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Gradient Gibbs measures for the SOS model with integer spin values on a Cayley tree

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Abstract. In the present paper we continue the investigation from Bovier and Külske (1996 *J. Stat. Phys.* **83** 751–59) and consider the SOS (solid-on-solid) model on the Cayley tree of order $k \geq 2$. In the ferromagnetic SOS case on the Cayley tree, we find three solutions to a class of period-4 height-periodic boundary law equations and these boundary laws define up to three periodic gradient Gibbs measures.

Keywords: generalized Gibbs ensemble, integrable spin chains and vertex models, spin glasses, stochastic thermodynamics

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1. Introduction

A solid-on-solid (SOS) model can be considered as a generalization of the Ising model, which corresponds to $E = \{-1, 1\}$, or a less symmetric variant of the Potts model with non-compact state space. SOS-models on the cubic lattice were analyzed in [2, 3] where an analogue of the so-called Dinaburg–Mazel–Sinai theory was developed. Besides interesting phase transitions in these models, the attention to them is motivated by applications, in particular in the theory of communication networks; see, e.g., [1, 5, 7, 11].

In [6] it is shown that on the Cayley tree there are several tree automorphism invariant gradient Gibbs measures and the existence of q different gradient Gibbs measures for q -component models on the Cayley tree of order $k \geq 2$. To the best of our knowledge, the first paper devoted to the SOS model on the Cayley tree is [10]. In [10] the case of arbitrary $m \geq 1$ is treated and a vector-valued functional equation for possible boundary laws of the model is obtained. Recall that each solution to this functional equation determines a splitting Gibbs measure (SGM), in other words a tree-indexed Markov chain which is also a Gibbs measure. Such measures can be obtained by propagating spin values along the edges of the tree, from any site singled out to be the root to the outside, with a transition matrix depending on initial Hamiltonian and the boundary law solution. In particular the homogeneous (site-independent) boundary laws then define translation-invariant SGMS. For a recent investigation of the influence of weakly non-local perturbations in the interaction to the structure of Gibbs measures, see [9] in the context of the Ising model.

The present paper is organized as follows. In section 2 we present the preliminaries of the model. In the third section we construct gradient Gibbs measures for period 4 height-periodic boundary laws on the Cayley tree of order $k \geq 2$. Note that the results in [6] are proved only on the Cayley tree of order two.

2. Preliminaries

Cayley tree. The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k + 1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L the set of edges. Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them. We will use the notation $l = \langle x, y \rangle$. A collection of nearest neighbor pairs $\langle x, x_1 \rangle, \langle x_1, x_2 \rangle, \dots, \langle x_{d-1}, y \rangle$ is called a *path* from x to y . The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x to y .

For a fixed $x^0 \in V$, called the root, we set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^n W_m$$

and denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n,$$

the set of *direct successors* of x .

SOS model. We consider a model where the spin takes values in the set of all integer numbers $Z := \{\dots, -1, 0, 1, \dots\}$, and is assigned to the vertices of the tree. A configuration σ on V is then defined as a function $x \in V \mapsto \sigma(x) \in Z$; the set of all configurations is $\Omega := Z^V$.

The (formal) Hamiltonian of the SOS model is :

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} |\sigma(x) - \sigma(y)|, \quad (2.1)$$

where $J \in \mathbb{R}$ is a constant and $\langle x, y \rangle$ stands for nearest neighbor vertices.

Note that the Hamiltonian is invariant under the spin-translation/height-shift $t : (t\omega)_i = \omega_i + t$. This suggests reducing the complexity of the configuration space by considering it gradient configurations instead of height configurations as will be explained in the following:

Gradient configuration. We may induce an orientation on Γ^k relative to an arbitrary site ρ (which we may call the root) by calling an edge $\langle x, y \rangle$ oriented iff it points away from the ρ . More precisely, the set of oriented edges is defined by

$$\vec{L} := \vec{L}_\rho := \{\langle x, y \rangle \in L : d(\rho, y) = d(\rho, x) + 1\}.$$

Note that the oriented graph $(V, \vec{L})$ also possesses all tree-properties, namely connectedness and absence of loops.

