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On the degenerations of solvable Leibniz algebras

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ABSTRACT

The present paper is devoted to the description of rigid solvable Leibniz algebras. In particular, we prove that solvable Leibniz algebras under some conditions on the nilradical are rigid and we describe four-dimensional solvable Leibniz algebras with three-dimensional rigid nilradical. We show that the Grunewald–O'Halloran's conjecture "any n-dimensional nilpotent Lie algebra is a degeneration of some algebra of the same dimension" holds for Leibniz algebras of dimensions less than four. The algebra of level one, which is omitted in the 1991 Gorbatsevich's paper, is indicated.

1. Introduction

Largely because of their importance to string theory, quantum field theory and other branches of fundamental research in mathematical physics, noncommutative analogs of many classical constructions have received much attention in the past few years [1,2].

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The noncommutative analog of Lie algebras are Leibniz algebras, discovered by Loday when he handled periodicity phenomena in algebraic K-theory [3]. This algebraic structure found applications in several fields as Physics and Geometry [4–7].

Important subjects playing a relevant role in Mathematics and Physics are degenerations, contractions and deformations of Lie and Leibniz algebras. Namely, in [8] the notion of contractions of Lie algebras on physical grounds was introduced: if two physical theories (like relativistic and classical mechanics) are related by a limiting process, then the associated invariance groups (like the Poincaré and Galilean groups) should also be related by some limiting process. If the velocity of light is assumed to go to infinity, relativistic mechanics "transforms" into classical mechanics. This also induces a singular transition from the Poincaré algebra to the Galilean one.

Other example is a limiting process from quantum mechanics to classical mechanics under $\hbar \to 0$ of the Planck constant, that corresponds to the contraction of the Heisenberg algebras to the abelian ones of the same dimensions [9].

Nevertheless, as it was proved in [10], the notions of deformations, contractions and degenerations are isomorphic over the fields $\mathbb{R}$ or $\mathbb{C}$. Degenerations of Lie and Leibniz algebras were the subject of numerous papers, see for instance [10–16] and references given therein, and their research continues actively. These facts motivate that we focus our attention in the study of degenerations of solvable Leibniz algebras.

In order to do so, we know that an n-dimensional Leibniz algebra may be considered as an element λ of the affine variety $\mathrm{Hom}(V \otimes V, V)$ via the mapping $\lambda : V \otimes V \to V$ defining the Leibniz bracket on a vector space V of dimension n over a field F. Since Leibniz algebras are defined via polynomial identities, the set of n-dimensional Leibniz algebra structures, $\mathcal{L}eib_n$, forms an algebraic subset of the variety $\mathrm{Hom}(V \otimes V, V)$ and the linear reductive group $\mathrm{GL}_n(F)$ acts on $\mathcal{L}eib_n$ via change of basis, i.e.,

$$(g * \lambda)(x, y) = g\left(\lambda\left(g^{-1}(x), g^{-1}(y)\right)\right), \quad g \in \mathrm{GL}_n(F),\ \lambda \in \mathcal{L}eib_n\,.$$

The orbits $\mathrm{Orb}(-)$ under this action are the isomorphism classes of algebras. Note that solvable (respectively, nilpotent) Leibniz algebras of the same dimension also form an invariant subvariety of the variety of Leibniz algebras under the mentioned action.

Let V be an n-dimensional vector space over a field F. The bilinear maps $V \times V \to V$ form an F^{n^3}-dimensional affine space. We shall consider the Zariski topology on this space. Recall, a set is called irreducible if it cannot be represented as a union of two nontrivial closed subsets, otherwise it is called reducible. The maximal irreducible closed subset of a variety is called an irreducible component. From algebraic geometry we know that an algebraic variety is a union of irreducible components and that closures of open sets produce irreducible components. Therefore, for the description of a variety it is very important to find all open sets. Since under the above action the variety of Leibniz algebras consists of orbits of algebras, the description of the variety is reduced to find the open orbits. By Noetherian consideration there are a finite number of open orbits. In any variety of algebras there are algebras with open orbits (so-called rigid algebras). Thus, the closure of orbits of rigid algebras gives irreducible components of the variety. Hence, to describe the variety of algebras it is enough to describe all rigid algebras.

A powerful tool in the study of a variety of algebras is that for constructible subsets of algebraic varieties, the closures with respect to the Euclidean and the Zariski topologies coincide. In particular, for an algebraically closed field F, the limit in usual Euclidean topology leads to the same limit as in the Zariski topology. It has lead to consideration of such notions as deformations and degenerations of algebras. Existence or absence of degeneration in a given variety of algebras is revealed by construction or by using invariant arguments. This approach is very effective in case of nilpotent and solvable algebras.

The description of a variety of any class of algebras is a very difficult problem. Note that for the description of the variety of nilpotent Lie algebras with dimensions less than eight the works [12,16,17] are devoted. The complete description of orbits closure of four-dimensional Lie algebras is given in [13]. To the investigation of the variety of Leibniz algebras the work [18] is devoted. In particular, in [18] it is described all irreducible components of the varieties of complex nilpotent Leibniz algebras of dimension less than 5.

On the other hand, Grunewald and O'Halloran in [19] proposed the following:

Conjecture: Any n-dimensional nilpotent Lie algebra is a degeneration of some algebra of the same dimension.

In other words, there is not nilpotent rigid algebra in the variety of Lie algebras, although a rigid Lie algebra exists in the subvariety of nilpotent Lie algebras. The statement is based on the fact that second cohomology groups of rigid algebras are trivial, while for nilpotent Lie algebras, they are always nontrivial. Similarly to the case of Lie algebras, Balavoine proved the general principles for deformations and rigidity of Leibniz algebras [11].

In this paper we prove that solvable Leibniz algebras, whose nilradical is rigid in the variety of nilpotent Leibniz algebras, cannot be obtained as a degeneration of a solvable Leibniz algebra with different nilradical. In other words, any solvable Leibniz algebra with a given rigid nilradical, such that there is not other solvable Leibniz algebra with the same nilradical, is rigid. The description of solvable Leibniz algebras with three-dimensional rigid nilradical is obtained. Moreover, we prove that for the case of Leibniz algebras the Conjecture above is true for dimension less than four. Finally, we find one algebra which was omitted in the work [14].

Throughout the paper we consider finite-dimensional vector spaces and algebras over the field $\mathbb{C}$. Moreover, in the multiplication table of an algebra omitted products are assumed to be zero and if it is not noticed we shall consider non-nilpotent solvable algebras.

2. Preliminaries

In this section we give necessary definitions and results for understanding main parts of the work.

Definition 2.1 ([3])**.** A vector space L over a field F with a binary operation $[-,-]$ is called a Leibniz algebra, if for any $x, y, z \in L$ the so-called Leibniz identity holds

$$[x,[y,z]] = [[x,y],z] - [[x,z],y] .$$

Every Lie algebra is a Leibniz algebra, but the bracket in a Leibniz algebra does not necessarily need to be skew-symmetric.

For a Leibniz algebra L consider the following lower central and derived series:

$$L^1 = L, \qquad L^{k+1} = [L^k, L^1], \qquad k \geqslant 1,$$
$$L^{[1]} = 1, \qquad L^{[s+1]} = [L^{[s]}, L^{[s]}], \qquad s \geqslant 1.$$

Definition 2.2. A Leibniz algebra L is said to be nilpotent (respectively, solvable), if there exists $p \in \mathbb{N}$ ($q \in \mathbb{N}$) such that $L^p = 0$ (respectively, $L^{[q]} = 0$).

