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PHASE TRANSITIONS FOR MODELS WITH A CONTINUUM SET OF SPIN VALUES ON A BETHE LATTICE

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We consider a model with nearest-neighbor interactions and the set $[0,1]$ *of spin values on a Bethe lattice (Cayley tree) of arbitrary order. This model depends on a continuous parameter* θ *and is a generalization of known models. For all values of* θ*, we give a complete description of the set of translation-invariant Gibbs measures of this model.*

Keywords: Cayley tree, spin value, Gibbs measure, Hammerstein's equation, fixed point

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1. Introduction

Spin models on a graph or in continuous spaces form a large class of systems considered in statistical mechanics. Some of them have a real physical meaning, and others have been proposed as suitably simplified models of more complicated systems. The geometric structure of a graph or physical space plays an important role in such investigations. For example, to study the phase transition problem on a cubic lattice $\mathbb{Z}^d$ or in space, we essentially use the Pirogov–Sinai theory [1]. A general methodology of phase transitions in $\mathbb{Z}^d$ or $\mathbb{R}^d$ was developed in [2]. On the other hand, on the Bethe lattice (Cayley tree) of order k, we use the theory of Markov splitting random fields based upon the corresponding recurrence relations. In particular, in [3]–[6], Gibbs measures on the graph Γ_k were described in terms of solutions of the recurrence relations.

During the last ten years, increasing attention has been given to models with a *continuum* set of spin values on Bethe lattices. Up to now, four competing interactions for models with an uncountable set of spin values have been studied, and the following results were obtained. Describing the splitting Gibbs measures of the model was reduced to analyzing solutions of a certain nonlinear integral equation. The uniqueness of splitting Gibbs measures of the models on the one-dimensional Bethe lattice was shown. For arbitrary $k \in \mathbb{N}$, a sufficient condition for the uniqueness of periodic splitting Gibbs measure was found (see [7]–[11]).

The existence of phase transitions on a Cayley tree Γ_k of order $k \geq 2$ was proved in [12], i.e., some examples were given of the Hamiltonian of a model for which there exists phase transitions. There are also several methods for constructing models with a continuum set of spin values for which there exists a phase transition on Cayley trees [12]–[14]. In addition, models with the parameters $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$

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were considered in [15], and it was proved that there exists a unique translation-invariant Gibbs measure if $0 \le \theta \le (2n+3)/2(2n+1)$ and there exist three translation-invariant Gibbs measures (i.e., a phase transition occurs) if $(2n+3)/2(2n+1) < \theta < 1$.

Here, we consider a model with a continuous parameter that is a generalization of the models in [12]–[15]. For all values of the parameter, we completely describe the set of translation-invariant Gibbs measures of the model.

2. Preliminaries

We briefly review definitions, notation, and other known results. Detailed information can be found in [10].

A Cayley tree $\Gamma_k = (V, L)$ of order $k \in \mathbb{N}$ is an infinite homogeneous tree, i.e., a graph without cycles, with exactly $k+1$ edges incident to each vertex. Here, V is the set of vertices, and L is the set of edges. Two vertices x and y are called nearest neighbors if there exists an edge $l \in L$ connecting them. We use the notation $l = \langle x, y\rangle$. The distance $d(x, y)$ for $x, y \in V$ on the Cayley tree is the length of the path from x to y.

Let $x^0 \in V$ be fixed. We set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \qquad V_n = \{x \in V \mid d(x, x^0) \le n\},$$
$$L_n = \{l = \langle x, y\rangle \in L \mid x, y \in V_n\}.$$

The set of direct successors of x is denoted by $S(x)$,

$$S(x) = \{y \in W_{n+1} \mid d(x, y) = 1\}, \quad x \in W_n.$$

We note that any vertex $x \ne x^0$ has k direct successors and x^0 has $k+1$ direct successors. Vertices x and y are called second neighbors, denoted by $\rangle x, y\langle$, if there exists a vertex $z \in V$ such that x and z are nearest neighbors and y and z are nearest neighbors. We consider only second neighbors $\rangle x, y\langle$ for which there exist n such that $x, y \in W_n$. Three vertices x, y, and z are called a triple of neighbors, denoted by $\langle x, y, z\rangle$, if each of the pairs x, y and y, z are nearest neighbors (there exist edges $\langle x, y\rangle$ and $\langle y, z\rangle$) and $x, z \in W_n$ and $y \in W_{n-1}$ for some $n \in \mathbb{N}$.

