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Classification of complex naturally graded quasi-filiform Zinbiel algebras

J.Q. Adashev
Institute of Mathematics and Information Technologies,
Uzbek Academy of Sciences, Tashkent, Uzbekistan, e-mail: adashevjq@mail.ru

A.Kh. Khudoyberdiyev
Institute of Mathematics and Information Technologies,
Uzbek Academy of Sciences, Tashkent, Uzbekistan, e-mail: abror_alg_85@mail.ru

B.A. Omirov [1]
Institute of Mathematics and Information Technologies,
Uzbek Academy of Sciences, Tashkent, Uzbekistan, e-mail: omirovb@mail.ru

Abstract

In this work the description up to isomorphism of complex naturally graded quasi-filiform Zinbiel algebras is obtained.

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This work is dedicated to the 60-th anniversary of Professor Shestakov I.P.

1 Introduction

In the present paper we investigate algebras which are Koszul dual to Leibniz algebras. The Leibniz algebras were introduced in the work [6] and they present a "non commutative" (to be more precise, a "non antisymmetric") analogue of Lie algebras. Many works, including [6]-[7], were devoted to the investigation of cohomological and structural properties of Leibniz algebras. Ginzburg and Kapranov introduced and studied the concept of Koszul dual operads [4]. Following this concept, it was shown in [5] that the category of dual algebras to the category of Leibniz algebras is defined by the identity:

$$(x \circ y) \circ z = x \circ (y \circ z) + x \circ (z \circ y).$$

In this paper, dual Leibniz algebras will called Zinbiel algebras (Zinbiel is obtained from Leibniz written in inverse order). Some interesting properties of Zinbiel algebras were obtained in [1], [2], and [3]. In particular, the nilpotency of an arbitrary finite-dimensional complex Zinbiel algebra was proved in [3], and zero-filiform and filiform Zinbiel algebras were classified in [1]. The classification of complex Zinbiel algebras up to dimension 4 is obtained in works [3] and [8]. The present paper is devoted to the investigation of the next stage — description of quasi-filiform complex Zinbiel algebras.

Examples of Zinbiel algebras can be found in [1], [3] and [5].

We consider below only complex algebras and, for convenience, we will omit zero products the algebra's multiplication table.

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2 Preliminaries

Definition 2.1. *An algebra A over a field F is called a Zinbiel algebra if for any $x, y, z \in A$ the identity*

$$(x \circ y) \circ z = x \circ (y \circ z) + x \circ (z \circ y) \tag{1}$$

holds.

For an arbitrary Zinbiel algebras define the lower central series

$$A^1 = A, \quad A^{k+1} = A \circ A^k, \quad k \geq 1.$$

Definition 2.2. *A Zinbiel algebra A is called nilpotent if there exists an $s \in N$ such that $A^s = 0$. The minimal number s satisfying this property (i.e. $A^{s-1} \neq 0$ and $A^s = 0$) is called nilindex of the algebra A.*

It is not difficult to see that nilindex of an arbitrary n-dimensional nilpotent algebra does not exceed the number $n + 1$.

Definition 2.3. *An n-dimensional Zinbiel algebra A is called zero-filiform if $dim A^i = (n + 1) - i$, $1 \leq i \leq n + 1$.*

Clearly, the definition of a zero-filiform algebra A amounts to requiring that A has a maximal nilindex.

Theorem 2.1. [1] *An arbitrary n-dimensional zero-filiform Zinbiel algebra is isomorphic to the algebra*

$$e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad \text{for } \; 2 \leq i + j \leq n \tag{2}$$

where symbol C^t_s is a binomial coefficient defined as $C^t_s = \frac{s!}{t!(s-t)!}$, and $\{e_1, e_2, \ldots, e_n\}$ is a basis of the algebra.

We denote the algebra from Theorem 2.1 as NF_n.

It is easy to see that an n-dimensional Zinbiel algebra is one-generated if and only if it is isomorphic to NF_n.

Definition 2.4. *An n-dimensional Zinbiel algebra A is called filiform if $dim A^i = n - i$, $2 \leq i \leq n$.*

The following theorem gives classification of filiform Zinbiel algebras.

Theorem 2.2. *Any n-dimensional ($n \geq 5$) filiform Zinbiel algebra is isomorphic to one of the following three pairwise non isomorphic algebras:*

$$\begin{array}{l}
F^1_n : e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad 2 \leq i + j \leq n - 1; \\
F^2_n : e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad 2 \leq i + j \leq n - 1, \quad e_n \circ e_1 = e_{n-1}; \\
F^3_n : e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad 2 \leq i + j \leq n - 1, \quad e_n \circ e_n = e_{n-1}.
\end{array}$$

Since the direct sum of nilpotent Zinbiel algebras is nilpotent one, we shall consider only non split algebras.

Summarizing the results of [1], [3], and [8], we give the classification of complex Zinbiel algebras up to dimension ≤ 4.

Theorem 2.3. *An arbitrary non split Zinbiel algebra is isomorphic to the following pairwise non isomorphic algebras:*
$Dim A = 1:$ *Abelian*
$Dim A = 2: e_1 \circ e_1 = e_2$
$Dim A = 3:$

$Z_3^1: \ e_1 \circ e_1 = e_2, \quad e_1 \circ e_2 = \frac{1}{2} e_3, \quad e_2 \circ e_1 = e_3;$

$Z_3^2: \ e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = -e_3;$

$Z_3^3: \ e_1 \circ e_1 = e_3, \quad e_1 \circ e_2 = e_3, \quad e_2 \circ e_2 = \alpha e_3, \quad \alpha \in C;$

$Z_3^4: \ e_1 \circ e_1 = e_3, \quad e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = e_3.$

$Dim A = 4:$

$Z_4^1: \ e_1 \circ e_1 = e_2, \quad e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = 2e_3, \quad e_1 \circ e_3 = e_4, \quad e_2 \circ e_2 = 3e_4, \quad e_3 \circ e_1 = 3e_4;$

$Z_4^2: \ e_1 \circ e_1 = e_3, \quad e_1 \circ e_2 = e_4, \quad e_1 \circ e_3 = e_4, \quad e_3 \circ e_1 = 2e_4;$

$Z_4^3: \ e_1 \circ e_1 = e_3, \quad e_1 \circ e_3 = e_4, \quad e_2 \circ e_2 = e_4, \quad e_3 \circ e_1 = 2e_4;$

$Z_4^4: \ e_1 \circ e_2 = e_3, \quad e_1 \circ e_3 = e_4, \quad e_2 \circ e_1 = -e_3;$

$Z_4^5: \ e_1 \circ e_2 = e_3, \quad e_1 \circ e_3 = e_4, \quad e_2 \circ e_1 = -e_3, \quad e_2 \circ e_2 = e_4;$

$Z_4^6: \ e_1 \circ e_1 = e_4, \quad e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = -e_3, \quad e_2 \circ e_2 = -2e_3 + e_4;$

$Z_4^7: \ e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = e_4, \quad e_2 \circ e_2 = -e_3;$

$Z_4^8(\alpha): \ e_1 \circ e_1 = e_3, \quad e_1 \circ e_2 = e_4, \quad e_2 \circ e_1 = -\alpha e_3, \quad e_2 \circ e_2 = -e_4, \quad \alpha \in C$

$Z_4^9(\alpha): \ e_1 \circ e_1 = e_4, \quad e_1 \circ e_2 = \alpha e_4, \quad e_2 \circ e_1 = -\alpha e_4, \quad e_2 \circ e_2 = e_4, \quad e_3 \circ e_3 = e_4, \quad \alpha \in C$

$Z_4^{10}: \ e_1 \circ e_2 = e_4, \quad e_1 \circ e_3 = e_4, \quad e_2 \circ e_1 = -e_4, \quad e_2 \circ e_2 = e_4, \quad e_3 \circ e_1 = e_4;$

$Z_4^{11}: \ e_1 \circ e_1 = e_4, \quad e_1 \circ e_2 = e_4, \quad e_2 \circ e_1 = -e_4, \quad e_3 \circ e_3 = e_4;$

$Z_4^{12}: \ e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = e_4;$

$Z_4^{13}: \ e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = -e_3, \quad e_2 \circ e_2 = e_4;$

$Z_4^{14}: \ e_2 \circ e_1 = e_4, \quad e_2 \circ e_2 = e_3;$

$Z_4^{15}(\alpha): \ e_1 \circ e_2 = e_4, \quad e_2 \circ e_2 = e_3, \quad e_2 \circ e_1 = \frac{1+\alpha}{1-\alpha} e_4, \quad \alpha \in C \setminus \{1\};$

$Z_4^{16}: \ e_1 \circ e_2 = e_4, \quad e_2 \circ e_1 = -e_4, \quad e_3 \circ e_3 = e_4;$

Let us introduce some definitions and notations.