It is well known [20] that in Leibniz algebras case, in each dimension, there exists a unique (up to isomorphism) algebra with maximal index of nilpotency whose multiplication table is:

$$\mathbf{NF_n}: \ [e_i, e_1] = e_{i+1}, \quad 1 \leqslant i \leqslant n-1.$$

Denote by $\mathcal{L}eib_n$ (respectively, by $\mathcal{L}N_n$ and $\mathcal{L}R_n$) the set of all n-dimensional (respectively, nilpotent and solvable) Leibniz algebras.

Remark 2.3. Null-filiform Leibniz algebras of dimension n can be characterized as n-dimensional nilpotent Leibniz algebras such that the n-th term in the lower central series is nontrivial. This means that their orbits are open sets in the variety of n-dimensional nilpotent Leibniz algebras with respect to the Zariski topology, hence null-filiform Leibniz algebras of dimension n are rigid.

Let λ and μ be Leibniz algebras of the same dimension over a field F.

Definition 2.4. It is said that an algebra λ degenerates to an algebra μ, if $\mathrm{Orb}(\mu)$ lies in the Zariski closure of $\mathrm{Orb}(\lambda)$, $\overline{\mathrm{Orb}(\lambda)}$. We denote this by $\lambda \to \mu$.

The degeneration $\lambda \to \mu$ is called a direct degeneration if there is not a chain of nontrivial degenerations of the form: $\lambda \to \nu \to \mu$.

The level of an algebra λ, denoted by $\mathrm{lev}_n(\lambda)$, is the maximum length of a chain of direct degenerations, which, of course, ends with the algebra $\mathbf{a_n}$ (the algebra with zero multiplication).

Remark 2.5. Recall that any n-dimensional algebra degenerates to the algebra $\mathbf{a_n}$.

The following important result is due to Borel [21, 1.8 closed orbit lemma]:

Proposition 2.6. If G is a complex algebraic group and X is a complex algebraic variety with regular action, then each orbit $\mathrm{Orb}(x)$, $x \in X$, is a smooth algebraic variety, open in its closure $\overline{\mathrm{Orb}(x)}$. Its boundary $\overline{\mathrm{Orb}(x)} \setminus \mathrm{Orb}(x)$ is a union of orbits of strictly lower dimension. In particular, the orbits of minimal dimension are closed.

Recall that a subset $Y \subseteq X$ is called constructible if it is a finite union of locally closed sets. By the previous proposition, each orbit $\mathrm{Orb}(x)$ is a constructible set, and so its closures relative to the Euclidean and the Zariski topologies coincide [22, I. Corollary 1]. Therefore, the usual Euclidean topology on $\mathbb{C}^{n^3}$ leads to the same degenerations as does the Zariski topology.

In the case of the field F be the complex numbers $\mathbb{C}$, we give an equivalent definition of degeneration.

Definition 2.7. Let $g\colon (0,1] \to \mathrm{GL}_n(\mathbb{C})$, $t \mapsto g(t) = g_t$, be a continuous map. We construct a parameterized family of Leibniz algebras $\lambda_t = (V, [-,-]_t)$, $t \in (0,1]$, isomorphic to λ. For each t the new Leibniz bracket $[-,-]_t$ on V is defined via the old one as follows:

$$[x,y]_t = g_t[g_t^{-1}(x), g_t^{-1}(y)], \quad x, y \in V.$$

If for any $x, y \in V$ there exists the limit

$$\lim_{t\to 0^+}[x,y]_t = \lim_{t\to 0^+} g_t[g_t^{-1}(x), g_t^{-1}(y)] =: [x,y]_0,$$

then $[-,-]_0$ is a well-defined Leibniz bracket. The Leibniz algebra $\lambda_0 = (V, [-,-]_0)$ is called a degeneration of the algebra λ.

A constructible set in $\mathbb{C}^n$ becomes a semialgebraic set (a finite Boolean combination of solution sets to polynomial equations and polynomial inequalities) in $\mathbb{R}^{2n}$ and by the curve selection lemma [23] we have the powerful result:

Theorem 2.8 ([23])**.** (Analytic curve selection) Let A be a semialgebraic subset of $\mathbb{R}^n$ and $x \in \mathbb{R}^n$ a point belonging to the closure of A, $x \notin A$. Then there exists a Nash mapping $\gamma\colon (-1,1) \to \mathbb{R}^n$ such that $\gamma(0) = x$ and $\gamma((0,1)) \subset A$.

Replacing A by $\mathrm{Orb}(\lambda) = \mathrm{GL}_n(\mathbb{C}) * \lambda$ and $x = \lambda_0$, and since $\mathrm{Orb}(\lambda)$ is a semialgebraic set in $\mathbb{R}^{2n^2}$ and $\lambda_0 \in \overline{\mathrm{Orb}(\lambda)}$, due to Theorem 2.8 we can assume the mapping g is a Nash mapping (which is analytic and semialgebraic mapping).

Remark 2.9. It is easy to note that a rigid nilpotent (solvable) algebra cannot be obtained by degeneration of any other nilpotent (solvable) algebra.

Further we shall need the following results.

Proposition 2.10 ([17])**.** Let G be a reductive algebraic group over $\mathbb{C}$ with Borel subgroup B and let X be an algebraic set on which G acts rationally. Then

$$\overline{G * x} = G * \overline{(B * x)} \quad \text{for all } x \in X.$$

Note that for the classification of solvable Leibniz algebras with given nilradical, the number of nil-independent derivations of the nilradical is important. Namely, for a solvable Leibniz algebra with nilradical N, the dimension of the complementary vector space to N is not greater than the maximal number of nil-independent derivations of N.

Theorem 2.11 ([24]). Let R be a solvable Leibniz algebra whose nilradical is $\mathbf{NF_n}$. Then there exists a basis $\{e_1, e_2, \dots, e_n, x\}$ of the algebra R such that the multiplication table of R with respect to this basis has the following form:

$$\mathbf{RNF_n}: \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leqslant i \leqslant n-1, \\ [x, e_1] = e_1, \\ [e_i, x] = -ie_i, & 1 \leqslant i \leqslant n. \end{cases}$$

In [16,17] it was shown that the rigid nilpotent Lie algebras in dimensions less than seven are the following:

$$\begin{aligned}
&\mathbf{n_3}: [e_1, e_2] = -[e_2, e_1] = e_3;\\
&\mathbf{n_4}: [e_1, e_2] = -[e_2, e_1] = e_3, \quad [e_1, e_3] = -[e_3, e_1] = e_4;\\
&\mathbf{n_5}: [e_1, e_2] = -[e_2, e_1] = e_3, \quad [e_1, e_3] = -[e_3, e_1] = e_4,\\
&\qquad [e_1, e_4] = -[e_4, e_1] = e_5, \quad [e_2, e_3] = -[e_3, e_2] = e_5;\\
&\mathbf{n_6}: [e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad [e_1, e_4] = e_5,\\
&\qquad [e_2, e_3] = e_5, \quad [e_2, e_5] = e_6, \quad [e_3, e_4] = -e_6.
\end{aligned}$$

Due to [18] we can present the list of three-dimensional nilpotent rigid Leibniz algebras:

$$\begin{aligned}
&\boldsymbol{\lambda_4}(\alpha): [e_1, e_1] = e_3, \quad [e_2, e_2] = \alpha e_3, \quad [e_1, e_2] = e_3, \quad \alpha \neq 0;\\
&\boldsymbol{\lambda_5}: [e_2, e_1] = e_3, \quad [e_1, e_2] = e_3;\\
&\boldsymbol{\lambda_6}: [e_1, e_1] = e_2, \quad [e_2, e_1] = e_3.
\end{aligned}$$

Proposition 2.12 ([18]). Let λ be a complex non Lie algebra of $\mathcal{L}eib_n$. Then $\lambda \to \mathbf{n_2} \oplus \mathbb{C}^{n-2}$, where $\mathbf{n_2}: [e_1, e_1] = e_2$ is a two-dimensional non-abelian nilpotent Leibniz algebra.