We consider models where the spin takes values in the set $[0, 1]$ and is assigned to the vertices of the tree. For $A \subset V$, a configuration σ_A on A is an arbitrary function $\sigma_A\colon A \mapsto [0, 1]$. We let $\Omega_A = [0, 1]^A$ denote the set of all configurations on A. A configuration σ on V is then defined as a function $x \in V \mapsto \sigma(x) \in [0, 1]$. The set of all configurations is denoted by $[0, 1]^V$.

The (formal) Hamiltonian of the model is

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} \xi_{\sigma(x),\sigma(y)}, \tag{2.1}$$

where $J \in \mathbb{R}\backslash\{0\}$ and $\xi\colon (u, v) \in [0, 1]^2 \mapsto \xi_{u,v} \in \mathbb{R}$ is a given bounded and measurable function.

Let $h\colon x \in V \mapsto h_x = (h_{t,x}, t \in [0, 1]) \in \mathbb{R}^{[0,1]}$ be a map of the vertex $x \in V \setminus \{x^0\}$. For given $n = 1, 2, \ldots$, we consider the probability distribution $\mu^{(n)}$ on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\biggl(-\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x),x}\biggr). \tag{2.2}$$

Here, as above, $\sigma_n\colon x \in V_n \mapsto \sigma(x)$, and Z_n is the corresponding partition function,

$$Z_n = \int_{\Omega_{V_n}} \exp\biggl(-\beta H(\tilde\sigma_n) + \sum_{x\in W_n} h_{\tilde\sigma(x),x}\biggr)\lambda_{V_n}(d\tilde\sigma_n). \tag{2.3}$$

A family of probability distributions $\mu^{(n)}$ is said to be compatible if for any $n \ge 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$, it satisfies the condition

$$\int_{\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1}\vee\omega_n)\lambda_{W_n}(d(\omega_n)) = \mu^{(n-1)}(\sigma_{n-1}). \tag{2.4}$$

Here, $\sigma_{n-1}\vee\omega_n \in \Omega_{V_n}$ is the concatenation of the configurations σ_{n-1} and ω_n. In this case, there exists a unique measure μ on Ω_V such that for any n and $\sigma_n \in \Omega_{V_n}$, we have $\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n)$.

Definition. The measure μ is called a *splitting Gibbs measure* corresponding to Hamiltonian (2.1) and the function $x \mapsto h_x$, $x \ne x^0$.

The following statement [10] describes conditions on h_x guaranteeing the compatibility of the corresponding distributions $\mu^{(n)}(\sigma_n)$.

Proposition 2.1. *The probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \dots$, in* (2.2) *are compatible iff for any $x \in V \setminus \{x^0\}$, the equation*

$$f(t,x) = \prod_{y\in S(x)} \frac{\int_0^1 e^{J\beta\xi_{t,u}} f(u,y)\,du}{\int_0^1 e^{J\beta\xi_{0u}} f(u,y)\,du} \tag{2.5}$$

is satisfied. Here and hereafter, $f(t,x) = e^{h_{t,x}-h_{0,x}}$, $t \in [0,1]$, and $du = \lambda(du)$ is the Lebesgue measure.

3. Main results

In this section, we consider the model

$$H(\sigma) = -\frac{1}{\beta}\sum_{\langle x,y\rangle\in L} \log\bigl(1 + \theta \sqrt[2n+1]{(\sigma(x)-1/2)(\sigma(y)-1/2)}\,\bigr), \tag{3.1}$$

where $\theta \in [-4^{1/(2n+1)}, 4^{1/(2n+1)}]$ (it is easy to see that model (3.1) is not defined in the case $\theta \notin [-4^{1/(2n+1}, 4^{1/(2n+1)}]$) and β is the inverse temperature, i.e., $\beta = 1/T > 0$. We note that model (3.1) is a generalization of some models (see [12]–[15]) in statistical physics, for example, the Ising–Vanniminus model.

