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Nonlinear Inverse Problem for a Sixth Order Differential Equation with Two Redefinition Functions

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Abstract—In this paper, we consider an inhomogeneous sixth-order partial differential equation with two redefinition functions at the end point of the given segment. These redefinition functions enter nonlinearly into partial differential equation. The Fourier method of separation of variables is applied. Absolutely and uniformly convergence of Fourier series are proved. The Cauchy–Schwartz inequalities and the Bessel inequality are used. Theorems on the unique solvability of inverse problems are proved. The method of successive approximations is used in combination with the method of contraction mappings.

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1. INTRODUCTION

In recent years the interest to study the differential equations with local and nonlocal boundary conditions is increasing (see, for example, [1–16]. In [17] a physical situation, in which a non-metallic conductor is in contact with a perfect conductor, is studied. In [18], the problems of mathematical models in reaction-diffusion systems are considered. In [19], the nonlocal conditions are used in the theory of phase transitions. Inverse problems for differential equations find many applications in modern science and technology. Therefore, a large number of research works are devoted to the study of various kinds of inverse problems (see, for example [20–29]).

In this paper, we study an inverse boundary value problem for an inhomogeneous sixth-order partial differential equation with two redefinition functions at the end point of the given segment. The questions of the existence and uniqueness of the solution to the inverse boundary value problem are investigated.

2. FORMULATION OF THE PROBLEM STATEMENT

In the rectangular domain $\Omega = \{0 < t < T, \ 0 < x < l\}$ we consider the following partial differential equation

$$U_{tt}(t,x) - a(t)\big(U_{ttxx}(t,x) - U_{ttxxxx}(t,x)\big) + b(t)\big(U_{xx}(t,x) - U_{xxxx}(t,x)\big)$$

$$= f\left(t, x, \int_0^l \varphi_1(y)\,dy, \int_0^l \varphi_2(y)\,dy\right) \tag{1}$$

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with Dirichlet boundary value conditions

$$U(t,0)=U(t,l)=U_{xx}(t,0)=U_{xx}(t,l)=0, \quad 0\le t\le T \tag{2}$$

and conditions at the endpoint of the given segment:

$$U(T,x)=\varphi_1(x), \quad 0\le x\le l, \tag{3}$$

$$U_t(T,x)=\varphi_2(x), \quad 0\le x\le l, \tag{4}$$

where $f(t,x,\,\cdot\,,\,\cdot\,)\in C\,(\Omega\times\mathbb{R}\times\mathbb{R})$, $0<a(t)\in C[0,T]$, $0\neq b(t)\in C[0,T]$, functions $\varphi_1(x)$ and $\varphi_2(x)$ are redefinition functions, T, l are given positive numbers.

Remark 1. *The function on the right-hand side of the equation* (1) *depends from the variable* t *and this function is not zero for* $0<x<l$.

Remark 2. *For the functions* $\varphi_1(x)$ *and* $\varphi_2(x)$ *the following periodical conditions are fulfilled*

$$\varphi_i(0)=\varphi_i(l)=\varphi_i''(0)=\varphi_i''(l)=0, \quad \varphi_i(x)=\varphi_i(x)\neq 0,\ x\in(0,l),\ i=1,2.$$

In order to determine the unknown functions of redefinition we use the following two additional conditions

$$U(t_1,x)=\psi_1(x), \quad 0\le x\le l, \tag{5}$$

$$U(t_2,x)=\psi_2(x), \quad 0\le x\le l, \tag{6}$$

where $\psi_1(x)$ and $\psi_2(x)$ are known enough smooth on the segment $[0,l]$ functions, $0<t_1<t_2<T$.

Remark 3. *For the functions* $\psi_1(x)$ *and* $\psi_2(x)$ *the following periodical conditions are fulfilled*

$$\psi_i(0)=\psi_i(l)=\psi_i''(0)=\psi_i''(l)=0; \psi_i(x)\neq 0,\ x\in(0,l),\ i=1,2.$$

Problem statement. To find three functions

$$\left\{U(t,x)\in C\,(\overline{\Omega})\cap C^{2,4}_{t,x}(\Omega)\cap C^{2+4}_{t,x}(\Omega), \quad \varphi_i(x)\in C[0,1], \quad i=1,2\right\},$$

the first of which satisfies the differential equation (1) and the specified conditions (2)–(6), where $\overline{\Omega}=\{0\le t\le T,\ 0\le x\le l\}$.