Consider the following algebras:

$$\begin{aligned}
&\mathbf{p_n^{\pm}}: [e_1, e_i] = e_i, \qquad [e_i, e_1] = \pm e_i, \qquad i \geqslant 2,\\
&\mathbf{n_3^{\pm}}: [e_1, e_2] = e_3, \qquad [e_2, e_1] = \pm e_3.
\end{aligned}$$

Theorem 2.13 ([14]). Let λ be an n-dimensional algebra. Then

1. If the algebra λ is skew-commutative, then $\mathrm{lev}_n(\lambda) = 1$ if and only if it is isomorphic to $\mathbf{p_n^-}$ or to the algebra $\mathbf{n_3^-} \oplus \mathbf{a_{n-3}}$ ($n \geqslant 3$). In particular, the algebra λ is a Lie algebra.
2. If the algebra λ is commutative, then $\mathrm{lev}_n(\lambda) = 1$ if and only if it is isomorphic to $\mathbf{p_n^+}$ or to the algebra $\mathbf{n_3^+} \oplus \mathbf{a_{n-3}}$ ($n \geqslant 3$). In particular, the algebra λ is a Jordan algebra.

Remark 2.14. We note that the algebra $\mathbf{p_n^+}$ is not a Jordan algebra.

3. Main results

We divide the main section into three subsections where we study the rigidness of solvable Leibniz algebras with rigid nilradical, describe such four-dimensional algebras with three-dimensional radical and present one algebra of level one, which was omitted in the work [14].

3.1. Rigidity of solvable Leibniz algebras with rigid nilradical

In this subsection we investigate the rigidity of solvable Leibniz algebras with rigid nilradical.

Definition 3.1. The algebras whose orbits are open sets in the variety $\mathcal{L}eib_n$ with respect to the Zariski topology are said to be rigid.

Remark 3.2. The notion of rigidity is characterized by the fact that the orbit of a rigid algebra does not belong to the closure of the orbit of any other algebra.

Let N be a nilpotent Leibniz algebra. Denote by $\mathcal{L}R_n(N)$ the set of all n-dimensional solvable Leibniz algebras whose nilradical is N.

For any m ($1 \leqslant m \leqslant n$) define the subset $\wedge_m \subset \mathcal{L}R_n$ such that $\wedge_m = \{\lambda = (c_{i,j}^k)\}$ with the properties:

$$\sum_{k_1=n-m+1}^{n} \sum_{k_2=n-m+1}^{n} \cdots \sum_{k_{s-1}=n-m+1}^{n} c_{i_1,i_2}^{k_1} c_{k_1,i_3}^{k_3} \ldots c_{k_{s-1},i_s}^{k_s} = 0,$$
$$n-m+1 \leqslant i_1, i_2, \ldots, i_s \leqslant n, \quad c_{i,j}^k = 0,\ 1 \leqslant i,j \leqslant n,\ 1 \leqslant k \leqslant n-m,$$

where $c_{i,j}^k$ are structural constants and s any fixed number.

Let us observe that $R \in \wedge_m$ if and only if R contains the nilpotent ideal $N = \langle \{e_{n-m+1}, e_{n-m+2}, \ldots, e_n\} \rangle$ satisfying $R^2 \subseteq N$.

It is not difficult to see that $\wedge_m$ is a Zariski closed subset of $\mathcal{L}R_n$, but it is not $\mathrm{GL}_n(\mathbb{C})$-stable. However, the set $\wedge_m$ is B-stable, where B is the Borel subgroup of $\mathrm{GL}_n(\mathbb{C})$ consisting of upper triangular matrices.

Proposition 3.3. Let $R_1, R_2 \in \mathcal{L}R_n$ and let $R_1 \in \mathcal{L}R_n(N_1),\ R_2 \in \mathcal{L}R_n(N_2)$. If $R_1 \to R_2$, then $\dim N_2 \geqslant \dim N_1$.

Proof. Let $\dim N_1 = m$, then choose $g \in \mathrm{GL}_n(\mathbb{C})$ such that $R' = g * R_1 \in \wedge_m$. Since $B * R' \in \wedge_m$ and $\wedge_m$ is a closed set, then $\overline{B * R'} \in \wedge_m$. By Proposition 2.10 and by condition $R_1 \to R_2$ we conclude that $R_2 \in \mathrm{GL}_n(\mathbb{C}) * \wedge_m$. Therefore, the algebra R_2 contains a nilpotent ideal of dimension m. Since N_2 is the nilradical of R_2, we get $\dim N_2 \geqslant m$. □

Corollary 3.4. Let $R_1 \in \mathcal{L}R_n(N_1)$ and $R_2 \in \mathcal{L}R_n(N_2)$. If $\dim N_1 = \dim N_2$ and $R_1 \to R_2$, then $N_1 \to N_2$.

Proof. Let g_t be a family such that $\lim\limits_{t\to 0} g_t * R_1 = R_2$. By Proposition 3.3 we have that $\lim\limits_{t\to 0} g_t * N_1$ is a nilpotent ideal of R_2. Therefore, we get $\dim N_1 = \dim(\lim\limits_{t\to 0} g_t * N_1) = \dim N_2$. Since N_2 is the nilradical of R_2, then $\lim\limits_{t\to 0} g_t * N_1 = N_2$, i.e., $N_1 \to N_2$. □

Consider now a solvable Leibniz algebra R with rigid nilradical N.

Proposition 3.5. Let $R^2 = N$ and suppose that there exists a solvable Leibniz algebra R_1 such that $R_1 \to R$. Then $R_1 \in \mathcal{L}R_n(N)$.

Proof. Let N_1 be the nilradical of the algebra R_1. Note that by the Proposition 3.3 $\dim N_1 \leqslant \dim N$.

If $\dim N_1 < \dim N$, then we have $\dim R_1^2 \leqslant \dim N_1 < \dim N = \dim R^2$, which is a contradiction to the condition $R_1 \to R$ by a consequence of [17, Theorem 1.4] (see also [18, Corollary]).

If $\dim N_1 = \dim N$, then by Corollary 3.4 we conclude that $N_1 \to N$. Since N is a rigid algebra, then we get $N_1 \cong N$. $\square$

Corollary 3.6. The algebra $\mathbf{RNF_n}$ is a rigid algebra of the variety $\mathcal{L}R_{n+1}$.

From the above, we conclude that for a rigid nilpotent Leibniz algebra N in the variety $\mathcal{L}N_s$ and for $R \in \mathcal{L}R_n(N)$ there are only two possibilities: R is rigid in $\mathcal{L}R_n$ or there exists a rigid algebra $R_1 \in \mathcal{L}R_n(N)$ such that $R_1 \to R$.

Next proposition establishes a relationship between a solvable algebra and its nilradical.

Proposition 3.7. For any solvable algebra R with nilradical N there exists a degeneration: $R \to N \oplus \mathbb{C}^k$, where $k = \dim R/N$.