Let $\xi_{t,u}$ be a continuous function. We consider solutions of (2.5) in the class of translation-invariant functions $f(t,x)$, i.e., $f(t,x) = f(t)$ for any $x \in V$. For such functions, we can write (2.5) as

$$f(t) = \left(\frac{\int_0^1 K(t,u)f(u)\,du}{\int_0^1 K(0,u)f(u)\,du}\right)^k := (A_k f)(t), \tag{3.2}$$

where $K(t,u) = e^{J\beta\xi_{t,u}}$, $f(t) > 0$, $t,u \in [0,1]$. We completely describe the set of translation-invariant Gibbs measures for this model for all values of θ.

Let $C^+[0,1] = \{f \in C[0,1]\colon f(x) \ge 0\}$. For every $k \in \mathbb{N}$, we consider an integral operator H_k acting in the cone $C^+[0,1]$ as

$$(H_k f)(t) = \int_0^1 K(t,u)f^k(u)\,du, \quad k \in \mathbb{N}. \tag{3.3}$$

The operator H_k is called Hammerstein's integral operator of order k. It is known that translation-invariant Gibbs measures of model (2.1) can be described by positive fixed points of Hammerstein's integral operator, namely, we have the following proposition [7].

Proposition 3.1. *Let $k \geq 2$. Then the equation $A_k f = f$ has a strictly positive solution iff the equation $H_k f = f$ has a strictly positive solution.*

From (3.1), we obtain

$$\xi_{t,u} = \xi_{t,u}(\theta,\beta) = \frac{1}{J\beta}\log(1+\theta \sqrt[2n+1]{(t-1/2)(u-1/2)}\,), \quad t,u \in [0,1],$$

where $-4^{1/(2n+1)} < \theta < 4^{1/(2n+1)}$. Because $K(t,u) = e^{J\beta\xi_{t,u}}$, we also obtain

$$K(t,u) = 1+\theta \sqrt[2n+1]{(t-1/2)(u-1/2)}. \tag{3.4}$$

Hence, based on Proposition 3.1, instead of seeking translation-invariant Gibbs measures for model (3.1) on a Cayley tree of order k, we study positive fixed points of Hammerstein's integral operator of order k for kernel (3.4), i.e., positive solutions of the equation

$$\int_0^1 (1+\theta \sqrt[2n+1]{(t-1/2)(u-1/2)}\,) f^k(u)\,du = f(t). \tag{3.5}$$

Let $t_1 = t-1/2$ and $u_1 = u-1/2$. We can then write (3.5) as

$$\int_{-1/2}^{1/2} f^k(u_1)\,du_1 + \theta \sqrt[2n+1]{t_1}\int_{-1/2}^{1/2} f^k(u_1)\sqrt[2n+1]{u_1}\,du_1 = f(t_1).$$

We set

$$\mathcal{C}_1 = \int_{-1/2}^{1/2} f^k(u_1)\,du_1, \qquad \mathcal{C}_2 = \int_{-1/2}^{1/2} f^k(u_1)\sqrt[2n+1]{u_1}\,du_1. \tag{3.6}$$

Then $f(t_1) = \mathcal{C}_1 + \theta\mathcal{C}_2 \sqrt[2n+1]{(t_1)}$. Consequently, (3.6) is equivalent to

$$\mathcal{C}_1 = \int_{-1/2}^{1/2} (\mathcal{C}_1 + \theta\mathcal{C}_2 \sqrt[2n+1]{u_1}\,)^k\,du_1, \qquad \mathcal{C}_2 = \int_{-1/2}^{1/2} (\mathcal{C}_1 + \theta\mathcal{C}_2 \sqrt[2n+1]{u_1}\,)^k \sqrt[2n+1]{u_1}\,du_1.$$

Namely,

$$\begin{aligned} \mathcal{C}_1 &= \sum_{i=0}^{k} \theta^i \left(C_k^{k-i} \int_{-1/2}^{1/2} \sqrt[2n+1]{u_1^i}\,du_1 \right) \mathcal{C}_1^{k-i}\mathcal{C}_2^i, \\ \mathcal{C}_2 &= \sum_{i=0}^{k} \theta^i \left(C_k^{k-i} \int_{-1/2}^{1/2} \sqrt[2n+1]{u_1^{i+1}}\,du_1 \right) \mathcal{C}_1^{k-i}\mathcal{C}_2^i. \end{aligned} \tag{3.7}$$