3. FORMAL SOLUTION OF THE PROBLEM

Note that the functions $\vartheta_n(x)=\sqrt{\dfrac{2}{l}}\sin\lambda_n x$, where $\lambda_n=\dfrac{n\,\pi}{l}$, $n\in\mathbb{N}$, form a complete system of orthonormal eigenfunctions in the space $L_2[0,l]$. Therefore, we seek nontrivial solutions to the inhomogeneous differential equation (1) in the form of a Fourier series in sine

$$U(t,x)=\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}u_n(t)\,\sin\lambda_n x, \tag{1}$$

$$u_n(t)=\sqrt{\frac{2}{l}}\int_0^l U(t,x)\,\sin\lambda_n x\,dx. \tag{2}$$

We require that the function $f(t,x,\,\cdot,\,\cdot)$ can also be expanded in a Fourier series

$$f(t,x,\,\cdot,\,\cdot)=\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}f_n(t,\,\cdot,\,\cdot)\,\sin\lambda_n x, \tag{3}$$

$$f_n(t,\,\cdot,\,\cdot)=\sqrt{\frac{2}{l}}\int_0^l f(t,x,\,\cdot,\,\cdot)\,\sin\lambda_n x\,dx. \tag{4}$$

Substituting the Fourier series (1) and (3) into the given nonlinear differential equation (1), we obtain a countable system of ordinary differential equations of the second order

$$u''_n(t) = h_n(t)\, u_n(t) + f_n(t, \cdot, \cdot), \quad \text{where} \quad h_n(t) = \frac{\left(\lambda_n^2 + \lambda_n^4\right) b(t)}{1 + (\lambda_n^2 + \lambda_n^4)\; a(t)}. \tag{5}$$

By integrating twice the countable system of differential equations (5), we obtain the countable system of integral equations

$$u_n(t) = A_{1n} + A_{2n}\, t + \int\limits_0^t (t-s)\; [h_n(s) u_n(s) + f_n(s, \cdot, \cdot)]\, ds, \tag{6}$$

where $A_{1\,n}$ and $A_{2\,n}$ are unknown coefficients, which will be determined.

Now, suppose the redefinition functions $\varphi_1(x)$ and $\varphi_2(x)$ expand into a Fourier series. Then, using the Fourier coefficients (2), the integral conditions (3) and (4) are written in the following form

$$u_{\,n}(T) = \sqrt{\frac{2}{l}} \int\limits_0^l U(t,x)\; \sin \lambda_{\,n} x\, dx = \sqrt{\frac{2}{l}} \int\limits_0^l \varphi_1(x)\; \sin \lambda_{\,n} x\, dx = \varphi_{1\,n}, \tag{7}$$

$$u'_n(T) = \sqrt{\frac{2}{l}} \int\limits_0^l U_t(t,x)\; \sin \lambda_{\,n} x\, dx = \sqrt{\frac{2}{l}} \int\limits_0^l \varphi_2(x)\; \sin \lambda_{\,n} x\, dx = \varphi_{2\,n}. \tag{8}$$

To find the unknown coefficients $A_{1\,n}$ and $A_{2\,n}$ in the integral equation (6), we use the boundary conditions (7) and (8). Then from (6) we have

$$A_{1n} = \varphi_{1n} - \varphi_{2n}\, T + \int\limits_0^T s\, [h_n(s) u_n(s) + f_n(s, \cdot, \cdot)]\, ds,$$

$$A_{2n} = \varphi_{2n} - \int\limits_0^T \; [h_n(s) u_n(s) + f_n(s, \cdot, \cdot)]\, ds.$$

Substituting these values of $A_{1\,n}$ and $A_{2\,n}$ into representation (6), we obtain a countable system of Volterra integral equations

$$u_n(t) = \varphi_{1\,n} + \varphi_{2n}(t-T) + \int\limits_t^T (s-t)\, [h_n(s) u_n(s) + f_n(s, \cdot, \cdot)]\, ds. \tag{9}$$

Substituting representation (9) in the Fourier series (1), we obtain

$$U(t,x) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left[\varphi_{1\,n} + \varphi_{2\,n}(t-T) + \int\limits_t^T (s-t)\, [h_n(s) u_n(s) + f_n(s, \cdot, \cdot)]\, ds \right] \sin \lambda_n x. \tag{10}$$

We will now formally define the redefinition functions $\varphi_1(x)$ and $\varphi_2(x)$. For this purpose, we subordinate function (10) to conditions (3) and (4):

$$\psi_1(x) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left[\varphi_{1n} + \varphi_{2n}(t_1 - T) + \int\limits_{t_1}^T (s-t_1)\, [h_n(s) u_n(s) + f_n(s, \cdot, \cdot)]\, ds \right] \sin \lambda_{\,n} x,$$

$$\psi_2(x) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left[\varphi_{1n} + \varphi_{2n}(t_2 - T) + \int\limits_{t_2}^T (s-t_2)\, [h_n(s) u_n(s) + f_n(s, \cdot, \cdot)]\, ds \right] \sin \lambda_{\,n} x.$$

Expanding the functions $\psi_1(x)$ and $\psi_2(x)$ in a Fourier series, we obtain a system of countable systems of functional equations

$$\psi_{i\,n} = \varphi_{1\,n} + \varphi_{2\,n}(t_i - T) + \int_{t_i}^{T} (s - t_i)\left[h_n(s)u_n(s) + f_n(s,\,\cdot,\,\cdot)\right] ds, \quad i = 1, 2, \tag{11}$$

where

$$\psi_{i\,n} = \sqrt{\frac{2}{l}} \int_0^l \psi_i(x)\, \sin\lambda_n x\, dx, \quad i = 1, 2. \tag{12}$$