The set $R(A) = \{a \in A| \ \ b \circ a = 0 \text{ for any } b \in A\}$ is called the right annihilator of the Zinbiel algebra A.

The set $L(A) = \{a \in A| \ \ a \circ b = 0 \text{ for any } b \in A\}$ is called the left annihilator of the Zinbiel algebra A.

$Z(a, b, c)$ denotes the following polynomial:

$$Z(a, b, c) = (a \circ b) \circ c - a \circ (b \circ c) - a \circ (c \circ b).$$

It is obvious that Zinbiel algebras are determined by the identity $Z(a, b, c) = 0$.

3 Classification of naturally graded quasi-filiform Zinbiel algebras

Definition 3.1. *A Zinbiel algebra A is called quasi-filiform if $A^{n-2} \neq 0$ and $A^{n-1} = 0$, where $dimA = n$.*

Let A be a quasi-filiform Zinbiel algebra. Putting $A_i = A^i/A^{i+1}$, $1 \leq i \leq n-2$, we obtain the graded Zinbiel algebra

$$GrA = A_1 \oplus A_2 \oplus \ldots \oplus A_{n-2}, \quad \text{where} \quad A_i \circ A_j \subseteq A_{i+j}.$$

An algebra A is called naturally graded if $A \cong GrA$. It is not difficult to see that $A_1 \circ A_j = A_{j+1}$ in the naturally graded algebra A. Let A be an n-dimensional graded quasi-filiform algebra. Then there exists a basis $\{e_1, e_2, \ldots, e_n\}$ of the algebra A such that $e_i \in A_i$, $1 \leq i \leq n-2$. It is evident that $dimA_1 > 1$. In fact, if $dimA_1 = 1$, then the algebra A is one-degenerated and therefore it is the zero-filiform algebra, but it is not quasi-filiform. Without loss of generality, one can assume $e_{n-1} \in A_1$. If for a Zinbiel algebra A, the condition $e_n \in A_r$ holds, the algebra is said to be of type $A_{(r)}$.

3.1 The case $r = 1$

Theorem 3.1. *Any n-dimensional ($n \geq 6$) naturally graded quasi-filiform Zinbiel algebra of the type $A_{(1)}$ is isomorphic to the algebra:*

$$KF_n^1 : e_i \circ e_j = C_{i+j-1}^j e_{i+j} \quad \textit{for} \quad 2 \leq i+j \leq n-2. \tag{3}$$

Proof. Let an algebra A satisfy the conditions of the theorem. Then there exists a basis $\{e_1, e_2, \ldots, e_n\}$ such that

$$A_1 = \langle e_1, e_{n-1}, e_n \rangle, \quad A_2 = \langle e_2 \rangle, \quad A_3 = \langle e_3 \rangle, \quad \ldots, \quad A_{n-2} = \langle e_{n-2} \rangle.$$

Using arguments similar to the ones from [9], we obtain

$$e_1 \circ e_i = e_{i+1} \quad \text{for} \quad 2 \leq i \leq n-3.$$

Now introduce the notations

$$\begin{array}{lll} e_1 \circ e_1 = \alpha_{1,1} e_2, & e_1 \circ e_{n-1} = \alpha_{1,2} e_2, & e_1 \circ e_n = \alpha_{1,3} e_2, \\ e_{n-1} \circ e_1 = \alpha_{2,1} e_2, & e_{n-1} \circ e_{n-1} = \alpha_{2,2} e_2, & e_{n-1} \circ e_n = \alpha_{2,3} e_2, \\ e_n \circ e_1 = \alpha_{3,1} e_2, & e_n \circ e_{n-1} = \alpha_{3,2} e_2, & e_n \circ e_n = \alpha_{3,3} e_2. \end{array}$$

We consider the following cases:

Case 1. Let $(\alpha_{1,1}, \alpha_{2,2}, \alpha_{3,3}) \neq (0,0,0)$. Then without loss of generality, one can take $\alpha_{1,1} \neq 0$. Moreover, making the change $e_2' = \alpha_{1,1} e_2, \quad e_3' = \alpha_{1,1} e_3, \quad \ldots, \quad e_{n-2}' = \alpha_{1,1} e_{n-2}$, we can assume $\alpha_{1,1} = 1$.

Obviously, the linear span $lin\langle e_1, e_2, \ldots, e_{n-2} \rangle$ forms zero-filiform Zinbiel algebra. Hence

$$e_i \circ e_j = C_{i+j-1}^j e_{i+j} \quad \text{at} \quad 2 \leq i+j \leq n-2$$

and omitted products of the basic elements $\{e_1, e_2, \dots, e_{n-2}\}$ are equal to zero.
Taking into account equalities

$$Z(e_1, e_{n-1}, e_1) = Z(e_1, e_1, e_{n-1}) = Z(e_1, e_{n-1}, e_{n-1}) = Z(e_{n-1}, e_1, e_1) = 0$$

we obtain

$$\alpha_{1,2} = \alpha_{2,1}, \quad e_2 \circ e_{n-1} = 2\alpha_{1,2}e_3, \quad \alpha_{1,2}^2 = \alpha_{2,2}, \quad e_{n-1} \circ e_2 = \alpha_{1,2}e_3.$$

Substituting $e'_{n-1} = -\alpha_{1,2}e_1 + e_{n-1}$, $e'_i = e_i$ for $i \neq n-1$, one can suppose

$$\alpha_{1,2} = \alpha_{2,1} = \alpha_{2,2} = 0.$$

Analogously, we have

$$e_1 \circ e_n = e_n \circ e_1 = e_n \circ e_n = e_2 \circ e_n = e_n \circ e_2 = 0 \quad \text{and} \quad e_n \circ e_{n-1} = e_{n-1} \circ e_n = 0.$$

Hence, using induction and the following chain of equalities

$$e_{n-1} \circ e_i = e_{n-1} \circ (e_1 \circ e_{i-1}) = (e_{n-1} \circ e_1) \circ e_{i-1} - e_{n-1} \circ (e_{i-1} \circ e_1) = -(i-1)e_{n-1} \circ e_i$$

we prove validity of the equality $e_{n-1} \circ e_i = 0$ for $1 \leq i \leq n$.
In the same way, equalities

$$e_{i+1} \circ e_{n-1} = (e_1 \circ e_i) \circ e_{n-1} = e_1 \circ (e_i \circ e_{n-1}) + e_1 \circ (e_{n-1} \circ e_i) = 0$$

lead to $e_{i+1} \circ e_{n-1} = 0$, $1 \leq i \leq n$. But it means that the algebra A is split, i.e. $A = F^1_{n-1} \oplus C$, where $F^1_{n-1} = NF_{n-2} \oplus C$. Hence, $A = KF^1_n$.
Case 2. Let $(\alpha_{1,1}, \alpha_{2,2}, \alpha_{3,3}) = (0,0,0)$. Then

$$(\alpha_{1,2}, \alpha_{2,1}, \alpha_{1,3}, \alpha_{3,1}, \alpha_{2,3}, \alpha_{3,2}) \neq (0,0,0,0,0,0).$$

Put $e'_1 = ae_1 + be_{n-1} + ce_n$. Then

$$e'_1 \circ e'_1 = [ab(\alpha_{1,2} + \alpha_{2,1}) + ac(\alpha_{1,3} + \alpha_{3,1}) + bc(\alpha_{2,3} + \alpha_{3,2})]e_2.$$

From this it follows $\alpha_{1,2} + \alpha_{2,1} = 0$, $\alpha_{1,3} + \alpha_{3,1} = 0$, and $\alpha_{2,3} + \alpha_{3,2} = 0$. In fact, otherwise revert to the conditions of the case 1.
Without loss of generality, we can assume $\alpha_{1,2} = 1$. The equality $Z(e_1, e_{n-1}, e_1) = 0$ implies $e_2 \circ e_1 = 0$. In addition,

$$0 = (e_1 \circ e_1) \circ e_2 = e_1 \circ (e_1 \circ e_2) + e_1 \circ (e_2 \circ e_1) = e_1 \circ e_3 = e_4.$$