We define an operator $V_k\colon (x,y) \mapsto (x',y')$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ as

$$V_k : \begin{aligned} x' &= \sum_{i=0}^{[k/2]} \frac{(2n+1)C_k^{k-2i}\theta^{2i}}{(2n+2i+1)\sqrt[2n+1]{4^i}} x^{k-2i}y^{2i}, \\ y' &= \sum_{i=0}^{[k/2]} \frac{(2n+1)C_k^{k-2i-1}\theta^{2i+1}}{2(n+i+1)\sqrt[2n+1]{2\cdot 4^i}} x^{k-2i}y^{2i}. \end{aligned} \tag{3.8}$$

We then write Eq. (3.7) as $V_k(\mathcal{C}) = \mathcal{C}$, where $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$. Based on the obtained results, we have the following statement.

Lemma 3.1. *A function $\varphi \in C[0,1]$ is a fixed point of Hammerstein's equation iff $\varphi(t)$ has the form*

$$\varphi(t) = \mathcal{C}_1 + \mathcal{C}_2 \theta \sqrt[2n+1]{4(t-1/2)},$$

where $(\mathcal{C}_1, \mathcal{C}_2) \in \mathbb{R}^2$ is a fixed point of the operator V_k.

We now introduce some notation. Let

$$\alpha_{2i} = \frac{2n+1}{2n+2i+1}\left(\frac{1}{2}\right)^{2i/(2n+1)}, \quad i, n \in \mathbb{N}.$$

If $k = 2s$, $s \in \mathbb{N}$, then for $i, n \in \mathbb{N}$, we set

$$\beta_i = C_{2s}^{i}\alpha_{i+1}, \qquad \theta_{2i+1} = \frac{(2i+1)(2n+2i+3)4^{1/(2n+1)}}{(2s-2i)(2n+2i+1)}.$$

If $k = 2s+1$, $s \in \mathbb{N}$, then for $i, n \in \mathbb{N}$, we set

$$\beta_i = C_{2s+1}^{i+1}\alpha_{i+1}, \qquad \theta_{2i+1} = \frac{(2i+1)(2s+2i+3)4^{1/(2n+1)}}{(2s-2i+1)(2n+2i+1)}.$$

Remark 3.1. We consider the function

$$\theta_x = \frac{x^2 + (2n+2)x}{(2s+1-x)(x+2n)}.$$

It is easy to see that

$$\theta_x' = \frac{2x^2 + (2s+2)(x+2n)^2 + 4n(2s+1)}{(2s+1-x)^2(x+2n)^2} > 0.$$

Therefore, we can conclude that

$$\theta_1 < \theta_3 < \cdots < \theta_{2s-1}. \tag{3.9}$$

Lemma 3.2. *Let $k = 2s$, $s \in \mathbb{N}$. If the point $(x_0, y_0) \in \mathbb{R}_+^2$ is a fixed point of* (3.8), *then $\lambda = y_0/x_0$ is a root of the equation*

$$P(\lambda) := \beta_1(\theta_1 - \theta) + \beta_3(\theta_3 - \theta)\lambda^2 + \cdots + \beta_{2s-1}(\theta_{2s-1} - \theta)\lambda^{2s-2} + \alpha_{2s}\lambda^{2s} = 0. \tag{3.10}$$

Proof. Let (x_0, y_0) be a fixed point of (3.8). Dividing the second equation in system (3.8) by the first, we obtain

$$\frac{\lambda}{\theta} = \frac{C_{2s}^{1}\alpha_2\lambda + C_{2s}^{2}\alpha_3\lambda^2 + \cdots + C_{2s}^{2s}\alpha_{2s+1}\lambda^{2s}}{1 + C_{2s}^{1}\alpha_1\lambda + C_{2s}^{2}\alpha_2\lambda^2 + \cdots + C_{2s}^{2s}\alpha_{2s}\lambda^{2s}},$$

where $\lambda = y/x$. Let $\lambda \neq 0$. After some calculations, we then have

$$1 - C_{2s}^{1}\alpha_2\theta + (C_{2s}^{2}\alpha_2 - C_{2s}^{3}\alpha_4\theta)\lambda^2 + \cdots + (C_{2s}^{2s-2}\alpha_{2s-2} - C_{2s}^{2s-1}\alpha_{2s}\theta)\lambda^{2s-2} + C_{2s}^{2s}\alpha_{2s}\lambda^{2s} = 0.$$