Solving system (11), we find the Fourier coefficients for the redefinition functions

$$\varphi_{1\,n} = I_1\,\left(t; u_n, \varphi_{1\,n}, \varphi_{2n}\right) \equiv \psi_{1n}\frac{t_2 - T}{t_2 - t_1} + \psi_{2n}\frac{T - t_1}{t_2 - t_1}$$

$$+ \int_{t_1}^{T} K_1(s)\left[h_n(s)u_n(s) + f_n\left(s, \int_0^l \varphi_1(y)\,dy, \int_0^l \varphi_2(y)\,dy\right)\right] ds, \tag{13}$$

$$\varphi_{2\,n} = I_2\,\left(t; u_n, \varphi_{1n}, \varphi_{2n}\right) \equiv -\psi_{1n}\frac{1}{t_2 - t_1} + \psi_{2n}\frac{1}{t_2 - t_1}$$

$$+ \int_{t_1}^{T} K_2(s)\left[h_n(s)u_n(s) + f_n\left(s, \int_0^l \varphi_1(y)\,dy, \int_0^l \varphi_2(y)\,dy\right)\right] ds, \tag{14}$$

where

$$K_1(s) = \begin{cases} \dfrac{t_1 - s}{t_2 - t_1} - s + t_1, & t_1 \le s < t_2, \\ t_1 - s - 1, & t_2 \le s \le T, \end{cases} \qquad K_2(s) = \begin{cases} \dfrac{s - t_1}{t_2 - t_1}, & t_1 \le s < t_2, \\ 1, & t_2 \le s \le T. \end{cases}$$

Substituting representations (13) and (14) in the Fourier series, we have the formal series

$$\varphi_1(x) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \sin\lambda_n x \Bigg[\psi_{1\,n}\frac{t_2 - T}{t_2 - t_1} + \psi_{2n}\frac{T - t_1}{t_2 - t_1}$$

$$+ \int_{t_1}^{T} K_1(s)\,\left[h_n(s)u_n(s) + f_n\left(s, \int_0^l \varphi_1(y)\,dy, \int_0^l \varphi_2(y)\,dy\right)\right] ds\Bigg], \tag{15}$$

$$\varphi_2(x) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \sin\lambda_n x \Bigg[-\psi_{1n}\frac{1}{t_2 - t_1} + \psi_{2n}\frac{1}{t_2 - t_1}$$

$$+ \int_{t_1}^{T} K_2(s)\,\left[h_n(s)u_n(s) + f_n\left(s, \int_0^l \varphi_1(y)\,dy, \int_0^l \varphi_2(y)\,dy\right)\right] ds\Bigg]. \tag{16}$$

Substituting representations (13) and (14) into formulas (9) and (10), we obtain the following countable system of integral equations

$$u_n(t) = I_3\,\left(t; u_n, \varphi_{1n}, \varphi_{2n}\right) \equiv \psi_{1n}\frac{t_2 - t}{t_2 - t_1} + \psi_{2n}\frac{t - t_1}{t_2 - t_1}$$

$$+\int\limits_t^T K_3(s)\left[h_n(s)u_n(s)+f_n\left(s,\int\limits_0^l \varphi_1(y)\,dy,\int\limits_0^l \varphi_2(y)\,dy\right)\right]ds \tag{17}$$

and the next Fourier series for the main unknown function

$$U(t,x)=\sqrt{\frac{2}{l}}\sum_{n=1}^{\infty}\sin\lambda_n x\Bigg[\psi_{1n}\frac{t_2-t}{t_2-t_1}+\psi_{2n}\frac{t-t_1}{t_2-t_1}$$

$$+\int\limits_t^T K_3(s)\left[h_n(s)u_n(s)+f_n\left(s,\int\limits_0^l \varphi_1(y)\,dy,\int\limits_0^l \varphi_2(y)\,dy\right)\right]ds\Bigg], \tag{18}$$

where

$$K_3(s)=\begin{cases} s-t_1, & 0\le s\le t_1,\\ K_1(s)+(t_1-T)K_2(s)+s-t_1, & t_1<s\le T.\end{cases}$$

4. SOLVABLE OF THE SYSTEM OF COUNTABLE SYSTEM OF FUNCTIONAL AND INTEGRAL EQUATIONS

First, we present the following well-known Banach spaces, which we will use in our further actions.