Proof. We choose a basis $\{e_1, e_2, \dots, e_k, e_{k+1}, \dots, e_n\}$ of R such that $N = \langle\{e_{k+1}, \dots, e_n\}\rangle$. A degeneration is given by the family g_t defined as follows:

$$g_t(e_i) = \begin{cases} t^{-1}e_i & \text{if } \ 1 \leqslant i \leqslant k, \\ e_i & \text{if } \ k+1 \leqslant i \leqslant n. \end{cases}$$

Indeed,

$$\begin{aligned} g_t*[e_i, e_j] &= g_t([g_t^{-1}(e_i), g_t^{-1}(e_j)]) = t^2 g_t([e_i, e_j]) = t^2[e_i, e_j] \to 0, \ 1 \leqslant i, j \leqslant k, \\ g_t*[e_i, e_j] &= g_t([g_t^{-1}(e_i), g_t^{-1}(e_j)]) = t g_t([e_i, e_j]) = t[e_i, e_j] \to 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad 1 \leqslant i \leqslant k, \ k+1 \leqslant j \leqslant n, \\ g_t*[e_i, e_j] &= g_t([g_t^{-1}(e_i), g_t^{-1}(e_j)]) = g_t([e_i, e_j]) = [e_i, e_j], \ \ k+1 \leqslant i, j \leqslant n. \qquad \square \end{aligned}$$

Now we present a family which will be useful in the sequel

$$g_t(e_1) = t^{-1}e_1, \quad g_t(e_2) = t^{-1}e_2, \quad g_t(e_i) = t^{-i+1}e_i, \quad 3 \leqslant i \leqslant n,$$

that degenerates the algebra $\mathbf{NF_n}$ to the so-called filiform algebra

$\mathbf{F_n} : [e_i, e_1] = e_{i+1}, \ 2 \leqslant i \leqslant n-1$ (see [20]).

3.2. Classification of four-dimensional solvable Leibniz algebras with three-dimensional rigid nilradicals

In this subsection we classify four-dimensional solvable Leibniz algebras whose nilradical is rigid and three-dimensional.

First of all, in the following proposition we describe the derivations of the three-dimensional nilpotent rigid Leibniz algebras $\boldsymbol{\lambda_4}(\alpha)$, $\boldsymbol{\lambda_5}$ and $\boldsymbol{\lambda_6}$. Recall that a derivation of a Leibniz algebra $(L, [-,-])$ is a F-linear map $d\colon L \to L$ such that $d[x, y] = [d(x), y] + [x, d(y)]$, for all $x, y \in L$.

Proposition 3.8. In the algebras $\boldsymbol{\lambda_4}(\alpha)$, $\boldsymbol{\lambda_5}$ and $\boldsymbol{\lambda_6}$ there exist bases such that their derivations have the following forms:

$$\mathrm{Der}\,(\boldsymbol{\lambda_4}(\alpha)) = \begin{pmatrix} a_1 & 0 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & a_1 + b_2 \end{pmatrix}, \ \alpha \neq \frac{1}{4}; \ \mathrm{Der}\left(\boldsymbol{\lambda_4}\left(\frac{1}{4}\right)\right) = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & b_3 \\ 0 & 0 & 2a_1 \end{pmatrix};$$

$$\mathrm{Der}(\boldsymbol{\lambda_5}) = \begin{pmatrix} a_1 & 0 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & a_1 + b_2 \end{pmatrix}; \qquad \mathrm{Der}(\boldsymbol{\lambda_6}) = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & 2a_1 & a_2 \\ 0 & 0 & 3a_1 \end{pmatrix}.$$

Proof. Taking the following change of basis in the algebra $\boldsymbol{\lambda_4}(\alpha)$:

$$f_1 = e_1, \quad f_2 = e_2 + \beta e_1, \quad f_3 = e_3,$$

with $\beta = -\frac{1+\sqrt{1-4\alpha}}{2}$, we deduce that the multiplications of $\boldsymbol{\lambda_4}(\alpha)$ become of the form:

$$[f_1, f_1] = f_3, \quad [f_2, f_1] = \beta f_3, \quad [f_1, f_2] = (1+\beta) f_3.$$

If $\beta \neq -\frac{1}{2}$ (i.e., $\alpha \neq \frac{1}{4}$), then setting $f_1' = f_1 - \frac{1}{2\beta+1} f_2$, $f_2' = \frac{1}{\beta} f_2$, we derive

$$[f_2, f_1] = f_3, \quad [f_1, f_2] = \beta' f_3, \tag{1}$$

where $\beta' = \frac{\sqrt{1-4\alpha}-1}{\sqrt{1-4\alpha}+1}$.

If $\beta = -\frac{1}{2}$ (i.e., $\alpha = \frac{1}{4}$), then putting $f_2' = -2f_2$, we get

$$\boldsymbol{\lambda_4}\left(\frac{1}{4}\right) : [f_1, f_1] = f_3, \quad [f_2, f_1] = f_3, \quad [f_1, f_2] = -f_3. \tag{2}$$

By checking the derivation property for algebras (1) and (2) we obtain

$$\mathrm{Der}\,(\boldsymbol{\lambda_4}(\alpha)) = \begin{pmatrix} a_1 & 0 & a_3 \\ 0 & b_2 & b_3 \\ 0 & 0 & a_1 + b_2 \end{pmatrix}, \ \alpha \neq \frac{1}{4}; \ \mathrm{Der}\left(\boldsymbol{\lambda_4}\left(\frac{1}{4}\right)\right) = \begin{pmatrix} a_1 & a_2 & a_3 \\ 0 & a_1 & b_3 \\ 0 & 0 & 2a_1 \end{pmatrix}.$$

The derivations of the algebras $\boldsymbol{\lambda_5}$ and $\boldsymbol{\lambda_6}$ are obtained directly applying the derivation property. □

Below, we prove that there do not exist four-dimensional solvable Leibniz algebras with nilradical $\boldsymbol{\lambda_4}(\frac{1}{4})$.

Proposition 3.9. There are no four-dimensional solvable Leibniz algebras with three-dimensional nilradical $\boldsymbol{\lambda_4}(\frac{1}{4})$.

Proof. Let us assume the contrary. Let $R \in \mathcal{L}R_4\left(\boldsymbol{\lambda_4}\left(\frac{1}{4}\right)\right)$. We choose a basis $\{x, f_1, f_2, f_3\}$ of R such that $\{f_1, f_2, f_3\}$ is the basis of $\boldsymbol{\lambda_4}(\frac{1}{4})$ chosen in the proof of Proposition 3.8. Since the algebra R is non-nilpotent, the restriction of the right multiplication operator $\mathcal{R}_x$ to $\boldsymbol{\lambda_4}\left(\frac{1}{4}\right)$ is a non-nilpotent derivation of $\boldsymbol{\lambda_4}\left(\frac{1}{4}\right)$. Then using the form of this derivation from Proposition 3.8 we have

$$\begin{array}{lll} [f_1, x] = a_1 f_1 + a_2 f_2 + a_3 f_3, & [f_2, x] = a_1 f_2 + b_3 f_3, & [f_3, x] = 2a_1 f_3, \\ [f_1, f_1] = f_3, & [f_2, f_1] = f_3, & [f_1, f_2] = -f_3. \end{array}$$

Since $\mathcal{R}_{x|\boldsymbol{\lambda_4}}$ is non-nilpotent, we can suppose $a_1 = 1$. It is easy to see that the right annihilator of the algebra R only consists of $\{f_3\}$. Therefore,

$$\begin{array}{lll} [f_1, x] = f_1 + a_2 f_2 + a_3 f_3, & [f_2, x] = f_2 + b_3 f_3, & [f_3, x] = 2f_3, \\ [x, f_1] = -f_1 - a_2 f_2 + \alpha_3 f_3, & [x, f_2] = -f_2 + \beta_3 f_3, & [x, x] = \gamma_3 f_3, \\ [f_1, f_1] = f_3, & [f_2, f_1] = f_3, & [f_1, f_2] = -f_3. \end{array}$$

Considering the Leibniz identity

$$\begin{aligned} 0 &= [x, [f_2, f_1]] = [[x, f_2], f_1] - [[x, f_1], f_2] \\ &= [-f_2 + \beta_3 f_3, f_1] - [-f_1 - a_2 f_2 + \alpha_3 f_3, f_2] = -f_3 - f_3 = -2f_3, \end{aligned}$$

we have a contradiction with the assumption. □

The following theorem gives the classification of four-dimensional solvable Leibniz algebras with three-dimensional rigid nilradicals.