Thus we obtain a contradiction with existence of an algebra in case 2. □

Proposition 3.1. *Let A be a five-dimensional naturally graded quasi-filiform Zinbiel algebra of type $A_{(1)}$. Then it is isomorphic to one of the following three pairwise non isomorphic algebras:*

$$KF_5^1 : e_1 \circ e_1 = e_2, \quad e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = e_3;$$
$$KF_5^2 : e_1 \circ e_4 = e_2, \quad e_1 \circ e_2 = e_3, \quad e_4 \circ e_1 = -e_3;$$
$$KF_5^3 : e_1 \circ e_4 = e_2, \quad e_1 \circ e_2 = e_3, \quad e_4 \circ e_1 = -e_3, \quad e_4 \circ e_1 = e_3.$$

Proof. Let an algebra A satisfy the conditions of the proposition. If the conditions of the case 1 of Theorem 3.1 hold, then A is isomorphic to the algebra KF_5^1. if the conditions of the case 2 of Theorem 3.1 are valid for A, we obtain the following multiplication in A :

$$\begin{array}{lll} e_1 \circ e_4 = e_2, & e_1 \circ e_5 = \alpha_{1,3} e_2, & e_1 \circ e_2 = e_3, \\ e_4 \circ e_1 = -e_2, & e_4 \circ e_5 = \alpha_{2,3} e_2, & e_4 \circ e_2 = \beta_1 e_3, \\ e_5 \circ e_1 = -\alpha_{1,3} e_2, & e_5 \circ e_4 = -\alpha_{2,3} e_2, & e_5 \circ e_2 = \beta_2 e_3. \end{array}$$

Changing basic elements by the rules

$$e_1' = e_1, \quad e_2' = e_2, \quad e_3' = e_3, \quad e_4' = -\beta_1 e_1 + e_4, \quad e_5' = \alpha_{2,3} e_1 - \alpha_{1,3} e_4 + e_5,$$

we can assume $\alpha_{1,3} = \alpha_{2,3} = \beta_1 = 0$. If $\beta_2 = 0$, then we obtain the algebra KF_5^2. But if $\beta_2 \neq 0$, then putting $e_5' = \frac{1}{\beta_2} e_5$, we have $\beta_2 = 1$ and obtain isomorphism to algebra KF_5^3. By virtue of $dimL(KF_5^2) = 3$, $dimL(KF_5^3) = 2$, and taking into account that generating elements of algebras KF_5^2 and KF_5^3 satisfy the identity $x \circ x = 0$, but this identity does not hold for generating elements of the algebra KF_5^1 (in particular, $e_1 \circ e_1 = e_2$), we obtain pairwise non isomorphic of obtained algebras. □

3.2 The case $r = 2$

Classification of naturally graded quasi-filiform Zinbiel algebras of type $A_{(2)}$ is given in the following theorem.

Theorem 3.2. *Any n-dimensional ($n \geq 8$) naturally graded quasi-filiform Zinbiel algebra of type $A_{(2)}$ is isomorphic to one of the following pairwise non isomorphic algebras:*

$$KF_n^1 : \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, & 2 \leq i+j \leq n-2, \\ e_1 \circ e_{n-1} = e_n, & e_{n-1} \circ e_1 = \alpha e_n; \end{cases}$$

$$KF_n^2 : \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq n-2, \\ e_1 \circ e_{n-1} = e_n, \quad e_{n-1} \circ e_1 = e_n, \quad e_{n-1} \circ e_{n-1} = e_n; \end{cases}$$

$$KF_n^3 : \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, & 2 \leq i+j \leq n-2, \\ e_1 \circ e_{n-1} = e_n, & e_{n-1} \circ e_{n-1} = e_n; \end{cases}$$

$$KF_n^4 : \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq n-2, \\ e_{n-1} \circ e_1 = e_n. \end{cases}$$

Proof. Let an algebra A satisfy the conditions of theorem and let $\{e_1, e_2, \ldots, e_n\}$ be a basis of A such that $A_1 = \langle e_1, e_{n-1} \rangle$, $A_2 = \langle e_2, e_n \rangle$, $A_i = \langle e_i \rangle$ for $3 \leq i \leq n-2$. Analogously as in the proof of Theorem 3.1, we obtain

$$\begin{array}{ll} e_1 \circ e_1 = \alpha_1 e_2 + \alpha_2 e_n, & e_1 \circ e_{n-1} = \alpha_3 e_2 + \alpha_4 e_n, \\ e_{n-1} \circ e_1 = \alpha_5 e_2 + \alpha_6 e_n, & e_{n-1} \circ e_{n-1} = \alpha_7 e_2 + \alpha_8 e_n. \end{array}$$

$$e_1 \circ e_i = e_{i+1} \quad \text{for} \quad 3 \leq i \leq n-3.$$

Without loss of generality one can suppose $e_1 \circ e_1 = e_2$. In fact, if there exists $x \in A_1$ such that $x \circ x \neq 0$, we can set $e_1 = x$ and $e_2 = x \circ x$. But if for any $x \in A_1$, we have $x \circ x = 0$, then $e_1 \circ e_1 = e_{n-1} \circ e_{n-1} = 0$, $e_1 \circ e_{n-1} = -e_{n-1} \circ e_1$ and in this case $dim A_2 = 1$, what contradicts to conditions of the theorem.

Thus,

$$\begin{array}{ll} e_1 \circ e_1 = e_2, & e_1 \circ e_{n-1} = \alpha_3 e_2 + \alpha_4 e_n, \\ e_{n-1} \circ e_1 = \alpha_5 e_2 + \alpha_6 e_n, & e_{n-1} \circ e_{n-1} = \alpha_7 e_2 + \alpha_8 e_n. \end{array}$$

Case 1. Let $\alpha_4 \neq 0$. Then we can assume $e_1 \circ e_{n-1} = e_n$. Let us introduce denotations

$$e_1 \circ e_2 = \beta_1 e_3, \quad e_{n-1} \circ e_2 = \beta_2 e_3, \quad e_1 \circ e_n = \beta_3 e_3, \quad e_{n-1} \circ e_n = \beta_4 e_5.$$

Case 1.1. Let $\beta_1 \neq 0$. Then we can assume $e_1 \circ e_2 = e_3$. Putting $e'_{n-1} = -\beta_2 e_1 + e_{n-1}$, we have $\beta_2 = 0$, i.e. $e_{n-1} \circ e_2 = 0$. Using the equality $e_1 \circ e_i = e_{i+1}$ for $1 \leq i \leq n-3$ and induction, we can get the following equality

$$e_i \circ e_j = C^j_{i+j-1} e_{i+j} \ \text{ for } \ 2 \leq i+j \leq n-2.$$

The equalities

$$\begin{array}{l} Z(e_1, e_{n-1}, e_1) = Z(e_1, e_1, e_{n-1}) = Z(e_n, e_1, e_1) = Z(e_1, e_n, e_1) = \\ Z(e_1, e_1, e_n) = Z(e_1, e_{n-1}, e_2) = Z(e_{n-1}, e_2, e_1) = Z(e_{n-1}, e_1, e_2) = \\ Z(e_1, e_{n-1}, e_{n-1}) = Z(e_{n-1}, e_{n-1}, e_2) = Z(e_1, e_n, e_{n-1}) = Z(e_1, e_{n-1}, e_n) = 0, \end{array}$$

deduce

$$\begin{array}{l} e_1 \circ e_n = e_n \circ e_1 = e_2 \circ e_n = e_n \circ e_2 = e_2 \circ e_{n-1} = e_{n-1} \circ e_3 = 0 \\ e_n \circ e_{n-1} = e_{n-1} \circ e_n = e_n \circ e_n = 0 \ \text{ and } \ \alpha_5 = \alpha_7 = \beta_3 = \beta_4 = 0. \end{array}$$