The space $B_2(T)$ of function sequences $\{\,u_n(t)\,\}_{n=1}^{\infty}$ on the segment $[0;\,T\,]$ with the norm

$$\|\,u\,(t)\,\|_{\,B_2(T)}=\left\{\sum_{n=1}^{\infty}\left(\max_{t\in\,[0;\,T]}\,|\,u_n(t)\,|\right)^2\right\}^{1/2}<\infty.$$

The Hilbert coordinate space ℓ_2 of number sequences $\{\varphi_n\}_{n=1}^{\infty}$ with the norm

$$\|\,\varphi\,\|_{\,\ell_2}=\left\{\sum_{n=1}^{\infty}|\,\varphi_n\,|^2\right\}^{1/2}<\infty.$$

The space $L_2\,[0,l]$ of square-integrable functions on an interval $[0,l]$ with norm

$$\|\,\vartheta\,(x)\,\|_{\,L_2[0,l]}=\left\{\int\limits_0^l|\,\vartheta\,(\eta)\,|^{\,2}\,d\eta\right\}^{1/2}<\infty.$$

Smoothness conditions. Let for the functions $\psi_i(x)\in C^{\,4}[0,l]\,,\quad i=1,2,\quad f\,(t,x,\,\cdot,\,\cdot)\in C^{\,0,\,4}\,(\Omega\times\mathbb{R}\times\mathbb{R})$ on the segments $[0,l]$ exist peace-wise continuous derivatives up fifth order on x. Then, after integration the functions (12) and (4) by part fifth time on the variable E, we obtain the following relations

$$|\,\psi_{i\,n}\,|=\left(\frac{l}{\pi}\right)^5\frac{|\,\psi_{i\,n}^{V}\,|}{n^{\,5}},\quad i=1,2, \tag{1}$$

$$|\,f_n(t,\,\cdot,\,\cdot)\,|=\left(\frac{l}{\pi}\right)^5\frac{|\,f_n^{\,V}(t,\,\cdot,\,\cdot)\,|}{n^{\,5}}, \tag{2}$$

where

$$\psi_{i\,n}^{V}=\int\limits_0^l\frac{\partial^{\,5}\,\psi_i\,(x)}{\partial\,x^{\,5}}\,\sin\lambda_n x\,d\,x,\quad i=1,2,\quad f_n^{\,V}(t,\,\cdot,\,\cdot)=\int\limits_0^l\frac{\partial^{\,5}\,f(t,x,\,\cdot,\,\cdot)}{\partial\,x^{\,5}}\sin\lambda_n x\,d\,x.$$

Here the Bessel inequalities are valid

$$\sum_{n=1}^{\infty}\left[\psi_{in}^{V}\right]^{2}\leq\left(\frac{2}{l}\right)^{5}\int\limits_{0}^{l}\left[\frac{\partial^{5}\psi_{i}(x)}{\partial x^{5}}\right]^{2}dx,\quad i=1,2, \tag{3}$$

$$\sum_{n=1}^{\infty}\left[f_{n}^{V}(t,\cdot,\cdot)\right]^{2}\leq\left(\frac{2}{l}\right)^{5}\int\limits_{0}^{l}\left[\frac{\partial^{5}f(t,x,\cdot,\cdot)}{\partial x^{5}}\right]^{2}dx. \tag{4}$$

Theorem 1. *Let the smoothness conditions and the following conditions be fulfilled:*

1) $|f(t,x,\varphi_{11},\varphi_{21})-f(t,x,\varphi_{12},\varphi_{22})|\leq M_0(t,x)\left[|\varphi_{11}-\varphi_{12}|+|\varphi_{21}-\varphi_{22}|\right]$,

2) $\rho=\max\left\{\alpha_2+\alpha_4+\alpha_6;\ \sqrt{l}\,(\alpha_3+\alpha_5+\alpha_7)\,\|M_0(t,x)\|_{L_2[0,l]}\right\}<1$, *where*

$$\alpha_2=\left[\sum_{n=1}^{\infty}\max_{t\in[0,T]}\left(\int\limits_{t}^{T}|K_3(s)\,h_n(s)|\,ds\right)^{2}\right]^{\frac{1}{2}},\quad \alpha_4=\left[\sum_{n=1}^{\infty}\max_{t\in[t_1,T]}\left(\int\limits_{t}^{T}|K_1(s)\,h_n(s)|\,ds\right)^{2}\right]^{\frac{1}{2}},$$

$$\alpha_6=\left[\sum_{n=1}^{\infty}\max_{t\in[t_1,T]}\left(\int\limits_{t}^{T}|K_2(s)\,h_n(s)|\,ds\right)^{2}\right]^{\frac{1}{2}},\quad \alpha_3=\max_{t\in[0,T]}\int\limits_{t}^{T}|K_3(s)|\,ds,$$

$$\alpha_5=\max_{t\in[t_1,T]}\int\limits_{t}^{T}|K_1(s)|\,ds,\quad \alpha_7=\max_{t\in[t_1,T]}\int\limits_{t}^{T}|K_2(s)|\,ds.$$