Theorem 3.10. Up to isomorphism, there exist three four-dimensional solvable Leibniz algebras with three-dimensional rigid nilradicals. Namely,

$$\mathbf{R_1^4}: \begin{cases} [e_2, e_1] = e_3, \quad [x, e_1] = -e_1, \\ [e_1, e_2] = \beta e_3, \ [x, e_2] = -\beta e_2, \\ \ [e_1, x] = e_1, \quad [e_2, x] = \beta e_2, \quad [e_3, x] = (\beta + 1)e_3, \end{cases}$$

where $\beta = \frac{\sqrt{1-4\alpha}-1}{\sqrt{1-4\alpha}+1}$ *for* $\alpha \neq 0, \frac{1}{4}$;

$$\mathbf{R_2^4}: \begin{cases} [e_2, e_1] = e_3, \ [x, e_1] = -e_1, \\ [e_1, e_2] = e_3, \ [x, e_2] = -e_2, \\ \ [e_1, x] = e_1, \ [e_2, x] = e_2, \quad [e_3, x] = 2e_3\,; \end{cases}$$

$$\mathbf{R_3^4}: \begin{cases} [e_1, e_1] = e_2, \ [e_2, e_1] = e_3, \quad [x, e_1] = -e_1, \\ \ [e_1, x] = e_1, \quad [e_2, x] = 2e_2, \ [e_3, x] = 3e_3. \end{cases}$$

Proof. Here we shall use the form of the algebra $\boldsymbol{\lambda_4}(\alpha)$ as in the proof of Proposition 3.8 after the change of basis, i.e., the form $\boldsymbol{\lambda_4}(\beta)$. Consider the class $\mathcal{L}R_4\ (\boldsymbol{\lambda_4}(\beta))$. Due to Proposition 3.8, we can choose a basis $\{x, f_1, f_2, f_3\}$ of the algebra of $\mathcal{L}R_4\ (\boldsymbol{\lambda_4}(\beta))$ such that $\mathcal{R}_{x|\boldsymbol{\lambda_4}(\beta)}$ is a non-nilpotent derivation of $\boldsymbol{\lambda_4}(\beta)$. Therefore, in the algebra of $\mathcal{L}R_4\ (\boldsymbol{\lambda_4}(\beta))$ we have the following products:

$$[f_2, f_1] = f_3, \qquad [f_1, f_2] = \beta f_3,$$
$$[f_1, x] = a_1 f_1 + a_3 f_3, \ [f_2, x] = b_2 f_2 + b_3 f_3, \ [f_3, x] = (a_1 + b_2)f_3.$$

It is easy to see that the right annihilator of the algebra consists of $\{f_3\}$. Hence we get

$$[f_1, x] = a_1 f_1 + a_3 f_3, \quad [f_2, x] = b_2 f_2 + b_3 f_3, \quad [f_3, x] = (a_1 + b_2)f_3,$$
$$[x, f_1] = -a_1 f_1 + \alpha_3 f_3, \ [x, f_2] = -b_2 f_2 + \beta_3 f_3, \ [x, x] = \gamma_3 f_3,$$
$$[f_2, f_1] = f_3, \qquad [f_1, f_2] = \beta f_3.$$

Applying the Leibniz identity

$$\begin{aligned} 0 &= [x, [f_2, f_1]] = [[x, f_2], f_1] - [[x, f_1], f_2] \\ &= [-b_2 f_2 + \beta_3 f_3, f_1] - [-a_1 f_1 + \alpha_3 f_3, f_2] = -b_2 f_3 + a_1 \beta f_3, \end{aligned}$$

we derive $b_2 = a_1 \beta$.

Since $\mathcal{R}_{x|\boldsymbol{\lambda_4}(\beta)}$ is non-nilpotent, we have $a_1 = b_2 \neq 0$. Consequently, we can assume $a_1 = 1, \ b_2 = \beta$.

Taking the change of basis:

$$e_1 = f_1 - \frac{a_3}{\beta} f_3, \quad e_2 = f_2 - b_3 f_3, \quad e_3 = f_3, \quad x' = x - \frac{\gamma_3}{\beta + 1} f_3,$$

we can suppose that $a_3 = b_3 = \gamma_3 = 0$ and the multiplication table has the form

$$[e_1, x] = e_1, \qquad [e_2, x] = \beta e_2, \qquad [e_3, x] = (1 + \beta)e_3,$$
$$[x, e_1] = -e_1 + \alpha_3 e_3, \quad [x, e_2] = -\beta e_2 + \beta_3 e_3,$$
$$[e_2, e_1] = e_3, \qquad [e_1, e_2] = \beta e_3.$$

Consider the chain of equalities

$$[x,[e_1,x]] = [[x,e_1],x] - [[x,x],e_1] = [-e_1+\alpha_3e_3,x] = -e_1+\alpha_3(1+\beta)e_3.$$

On the other hand, $[x,[e_1,x]] = [x,e_1] = -e_1+\alpha_3e_3$.
Comparing the coefficients at the basis elements, we obtain $\alpha_3\beta = 0$, which implies $\alpha_3 = 0$.
Similarly, from

$$\begin{aligned}[x,[e_2,x]] &= [[x,e_2],x] - [[x,x],e_2] = [-\beta e_2+\beta_3e_3,x]\\ &= -\beta^2e_2+\beta_3(1+\beta)e_3,\\ [x,[e_2,x]] &= [x,\beta e_2] = -\beta^2e_2+\beta\beta_3e_3,\end{aligned}$$

we deduce $\beta_3 = 0$. Thus the algebra $\mathbf{R_1^4}$ is obtained.
Applying the above arguments for the class $\mathcal{L}R_4(\boldsymbol{\lambda_5})$ we derive the multiplication table:

$$\begin{array}{lll} [f_1,x] = a_1f_1+a_3f_3, & [f_2,x] = b_2f_2+b_3f_3, & [f_3,x] = (a_1+b_2)f_3,\\ [x,f_1] = -a_1f_1+\alpha_3f_3, & [x,f_2] = -b_2f_2+\beta_3f_3, & [x,x] = \gamma_3f_3,\\ [f_2,f_1] = f_3, & [f_1,f_2] = f_3. & \end{array}$$

From the chain of equalities

$$\begin{aligned}0 &= [x,[f_2,f_1]] = [[x,f_2],f_1] - [[x,f_1],f_2]\\ &= [-b_2f_2+\beta_3f_3,f_1] - [-a_1f_1+\alpha_3f_3,f_2] = -b_2f_3+a_1f_3,\end{aligned}$$

we have $b_2 = a_1$.
Since the restriction of the right multiplication operator on the element x to $\boldsymbol{\lambda_5}$ is non-nilpotent, we have $a_1 = b_2 \neq 0$ and without loss of generality we can suppose $a_1 = b_2 = 1$.
Taking the change of basis

$$e_1 = f_1 - a_3f_3,\quad e_2 = f_2 - b_3f_3,\quad e_3 = f_3,\quad x' = x - \frac{\gamma_3}{2}f_3,$$

we can suppose that $a_3 = b_3 = \gamma_3 = 0$ and the multiplication table has the form

$$\begin{array}{lll} [e_1,x] = e_1, & [e_2,x] = e_2, & [e_3,x] = 2e_3,\\ [x,e_1] = -e_1+\alpha_3e_3, & [x,e_2] = -e_2+\beta_3e_3, & \\ [e_2,e_1] = e_3, & [e_1,e_2] = e_3. & \end{array}$$

Applying the Leibniz identity to the brackets $[x,[x,e_1]]$ and $[x,[e_1,x]]$ with respect to the above multiplication, we derive that $\alpha_3 = \beta_3 = 0$. Thus, we obtain the algebra $\mathbf{R_2^4}$.
Since an algebra of $\mathcal{L}R_4(\boldsymbol{\lambda_6})$ is nothing else but the algebra $\mathbf{RNF_3}$, the algebra $\mathbf{R_3^4}$ is directly followed from Theorem 2.11. □

It should be noted that thanks to Proposition 3.5 and Corollary 3.6 the algebras $\mathbf{R_1^4}$, $\mathbf{R_2^4}$ and $\mathbf{R_3^4}$ are rigid in the variety $\mathcal{L}R_4$.