In other words,

$$\beta_1(\theta_1 - \theta) + \beta_3(\theta_3 - \theta)\lambda^2 + \cdots + \beta_{2s-1}(\theta_{2s-1} - \theta)\lambda^{2s-2} + \alpha_{2s}\lambda^{2s} = 0, \tag{3.11}$$

where

$$\theta_{2i-1} = \frac{C_{2s}^{2i-2}\alpha_{2i-2}}{C_{2s}^{2i-1}\alpha_{2i}}.$$

It is easy to verify that if $\lambda = 0$, then this solution corresponds to the fixed point $(x_0, y_0) = (1, 0)$ of transformation (3.8).

Analogously, we obtain the following lemma.

Lemma 3.3. *Let $k = 2s+1$, $s \in \mathbb{N}$. If the point $(x_0, y_0) \in \mathbb{R}^2_+$ is a fixed point of transformation* (3.8), *then $\lambda = y_0/x_0$ is a root of the equation*

$$Q(\lambda) := \beta_1(\theta_1 - \theta) + \beta_3(\theta_3 - \theta)\lambda^2 + \cdots + \beta_{2s-1}(\theta_{2s-1} - \theta)\lambda^{2s-2} + \beta_{2s+1}(\theta_{2s+1} - \theta)\lambda^{2s} = 0. \tag{3.12}$$

Proposition 3.2. *Let $k = 2s$, $s \in \mathbb{N}$. If $\theta \le \theta_1$, then there is no nontrivial solution of* (3.10). *If $\theta > \theta_1$, then there are exactly two* (*nontrivial*) *solutions of* (3.10). *These solutions have the same absolute value but opposite signs.*

Proof. Because the proof for $\theta \le \theta_1$ is very simple, we consider only the case $\theta > \theta_1$. The number of sign changes of the coefficients of the polynomial $P(\lambda)$ is equal to one. Hence, $P(\lambda)$ has at most one positive solution. On the other hand, we have $P(0) < 0$ and $\lim_{\lambda\to\infty} P(\lambda) = +\infty$. It then follows from Rolle's theorem that $P(\lambda)$ has at least one positive solution. Therefore, there exists a unique $\lambda^* > 0$ such that $P(\lambda^*) = 0$. Because $P(\lambda)$ is an even function, there is only one negative solution $-\lambda^*$.

Proof of the next proposition is similar to the previous proof, and we therefore omit it.

Proposition 3.3. *Let $k = 2s+1$, $s \in \mathbb{N}$. If $\theta \le \theta_1$, then there is no nontrivial solution of* (3.12). *If $\theta > \theta_1$, then there are exactly two* (*nontrivial*) *solutions of* (3.12). *These solutions have the same absolute value but opposite signs.*

Proposition 3.4. *Let $k = 2s$, $s \in \mathbb{N}$. Then we have the following statements*:

1. *If θ satisfies the condition*
$$-4^{1/(2n+1)} < \theta \le \theta_1, \tag{3.13}$$
then (3.3) *has only one positive fixed point $f(t) = 1$.*

2. *If θ satisfies the condition*
$$\frac{\sum_{i=1}^{s} 2^{(2i-2)/(2n+1)}\beta_{2i-1}\theta_{2i-1} + \alpha_{2s}2^{2s/(2n+1)}}{\sum_{i=1}^{s} 2^{(2i-2)/(2n+1)}\beta_{2i-1}} \le \theta \le 4^{1/(2n+1)}, \tag{3.14}$$
then (3.3) *has exactly two positive fixed points*
$$f_1(t) = 1, \qquad f_2(t) = \overline{C}(1 + \lambda^* t^{1/(2n+1)}).$$