Thus, we obtain the following multiplications

$$e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad \text{for } \ 2 \leq i+j \leq n-2,$$

$$\begin{array}{llll} e_1 \circ e_{n-1} = e_n, & e_1 \circ e_n = 0, & e_{n-1} \circ e_1 = \alpha_6 e_n, & e_{n-1} \circ e_{n-1} = \alpha_8 e_n, \\ e_{n-1} \circ e_n = 0, & e_{n-1} \circ e_2 = 0, & e_n \circ e_1 = 0, & e_n \circ e_{n-1} = 0, \\ e_n \circ e_n = 0, & e_n \circ e_2 = 0, & e_2 \circ e_{n-1} = 0, & e_2 \circ e_n = 0. \end{array}$$

Applying the induction and the following chains of equalities

$$\begin{array}{l} e_n \circ e_{k+1} = e_n \circ (e_1 \circ e_k) = (e_n \circ e_1) \circ e_k - e_n \circ (e_k \circ e_1) = -k e_n \circ e_{k+1}, \\ e_{k+1} \circ e_n = (e_1 \circ e_k) \circ e_n = e_1 \circ (e_k \circ e_n) + e_1 \circ (e_n \circ e_k), \\ e_{n-1} \circ e_{k+1} = e_{n-1} \circ (e_1 \circ e_k) = (e_{n-1} \circ e_1) \circ e_k - e_{n-1} \circ (e_k \circ e_1) = \alpha_6 e_n \circ e_k - k e_{n-1} \circ e_{k+1}, \\ e_{k+1} \circ e_{n-1} = (e_1 \circ e_k) \circ e_{n-1} = e_1 \circ (e_k \circ e_{n-1}) + e_1 \circ (e_{n-1} \circ e_k), \end{array}$$

one can prove $e_i \circ e_{n-1} = e_{n-1} \circ e_i = e_i \circ e_n = e_n \circ e_i = 0$ for $2 \leq i \leq n-2$.
Thus we obtain the following multiplication in the algebra A :

$$\begin{array}{l} e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad \text{for } \ 2 \leq i+j \leq n-2, \\ e_1 \circ e_{n-1} = e_n, \quad e_{n-1} \circ e_1 = \alpha e_n, \quad e_{n-1} \circ e_{n-1} = \beta e_n. \end{array}$$

For the study the given family of algebras on isomorphity we consider the following change of generating basic elements

$$e_1' = ae_1 + be_{n-1},$$
$$e_{n-1}' = ce_1 + de_{n-1}$$

where $ad - bc \neq 0$.

Making the change of basis $e_2' = a^2e_2 + (ab + ab\alpha + b^2\beta)e_n$, $e_i' = a^ie_i$, for $3 \leq i \leq n-2$, consider the multiplication

$$e_2' \circ e_{n-1}' = 2a^2ce_3.$$

On the other hand, $e_2' \circ e_{n-1}' = 0$, hence $c = 0$.

Comparing in a similar way the coefficients in decomposition of the following multiplications

$$e_1' \circ e_{n-1}', \quad e_{n-1}' \circ e_1', \quad e_{n-1}' \circ e_{n-1}'$$

in bases $\{e_1, e_2, \dots, e_n\}$ and $\{e_1', e_2', \dots, e_n'\}$, we obtain

$$d(a + b\beta) \neq 0, \quad a\alpha + b\beta = \alpha'(a + b), \quad \beta' = \frac{d}{a + b\beta}\beta.$$

It should be noted validity of the equality

$$\alpha' - 1 = \frac{a}{a + b\beta}(a - 1).$$

If $\beta = 0$, then $\beta' = 0$ and we obtain the one-parametric family of algebras KF_n^1. If $\beta \neq 0$, then putting $d = \frac{a+b\beta}{\beta}$, we obtain $\beta' = 1$. In the case $\alpha = 1$ we have $\alpha' = 1$, i.e. the algebra A is isomorphic to the algebra KF_n^2. But if $\alpha \neq 1$, putting $b = -\frac{a\alpha}{\beta}$, we get $\alpha = 0$ and the algebra KF_n^3.

Case 1.2. Let $\beta_1 = 0$. Put $e_1' = ae_1 + be_{n-1}$. Then

$$e_2' = e_1' \circ e_1' = (a^2 + ab\alpha_5 + b^2\alpha_7)e_2 + (ab(1 + \alpha_6) + b^2\alpha_8)e_n.$$

Consider the multiplication

$$e_1' \circ e_2' = [(\beta_2 + (1 + \alpha_6)\beta_3)a^2b + (\alpha_5\beta_2 + \alpha_8\beta_3 + (1 + \alpha_6)\beta_4)ab^2 + (\alpha_7\beta_2 + \alpha_8\beta_4)b^3]e_3.$$

If at least one of the expressions $\beta_2 + (1 + \alpha_6)\beta_3$, $\alpha_5\beta_2 + \alpha_8\beta_3 + (1 + \alpha_6)\beta_4$, and $\alpha_7\beta_2 + \alpha_8\beta_4$ is different from zero, we have case 1.1. That is why we suppose

$$\beta_2 + (1 + \alpha_6)\beta_3 = 0,$$
$$\alpha_5\beta_2 + \alpha_8\beta_3 + (1 + \alpha_6)\beta_4 = 0,$$
$$\alpha_7\beta_2 + \alpha_8\beta_4 = 0.$$

Case 1.2.1. Let $\beta_3 \neq 0$. Then by scale of basis we obtain $\beta_3 = 1$. The equalities

$$Z(e_1, e_1, e_1) = Z(e_1, e_1, e_{n-1}) = Z(e_1, e_{n-1}, e_1) = Z(e_2, e_1, e_1) = 0$$

deduce

$$e_2 \circ e_1 = e_2 \circ e_2 = 0, \quad e_2 \circ e_{n-1} = e_n \circ e_1 = (1 + \alpha_6)e_3.$$

Since $\beta_2 + (1+\alpha_6) = 0$, we have $e_{n-1} \circ e_2 = -(1+\alpha_6)e_3$. Hence

$$(e_{n-1} \circ e_1) \circ e_1 = -2(1+\alpha_6)e_3.$$

On the other hand, $(e_{n-1} \circ e_1) \circ e_1 = \alpha_6(1+\alpha_6)e_3$, therefore

$$(1+\alpha_6)(2+\alpha_6) = 0, \tag{4}$$

One can easily obtain

$$e_i \circ e_1 = ((i-1)+\alpha_6)e_{i+1}, \quad e_i \circ e_2 = 0, \quad 3 \leq i \leq n-3.$$

On the other hand,

$$e_3 \circ e_2 = \frac{1}{2}(2+\alpha_6)(3+\alpha_6)e_5, \quad e_4 \circ e_2 = \frac{1}{2}(3+\alpha_6)(4+\alpha_6)e_6.$$

Thus, we obtain

$$(2+\alpha_6)(3+\alpha_6) = 0, \quad (3+\alpha_6)(4+\alpha_6) = 0 \tag{5}$$

It should be noted that for $n \geq 8$ the (4) and (5) lead to a contradiction with existence of an algebra in this case.

Case 1.2.2. Let $\beta_3 = 0$. Then

$$\alpha_8 = \beta_2 = 0, \quad \alpha_6 = -1, \quad \beta_4 \neq 0.$$

Note that taking as e_1' the expression $ae_1 + be_{n-1}$, we get into case 1.2.1. In fact, at $e_1' = ae_1 + be_{n-1}$ we have

$$e_n' = e_1' \circ e_{n-1}' = ae_n + b\alpha_7 e_2,$$

what follows $e_1' \circ e_n' = ab\beta_4 e_3$. If $ab \neq 0$, then we can assume $e_1' \circ e_n' = e_3'$, where $e_3' = ab\beta_4 e_3$ and we are in conditions of case 1.2.1.