It is well known that rigid nilpotent Lie algebras in the variety of solvable Lie algebras are characteristically nilpotent [25], and since they appear from dimension 7 and forward, we have that nilpotent Lie algebras of dimensions less than 7 are not rigid (see [26]).

Remark 3.11. Since in [16,17] the lists of rigid nilpotent Lie algebras in dimensions less than seven are presented (there are $\mathbf{n_3} - \mathbf{n_6}$), it is sufficient to show their non-rigidity in the variety of solvable Lie algebras $\mathcal{L}R_n$. We have found degenerations $\lambda \to \mu$ in dimensions less than seven by explicitly constructing $g_t \in \mathrm{GL}_n(\mathbb{C})$.

Consider the following solvable Lie algebras of dimension less than seven, whose multiplication tables are:

$$\begin{aligned}
\mathbf{r_3} &: [e_1, e_2] = -[e_2, e_1] = e_1 + e_3, \quad [e_3, e_2] = -[e_2, e_3] = e_3;\\
\mathbf{r_4} &: [e_1, e_2] = -[e_2, e_1] = e_1 + e_3, \quad [e_1, e_3] = -[e_3, e_1] = e_4,\\
&\quad [e_2, e_3] = -[e_2, e_3] = e_3;\\
\mathbf{r_5} &: [e_1, e_2] = -[e_2, e_1] = e_3, \quad [e_1, e_3] = -[e_3, e_1] = e_2,\\
&\quad [e_1, e_4] = -[e_4, e_1] = e_5, \quad [e_2, e_3] = -[e_3, e_2] = e_5;\\
\mathbf{r_6} &: [e_1, e_2] = e_2 + e_3, \quad [e_1, e_3] = e_3 + e_4, \quad [e_1, e_4] = 2e_4 + e_5,\\
&\quad [e_1, e_5] = 2e_5, \quad [e_1, e_6] = 3e_6, \quad [e_2, e_3] = e_5, \quad [e_2, e_5] = e_6,\\
&\quad [e_3, e_4] = -e_6.
\end{aligned}$$

It is easy to check that

$\mathbf{r_3} \to \mathbf{n_3}$ via the family g_t defined as follows

$$g_t(e_1) = t^{-1}e_1, \quad g_t(e_2) = t^{-1}e_2, \quad g_t(e_3) = t^{-2}e_3;$$

$\mathbf{r_4} \to \mathbf{n_4}$ via the family g_t defined as

$$g_t(e_1) = t^{-1}e_1, \quad g_t(e_2) = t^{-1}e_2, \quad g_t(e_3) = t^{-2}e_3, \quad g_t(e_4) = t^{-3}e_4;$$

$\mathbf{r_5} \to \mathbf{n_5}$ via the family g_t defined as

$$\begin{aligned}
&g_t(e_1) = t^{-1}e_1, \qquad g_t(e_2) = t^{-3}e_4, \quad g_t(e_3) = t^{-4}e_5,\\
&g_t(e_4) = -e_2 + t^{-2}e_4, \quad g_t(e_5) = -t^{-1}e_3 + t^{-3}e_5;
\end{aligned}$$

$\mathbf{r_6} \to \mathbf{n_6}$ via the family g_t defined as

$$\begin{aligned}
&g_t(e_1) = t^{-1}e_1, \quad g_t(e_2) = t^{-2}e_2, \quad g_t(e_3) = t^{-3}e_3,\\
&g_t(e_4) = t^{-4}e_4, \quad g_t(e_5) = t^{-5}e_5, \quad g_t(e_6) = t^{-7}e_6.
\end{aligned}$$

Remark 3.12. Consider the following n-dimensional solvable Leibniz algebra

$$\mathbf{R_n}: \ [e_1, e_1] = e_2, \quad [e_i, e_1] = e_i + e_{i+1}, \quad 2 \leqslant i \leqslant n-1.$$

It is known that the algebra $\mathbf{NF_n}$ is rigid in the variety of nilpotent Leibniz algebras [20]. However, this algebra it is not rigid in the variety of solvable Leibniz algebras. Indeed, the family of basis transformations

$$g_t(e_i) = t^{-i}e_i, \qquad 1 \leqslant i \leqslant n,$$

degenerates the algebra $\mathbf{R_n}$ to $\mathbf{NF_n}$.

Now we present a result which asserts that the Conjecture is true for the case of Leibniz algebras of dimensions less than four.

Theorem 3.13. Any nilpotent Leibniz algebra of dimension less than four is not rigid.

Proof. From [18] we have a unique two-dimensional rigid nilpotent Leibniz algebra $\mathbf{n_2}: [e_1, e_1] = e_2$. It is easy to check that the algebra $\mathbf{r_2}$ with the multiplication table $[e_2, e_1] = e_2$ degenerates to $\mathbf{n_2}$ via the family of transformations:

$$g_t: \quad g_t(e_1) = t^{-1}e_1 - t^{-2}e_2, \qquad g_t(e_2) = t^{-2}e_2.$$

For the three-dimensional case we have the rigid nilpotent algebras $\boldsymbol{\lambda_4}(\alpha)$, $\boldsymbol{\lambda_5}$ and $\boldsymbol{\lambda_6}$.

Let us consider the solvable Leibniz algebra

$$\mathbf{r_{3,2}}(\alpha):\begin{cases}[e_1,e_1]=e_3, & [e_1,e_2]=-(2+\beta)\alpha e_1+e_2+e_3,\\ [e_2,e_1]=(2+\beta)\alpha e_1-e_2, & [e_2,e_2]=\alpha e_3,\quad \alpha\neq 0,\\ [e_3,e_1]=\beta e_3, & [e_3,e_2]=(2+\beta)\beta\alpha e_3,\end{cases}$$

where $\beta=\frac{1-4\alpha+\sqrt{1-4\alpha}}{2\alpha}$.

Then g_t defined as

$$g_t(e_1)=t^{-1}e_1,\quad g_t(e_2)=t^{-1}e_2,\quad g_t(e_3)=t^{-2}e_3$$

degenerates the algebra $\mathbf{r_{3,2}}(\alpha)$ to the algebra $\boldsymbol{\lambda_4}(\alpha)$.