3. *If θ satisfies the condition*
$$\theta_1 < \theta < \frac{\sum_{i=1}^{s} 2^{(2i-2)/(2n+1)}\beta_{2i-1}\theta_{2i-1} + \alpha_{2s}2^{2s/(2n+1)}}{\sum_{i=1}^{s} 2^{(2i-2)/(2n+1)}\beta_{2i-1}}, \tag{3.15}$$
then (3.3) *has exactly three positive fixed points*
$$f_1(t) = 1, \qquad f_2(t) = \overline{C}(1 + \lambda^* t^{1/(2n+1)}), \qquad f_3(t) = \overline{C}(1 - \lambda^* t^{1/(2n+1)}).$$

Here, λ^* is the positive solution of (3.10).

Proof. We prove statement 2 (other statements are proved similarly). By virtue of the inequality

$$\theta_1 \le \frac{\sum_{i=1}^{s} 2^{(2i-2)/(2n+1)}\beta_{2i-1}\theta_{2i-1} + \alpha_{2s}2^{2s/(2n+1)}}{\sum_{i=1}^{s} 2^{(2i-2)/(2n+1)}\beta_{2i-1}}, \le \theta$$

Eq. (3.10) has exactly three solutions. They are $\lambda_1 = 0$, $\lambda_2 = \lambda^*$, and $\lambda_3 = -\lambda^*$. According to the definition of $f(t)$, we have the solutions

$$f_1(t) = 1, \qquad f_2(t) = C_1(1 + \lambda^* t^{1/(2n+1)}), \qquad f_3(t) = C_1(1 - \lambda^* t^{1/(2n+1)}).$$

We need only positive solutions. It is easy to verify that $f_1(t)$ and $f_2(t)$ are positive solutions. We check the third solution. The function $f_3(t)$ is negative iff $\lambda^* \ge 2^{1/(2n+1)}$. In this case, it suffices to verify that $P(2^{1/(2n+1)}) < 0$. This condition is equivalent to

$$\frac{\sum_{i=1}^{s} 2^{(2i-2)/(2n+1)}\beta_{2i-1}\theta_{2i-1} + \alpha_{2s}2^{2s/(2n+1)}}{\sum_{i=1}^{s} 2^{(2i-2)/(2n+1)}\beta_{2i-1}} \le \theta.$$

The last inequality exactly corresponds to the condition in statement 2.

Based on the obtained results, we have the following theorem.

Theorem 3.1. *Let $k = 2s$, $s \in \mathbb{N}$.*

1. *If θ satisfies condition* (3.13), *then there exists a unique translation-invariant Gibbs measure for model* (2.1) *on a Cayley tree of order k.*
2. *If θ satisfies condition* (3.14), *then there exist exactly two translation-invariant Gibbs measures for this model.*
3. *If θ satisfies condition* (3.15), *then there exist exactly three translation-invariant Gibbs measures for this model.*

The proofs of the following proposition and theorem are similar to the preceding proofs, and we omit them.

Proposition 3.5. *Let $k = 2s$, $s \in \mathbb{N}$.*

1. *If θ satisfies the condition*

$$-4^{1/(2n+1)} < \theta \le \theta_1 \qquad \text{or} \qquad \theta_{2s+1} \le \theta < 4^{1/(2n+1)}, \tag{3.16}$$

then (3.3) *has only one positive fixed point $f(t) = 1$.*

2. *If θ satisfies the condition*

$$\frac{\sum_{i=1}^{s+1} 2^{(2i-2)/(2n+1)}\beta_{2i-1}\theta_{2i-1}}{\sum_{i=1}^{s+1} 2^{(2i-2)/(2n+1)}\beta_{2i-1}} \le \theta < \theta_{2s+1}, \tag{3.17}$$

then (3.3) *has exactly two positive fixed points*

$$f_1(t) = 1, \qquad f_2(t) = \overline{C}(1 + \lambda^* t^{1/(2n+1)}).$$

3. *If θ satisfies the condition*

$$\theta_1 < \theta < \frac{\sum_{i=1}^{s+1} 2^{(2i-2)/(2n+1)}\beta_{2i-1}\theta_{2i-1}}{\sum_{i=1}^{s+1} 2^{(2i-2)/(2n+1)}\beta_{2i-1}}, \tag{3.18}$$

then (3.3) *has exactly three positive fixed points*

$$f_1(t) = 1, \qquad f_2(t) = \overline{C}(1 + \lambda^* t^{1/(2n+1)}), \qquad f_3(t) = \overline{C}(1 - \lambda^* t^{1/(2n+1)}).$$

Here, λ^* is the positive solution of (3.12).