Then the system of countable systems of equations (17), (13), (14) *is uniquely solvable in the spaces* $B_2(T)$, ℓ_2, *respectively. In this case, the desired solution can be found from the following iterative process:*

$$\begin{cases} u_n^0(t)=\psi_{1n}\dfrac{t_2-t}{t_2-t_1}+\psi_{2n}\dfrac{t-t_1}{t_2-t_1},\quad u_n^{m+1}(t)=I_3\left(t;u_n^m,\varphi_{1n}^m,\varphi_{2n}^m\right),\\ \varphi_{1n}^0=\psi_{1n}\dfrac{t_2-T}{t_2-t_1}+\psi_{2n}\dfrac{T-t_1}{t_2-t_1},\quad \varphi_{1n}^{m+1}=I_1\left(t;u_n^m,\varphi_{1n}^m,\varphi_{2n}^m\right),\\ \varphi_{2n}^0=-\psi_{1n}\dfrac{1}{t_2-t_1}+\psi_{2n}\dfrac{1}{t_2-t_1},\quad \varphi_{2n}^{m+1}=I_2\left(t;u_n^m,\varphi_{1n}^m,\varphi_{2n}^m\right), \end{cases} \tag{5}$$

$m=0,1,2,\ldots$

Proof. We use the method of contraction maps in combination with the method of successive approximations in spaces $B_2(T)$, ℓ_2. We apply the Cauchy–Schwartz inequality (1) and (2) and then the Bessel inequalities (3) and (4). Then we obtain from (5) that the following estimates are valid:

$$\sum_{n=1}^{\infty}\max_{t\in[0,T]}|u_n^0(t)|\leq\sum_{n=1}^{\infty}\max_{t\in[0,T]}\left[|\psi_{1n}|\left|\frac{t_2-t}{t_2-t_1}\right|+|\psi_{2n}|\left|\frac{t-t_1}{t_2-t_1}\right|\right]$$

$$\leq\alpha_1\left(\frac{l}{\pi}\right)^{5}\left[\sum_{n=1}^{\infty}\frac{|\psi_{1n}^{V}|}{n^{5}}+\sum_{n=1}^{\infty}\frac{|\psi_{2n}^{V}|}{n^{5}}\right]\leq\alpha_1\left(\frac{l}{\pi}\right)^{5}\left(\sqrt{\frac{2}{l}}\right)^{5}\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^{10}}}$$

$$\times\left[\left\|\frac{\partial^{5}\psi_1(x)}{\partial x^{5}}\right\|_{L_2[0,l]}+\left\|\frac{\partial^{5}\psi_2(x)}{\partial x^{5}}\right\|_{L_2[0,l]}\right]<\infty, \tag{6}$$

where $\alpha_1 = \max\limits_{t\in[0,T]} \left\{ \left| \dfrac{t_2 - t}{t_2 - t_1} \right|; \left| \dfrac{t - t_1}{t_2 - t_1} \right| \right\}$;

$$\sum_{n=1}^{\infty} |\varphi_{1n}^0| \le \sum_{n=1}^{\infty} \left[|\psi_{1n}| \left| \frac{t_2 - T}{t_2 - t_1} \right| + |\psi_{2n}| \left| \frac{T - t_1}{t_2 - t_1} \right| \right] \le \beta_1 \left(\frac{l}{\pi}\right)^5 \left[\sum_{n=1}^{\infty} \frac{|\psi_{1n}^V|}{n^5} + \sum_{n=1}^{\infty} \frac{|\psi_{2n}^V|}{n^5} \right]$$

$$\le \beta_1 \left(\frac{l}{\pi}\right)^5 \left(\sqrt{\frac{2}{l}}\right)^5 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{10}}} \left[\left\| \frac{\partial^5 \psi_1(x)}{\partial x^5} \right\|_{L_2[0,l]} + \left\| \frac{\partial^5 \psi_2(x)}{\partial x^5} \right\|_{L_2[0,l]} \right] < \infty, \tag{7}$$

where $\beta_1 = \max \left\{ \left| \dfrac{t_2 - T}{t_2 - t_1} \right|; \left| \dfrac{T - t_1}{t_2 - t_1} \right| \right\}$;

$$\sum_{n=1}^{\infty} |\varphi_{2n}^0| \le \sum_{n=1}^{\infty} \left[|\psi_{1n}| \left| -\frac{1}{t_2 - t_1} \right| + |\psi_{2n}| \left| \frac{1}{t_2 - t_1} \right| \right] \le \gamma_1 \left(\frac{l}{\pi}\right)^5 \left[\sum_{n=1}^{\infty} \frac{|\psi_{1n}^V|}{n^5} + \sum_{n=1}^{\infty} \frac{|\psi_{2n}^V|}{n^5} \right]$$

$$\le \gamma_1 \left(\frac{l}{\pi}\right)^5 \left(\sqrt{\frac{2}{l}}\right)^5 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{10}}} \left[\left\| \frac{\partial^5 \psi_1(x)}{\partial x^5} \right\|_{L_2[0,l]} + \left\| \frac{\partial^5 \psi_2(x)}{\partial x^5} \right\|_{L_2[0,l]} \right] < \infty, \tag{8}$$