Consider the solvable Leibniz algebra

$$\mathbf{r_{3,1}}:\ [e_2,e_1]=-e_2+e_3,\quad [e_3,e_1]=-2e_3,\quad [e_1,e_2]=e_2+e_3,\quad [e_2,e_2]=e_3.$$

Then $\mathbf{r_{3,1}}\to\boldsymbol{\lambda_5}$ via g_t, which is given by

$$g_t(e_1)=t^{-1}e_1,\qquad g_t(e_2)=t^{-2}e_2,\qquad g_t(e_3)=t^{-3}e_3.$$

Due to Remark 3.12, we get $\mathbf{R_3}\to\boldsymbol{\lambda_6}$. $\square$

3.3. *On the algebra of level one*

In this subsection we show that the result of Theorem 2.13 is not complete. Namely, the algebra $\mathbf{n_2}\oplus\mathbf{a_{n-2}}$ is also an algebra of level one and it is not isomorphic to the algebras $\mathbf{p_n^{\pm}}$ and $\mathbf{n_3^{\pm}}\oplus\mathbf{a_{n-3}}$.

Theorem 3.14. The n-dimensional commutative algebra $\mathbf{n_2}\oplus\mathbf{a_{n-2}}$ is of level one.

Proof. Firstly, we shall prove that the algebra $\mathbf{n_2}\oplus\mathbf{a_{n-2}}$ does not degenerate to $\mathbf{p_n^{\pm}}$ and $\mathbf{n_3^{\pm}}\oplus\mathbf{a_{n-3}}$. Since $\mathbf{n_2}\oplus\mathbf{a_{n-2}}$ is commutative, it is enough to prove it for $\mathbf{p_n^{+}}$ and $\mathbf{n_3^{+}}\oplus\mathbf{a_{n-3}}$.

(I) Let us assume the contrary, that there exists a family $g_t\in \mathrm{GL}_n(\mathbb{C})$ such that $\mathbf{n_2}\oplus\mathbf{a_{n-2}}\to\mathbf{p_n^{+}}$. Let g_t be of the form

$$g_t(e_i)=\sum_{s=1}^{n}\alpha_{i,s}(t)e_s,\quad g_t^{-1}(e_i)=\sum_{s=1}^{n}\beta_{i,s}(t)e_s.$$

Consider

$$g_t([g_t^{-1}(e_1),g_t^{-1}(e_p)])=\beta_{1,1}(t)\beta_{p,1}(t)g_t(e_2)=\beta_{1,1}(t)\beta_{p,1}(t)\sum_{i=1}^{n}\alpha_{2,i}(t)e_i. \tag{3}$$

Since in the algebra $\mathbf{p_n^{+}}$ we have $[e_1,e_p]=e_p$, for any p $(2\leqslant p\leqslant n)$, then

$$\lim_{t\to 0}g_t([g_t^{-1}(e_1),g_t^{-1}(e_p)])=e_p. \tag{4}$$

Therefore, taking into account (3) and (4), we obtain

$$\lim_{t\to 0}\beta_{1,1}(t)\beta_{p,1}(t)\alpha_{2,p}(t)=1,\qquad \lim_{t\to 0}\beta_{1,1}(t)\beta_{p,1}(t)\alpha_{2,q}(t)=0,\quad q\neq p.$$

In particular, for $p=2$, $q=3$, we have

$$\lim_{t\to 0}\beta_{1,1}(t)\beta_{2,1}(t)\alpha_{2,2}(t)=1,\qquad \lim_{t\to 0}\beta_{1,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t)=0. \tag{5}$$

On the other hand, for $p = 3$, $q = 2$, we have

$$\lim_{t\to 0} \beta_{1,1}(t)\beta_{3,1}(t)\alpha_{2,3}(t) = 1, \qquad \lim_{t\to 0} \beta_{1,1}(t)\beta_{3,1}(t)\alpha_{2,2}(t) = 0. \tag{6}$$

Now, taking into account (5) and (6), we have

$$\lim_{t\to 0} \frac{\alpha_{2,3}(t)}{\alpha_{2,2}(t)} = \lim_{t\to 0} \frac{\beta_{1,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t)}{\beta_{1,1}(t)\beta_{2,1}(t)\alpha_{2,2}(t)} = 0,$$

$$\lim_{t\to 0} \frac{\alpha_{2,3}(t)}{\alpha_{2,2}(t)} = \lim_{t\to 0} \frac{\beta_{1,1}(t)\beta_{3,1}(t)\alpha_{2,3}(t)}{\beta_{1,1}(t)\beta_{3,1}(t)\alpha_{2,2}(t)} = \infty.$$

This is a contradiction with the assumption of the existence of g_t, i.e., $\mathbf{n_2} \oplus \mathbf{a_{n-2}}$ does not degenerate to $\mathbf{p_n^+}$.

(II) Let us show that $\mathbf{n_2} \oplus \mathbf{a_{n-2}}$ does not degenerate to the algebra $\mathbf{n_3^+} \oplus \mathbf{a_{n-3}}$. Similarly as above we can assume the existence of a family g_t.

From (3) we get

$$g_t([g_t^{-1}(e_1), g_t^{-1}(e_1)]) = \beta_{1,1}(t)\beta_{1,1}(t) \sum_{i=1}^{n} \alpha_{2,i}(t)e_i.$$

Since in the algebra $\mathbf{n_3^+} \oplus \mathbf{a_{n-3}}$ we have the product $[e_1, e_1] = 0$, then

$$\lim_{t\to 0} g_t([g_t^{-1}(e_1), g_t^{-1}(e_1)]) = 0.$$

Consequently, $\lim_{t\to 0} \beta_{1,1}(t)\beta_{1,1}(t)\alpha_{2,3}(t) = 0$.

Similarly, from (3) with $p = 2$, i.e.,

$$g_t([g_t^{-1}(e_1), g_t^{-1}(e_2)]) = \beta_{1,1}(t)\beta_{2,1}(t) \sum_{i=1}^{n} \alpha_{2,i}(t)e_i$$

and of the product $[e_1, e_2] = e_3$ in $\mathbf{n_3^+} \oplus \mathbf{a_{n-3}}$, we conclude

$$\lim_{t\to 0} \beta_{1,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t) = 1.$$

Using the equalities

$$g_t([g_t^{-1}(e_2), g_t^{-1}(e_2)]) = g_t\left(\left[\sum_{s=1}^{n} \beta_{2,s}(t)e_s, \sum_{s=1}^{n} \beta_{2,s}(t)e_s\right]\right)$$
$$= \beta_{2,1}(t)\beta_{2,1}(t)g_t(e_2) = \beta_{2,1}(t)\beta_{2,1}(t) \sum_{i=1}^{n} \alpha_{2,i}(t)e_i$$

and $[e_2, e_2] = 0$, we derive $\lim_{t\to 0} \beta_{2,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t) = 0$.

Thus, we summarize

$$\lim_{t\to 0} \beta_{1,1}(t)\beta_{1,1}(t)\alpha_{2,3}(t) = \lim_{t\to 0} \beta_{2,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t) = 0,$$
$$\lim_{t\to 0} \beta_{1,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t) = 1.$$

However,

$$\lim_{t\to 0} (\beta_{1,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t))^2 = \lim_{t\to 0} \beta_{1,1}(t)\beta_{1,1}(t)\alpha_{2,3}(t) \cdot \lim_{t\to 0} \beta_{2,1}(t)\beta_{2,1}(t)\alpha_{2,3}(t) = 0.$$

Thus, the algebra $\mathbf{n_2} \oplus \mathbf{a_{n-2}}$ does not degenerate to $\mathbf{n_3^+} \oplus \mathbf{a_{n-3}}$.

Now we shall prove that $\mathrm{lev}_n(\mathbf{n_2} \oplus \mathbf{a_{n-2}}) = 1$. Assume that there exists a Leibniz algebra λ such that $\mathbf{n_2} \oplus \mathbf{a_{n-2}} \to \lambda$ is a direct degeneration. Then $\dim \mathrm{Orb}(\lambda) < \dim \mathrm{Orb}(\mathbf{n_2} \oplus \mathbf{a_{n-2}})$ (see [14]).

