Solvability of a problem for a time fractional differential equation with the Hilfer operator on metric graphs

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Solvability of a problem for a time fractional differential equation with the Hilfer operator on metric graphs

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Metrik graflarda Hilfer operatori qatnashgan vaqt bo'yicha kasr tartibli differensial tenglama uchun masalaning yechilishi

Ushbu maqolada biz yulduz ko'rinishidagi metrik grafdagi Hilfer operatori qatnashgan vaqt bo'yicha kasr tartibli differentialsial tenglama uchun bir lokal masalani o'rganamiz. O'zgaruvchilarni ajratish usilidan foydalanib, biz o'rganilayotgan masalaning Furye qatori shaklida aniq echimini topilgan.

Kalit so'zlar: Hilfer operatori; metrik graf; o'zgaruvchilarni ajratish usuli; Mittag-Leffler funksiyasi

Разрешимость краевой задачи для дифференциального уравнения дробного порядка с оператором Хилфера на метрическом графе

Мы исследуем локальную задачу для дробного по времени дифференциального уравнения, включающего дробную производную Хилфера на звездном метрическом графе. Используя метод разделения переменных, мы находим явное решение исследуемой задачи в виде ряда Фурье.

Ключевые слова: Оператор Хилфера; метрический граф; метод разделения переменных; функция Миттаг-Леффлера.

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Introduction

In recent years noticeable interest has been shown in the study of initial and initial-boundary value problems for equations of fractional order. This is due to the fact that fractional-integral calculus have applications in the study of diffusion and dispersion processes in various fields of science (see [1, [4, [22, [23, [25, [26] and others. Especially, the study of initial and boundary value problems on metric graphs for fractional equations can be called a very modern field. In [27], the Cauchy problem for the Airy equation with a fractional derivative on a star-graph is solved using the method of potentials. Using by numerical methods V. Mehandiratta, M. Mehra [21] was studied for

\[ C^{D}_{0t}u(x,t) - u_{xx}(x,t) = f(x,t) \]

on the star metric graph. Besides, for this equation on the star metric graph a direct and inverse problems investigated in [7, [10]. We can say that the problems for equations involving the Hilfer operator on metric graphs have not yet been studied. We refer readers to several works [9, [12, [13], on applications of graphs and to [14, [15, [18, [17, [19, [20] on investigations differential equations in graphs.

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In this paper, we consider the initial boundary value problem (IBVP) for a time fractional heat equation on metric graphs. Here, we use the following Cauchy problem

\[
\begin{align*}
D_{0+}^\alpha u(t) &= \lambda u(t) + f(t), t \in (0,T) \\
\lim_{t \to 0^+} D_0^{1-\gamma} u(t) &= u_0,
\end{align*}
\]

that involved Hilfer operator \([23]\), where \(f(t)\) is a known function, \(u_0(t) = \text{const.}\)

Preliminaries

Fractional derivatives and integrals

**Definition 1.** The Riemann-Liouville (R-L) fractional integral of a function \(f(x)\) of order \(\alpha\) is defined by \([1]\)

\[
(I_0^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, (x > a, \ \alpha > 0).
\]

**Definition 2.** The Riemann-Liouville (R-L) left-sided fractional derivative of order \(\alpha\) of a function \(f(x)\) is defined by \([1]\)

\[
(D_0^\alpha f)(x) := \left( \frac{d}{dx} \right)^n (I_{a+}^{\alpha-n} f)(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^n \int_0^x \frac{f(t)dt}{(x-t)^{n-\alpha+1}}, (n = [\alpha] + 1, n \in \mathbb{N}, x > a)
\]

**Definition 3.** The Caputo-Gerasimov left-sided fractional derivative \((c D_0^\alpha f)(x)\) of order \(\alpha\) is defined by \([1]\)

\[
(c D_0^\alpha f)(x) := \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{f^{(n)}(t)dt}{(x-t)^{n-\alpha+1}} =: \left( I_0^{\alpha-n} \left( \frac{d}{dx} \right)^n f \right)(x), \quad x > a,
\]

where \(n \in \mathbb{N}\) and \(n = [\alpha] + 1\).