Case 2. Let $\alpha_4 = 0$ and $\alpha_6 \neq 0$. Then we can assume $e_{n-1} \circ e_1 = e_n$. Hence

$$\begin{aligned} e_1 \circ e_1 &= e_2, & e_1 \circ e_{n-1} &= \alpha_3 e_2, \\ e_{n-1} \circ e_1 &= e_n, & e_{n-1} \circ e_{n-1} &= \alpha_7 e_2 + \alpha_8 e_n. \end{aligned}$$

Consider the following change of generating elements of the basis $\{e_1, e_2, \ldots, e_n\}$ in the form:

$$\begin{aligned} e_1' &= ae_1 + be_{n-1}, \\ e_{n-1}' &= ce_1 + de_{n-1} \end{aligned}$$

where $ad - bc \neq 0$. Then in the new basis we have from multiplication $e_1' \circ e_1' = e_2'$ that

$$e_2' = (a^2 + ab\alpha_3 + b^2\alpha_7)e_2 + (ab + b^2\alpha_8)e_n.$$

Moreover, we have

$$e_1' \circ e_{n-1}' = (ac + ad\alpha_3 + bd\alpha_7)e_2 + (bc + bd\alpha_8)e_n.$$

Note that we can choose the numbers a, b, c, d such that they satisfy the conditions:

$$a^2 + ab\alpha_3 + b^2\alpha_7 \neq 0, \;\; bc + bd\alpha_8 \neq 0. \tag{6}$$

If $\frac{a^2+ab\alpha_3+b^2\alpha_7}{ac+ad\alpha_3+bd\alpha_7} \neq \frac{ab+b^2\alpha_8}{bc+bd\alpha_8}$, then denoting

$$e'_n = (ac + ad\alpha_3 + bd\alpha_7)e_2 + (bc + bd\alpha_8)e_n$$

we get into case 1. Let $\frac{a^2+ab\alpha_3+b^2\alpha_7}{ac+ad\alpha_3+bd\alpha_7} = \frac{ab+b^2\alpha_8}{bc+bd\alpha_8}$ for any values of a, b, c, d satisfying (6) at fixed α_3, α_7, and α_8. Then

$$(a^2 + ab\alpha_3 + b^2\alpha_7)(bc + bd\alpha_8) - (ac + ad\alpha_3 + bd\alpha_7)(ab + b^2\alpha_8) = 0.$$

$\Rightarrow a(ad - bc)(\alpha_3 - \alpha_8) + b(bc - ad)\alpha_7 = 0 \Rightarrow \alpha_8 = \alpha_3$ and $\alpha_7 = 0$. Taking into account that $\alpha_8 = \alpha_3$ and $\alpha_7 = 0$, we obtain $e'_2 = (a^2 + ab\alpha_3)e_2 + (ab + b^2\alpha_3)e_n \quad \Rightarrow a \neq -b\alpha_3$.

Express in the basis $\{e'_1, e'_2, \dots, e'_n\}$ the following multiplications:

$$\begin{aligned}
e'_1 \circ e'_{n-1} &= \frac{c + d\alpha_3}{a + b\alpha_3} e'_2, \\
e'_{n-1} \circ e'_1 &= (ac + bc\alpha_3)e_2 + (ad + bd\alpha_3)e_n = e'_n, \\
e'_{n-1} \circ e'_{n-1} &= \frac{c + d\alpha_3}{a + b\alpha_3} e'_n.
\end{aligned}$$

Choosing $c = -d\alpha_3$ $(d \neq 0) \Rightarrow e'_1 \circ e'_{n-1} = e'_{n-1} \circ e'_{n-1} = 0$, we obtain

$$\begin{aligned}
&e_1 \circ e_1 = e_2, \quad e_{n-1} \circ e_1 = e_n, \\
&e_1 \circ e_2 = \beta_1 e_3, \quad e_{n-1} \circ e_2 = \beta_2 e_3, \quad e_1 \circ e_n = \beta_3 e_3, \quad e_{n-1} \circ e_n = \beta_4 e_3.
\end{aligned}$$

Case 2.1. Let $\beta_1 \neq 0$. Then by scale of basis we obtain $e_1 \circ e_i = e_{i+1}$ for $1 \leq i \leq n-3$, what allows easily to obtain validity of the equality

$$e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad \text{for} \quad 2 \leq i + j \leq n - 2.$$

The equalities

$$\begin{aligned}
&Z(e_1, e_1, e_1) = Z(e_1, e_{n-1}, e_1) = Z(e_1, e_1, e_{n-1}) = Z(e_{n-1}, e_1, e_1) = \\
&= Z(e_1, e_n, e_1) = Z(e_{n-1}, e_{n-1}, e_1) = Z(e_{n-1}, e_1, e_n) = 0,
\end{aligned}$$

yield

$$\beta_2 = \beta_3 = \beta_4 = 0, \quad e_2 \circ e_1 = 2e_3, \quad e_2 \circ e_{n-1} = e_n \circ e_n = e_n \circ e_1 = 0.$$

Sum up multiplications being in the presence:

$$\begin{aligned}
&e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad \text{for} \quad 2 \leq i + j \leq n - 2, \quad e_{n-1} \circ e_1 = e_n, \\
&e_1 \circ e_{n-1} = e_1 \circ e_n = e_2 \circ e_{n-1} = e_2 \circ e_n = e_{n-1} \circ e_2 = 0, \\
&e_{n-1} \circ e_{n-1} = e_{n-1} \circ e_n = e_n \circ e_1 = e_n \circ e_2 = e_n \circ e_{n-1} = e_n \circ e_n = 0.
\end{aligned}$$

In a similar way as in case 1.1 one can prove

$$e_i \circ e_{n-1} = e_{n-1} \circ e_i = e_i \circ e_n = e_n \circ e_i = 0 \quad \text{at} \quad 2 \leq i \leq n - 2.$$

In the conclusion of this case we have that A is isomorphic to KF^4_n.

Case 2.2. Let $\beta_1 = 0$. Then $e_1 \circ e_2 = 0$. Set $e'_1 = ae_1 + be_{n-1}$. Then the condition $e'_1 \circ e'_1 = e'_2$, yield $e'_2 = a^2 e_2 + abe_n$. Consider the multiplication

$$e'_1 \circ e'_2 = ab(a(\beta_2 + \beta_3) + b\beta_4)e_3.$$

If either $\beta_2+\beta_3$ or β_4 doesn't equal zero, we get into case 2.1. Consider the case of $\beta_2+\beta_3=0$ and $\beta_4=0$. Then we have $\beta_2\neq 0$ (since otherwise, i.e. at $\beta_i=0,\ 1\leq i\leq 4$ we have $dimA_3=0$). Putting $e_3'=\beta_2 e_3$, one can assume $\beta_2=1$. $Z(e_1,e_{n-1},e_1)=0\Rightarrow e_3=0$, what contradicts to the existence condition of an algebra in this case.

Case 3. Let $\alpha_4=\alpha_6=0$. Then $\alpha_8\neq 0$ and putting $e_1'=ae_1+be_{n-1}$, where a and b are such that

$$ab\neq 0 \quad \text{and} \quad det\begin{pmatrix} a^2+ab\alpha_3+ab\alpha_5 & b^2 \\ \alpha_3 & b\alpha_8 \end{pmatrix}\neq 0,$$

we reduce this case to the case 1.

It should be noted that we consider a fortiori non isomorphic cases. □

Theorem 3.3. *Any five-dimensional naturally graded quasi-filiform Zinbiel algebra of type $A_{(2)}$ is isomorphic to one of the following pairwise non isomorphic algebras:*

$$KF_5^1:\begin{cases} e_1\circ e_1=e_2, \quad e_4\circ e_4=e_5, \quad e_1\circ e_2=e_3, \quad e_4\circ e_5=e_3, \\ e_2\circ e_1=2e_3, \quad e_5\circ e_4=2e_3; \end{cases}$$

$$KF_5^2:\begin{cases} e_1\circ e_1=e_2, \quad e_1\circ e_2=e_3, \quad e_1\circ e_4=e_5, \quad e_1\circ e_5=e_3, \\ e_2\circ e_1=2e_3, \quad e_4\circ e_1=-e_2, \quad e_4\circ e_2=-e_3, \quad e_4\circ e_5=-e_3; \end{cases}$$

$$KF_5^3(\beta):\begin{cases} e_1\circ e_1=e_2, \quad e_1\circ e_2=e_3, \quad e_1\circ e_4=e_5, \quad e_1\circ e_5=\beta e_3, \\ e_2\circ e_1=2e_3, \quad e_2\circ e_4=(\beta-1)e_3, \quad e_4\circ e_1=-e_2, \quad e_4\circ e_4=-e_5, \\ e_4\circ e_2=-e_3, \quad e_4\circ e_5=-\beta e_3, \quad e_5\circ e_1=(\beta-1)e_3, \quad e_5\circ e_4=-2\beta e_3, \quad \beta\in C; \end{cases}$$

$$KF_5^4:\begin{cases} e_1\circ e_1=e_2, \quad e_1\circ e_2=e_3, \quad e_1\circ e_4=e_5, \quad e_2\circ e_1=2e_3, \\ e_4\circ e_4=-e_5; \end{cases}$$