For any height configuration $\omega = (\omega(x))_{x \in V} \in Z^V$ and $b = \langle x, y \rangle \in \vec{L}$ the height difference along the edge b is given by $\nabla \omega_b = \omega_y - \omega_x$ and we also call $\nabla \omega$ the gradient field of ω . The gradient spin variables are now defined by $\eta_{\langle x, y \rangle} = \omega_y - \omega_x$ for each $\langle x, y \rangle \in \vec{L}$. Let us denote the space of *gradient configuration* by $\Omega^\nabla = Z^{\vec{L}}$. Note that in contrast to the notation used in [12] for the lattice Z^d , the gradient configurations defined above are indexed by the oriented edges of the tree and not by its vertices. Equip the integers Z with the power set as measurable structure. Having done this, the measurable structure on the space Ω^∇ is given by the product σ -algebra $\mathcal{F}^\nabla := \sigma(\{\nabla_b | b \in \vec{L}\})$. Clearly $\nabla : (\Omega, \mathcal{F}) \rightarrow (\Omega^\nabla, \mathcal{F}^\nabla)$ then becomes a measurable map.

3. Gradient Gibbs measures and tree-automorphism invariant solutions

3.1. Gibbs and gradient Gibbs measures

Recall that the set of height configurations $\Omega := Z^V$ was endowed with the product σ -algebra $\otimes_{i \in V} 2^Z$, where 2^Z denotes the power set of Z . Then, for any $\Lambda \subset V$, consider the coordinate projection map $\sigma_\Lambda : Z^V \rightarrow Z^\Lambda$ and the σ -algebra $\mathcal{F}_\Lambda := \sigma(\sigma_\Lambda)$ of cylinder sets on Z^V generated by the map σ_Λ .

We define Gibbs measures on the space of height-configurations for the model (2.1) on a Cayley tree. Let $\nu = \{\nu(i) > 0, i \in Z\}$ be σ -finite positive fixed *a priori* measure, which in the following we will always assume to be the counting measure.

Gibbs measures are built within the DLR framework by describing conditional probabilities w.r.t. the outside of finite sets, where a boundary condition is frozen. One introduces a so-called Gibbsian specification γ so that any Gibbs measure $\mu \in \mathcal{G}(\gamma)$ specified by γ verifies

$$\mu(A|\mathcal{F}_{\Lambda^c}) = \gamma_{\Lambda}(A|\cdot) \mu - \text{a.s} \quad (3.1)$$

for all $\Lambda \in \mathcal{S}$ and $A \in \mathcal{F}$. The Gibbsian specification associated to a potential Φ is given at any inverse temperature $\beta > 0$, for any boundary condition $\omega \in \Omega$ as

$$\gamma_{\Lambda}(A|\omega) = \frac{1}{Z_{\Lambda}^{\beta, \Phi}} \int e^{-\beta H_{\Lambda}^{\Phi}(\sigma_{\Lambda} \omega_{\Lambda^c})} \mathbf{1}_A(\sigma_{\Lambda} \omega_{\Lambda^c}) \nu^{\otimes \Lambda}(\mathrm{d}\sigma_{\Lambda}), \quad (3.2)$$

where the partition function $Z_{\Lambda}^{\beta, \Phi}$ -that has to be non-null and convergent in this countable infinite state-space context (this means that Φ is ν -admissible in the terminology of [8])-is the standard normalization whose logarithm is often related to pressure or free energy.

In our SOS-model on the Cayley tree, Φ is the unbounded nearest-neighbour potential with $\Phi_{\{x,y\}}(\omega_x - \omega_y) = |\omega_x - \omega_y|$ and $\Phi_x \equiv 0$, so γ is a *Markov specification* in the sense that

$$\gamma_{\Lambda}(\omega_{\Lambda} = \zeta|\cdot) \text{ is } \mathcal{F}_{\partial\Lambda} - \text{measurable for all } \Lambda \subset V \text{ and } \zeta \in Z^{\Lambda}. \quad (3.3)$$

In order to build up gradient specifications from the Gibbsian specifications defined in [6], we need to consider the following: due to the absence of loops in trees, for any finite subgraph Λ/Z , the complement Λ^c is not connected, but consists of at least two connected components where each of these contains at least one element of $\partial\Lambda$. This means that the gradient field outside Λ does not contain any information on the relative height of the boundary $\partial\Lambda$ (which is to be understood as an element of $Z^{\partial\Lambda} \setminus Z$). More precisely, let $\text{cc}(\Lambda^c)$ denote the number of connected components in Λ^c and note that $2 \leq \text{cc}(\Lambda^c) \leq |\partial\Lambda|$.