**Definition 4.** \([11]\) We consider the weighted spaces of continuous functions

\[
C_\gamma[a, b] = \{ f : (a, b) \to \mathbb{R} : (x - a)\gamma f(x) \in C[a, b] \}, \quad 0 \leq \gamma < 1
\]

and

\[
C_\gamma^n[a, b] = \{ f \in C^{n-1}[a, b] : f^{(n)} \in C_\gamma, n \in \mathbb{N},
\]

with the norms

\[
\| f \|_{C_\gamma} = \| (x - a)\gamma f(x) \|_C
\]

and

\[
\| f \|_{C_\gamma^n} = \sum_{k=0}^{n-1} \| f^{(k)} \|_C + \| f^{(n)} \|_{C_\gamma}.
\]

These spaces satisfy the following properties.

a) \(C_\gamma^0[a, b] = C[a, b]\).

b) \(C_\gamma^n[a, b] \subset AC^n[a, b]\).

c) \(C_\gamma[a, b] \subset C_\gamma^n[a, b], \quad 0 \leq \gamma_1 < \gamma_2 < 1\).

**Lemma 1.** (see \([1]\) p. 77) If \(\alpha > 0\), \(n = [\alpha] + 1\), and \(f(x) \in C_\gamma^n[a, b], 0 \leq \gamma < 1\), then the fractional derivatives \(D_0^\alpha f\) in Definition 1 and Definition 2 exist on \((a, b)\) and

\[
(D_0^\alpha f)(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(1 + k - \alpha)} (x - a)^{k-\alpha} + (c D_0^\alpha f)(x).
\]
**Definition 5.** Hilfer fractional derivative $D_{0+}^{\alpha,\mu}$ of order $\alpha$ and type $\mu$ with respect to $t$ is defined by

$$(D_{0+}^{\alpha,\mu}u)(t) = I_{0+}^{\mu(n-\alpha)}D^{(n)}(t) = I_{0+}^{\mu(n-\alpha)}\left(t^{(1-\mu)(n-\alpha)}u\right)(t),$$

$$n - 1 < \alpha < n, 0 \leq \mu \leq 1, n \in \mathbb{N}$$

everywhere the right-hand side exists.

The $D_{0+}^{\alpha,\mu}$ derivative is considered as an interpolation between the Riemann-Liouville and Caputo derivative:

$$D_{0+}^{\alpha,\mu} = \begin{cases} D_{0+}^{\alpha}, & \mu = 0, \\ C D_{0+}^{\alpha}, & \mu = 1. \end{cases}$$

**Lemma 2.** (1) Let $\alpha > 0, \beta > 0, 0 \leq \gamma < 1$. If $f(x) \in C_{\gamma}[a,b]$, then $\left(I_{a+}^{\alpha}, I_{a+}^{\beta}f\right)(x) = \left(I_{a+}^{\alpha + \beta}f\right)(x)$ for $x \in (a,b)$ and when $f(x) \in C[a,b]$, the equality holds at any point $x \in (a,b)$.

**Lemma 3.** (1) Let $\alpha > 0, 0 \leq \gamma < 1$. If $f(x) \in C_{\gamma}[a,b]$, then

$$(D_{a+}^{\alpha}, I_{a+}^{\alpha}f)(x) = f(x)$$

for $x \in (a,b)$ and when $f(x) \in C[a,b]$, the equality holds at any point $x \in (a,b)$.

**Lemma 4.** (1) Let $\alpha > 0, 0 \leq \gamma < 1, n = [\alpha] + 1$ and $f_{n-\alpha}(x) = (I_{a+}^{n-\alpha}f)(x)$. If $f(x) \in C_{\gamma}[a,b]$ and $f_{n-\alpha}(x) \in C_{\gamma}[a,b]$, then

$$(I_{a+}^{\alpha}, D_{a+}^{\alpha}f)(x) = f(x) - \sum_{j=1}^{n} f_{n-\alpha}^{(n-j)}(a+)(x-a)^{\gamma - j}, x \in (a,b),$$

where $f_{n-\alpha}^{(n-j)}(a+) = \lim_{x \to a+} f_{n-\alpha}^{(n-j)}(x)$. If $f(x) \in C[a,b]$ and $f_{n-\alpha}(x) \in C[a,b]$ then the equality holds at any point $x \in (a,b)$.

**Mittag-Leffler function**

A two-parametr of the Mittag-Leffler function at $\alpha > 0$, for all $\beta \in \mathbb{C}$ and $z \in \mathbb{C}$, represented as follows (1)

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}.$$

**Lemma 5.** (see (1) Theorem 1.6, p.35) if $\alpha < 2, \beta$ is arbitrary real number, $\mu$ is such that $\pi \alpha / 2 < \mu < \min\{\pi, \pi \alpha\}$ and $C_{1}$ is a real constant, then

$$|E_{\alpha,\beta}(z)| \leq \frac{C_{1}}{1+|z|}, (\mu \leq |\text{arg}(z)| \leq \pi), |z| \geq 0.$$