Theorem 3.2. *Let $k = 2s$, $s \in \mathbb{N}$. Then we have the following three cases.*

1. *If θ satisfies condition* (3.16)*, then there exists a unique translation-invariant Gibbs measure for model* (2.1) *on a Cayley tree of order k.*
2. *If θ satisfies condition* (3.17)*, then there exist exactly two translation-invariant Gibbs measures for this model.*
3. *If θ satisfies condition* (3.18)*, then there exist exactly three translation-invariant Gibbs measures for this model.*

Remark 3.2. If $k = 2$, then Theorem 3.1 coincides with Theorem 4.2 in [13]. If $k = 3$, then Theorem 3.2 coincides with Theorem 5.2 in [14].

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REFERENCES

1. S. A. Pirogov and Ya. G. Sinai, "Phase diagrams of classical lattice systems," *Theor. Math. Phys.*, **25**, 1185–1192 (1975); "Phase diagrams of classical lattice systems continuation," *Theor. Math. Phys.*, **26**, 39–49 (1976).
2. R. Kotecký and S. B. Shlosman, "First-order phase transition in large entropy lattice models," *Commun. Math. Phys.*, **83**, 493–515 (1982).
3. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Acad. Press, London (1982).
4. P. M. Bleher, J. Ruiz, and V. A. Zagrebnov, "On the purity of the limiting Gibbs state for the Ising model on the Bethe lattice," *J. Statist. Phys.*, **79**, 473–482 (1995).
5. C. Preston, *Gibbs States on Countable Sets*, Cambridge Univ. Press, Cambridge (1974).
6. F. Spitzer, "Markov random fields on an infinite tree," *Ann. Probab.*, **3**, 387–398 (1975).
7. Yu. Kh. Eshkabilov, F. H. Haydarov, and U. A. Rozikov, "Uniqueness of Gibbs measure for models with uncountable set of spin values on a Cayley tree," *Math. Phys. Anal. Geom.*, **16**, 1–17 (2013).
8. Yu. Kh. Eshkabilov, Sh. D. Nodirov, and F. H. Haydarov, "Positive fixed points of quadratic operators and Gibbs measures," *Positivity*, **20**, 929–943 (2016).
9. U. A. Rozikov and F. H. Haydarov, "Periodic Gibbs measures for models with uncountable set of spin values on a Cayley tree," *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **18**, 1550006 (2015).
10. U. A. Rozikov and Yu. Kh. Eshkabilov, "On models with uncountable set of spin values on a Cayley tree: Integral equations," *Math. Phys. Anal. Geom.*, **13**, 275–286 (2010).
11. U. A. Rozikov, F. Kh. Khaidarov, "Four competing interactions for models with an uncountable set of spin values on a Cayley tree," *Theor. Math. Phys.*, **191**, 910–923 (2017).

12. Yu. Kh. Eshkabilov, F. H. Haydarov, and U. A. Rozikov, "Non-uniqueness of Gibbs measure for models with uncountable set of spin values on a Cayley tree," *J. Statist. Phys.*, **147**, 779–794 (2012).
13. Yu. Kh. Eshkabilov, U. A. Rozikov, and G. I. Botirov, "Phase transition for a model with uncountable set of spin values on Cayley tree," *Lobachevskii J. Math.*, **34**, 256–263 (2013).
14. B. Jahnel, C. Külske, and G. I. Botirov, "Phase transition and critical value of nearest-neighbor system with uncountable local state space on Cayley tree," *Math. Phys. Anal. Geom.*, **17**, 323–331 (2014).
15. G. I. Botirov, "A model with uncountable set of spin values on a Cayley tree: Phase transitions," *Positivity*, **21**, 955–961 (2017).