Applying the general definition of gradient Gibbs measure (see [6]) we have

$$Z^{\Lambda^c}/Z = Z^{\{b \in \bar{L}\} | b \subset \Lambda^c} \times (Z^{\text{cc}(\Lambda^c)}/Z) \subset Z^{\{b \in \bar{L}\} | b \subset \Lambda^c} \times (Z^{\partial\Lambda}/Z) \quad (3.4)$$

where ‘=’ is in the sense of isomorphy between measurable spaces. For any $\eta \in \Omega^{\nabla} = Z^V/Z$, let $[\eta]_{\partial\Lambda}/Z$ denote the image of η under the coordinate projection $Z^V/Z \rightarrow Z^{\partial\Lambda}/Z$ with the latter set endowed with the final σ -algebra generated by the coset projection. Set

$$\mathcal{F}_{\Lambda}^{\nabla} := \sigma((\eta_b)_{b \subset \Lambda^c}) \subset \mathcal{T}_{\Lambda}^{\nabla} := \sigma((\eta_b)_{b \subset \Lambda^c}, [\eta]_{\partial\Lambda}). \quad (3.5)$$

Then $\mathcal{T}_{\Lambda}^{\nabla}$ contains all information on the gradient spin variables outside Λ and also information on the relative height of the boundary $\partial\Lambda$. By (3.4) we have that for any

event $A \in \mathcal{F}^\nabla$ the $\mathcal{F}_{\Lambda^c}$ -measurable function $\gamma_\Lambda(A|\cdot)$ is also measurable with respect to $\mathcal{T}_\Lambda^\nabla$, but in general not with respect to $\mathcal{F}_\Lambda^\nabla$. These observations lead to the following:

Definition 3.1. The gradient Gibbs specification is defined as the family of probability kernels $(\gamma'_\Lambda)_{\Lambda \subset \subset V}$ from $(\Omega^\nabla, \mathcal{T}_\Lambda^\nabla)$ to $(\Omega^\nabla, \mathcal{F}^\nabla)$ such that

$$\int F(\rho) \gamma'_\Lambda(d\rho|\zeta) = \int F(\nabla\varphi) \gamma_\Lambda(d\varphi|\omega) \quad (3.6)$$

for all bounded $\mathcal{F}^\nabla$ -measurable functions F , where $\omega \in \Omega$ is any height-configuration with $\nabla\omega = \zeta$.

Using the sigma-algebra $\mathcal{T}_\Lambda^\nabla$, this is now a proper and consistent family of probability kernels, i.e.

$$\gamma'_\Lambda(A|\zeta) = 1_A(\zeta) \quad (3.7)$$

for every $A \in \mathcal{T}_\Lambda^\nabla$ and $\gamma'_\Delta \gamma'_\Lambda = \gamma'_\Delta$ for any finite volumes $\Lambda, \Delta \subset V$ with $\Lambda \subset \Delta$. The proof is similar to the situation of regular (local) Gibbs specifications ([8], proposition 2.5).

Let $\mathcal{C}_b(\Omega^\nabla)$ be the set of bounded functions on Ω^∇ . Gradient Gibbs measures will now be defined in the usual way by having their conditional probabilities outside finite regions prescribed by the gradient Gibbs specification:

Definition 3.2. A measure $\nu \in \mathcal{M}_1(\Omega^\nabla)$ is called a gradient Gibbs measure (GGM) if it satisfies the DLR equation

$$\int \nu(d\zeta) F(\zeta) = \int \nu(d\zeta) \int \gamma'_\Lambda(d\tilde{\zeta}|\zeta) F(\tilde{\zeta}) \quad (3.8)$$

for every finite $\Lambda \subset V$ and for all $F \in \mathcal{C}_b(\Omega^\nabla)$. The set of gradient Gibbs measures will be denoted by $\mathcal{G}^\nabla(\gamma)$.