where $\gamma_1 = \left| \dfrac{1}{t_2 - t_1} \right|$. Taking into account estimates (6)–(8), applying the Cauchy–Schwartz inequality (1) and (2) and then the Bessel inequalities (3) and (4), for an arbitrary difference of approximation (5) we obtain

$$\sum_{n=1}^{\infty} \max_{t\in[0,T]} |u_n^{m+1}(t) - u_n^m(t)| \le \sum_{n=1}^{\infty} \max_{t\in[0,T]} \int_t^T |K_3(s)\, h_n(s)|\, |u_n^m(s) - u_n^{m-1}(s)|\, ds$$

$$+\alpha_3 \sum_{n=1}^{\infty} \left| \int_0^l M_0(t,x) \sin\lambda_n x\, dx \int_0^l \sum_{i=1}^{\infty} \left[|\varphi_{1i}^m - \varphi_{1i}^{m-1}| + |\varphi_{2i}^m - \varphi_{2i}^{m-1}| \right] \sin\lambda_i y\, dy \right|$$

$$\le \alpha_2 \| u^m(t) - u^{m-1}(t) \|_{B_2(T)}$$

$$+\alpha_3 \| M_0(t,x) \|_{L_2[0,l]} \left| \int_0^l \sum_{n=1}^{\infty} \left[|\varphi_{1n}^m - \varphi_{1n}^{m-1}| + |\varphi_{2n}^m - \varphi_{2n}^{m-1}| \right] \sin\lambda_n y\, dy \right|$$

$$\le \alpha_2 \| u^m(t) - u^{m-1}(t) \|_{B_2(T)} + \sqrt{l}\, \alpha_3 \| M_0(t,x) \|_{L_2[0,l]} \left[\| \varphi_1^m - \varphi_1^{m-1} \|_{\ell_2} + \| \varphi_2^m - \varphi_2^{m-1} \|_{\ell_2} \right], \tag{9}$$

where $\alpha_2 = \left[\sum\limits_{n=1}^{\infty} \max\limits_{t\in[0,T]} \left(\int\limits_t^T |K_3(s)\, h_n(s)|\, ds \right)^2 \right]^{\frac{1}{2}}$, $\alpha_3 = \max\limits_{t\in[0,T]} \int\limits_t^T |K_3(s)|\, ds$. Similarly, we find that the following estimates are also valid for the Fourier coefficients of the redefinition functions

$$\sum_{n=1}^{\infty} |\varphi_{1n}^{m+1} - \varphi_{1n}^m| \le \sum_{n=1}^{\infty} \max_{t\in[t_1,T]} \int_t^T |K_1(s)\, h_n(s)|\, |u_n^m(s) - u_n^{m-1}(s)|\, ds$$

$$+\alpha_5 \sum_{n=1}^{\infty} \left| \int_0^l M_0(t,x) \sin\lambda_n x\, dx \int_0^l \sum_{i=1}^{\infty} \left[|\varphi_{1i}^m - \varphi_{1i}^{m-1}| + |\varphi_{2i}^m - \varphi_{2i}^{m-1}| \right] \sin\lambda_i y\, dy \right|$$

$$\leq \alpha_4 \left\| u^m(t) - u^{m-1}(t) \right\|_{B_2(T)} + \sqrt{l}\,\alpha_5 \, \| M_0(t,x) \|_{L_2[0,l]} \left[\left\| \varphi_1^m - \varphi_1^{m-1} \right\|_{\ell_2} + \left\| \varphi_2^m - \varphi_2^{m-1} \right\|_{\ell_2} \right], \tag{10}$$

where $\alpha_4 = \left[\sum\limits_{n=1}^{\infty} \max\limits_{t \in [t_1,T]} \left(\int\limits_t^T | K_1(s)\, h_n(s) | \, ds \right)^2 \right]^{\frac{1}{2}}$, $\alpha_5 = \max\limits_{t \in [t_1,T]} \int\limits_t^T |K_1(s)| \, ds$;

$$\sum_{n=1}^{\infty} \left| \varphi_{2n}^{m+1} - \varphi_{2n}^{m} \right| \leq \alpha_6 \left\| u^m(t) - u^{m-1}(t) \right\|_{B_2(T)}$$

$$+\sqrt{l}\,\alpha_7 \, \| M_0(t,x) \|_{L_2[0,l]} \left[\left\| \varphi_1^m - \varphi_1^{m-1} \right\|_{\ell_2} + \left\| \varphi_2^m - \varphi_2^{m-1} \right\|_{\ell_2} \right], \tag{11}$$

where $\alpha_6 = \left[\sum\limits_{n=1}^{\infty} \max\limits_{t \in [t_1,T]} \left(\int\limits_t^T | K_2(s)\, h_n(s) | \, ds \right)^2 \right]^{\frac{1}{2}}$, $\alpha_7 = \max\limits_{t \in [t_1,T]} \int\limits_t^T |K_2(s)| \, ds$. From the estimates (9)–(11), we derive that