If λ is a non-Lie Leibniz algebra, then by Proposition 2.12 we have that $\lambda \to \mathbf{n_2} \oplus \mathbf{a_{n-2}}$. Then there exists a chain of direct degenerations $\lambda \to \lambda_1 \to \cdots \to \lambda_k \to \mathbf{n_2} \oplus \mathbf{a_{n-2}}$. Again by [14], we have that $\dim \mathrm{Orb}(\mathbf{n_2} \oplus \mathbf{a_{n-2}}) < \dim \mathrm{Orb}(\lambda_k) < \cdots < \dim \mathrm{Orb}(\lambda_1) < \dim \mathrm{Orb}(\lambda)$. This is a contradiction with $\dim \mathrm{Orb}(\lambda) < \dim \mathrm{Orb}(\mathbf{n_2} \oplus \mathbf{a_{n-2}})$.

Let λ be a Lie algebra, then by assumption there exists a family g_t such that

$$\lim_{t\to 0} g_t * (\mathbf{n_2} \oplus \mathbf{a_{n-2}}) = \lambda.$$

Then from the following equalities

$$g_t([g_t^{-1}(e_i), g_t^{-1}(e_j)]) = g_t\left(\left[\sum_{s=1}^{n} \beta_{i,s}(t)e_s, \sum_{s=1}^{n} \beta_{j,s}(t)e_s\right]\right) = \beta_{i,1}(t)\beta_{j,1}(t)g_t(e_2),$$

we deduce $\lambda(e_i, e_j) = \lambda(e_j, e_i)$. Since λ is a Lie algebra, it follows that it is abelian. Consequently, the algebra $\mathbf{n_2} \oplus \mathbf{a_{n-2}}$ is of level one. □

Remark 3.15. Another way of proving Theorem 3.14 would be the following one.

(I) We give other different reasons to establish that $\mathbf{n_2} \oplus \mathbf{a_{n-2}}$ does not degenerate to $\mathbf{p_n^+}$.

1. The algebra $\mathbf{n_2} \oplus \mathbf{a_{n-2}}$ is nilpotent, but the algebra $\mathbf{p_n^+}$ is not nilpotent (see [18]).
2. The projectivization of the group $\mathrm{Aut}(\mathbf{n_2} \oplus \mathbf{a_{n-2}})$ is parabolic in the group $\mathrm{GL}_n(\mathbb{C})$, then by arguments given in [14], the algebra $\mathbf{n_2} \oplus \mathbf{a_{n-2}}$ is of level one.
3. The invariants $c_{i,j}(\lambda) := \dfrac{\mathrm{tr}(\mathcal{R}_x)^i \cdot \mathrm{tr}(\mathcal{R}_y)^j}{tr\left((\mathcal{R}_x)^i \cdot (\mathcal{R}_y)^j\right)}$ do not coincide, because $c_{1,1}(\mathbf{n_2} \oplus \mathbf{a_{n-2}}) = 0$ and $c_{1,1}(\mathbf{p_n^+}) = \dfrac{1}{n-1}$ (see [27]), where $tr(\mathcal{R}_x)$ denotes the trace invariant of the right multiplication operator $\mathcal{R}_x$.
4. $\mathbf{n_2} \oplus \mathbf{a_{n-2}}$ is a Leibniz algebra, but $\mathbf{p_n^+}$ is not a Leibniz algebra, since $e_2 = [e_1, [e_1, e_2]] \neq [[e_1, e_1], e_2] - [[e_1, e_2], e_1] = -e_2$.
5. Since the derivations of $\mathbf{n_2} \oplus \mathbf{a_{n-2}}$ are

$$\begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \dots & \alpha_{n-1} & \alpha_n \\ 0 & 2\alpha_{1,1} & 0 & \dots & 0 & 0 \\ 0 & \alpha_{3,2} & \alpha_{3,3} & \dots & \alpha_{3,n-1} & \alpha_{3,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & \alpha_{n,2} & \alpha_{n,3} & \dots & \alpha_{n,n-1} & \alpha_{n,n} \end{pmatrix}$$

and the derivations of $\mathbf{p_n^+}$ are

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \alpha_{2,2} & \alpha_{2,3} & \dots & \alpha_{2,n-1} & \alpha_{2,n} \\ 0 & \alpha_{3,2} & \alpha_{3,3} & \dots & \alpha_{3,n-1} & \alpha_{3,n} \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & \alpha_{n,2} & \alpha_{n,3} & \dots & \alpha_{n,n-1} & \alpha_{n,n} \end{pmatrix},$$

then $\dim\left(\mathrm{Der}(\mathbf{n_2} \oplus \mathbf{a_{n-2}})\right) = n^2 - 2n + 2 > \dim\left(\mathrm{Der}(\mathbf{p_n^+})\right) = n^2 - 2n + 1$ (see [18]).

(II) We also give other different reasons to establish that $\mathbf{n_2} \oplus \mathbf{a_{n-2}}$ does not degenerate to $\mathbf{n_3^+} \oplus \mathbf{a_{n-3}}$.

1. Since $\mathrm{Ann}_r(\mathbf{n_2} \oplus \mathbf{a_{n-2}}) = \{e_2, e_3, \ldots, e_n\}$ and $\mathrm{Ann}_r(\mathbf{n_3^+} \oplus \mathbf{a_{n-3}}) = \{e_3, \ldots, e_n\}$, we have $\dim(\mathrm{Ann}_r(\mathbf{n_2} \oplus \mathbf{a_{n-2}})) > \dim\left(\mathrm{Ann}_r(\mathbf{n_3^+} \oplus \mathbf{a_{n-3}})\right)$ (see [18]). We might use Ann_l or Center instead of Ann_r, since in these two algebras, $\mathrm{Ann}_r(\mathbf{n_2} \oplus \mathbf{a_{n-2}}) = \mathrm{Ann}_l(\mathbf{n_2} \oplus \mathbf{a_{n-2}}) = \mathrm{Center}(\mathbf{n_2} \oplus \mathbf{a_{n-2}})$ and $\mathrm{Ann}_r(\mathbf{n_3^+} \oplus \mathbf{a_{n-3}}) = \mathrm{Ann}_l(\mathbf{n_3^+} \oplus \mathbf{a_{n-3}}) = \mathrm{Center}(\mathbf{n_3^+} \oplus \mathbf{a_{n-3}})$.
2. Similar arguments to the previous case 2 of **(I)** (see [14]).
3. Since the derivations of $\mathbf{n_3^+} \oplus \mathbf{a_{n-3}}$ are

$$\begin{pmatrix} \alpha_{1,1} & 0 & \alpha_{1,3} & \alpha_{1,4} & \ldots & \alpha_{1,n-1} & \alpha_{1,n} \\ 0 & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & \ldots & \alpha_{2,n-1} & \alpha_{2,n} \\ 0 & 0 & \alpha_{1,1}+\alpha_{2,2} & 0 & \ldots & 0 & 0 \\ 0 & 0 & \alpha_{4,3} & \alpha_{4,4} & \ldots & \alpha_{4,n-1} & \alpha_{3,n} \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \alpha_{n,3} & \alpha_{n,4} & \ldots & \alpha_{n,n-1} & \alpha_{n,n} \end{pmatrix},$$

then $\dim\left(\mathrm{Der}(\mathbf{n_3^+} \oplus \mathbf{a_{n-3}})\right) = n^2 - 3n + 4$ and so $\dim(\mathrm{Der}(\mathbf{n_2} \oplus \mathbf{a_{n-2}})) > \dim\left(\mathrm{Der}(\mathbf{n_3^+} \oplus \mathbf{a_{n-3}})\right)$ (see [18]).

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