3.2. Translation-invariant gradient Gibbs measures

In this subsection we construct gradient Gibbs measures for period 4 height-periodic boundary laws on the Cayley tree of order $k \geq 2$.

Proposition 4. ([6]). Probability distributions $\mu^{(n)}(\sigma_n)$, $n = 1, 2, \dots$, in (3.2) are compatible iff for any $x \in V \setminus \{x^0\}$ the following equation holds:

$$\mathbf{h}_x^* = \sum_{y \in S(x)} F(\mathbf{h}_y^*, \theta). \quad (4.1)$$

Here, $\theta = \exp(J\beta)$, $\mathbf{h}_x^* = (h_{i,x} - h_{0,x} + \ln \frac{\nu(i)}{\nu(0)}, \quad i \in Z_0)$ and the function $F(\cdot, \theta) : R^\infty \rightarrow R^\infty$ is $F(\mathbf{h}, \theta) = (F_i(\mathbf{h}, \theta), \quad i \in Z_0)$, with

$$F_i(\mathbf{h}, \theta) = \ln \frac{\nu(i)}{\nu(0)} + \ln \frac{\theta^{|i|} + \sum_{j \in Z_0} \theta^{|i-j|} \exp(h_j)}{1 + \sum_{j \in Z_0} \theta^{|j|} \exp(h_j)},$$

$$\mathbf{h} = (h_i, i \in Z_0).$$

Assume $\mathbf{h}_x = \mathbf{h} = (h_i, i \in Z_0)$ for any $x \in V$. In this case we obtain from (4.1):

$$z_i = \frac{\nu(i)}{\nu(0)} \left(\frac{\theta^{|i|} + \sum_{j \in Z_0} \theta^{|i-j|} z_j}{1 + \sum_{j \in Z_0} \theta^{|j|} z_j} \right)^k, \quad (4.2)$$

where $z_i = \exp(h_i)$, $i \in Z_0$.

Let $\mathbf{z}(\theta) = (z_i = z_i(\theta), i \in Z_0)$ be a solution to (4.2). Denote

$$l_i \equiv l_i(\theta) = \sum_{j=-\infty}^{-1} \theta^{|i-j|} z_j, \quad r_i \equiv r_i(\theta) = \sum_{j=1}^{\infty} \theta^{|i-j|} z_j, \quad i \in Z_0.$$

It is clear that each l_i and r_i can be a finite positive number or $+\infty$. We shall consider all possible cases.

Clearly, a solution $\mathbf{z} = (z_i, i \in Z_0)$ to (4.2) defines a tree-indexed Markov chain iff $r_0 + l_0 < +\infty$ (see [6]).

Let $\nu(i) = 1$ for any $i \in Z$ then we consider the solutions of (4.2) with $l_0 < +\infty$ and $r_0 < +\infty$.

Put $u_i = u_0 \sqrt[k]{z_i}$ for some $u_0 > 0$. Then (4.2) can be written as

$$u_i = C \left(\sum_{j=1}^{+\infty} \theta^j u_{i-j}^k + u_i^k + \sum_{j=1}^{+\infty} \theta^j u_{i+j}^k \right), \quad i \in Z. \quad (4.3)$$

Proposition 5. ([6]). *A vector $\mathbf{u} = (u_i, i \in Z)$, with $u_0 = 1$, is a solution to (4.3) if and only if for $u_i = \sqrt[k]{z_i}$ the following holds*

$$u_i^k = \frac{u_{i-1} + u_{i+1} - \tau u_i}{u_{-1} + u_1 - \tau}, \quad i \in Z, \quad (5.1)$$

where $\tau = \theta^{-1} + \theta$.