$$KF_5^5:\begin{cases} e_1\circ e_4=e_2, \quad e_1\circ e_2=e_3, \quad e_1\circ e_5=-e_3, \quad e_4\circ e_1=e_5, \\ e_4\circ e_2=e_3, \quad e_4\circ e_5=-e_3; \end{cases}$$

$$KF_5^6:\ e_1\circ e_4=e_2, \quad e_1\circ e_2=e_3, \quad e_1\circ e_5=-e_3, \quad e_4\circ e_1=e_5;$$

$$KF_5^7:\begin{cases} e_1\circ e_1=e_2, \quad e_1\circ e_2=e_3, \quad e_1\circ e_4=e_5, \quad e_1\circ e_5=e_3, \\ e_2\circ e_1=2e_3, \quad e_2\circ e_4=e_3, \quad e_5\circ e_1=e_3; \end{cases}$$

$$KF_5^8:\ e_1\circ e_1=e_2, \quad e_1\circ e_2=e_3, \quad e_1\circ e_4=e_5, \quad e_2\circ e_1=2e_3;$$

$$KF_5^9:\begin{cases} e_1\circ e_1=e_2, \quad e_1\circ e_2=e_3, \quad e_1\circ e_4=-e_5, \quad e_2\circ e_1=2e_3, \\ e_4\circ e_1=e_5, \quad e_4\circ e_5=e_3; \end{cases}$$

$$KF_5^{10}:\ e_1\circ e_1=e_2, \quad e_1\circ e_4=-e_5, \quad e_4\circ e_1=e_5, \quad e_4\circ e_5=e_3;$$

$$KF_5^{11}:\begin{cases} e_1\circ e_1=e_2, \quad e_1\circ e_2=e_3, \quad e_1\circ e_4=-e_5, \quad e_1\circ e_5=e_3, \\ e_2\circ e_1=2e_3, \quad e_4\circ e_1=e_5; \end{cases}$$

$$KF_5^{12}:\begin{cases} e_1\circ e_1=e_2, \quad e_1\circ e_2=e_3, \quad e_1\circ e_4=-e_5, \quad e_2\circ e_1=2e_3, \\ e_4\circ e_1=e_5; \end{cases}$$

$$KF_5^{13}:\ e_1\circ e_1=e_2, \quad e_1\circ e_4=-e_5, \quad e_1\circ e_5=e_3, \quad e_4\circ e_1=e_5;$$

$$KF_5^{14}(\alpha): \begin{cases} e_1\circ e_1=e_2, \quad e_1\circ e_2=e_3, \quad e_1\circ e_4=\alpha e_5, \quad e_2\circ e_1=2e_3, \\ e_4\circ e_1=e_5, \quad \alpha\in C\backslash\{-1\}; \end{cases}$$

$$KF_5^{15}(\alpha): \begin{cases} e_1\circ e_1=e_2, \quad e_1\circ e_4=\alpha e_5, \quad e_1\circ e_5=\frac{2\alpha}{1+\alpha}e_3, \quad e_2\circ e_4=2\alpha e_3, \\ e_4\circ e_1=e_5, \quad e_4\circ e_2=e_3, \quad e_5\circ e_1=2e_3, \quad \alpha\in C\backslash\{-1,-\frac{1}{2}\}; \end{cases}$$

$$KF_5^{16}: \begin{cases} e_1\circ e_1=e_2, \quad e_1\circ e_2=e_3, \quad e_1\circ e_4=-\frac{1}{2}e_5, \quad e_1\circ e_5=-2e_3, \\ e_2\circ e_1=2e_3, \quad e_2\circ e_4=-e_3, \quad e_4\circ e_1=e_5, \quad e_4\circ e_2=e_3, \\ e_5\circ e_1=2e_3; \end{cases}$$

$$KF_5^{17}: \begin{cases} e_1\circ e_1=e_2, \quad e_1\circ e_4=-\frac{1}{2}e_5, \quad e_1\circ e_5=-2e_3, \quad e_2\circ e_4=-e_3, \\ e_4\circ e_1=e_5, \quad e_4\circ e_2=e_3, \quad e_5\circ e_1=2e_3. \end{cases}$$

Proof. Let A be an algebra satisfying the conditions of the theorem and let $\{e_1, e_2, e_3, e_4, e_5\}$ be a basis of the algebra satisfying the natural gradating,

$$A_1=\langle e_1, e_4\rangle, \quad A_2=\langle e_2, e_5\rangle, \quad A_3=\langle e_3\rangle.$$

Write the multiplication of the basic elements in the form

$$\begin{array}{l} e_1\circ e_1=\alpha_1e_2+\alpha_2e_5, \quad e_1\circ e_4=\alpha_3e_2+\alpha_4e_5, \\ e_4\circ e_1=\alpha_5e_2+\alpha_6e_5, \quad e_4\circ e_4=\alpha_7e_2+\alpha_8e_5, \\ e_1\circ e_2=\beta_1e_3, \quad e_1\circ e_5=\beta_2e_3, \quad e_4\circ e_2=\beta_3e_3, \quad e_4\circ e_5=\beta_4e_3, \end{array}$$

where $(\beta_1, \beta_2, \beta_3, \beta_4)\neq(0,0,0,0)$.
It is easy to see that the linear span $\langle e_3\rangle$ is an ideal of A. Consider now the quotient algebra $A/I=\{\overline{e}_1, \overline{e}_4, \overline{e}_2, \overline{e}_5\}$. It is a four-dimensional Zinbiel algebra for which conditions $dim(A/I)^2=2$ and $dim(A/I)^3=0$ hold. Using classification of four-dimensional Zinbiel algebras according to Theorem 2.3, we conclude that A/I is isomorphic to the following pairwise non isomorphic algebras:

$M_1:\ e_1\circ e_1=e_2, \quad e_4\circ e_4=e_5;$

$M_2:\ e_1\circ e_1=e_2, \quad e_1\circ e_4=e_5, \quad e_4\circ e_1=-e_5, \quad e_4\circ e_4=e_2-2e_5;$

$M_3:\ e_1\circ e_1=e_2, \quad e_1\circ e_4=e_5, \quad e_4\circ e_1=-e_2;$

$M_4(\alpha):\ e_1\circ e_1=e_2, \quad e_1\circ e_4=e_5, \quad e_4\circ e_1=\alpha e_2, \quad e_4\circ e_4=-e_5, \quad \alpha\in C;$

$M_5:\ e_1\circ e_4=e_2, \quad e_4\circ e_1=e_5;$

$M_6:\ e_1\circ e_1=e_2, \quad e_1\circ e_4=e_5;$

$M_7(\alpha):\ e_1\circ e_1=e_2, \quad e_1\circ e_4=\alpha e_5, \quad e_4\circ e_1=e_5 \quad \alpha\in C.$

Hence we can get the values for structural constants α_i of the algebra A, namely by equating the values of α_i to the corresponding ones in the algebra from the above list. Applying standard classification methods in each of the seven cases complete the proof of the theorem. □

Theorem 3.4. *Any six-dimensional naturally graded quasi-filiform Zinbiel algebra of type $A_{(2)}$ is isomorphic to one of the following pairwise non isomorphic algebras:*

$$KF_6^1: \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, & 2 \le i+j \le 4, \\ e_1 \circ e_5 = e_6, & e_5 \circ e_1 = e_6; \end{cases}$$

$$KF_6^2: \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \le i+j \le 4, \\ e_1 \circ e_5 = e_6, \quad e_5 \circ e_1 = e_6, \quad e_5 \circ e_5 = e_6; \end{cases}$$

$$KF_6^3: \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, & 2 \le i+j \le 4, \\ e_1 \circ e_5 = e_6, & e_5 \circ e_5 = e_6; \end{cases}$$

$$KF_6^4: \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \le i+j \le 4, \\ e_5 \circ e_1 = e_6; \end{cases}$$

$$KF_6^5: \begin{cases} e_1 \circ e_1 = e_2, \quad e_1 \circ e_3 = e_4, \quad e_1 \circ e_5 = e_6, \quad e_1 \circ e_6 = e_3, \\ e_2 \circ e_5 = -e_3, \quad e_5 \circ e_1 = -3e_2 - 2e_6, \quad e_5 \circ e_2 = e_3, \quad e_5 \circ e_5 = 2e_2 + e_6, \\ e_5 \circ e_6 = -2e_3, \quad e_6 \circ e_1 = -e_3, \quad e_6 \circ e_5 = e_3; \end{cases}$$

$$KF_6^6: \begin{cases} e_1 \circ e_1 = e_2, \quad e_1 \circ e_3 = e_4, \quad e_1 \circ e_5 = e_6, \quad e_1 \circ e_6 = e_3, \\ e_2 \circ e_5 = -e_3, \quad e_5 \circ e_1 = -2e_6, \quad e_5 \circ e_2 = e_3, \quad e_6 \circ e_1 = -e_3; \end{cases}$$

$$KF_6^7: \begin{cases} e_1 \circ e_1 = e_2, \quad e_1 \circ e_3 = e_4, \quad e_1 \circ e_5 = e_6, \quad e_1 \circ e_6 = e_3, \\ e_3 \circ e_1 = e_4, \quad e_5 \circ e_1 = -e_6. \end{cases}$$