$$V^m \leq \rho V^{m-1}, \ \ m = 1, 2, 3, \ldots, \tag{12}$$

where

$$V^m = \left\| u^{m+1}(t) - u^m(t) \right\|_{B_2(T)} + \left\| \varphi_1^{m+1} - \varphi_1^m \right\|_{\ell_2} + \left\| \varphi_2^{m+1} - \varphi_2^m \right\|_{\ell_2},$$

$$\rho = \max \left\{ \alpha_2 + \alpha_4 + \alpha_6; \ \sqrt{l}\ (\alpha_3 + \alpha_5 + \alpha_7) \, \| M_0(t,x) \|_{L_2[0,l]} \right\}.$$

According to the second condition of the theorem, $\rho < 1$. Consequently, it follows from estimate (12) that the operators on the right-hand sides of (17), (13), (14) are contracting. It follows from estimates (6)–(8) that there is a unique triple of fixed points, which is a solution to systems of countable systems of functional and integral equations (17), (13), (14) in spaces $B_2(T)$, ℓ_2. Theorem 1 is proved. □

Remark 4. Since $\lambda_n = \frac{n\pi}{l}$, $n \in \mathbb{N}$, we consider the function $h_n(t) = \frac{(\lambda_n^2 + \lambda_n^4)\, b(t)}{1 + (\lambda_n^2 + \lambda_n^4)\ a(t)}$ for the large values of n:

$$\lim_{n \to \infty} h_n(t) = \lim_{n \to \infty} \frac{(\lambda_n^2 + \lambda_n^4)\, b(t)}{1 + (\lambda_n^2 + \lambda_n^4)\ a(t)} = \frac{b(t)}{a(t)}.$$

If, for example, we choose $b(t)$ such that $b(t) = b_n(t) = \frac{a(t)}{n}$, then, it is obvious that, the following series are convergence:

$$\alpha_2 = \left[\sum_{n=1}^{\infty} \max_{t \in [0,T]} \left(\int_t^T | K_3(s)\, h_n(s) | \, ds \right)^2 \right]^{\frac{1}{2}} = \left[\sum_{n=1}^{\infty} \max_{t \in [0,T]} \left(\int_t^T | K_3(s) |^2 \, ds \int_t^T | h_n(s) |^2 \, ds \right) \right]^{\frac{1}{2}},$$

$$\alpha_4 = \left[\sum_{n=1}^{\infty} \max_{t \in [t_1,T]} \left(\int_t^T | K_1(s)\, h_n(s) | \, ds \right)^2 \right]^{\frac{1}{2}} = \left[\sum_{n=1}^{\infty} \max_{t \in [t_1,T]} \left(\int_t^T | K_1(s) |^2 \, ds \int_t^T | h_n(s) |^2 \, ds \right) \right]^{\frac{1}{2}},$$

$$\alpha_6 = \left[\sum_{n=1}^{\infty} \max_{t \in [t_1,T]} \left(\int_t^T | K_2(s)\, h_n(s) | \, ds \right)^2 \right]^{\frac{1}{2}} = \left[\sum_{n=1}^{\infty} \max_{t \in [t_1,T]} \left(\int_t^T | K_2(s) |^2 \, ds \int_t^T | h_n(s) |^2 \, ds \right) \right]^{\frac{1}{2}}.$$

5. UNIFORM CONVERGENCE OF SERIES

Theorem 2. *The function $U(t, x)$ is defined using the Fourier series* (18). *If*

$$\bar{\alpha}_2 = \left[\sum_{n=1}^{\infty} \max_{t\in[0,T]} \left(n^4 \int_t^T | K_3(s) h_n(s) | ds \right)^2 \right]^{\frac{1}{2}} < \infty,$$

$$\bar{\alpha}_4 = \left[\sum_{n=1}^{\infty} \max_{t\in[t_1,T]} \left(n^4 \int_t^T | K_1(s) h_n(s) | ds \right)^2 \right]^{\frac{1}{2}} < \infty,$$

$$\bar{\alpha}_6 = \left[\sum_{n=1}^{\infty} \max_{t\in[t_1,T]} \left(n^4 \int_t^T | K_2(s) h_n(s) | ds \right)^2 \right]^{\frac{1}{2}} < \infty,$$

then series (18) *converges absolutely and uniformly. Moreover, function* (18) *is continuously differentiable with respect to the variables included in equation* (1). *In addition, series* (15) *and* (16)*, which determine the redefinition functions, also converge.*