By this lemma we have

$$1 + l_0 + r_0 = \frac{\theta - \theta^{-1}}{u_{-1} + u_1 - \tau}. \quad (5.2)$$

Equations of system (4.2) for $i = -1$ and $i = 1$ are satisfied independently on values of u_{-1} and u_1 and the equation (5.1) can be separated to the following independent recurrent equations

$$u_{-i-1} = (u_{-1} + u_1 - \tau) u_{-i}^k + \tau u_{-i} - u_{-i+1}, \quad (5.3)$$

$$u_{i+1} = (u_{-1} + u_1 - \tau) u_i^k + \tau u_i - u_{i-1}, \quad (5.4)$$

where $i \geq 1$, $u_0 = 1$ and u_{-1} , u_1 are some initial numbers (see again [6]).

So, if u_i is a solution to (5.4) then u_{-i} will be a solution for (5.3). Hence we can consider only equation (5.4).

Let us consider the periodic solutions of (5.1) i.e., we describe solutions of (5.1) which have the form

$$u_n = \begin{cases} 1, & \text{if } n = 2m, \\ a, & \text{if } n = 4m - 1, \\ b, & \text{if } n = 4m + 1, \end{cases} \quad m \in Z \quad (5.5)$$

where a and b some positive numbers. In this case (5.4) is equivalent to the following system of equations

$$\begin{aligned} (a + b - \tau)b^k + \tau b - 2 &= 0 \\ (a + b - \tau)a^k + \tau a - 2 &= 0. \end{aligned} \quad (5.6)$$

We describe positive solutions of (5.6)

Case $a \neq b$. We multiply the first equation of (5.6) by a^k and the second equation of (5.6) by b^k . And after that, subtract the first equation from the second and we obtain the following equation:

$$\tau ab(a^{k-1} - b^{k-1}) - 2(a^k - b^k) = 0$$

Dividing both sides by $a - b$ we get

$$(a^{k-1} + a^{k-2}b + \dots + a^2b^{k-3} + ab^{k-2})(\tau b - 2) - 2b^{k-1} = 0.$$

Put $x := \frac{a}{b}$, then the last equation can be written as

$$(x^{k-1} + x^{k-2} + \dots + x^2 + x)(\tau b - 2) - 2 = 0.$$

If $\tau b - 2 \leq 0$ then $(x^{k-1} + x^{k-2} + \dots + x^2 + x)(\tau b - 2) - 2 < 0$, i.e., there is not any solution (a, b) of (5.6) such that $a \neq b$.

Let $\tau b - 2 > 0$ then for any positive fixed b we consider the following polynomial $P_b(x) := (x^{k-1} + x^{k-2} + \dots + x^2 + x)(\tau b - 2) - 2$. For $x > 0$ it is easy to check that $P'_b(x) > 0$ and $P_b(0) < 0$, $\lim_{x \rightarrow \infty} P_b(x) > 0$. Thus, $P_b(x)$ has exactly one positive solution.

If $\tau b = 2 + \frac{2}{k-1}$ then there is not any solution (a, b) to (5.6) such that $a \neq b$. In other cases, from $P_b(1) \neq 0$ for any positive b there exists a unique $a(b) \neq b$ such that (a, b) is solution to (5.6).

Case $a = b$. In this case it is sufficient to consider one of equations of (5.6), i.e.,

$$2a^{k+1} - \tau a^k + \tau a - 2 = 0.$$

Last equation has the solution $a = 1$ independently on the parameters (τ, k) . Dividing both sides by $a - 1$ we get

$$Q(a) := 2a^k + (2 - \tau)(a^{k-1} + a^{k-2} + \dots + a) + 2 = 0 \quad (5.7)$$

By definition of τ we get $\tau \geq 2$ i.e., From $2 - \tau < 0$ and Descartes' rule of signs, $Q(a)$ has at most two positive roots. Since $Q'(a) = 2ka^{k-1} + (2 - \tau)((k-1)a^{k-2} + \dots + 2a + 1)$ and $Q'(0) < 0$, $Q'(\infty) > 0$ there is a unique a_c such that $Q'(a_c) = 0$. Consequently, if $\tau_c := Q(a_c) < 0$ then the polynomial $Q(a)$ has exactly two positive solutions. Let

$Q(a_c) = 0$ then $Q(a)$ has exactly one positive solution. Finally, if $Q(a_c) > 0$ then the polynomial $Q(a)$ has not any positive solution.