Proof. Similar to the cases 1.1 and 2.1 of Theorem 3.2 we can get the existence (pairwise non isomorphic) of the following algebras

$$KF_6^1, \quad KF_6^2, \quad KF_6^3, \quad KF_6^4.$$

Analogously to the case 1.2 we have for the algebra A with the basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$

$$e_1 \circ e_1 = e_2, \quad e_1 \circ e_5 = e_6, \quad e_1 \circ e_2 = 0, \quad e_1 \circ e_6 = e_3, \quad e_1 \circ e_3 = e_4,$$
$$e_5 \circ e_1 = \alpha_5 e_2 + \alpha_6 e_6, \quad e_5 \circ e_5 = \alpha_7 e_2 + \alpha_8 e_6, \quad e_5 \circ e_2 = -(1+\alpha_6)e_3,$$
$$e_5 \circ e_6 = \beta_4 e_3, \quad e_5 \circ e_3 = \gamma e_4$$

where for parameters $\alpha_5, \alpha_6, \alpha_7, \alpha_8, \beta_4, \gamma$, the relations

$$-5(1+\alpha_6) + \alpha_8 + (1+\alpha_6)\beta_4 = 0,$$
$$-7(1+\alpha_6) + \alpha_8\beta_4 = 0$$

hold. Consideration of the identity (1) for the basic elements reduces to the following restrictions:

$$\begin{cases} \alpha_8(2+\alpha_6)=0, \\ \alpha_8(\beta_4+2\alpha_8)=0, \\ (1+\alpha_6)(2+\alpha_6)=0, \\ (1+\alpha_6)(\beta_4+2\alpha_8)=0, \\ -\alpha_7(1+\alpha_6)+\alpha_8\beta_4=0, \\ \beta_4(2+\alpha_6)=(2+\alpha_6)\gamma, \\ \alpha_5(1+\alpha_6)+2\alpha_6\alpha_8=-\alpha_8, \\ \beta_4(\beta_4+2\alpha_8)=(\beta_4+2\alpha_8)\gamma, \\ -\alpha_5(1+\alpha_6)+\alpha_8+(1+\alpha_6)\beta_4=0, \\ \alpha_5(2+\alpha_6)+\alpha_6(\beta_4+2\alpha_8)=(2+\alpha_6)\gamma, \\ \alpha_7(2+\alpha_6)+\alpha_8(\beta_4+2\alpha_8)=(\beta_4+2\alpha_8)\gamma. \end{cases} \tag{7}$$

Case 1. Let $\alpha_6=-2$. Then (7) becomes to the form

$$\begin{cases} \beta_4=-2\alpha_8, \\ \alpha_5=-3\alpha_8, \\ \alpha_7=2\alpha_8^2. \end{cases}$$

Multiplication in the algebra in this case has the form

$$\begin{array}{llll} e_1\circ e_1=e_2, & e_1\circ e_5=e_6, & e_1\circ e_6=e_3, & e_1\circ e_3=e_4, \\ e_5\circ e_1=-3\alpha e_2-2e_6, & e_5\circ e_5=2\alpha^2e_2+\alpha e_6, & e_5\circ e_2=e_3, & e_5\circ e_6=-2\alpha e_3, \\ e_5\circ e_3=\gamma e_4, & e_2\circ e_5=-e_3, & e_6\circ e_1=-e_3, & e_6\circ e_5=\alpha e_3, \end{array}$$

If $\gamma+\alpha\neq 0$, then substituting

$$\begin{array}{lll} e_1'=(\gamma+\alpha)e_1, & e_2'=(\gamma+\alpha)^2e_2, & e_3'=(\gamma+\alpha)^2e_3, \\ e_4'=(\gamma+\alpha)^3e_4, & e_5'=-\gamma e_1+e_5, & e_6'=(\gamma+\alpha)(-\gamma e_2+e_6), \end{array}$$

we can assume $\gamma=0$ and $\alpha=1$, i.e. we obtain the algebra KF_6^5. But if $\gamma+\alpha=0$, then substituting

$$e_1'=e_1,\quad e_2'=e_2,\quad e_3'=e_3,\quad e_4'=e_4,\quad e_5'=-\gamma e_1+e_5,\quad e_6'=-\gamma e_2+e_6,$$

we obtain $\gamma=0$ and $\alpha=0$, hence we have the algebra KF_6^6.
Case 2. Let $\alpha_6=-1$. Then (7) become the form

$$\begin{cases} \alpha_8=0, \\ \alpha_5=2\gamma, \\ \alpha_7=\gamma^2, \\ \beta_4=\gamma. \end{cases}$$

Assume obtained table of multiplication:

$$\begin{array}{llll} e_1\circ e_1=e_2, & e_1\circ e_5=e_6, & e_1\circ e_2=0, & e_1\circ e_6=e_3, \\ e_1\circ e_3=e_4, & e_5\circ e_1=2\beta e_2-e_6, & e_5\circ e_5=\beta^2e_2, & e_5\circ e_6=\beta e_3, \\ e_5\circ e_3=\beta e_4, & e_6\circ e_6=\beta e_4, & e_3\circ e_1=e_4, & e_3\circ e_5=\beta e_4. \end{array}$$

Making the change of the basis:

$$e'_1 = e_1, \quad e'_2 = e_2, \quad e'_3 = e_3, \quad e'_4 = e_4, \quad e'_5 = -\beta e_1 + e_5, \quad e'_6 = -\beta e_2 + e_6,$$

we obtain the algebra KF_6^7.

Since consideration of cases 1.1, 1.2, and 2.1 was chosen such that algebras satisfying these various cases were non isomorphic, the sets of algebras $\{KF_6^1, KF_6^2, KF_6^3\}$, $\{KF_6^4\}$, $\{KF_6^5, KF_6^6, KF_6^7\}$ are pairwise not disjoint (up to isomorphism). Pairwise non isomorphity of algebras KF_6^1, KF_6^2, KF_6^3 follows from Theorem 3.2. By virtue of $dimR(KF_6^5) = dimR(KF_6^6) = 1$, and $dimR(KF_6^7) = 2$, the algebra KF_6^7 is not isomorphic to algebras KF_6^i $(i = 5, 6)$. Non isomorphity of algebras KF_6^5 and KF_6^6 can be easily checked by consideration of general change of the basis. Thus, we obtain pairwise non isomorphic algebras KF_6^i, $1 \leq i \leq 7$. $\square$

The following theorem can be proved in the same manner.