Proof. Taking into account that, $u(t) \in B_2(T)$, similarly to (6) and (9), from (18) we obtain the estimate

$$|U(t, x)| \le \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} |\sin \lambda_n x| \left\{ \alpha_1 \left(\frac{l}{\pi}\right)^5 \left[\sum_{n=1}^{\infty} \frac{|\psi_{1n}^V|}{n^5} + \sum_{n=1}^{\infty} \frac{|\psi_{2n}^V|}{n^5} + \sum_{n=1}^{\infty} \max_{t\in[0,T]} \frac{|f_n^V(t, \cdot, \cdot)|}{n^5} \right] \right.$$

$$\left. + \sum_{n=1}^{\infty} \max_{t\in[0,T]} \int_t^T |K_3(s) h_n(s)| \cdot \left| u_n^m(s) - u_n^{m-1}(s) \right| ds \right\}$$

$$\le \sqrt{\frac{2}{l}} \left\{ \alpha_2 \| u(t) \|_{B_2(T)} + \alpha_1 \left(\frac{l}{\pi}\right)^5 \left(\sqrt{\frac{2}{l}}\right)^5 \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{10}}} \right.$$

$$\left. \times \left[\left\| \frac{\partial^5 \psi_1(x)}{\partial x^5} \right\|_{L_2[0,l]} + \left\| \frac{\partial^5 \psi_2(x)}{\partial x^5} \right\|_{L_2[0,l]} + \max_{t\in[0,T]} \left\| \frac{\partial^5 f(t, x, \cdot, \cdot)}{\partial x^5} \right\|_{L_2[0,l]} \right] \right\} < \infty. \quad (1)$$

The absolute and uniform convergence of the Fourier series (18) follows from (1). The convergence of series (15) and (16) is proved similarly. Now we differentiate function (18) the required number of times

$$U_{tt}(t, x) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \sin \lambda_n x \left[-\psi_{1n} \frac{1}{t_2 - t_1} + \psi_{2n} \frac{1}{t_2 - t_1} - K_3(t) \left[h_n(t) u_n(t) + f_n(t, \cdot, \cdot) \right] \right], \quad (2)$$

$$\frac{\partial^4}{\partial x^4} U(t, x) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left(\frac{\pi n}{l}\right)^4 \sin \lambda_n x$$

$$\times \left[\psi_{1n} \frac{t_2 - t}{t_2 - t_1} + \psi_{2n} \frac{t - t_1}{t_2 - t_1} + \int_t^T K_3(s) \left[h_n(s) u_n(s) + f_n(s, \cdot, \cdot) \right] ds \right]. \quad (3)$$

Similarly to (2) and (3), we define the following function in the form of an expansion in Fourier series $\frac{\partial^6 U(t,\,x)}{\partial t^2\,\partial x^4}$. The proof of the convergence of the Fourier series (2) converges with the proof of the convergence of the series (18). We will show the absolute and uniform convergence of series (3). For this purpose, we use formulas (1)–(4). We apply the Cauchy–Schwartz inequality and the Bessel inequality. Then we have

$$\left|\frac{\partial^4}{\partial x^4}U(t,\,x)\right| \le \sqrt{\frac{2}{l}}\frac{\pi^4}{l^4}\sum_{n=1}^{\infty} n^4\,|\,u_n(t)\,|\cdot|\sin\lambda_n x\,|$$

$$\le \sqrt{\frac{2}{l}}\frac{\pi^4}{l^4}\left\{\alpha_3\,\|\,u(t)\,\|_{B_2(T)} + \alpha_1\left(\frac{l}{\pi}\right)^5\left(\sqrt{\frac{2}{l}}\right)^5\sqrt{\sum_{n=1}^{\infty}\frac{1}{n^2}}\right.$$

$$\left.\times\left[\left\|\frac{\partial^5\psi_1(x)}{\partial x^5}\right\|_{L_2[0,l]} + \left\|\frac{\partial^5\psi_2(x)}{\partial x^5}\right\|_{L_2[0,l]} + \max_{t\in[0,T]}\left\|\frac{\partial^5 f(t,x,\cdot,\cdot)}{\partial x^5}\right\|_{L_2[0,l]}\right]\right\} < \infty. \qquad (4)$$

Similarly to (4), the following statements can be easy established $\left|\frac{\partial^6 U(t,\,x)}{\partial t^2\,\partial x^4}\right| < \infty$. The theorem 2 is proved. □

6. CONCLUSION

The theory of differential equations plays an important role in solving applied problems. Especially, inverse boundary value problems for partial differential equations have many applications in mathematical physics, mechanics and technology, in particular in nanotechnology.

In this paper, we investigated an inverse boundary value problem for the inhomogeneous sixth-order partial differential equation (1) with Dirichlet boundary value conditions (2) and two redefinition functions at the end point of the given segment (3), (4). The nonlinear right-hand side of this equation consists the integrals of two redefinition functions. In determining the functions of redefinition we used the additional conditions (5), (6). The questions of the existence and uniqueness of the solution of the inverse boundary value problem (1)–(6) are studied.

The results obtained in this work allow us in the future to investigate nonlocal inverse boundary value problems for the heat equation and the wave equation with many redefinition functions. We hope that our work will stimulate the study of various kind of inverse boundary value problems for partial differential and integro-differential equations with many redefinition functions.

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