nabla_2, \nabla_3\rangle$.

(3) if $\alpha_3 = 0$, then we have the representative $\langle \nabla_1, \nabla_2\rangle$.

It is easy to verify that all previous orbits are different, and so we obtain

$$T_2(F_n^3) = \mathrm{Orb}\langle \nabla_1, \nabla_2\rangle \cup \mathrm{Orb}\langle \nabla_1, \nabla_3\rangle \cup \mathrm{Orb}\langle \nabla_1 + \nabla_2, \nabla_1 + \nabla_3\rangle \cup \mathrm{Orb}\langle \alpha\nabla_1 + \nabla_2, \nabla_3.\rangle$$

3.3.3. 3-dimensional central extensions of F_n^3

There is only one 3-dimensional non-split central extension of the algebra F_n^3. It is defined by $\langle \nabla_1, \nabla_2, \nabla_3\rangle$.

3.3.4. Non-split central extensions of F_n^3

So we have the next theorem.

Theorem 3.5: *An arbitrary non-split central extension of the algebra F_n^3 is isomorphic to one of the following pairwise non-isomorphic algebras*

- *one-dimensional central extensions*:

$$\mu_7^{n+1},\ \mu_{11}^{n+1}(\alpha),\ \mu_{12}^{n+1},\ \mu_3^{n+1}$$

- *two-dimensional central extensions*:

$$\mu_{15}^{n+2},\ \mu_6^{n+2},\ \mu_9^{n+2},\ \mu_{10}^{n+2}(\alpha)$$

- *three-dimensional central extensions*:

$$\mu_{14}^{n+3}$$

with $\alpha \in \mathbb{C}$.

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Appendix. The list of the algebras

$$\mu_1^n : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-2, \quad e_1 \circ e_n = e_{n-1},$$

$$\mu_2^n(\alpha) : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-2, \quad e_1 \circ e_n = \alpha e_{n-1}, \quad e_n \circ e_1 = e_{n-1},$$

$$\mu_3^n : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-2, \quad e_n \circ e_n = e_{n-1},$$

$$\mu_4^n : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-2, \quad e_1 \circ e_n = e_{n-1}, \quad e_n \circ e_n = e_{n-1},$$

$$\mu_5^n : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-3, \quad e_1 \circ e_n = e_{n-1}, \quad e_n \circ e_1 = e_{n-2},$$

$$\mu_6^n : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-3, \quad e_1 \circ e_n = e_{n-1}, \quad e_n \circ e_n = e_{n-2},$$

$$\mu_7^n : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-2, \quad e_1 \circ e_n = e_{n-1}, \quad e_n \circ e_n = e_{n-2},$$

$$\mu_8^n : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-2, \quad e_1 \circ e_n = \frac{1}{n-3} e_{n-1}, \quad e_n \circ e_1 = e_{n-2} + e_{n-1},$$

$$\mu_9^n : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-3, \quad e_1 \circ e_n = e_{n-2} + e_{n-1}, \quad e_n \circ e_1 = e_{n-1}, \quad e_n \circ e_n = e_{n-2}$$

$$\mu_{10}^n(\alpha) : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-3, \quad e_1 \circ e_n = \alpha e_{n-1}, \quad e_n \circ e_1 = e_{n-1}, \quad e_n \circ e_n = e_{n-2},$$

$$\mu_{11}^n(\alpha) : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-2, \quad e_1 \circ e_n = \alpha e_{n-1}, \quad e_n \circ e_1 = e_{n-1}, \quad e_n \circ e_n = e_{n-2},$$

$$\mu_{12}^n : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-2, \quad e_n \circ e_1 = e_{n-2} + e_{n-1}, \quad e_n \circ e_n = e_{n-1},$$

$$\mu_{13}^n : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-2, \quad e_n \circ e_1 = e_{n-2}, \quad e_n \circ e_n = e_{n-1},$$

$$\mu_{14}^n : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-4, \quad e_1 \circ e_n = e_{n-2}, \quad e_n \circ e_1 = e_{n-1}, \quad e_n \circ e_n = e_{n-3},$$

$$\mu_{15}^n : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-3, \quad e_1 \circ e_n = e_{n-2}, \quad e_n \circ e_1 = e_{n-1}, \quad e_n \circ e_n = e_{n-3},$$

$$\mu_{16}^n : e_i \circ e_j = C_{i+j-1}^j, \quad 2 \le i+j \le n-3, \quad e_1 \circ e_n = \frac{1}{n-4} e_{n-1}, \quad e_n \circ e_1 = e_{n-3} + e_{n-1}, \quad e_n \circ e_n = e_{n-2}.$$