Theorem 3.5. *Any seven-dimensional naturally graded quasi-filiform Zinbiel algebra of type $A_{(2)}$ is isomorphic to one of the following pairwise non isomorphic algebras:*

$$KF_7^1 : \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq 5, \\ e_1 \circ e_6 = e_7, \quad e_6 \circ e_1 = \alpha e_6; \end{cases}$$

$$KF_7^2 : \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq 5, \\ e_1 \circ e_6 = e_7, \quad e_6 \circ e_1 = e_7, \quad e_6 \circ e_6 = e_7; \end{cases}$$

$$KF_7^3 : \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq 5, \\ e_1 \circ e_6 = e_7, \quad e_6 \circ e_6 = e_7; \end{cases}$$

$$KF_7^4 : \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \leq i+j \leq 5, \\ e_6 \circ e_1 = e_7; \end{cases}$$

$$KF_7^5 : \begin{cases} e_1 \circ e_1 = e_2, \quad e_1 \circ e_3 = e_4, \quad e_1 \circ e_4 = e_5, \quad e_1 \circ e_6 = e_7, \\ e_1 \circ e_7 = e_3, \quad e_2 \circ e_6 = -e_3, \quad e_3 \circ e_1 = -e_4, \quad e_6 \circ e_1 = -2e_7, \\ e_6 \circ e_2 = e_3, \quad e_7 \circ e_1 = -e_3; \end{cases}$$

$$KF_7^6 : \begin{cases} e_1 \circ e_1 = e_2, \quad e_1 \circ e_3 = e_4, \quad e_1 \circ e_4 = e_5, \quad e_1 \circ e_6 = e_7, \\ e_1 \circ e_7 = e_3, \quad e_2 \circ e_6 = -e_3, \quad e_3 \circ e_1 = -e_4, \quad e_6 \circ e_1 = -2e_7, \\ e_6 \circ e_2 = e_3, \quad e_6 \circ e_4 = e_5, \quad e_7 \circ e_1 = -e_3. \end{cases}$$

3.3 The case $r > 2$

The proving the remaining cases, we need the following lemmas.

Lemma 3.1. *Let A be a naturally graded quasi-filiform Zinbiel algebra of type $A_{(r)}$ $(r > 2)$. Then $x \circ x = 0$ for any $x \in A_1$.*

Proof. Assume to the contrary, that is there exists $x \in A_1$ such that $x \circ x \neq 0$. Then we choose a basis $\{e_1, e_2, \dots, e_n\}$ of A such that $e_1 = x$, $e_2 = x \circ x$, and $A_1 = \langle e_1, e_{n-1}\rangle$, $A_2 = \langle e_2\rangle$, $\dots$, $A_r = \langle e_r, e_n\rangle$, $A_{r+1} = \langle e_{r+1}\rangle$, $\dots$, $A_{n-2} = \langle e_{n-2}\rangle$.
Thus, we can assume

$$e_1 \circ e_i = e_{i+1} \quad \text{for} \quad 2 \leq i \leq r-1,$$
$$e_{n-1} \circ e_{r-1} = e_n.$$

On the other hand, similar to the case of a filiform Zinbiel algebra in [1], we obtain $e_{n-1} \circ e_i = 0$ for $2 \leq i \leq r-1$, which contradicts the existence of an element x such that $x \circ x \neq 0$. □

Lemma 3.2. *Let A be a naturally graded quasi-filiform Zinbiel algebra of type $A_{(r)}$. Then $r \leq 3$.*

Proof. Assume to the contrary, i.e. $r > 3$. By Lemma 3.1, $x \circ x = 0$ for any $x \in A_1$. Choose a basis $\{e_1, e_2, \dots, e_n\}$ of the algebra A such that

$$e_1 \circ e_i = e_{i+1} \quad \text{for} \quad 2 \leq i \leq r-1$$
$$e_1 \circ e_1 = e_{n-1} \circ e_{n-1} = 0, \quad e_1 \circ e_{n-1} = -e_{n-1} \circ e_1 = e_2.$$

We get a contradiction from the equalities

$$e_2 \circ e_1 = (e_1 \circ e_{n-1}) \circ e_1 = e_1 \circ (e_{n-1} \circ e_1) + e_1 \circ (e_1 \circ e_{n-1}) = e_1 \circ (-e_2 + e_2) = 0,$$
$$0 = (e_1 \circ e_1) \circ e_2 = e_1 \circ (e_1 \circ e_2) + e_1 \circ (e_2 \circ e_1) = e_1 \circ e_3 = e_4,$$

thus completing the proof of the lemma. □

Lemma 3.3. *Let A be a naturally graded quasi-filiform Zinbiel algebra of type $A_{(3)}$. Then $dimA \leq 7$.*

Proof. Suppose that $dimA > 7$ and let $\{e_1, e_2, \dots, e_n\}$ be a basis satisfying the conditions $A_1 = \langle e_1, e_{n-1}\rangle$, $A_2 = \langle e_2\rangle$, $A_3 = \langle e_3, e_n\rangle$, $A_4 = \langle e_4\rangle$, $\dots$, $A_{n-2} = \langle e_{n-2}\rangle$,

$$e_1 \circ e_1 = e_{n-1} \circ e_{n-1} = 0,$$
$$e_1 \circ e_{n-1} = -e_{n-1} \circ e_1 = e_2,$$
$$e_1 \circ e_2 = e_3, \quad e_{n-1} \circ e_2 = e_n.$$

The equalities

$$Z(e_1, e_{n-1}, e_1) = Z(e_1, e_{n-1}, e_{n-1}) = Z(e_1, e_1, e_2) = Z(e_{n-1}, e_{n-1}, e_2) =$$
$$Z(e_1, e_{n-1}, e_2) = Z(e_{n-1}, e_1, e_2) = Z(e_1, e_2, e_{n-1}) = Z(e_{n-1}, e_2, e_1) =$$
$$Z(e_1, e_2, e_1) = Z(e_{n-1}, e_2, e_{n-1}) = 0,$$

lead to

$$e_2 \circ e_1 = e_2 \circ e_{n-1} = e_1 \circ e_3 = e_{n-1} \circ e_n = 0$$
$$e_3 \circ e_1 = e_1 \circ e_3 = e_n \circ e_{n-1} = e_{n-1} \circ e_n = 0,$$
$$e_2 \circ e_2 = e_1 \circ e_n = e_3 \circ e_{n-1} = -e_{n-1} \circ e_3 = -e_n \circ e_1 = \gamma e_4.$$

Note that $\gamma \neq 0$. Otherwise $A_4 = 0$, i.e. $dimA \leq 5$.
Without loss of generality we can assume that $\gamma = 1$ and

$$e_1 \circ e_i = e_{i+1} \quad \text{for} \quad 4 \leq i \leq n-2,$$
$$e_1 \circ e_n = e_4.$$

Using the identity (1) for elements $\{e_1, e_n, e_1\}$, we obtain $e_4 \circ e_1 = 0$.
On the other hand, the equalities

$$(e_1 \circ e_1) \circ e_4 = e_1 \circ (e_1 \circ e_4) + e_1 \circ (e_4 \circ e_1)$$

imply $0 = e_1 \circ e_5 = e_6$, i.e. we arrive at a contradiction, which completes the proof of the lemma. $\square$

From Lemma 3.3 one can easily derive the following corollaries.

Corollary 3.1. *Any five-dimensional naturally graded quasi-filiform Zinbiel algebra of type $A_{(3)}$ is isomorphic to the algebra*

$$e_1 \circ e_2 = e_3, \quad e_2 \circ e_1 = -e_3, \quad e_1 \circ e_3 = e_4, \quad e_2 \circ e_3 = e_5.$$

Corollary 3.2. *Any six-dimensional naturally graded quasi-filiform Zinbiel algebra of type $A_{(3)}$ is isomorphic to the algebra*

$$\begin{aligned} &e_1 \circ e_2 = e_3, \quad e_1 \circ e_3 = e_4, \quad e_1 \circ e_5 = e_6, \quad e_2 \circ e_1 = -e_3, \quad e_2 \circ e_3 = e_5, \\ &e_2 \circ e_4 = -e_6, \quad e_3 \circ e_3 = e_6, \quad e_4 \circ e_2 = e_6, \quad e_5 \circ e_1 = -e_6. \end{aligned}$$

Corollary 3.3. *Any seven-dimensional naturally graded quasi-filiform Zinbiel algebra of type $A_{(3)}$ is isomorphic to the algebra*

$$\begin{array}{llll} e_1 \circ e_2 = e_3, & e_1 \circ e_3 = e_4, & e_1 \circ e_5 = e_6, & e_1 \circ e_6 = e_7, \\ e_2 \circ e_1 = -e_3, & e_2 \circ e_3 = e_5, & e_2 \circ e_4 = -e_6, & e_3 \circ e_3 = e_6, \\ e_4 \circ e_2 = e_6, & e_4 \circ e_3 = 2e_7, & e_5 \circ e_1 = -e_6. & \end{array}$$

Thus, we obtain the classification of complex naturally graded quasi-filiform Zinbiel algebras of an arbitrary dimension. In fact, the results of this paper complete the classification of n-dimensional nilpotent naturally graded algebras A satisfying the condition $A^{n-2} \neq 0$.

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