Theorem 5.1. (theorem 4.1, remark 4.2 in [4]). Let l be any spatially homogenous period- q height-periodic boundary law to a tree-automorphism invariant gradient interaction potential on the Cayley tree. Let $\Lambda \subset V$ be any finite connected set and let $\omega \in \Lambda$ be any vertex. Then the measure ν with marginals given by

$$\nu(\eta_{\Lambda \cup \partial \Lambda} = \zeta_{\Lambda \cup \partial \Lambda}) = Z_{\Lambda} \left(\sum_{s \in Z_q} \prod_{y \in \partial \Lambda} l \left(s + \sum_{b \in \Gamma(\omega, y)} \zeta_b \right) \right) \prod_{b \cap \Lambda \neq \emptyset} Q(\zeta_b), \quad (5.8)$$

From above results and theorem 5.1 we can conclude the following theorems:

Theorem 5.2. Let $k \geq 2$ and $a = b$. For the SOS-model (2.1) on the k -regular tree, with parameter $\tau = 2\cosh(\beta)$ there numbers $\tau_c > 0$ such that the following assertions hold:

- (a) If $\tau < \tau_c$ then there is a unique GGM corresponding to nontrivial period-3 height-periodic boundary laws of the type (5.5) via theorem (5.8).
- (b) At $\tau = \tau_c$ there are exactly two GGMs corresponding to a nontrivial period-3 height-periodic boundary law of the type (5.5) via theorem.
- (c) For $\tau > \tau_c$ there are exactly three such (resp. one) gradient GMs.

Theorem 5.3. Let $k \geq 2$ and $a \neq b$. For the SOS-model (2.1) on the k -regular tree, with parameter $\tau = 2\cosh(\beta)$ the following assertions hold:

- (a) For any positive fixed b , if $\tau \leq \frac{2}{b}$ then there is no any GGM corresponding to nontrivial period-3 height-periodic boundary laws of the type (5.5) via theorem (5.8).
- (b) For any positive fixed b , if $\tau > \frac{2}{b}$ then there is a unique GGM corresponding to nontrivial period-3 height-periodic boundary laws of the type (5.5) via theorem (5.8).

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References

- [1] Bovier A and Külske C 1996 There are no nice interfaces in $(2 + 1)$ -dimensional SOS models in random media *J. Stat. Phys.* **83** 751–9
- [2] Mazel A E and Suhov Y M 1991 Random surfaces with two-sided constraints: an application of the theory of dominant ground states *J. Stat. Phys.* **64** 111–34
- [3] Brandenberger R and Wayne C E 1982 Decay of correlations in surface models *J. Stat. Phys.* **27** 425–40
- [4] Kulske C and Schrieffer P 2017 Gradient Gibbs measures and fuzzy transformations on trees *Markov Process Relat. Field* **23** 553–90
- [5] Gandolfo D, Haydarov F H, Rozikov U A and Ruiz J 2013 New phase transitions of the Ising model on Cayley trees *J. Stat. Phys.* **153** 400–11
- [6] Henning F, Külske C and Le Ny A Rozikov U 2019 Gradient Gibbs measures for the SOS model with countable values on a Cayley tree *Electron. J. Probab.* **24** 1–23

- [7] Botirov G I and Rozikov U A 2006 On q -component models on the Cayley tree: the general case *J. Stat. Mech.* **P10006**
- [8] Georgii H O 2011 *Gibbs Measures and Phase Transitions* (*de Gruyter Studies in Mathematics* vol 9) 2nd edn (Berlin: de Gruyter & Co)
- [9] Bissacot R, Endo E O and van Enter A C D 2017 Stability of the phase transition of critical-field Ising model on Cayley trees under inhomogeneous external fields *Stoch. Process. Appl.* **127** 4126–38
- [10] Rozikov U A and Suhov Y M 2006 Gibbs measures for SOS models on a Cayley tree *Infinite Dimens. Anal., Quantum Probab. Relat. Top.* **9** 471–88
- [11] Rozikov U A 2013 *Gibbs Measures on Cayley Trees* (Singapore: World Scientific)
- [12] Sheffield S 2006 Random surfaces *Am. Math. Soc.* MR-2251117 2005 Astérisque 304, SM F