**Definition 6.** We introduce the inner scalar product and the norm on graph as follows:

$$(f(x), g(x))_{\Gamma} = \sum_{k} \int_{\Gamma} f^{(k)}(x_{k})g^{(k)}(x_{k})dx_{k},$$

$$||f||_{\Gamma} = \sqrt{(f,f)_{\Gamma}}.$$

Here we understand the functions $f(x)$ and $g(x)$ in the following forms:

$$f(x) = \begin{pmatrix} f^{(1)}(x_{1}) \ \\
 f^{(2)}(x_{2}) \ \\
 \vdots \ \\
 f^{(k)}(x_{k}) \end{pmatrix}, \ g(x) = \begin{pmatrix} g^{(1)}(x_{1}) \ \\
 g^{(2)}(x_{2}) \ \\
 \vdots \ \\
 g^{(k)}(x_{k}) \end{pmatrix}.$$
Formulation of a problem

Let us consider simple star graph $\Gamma$ with three semi-finite bonds connected at the point $O$. The point $O$ is the vertex of the graph. We label bonds of the graph as $B_k, k = 1, 2, 3$. Let us define coordinate $x_k$ on the bond $B_k, k = 1, 2, 3$, and $x_k \in (0, M_k)$. At each bond the coordinate of the vertex point $O$ is equal to zero. Further, we will use $x$ instead of $x_k$.

On the each edges of the over defined graph, we consider fractional differential equations

$$D^{\alpha, \mu}_{0+} u^{(k)}(x, t) - u^{(k)}_{xx}(x, t) = f^{(k)}(x, t), \quad x \in B_k,$$

where $D^{\alpha, \mu}_{0+}$ is Hilfer operator, $0 < \alpha < 1, \mu$, $0 \leq \mu < 1$, $f^{(k)}(x, t)$ ($k = 1, 2, 3$) are known functions.

We will study the following problem for equation (1) in $\Gamma$.

**Problem.** To find functions $u^{(k)}(x, t)$ in the domain $B_k \times (0, T)$, satisfy an equation (1) for $0 < \alpha < 1$ with the following properties:

1. $t^{1-\alpha-\mu+\alpha\mu}u^{(k)}(x, t) \in C([0, M_k] \times [0, T]),$
2. local conditions:
   $I^{(1-\mu)(1-\alpha)}_{0+} u^{(k)}(x, t) \big|_{t=0} = \phi^{(k)}(x), \quad k = 1, 2, 3, \quad x \in B_k;$
3. vertex conditions
   $u^{(1)}(0, t) = u^{(2)}(0, t) = u^{(3)}(0, t), \quad t \in [0, T],$
   $u^{(1)}_x(0, t) + u^{(2)}_x(0, t) + u^{(3)}_x(0, t) = 0, \quad t \in [0, T], \quad k = 1, 2, 3$

and boundary conditions

$$u^{(k)}(M_k, t) = 0, \quad t \in [0, T], \quad k = 1, 2, 3.$$ 

where $\phi^{(k)}(x)$ are sufficiently smooth given functions, moreover

$$\phi^{(1)}(0) = \phi^{(2)}(0) = \phi^{(3)}(0), \quad \frac{d}{dx} \phi^{(1)}(0) + \frac{d}{dx} \phi^{(2)}(0) + \frac{d}{dx} \phi^{(3)}(0) = 0$$

$$\phi^{(k)}(M_k) = 0, \quad k = 1, 2, 3.$$ 

Main Result

**Theorem.** If $\phi^{(k)}(x), \quad C^1[0, M_k], \quad f^{(k)}(x, t) \in C^1(\partial B_k \times [0, T])$ and

$$\frac{d}{dx} \phi^{(k)}(x) \quad \text{and} \quad \frac{d^2}{dx^2} f^{(k)}(x, t)$$

absolutely integrable functions in $(0, M_k)$ and $(B_k \times (0, T))$ such that

$$\phi^{(1)}(0) = \phi^{(2)}(0) = \phi^{(3)}(0), \quad \phi^{(1)}_x(0) + \phi^{(2)}_x(0) + \phi^{(3)}_x(0) = 0 \quad \text{and} \quad f^{(1)}(0, t) = \frac{d}{dx} f^{(2)}(0, t) = \frac{d}{dx} f^{(3)}(0, t), \quad \frac{d}{dx} f^{(1)}(0, t) + \frac{d}{dx} f^{(2)}(0, t) + \frac{d}{dx} f^{(3)}(0, t) = 0, \quad f^{(k)}(M_k, t) = 0 (k = 1, 2, 3)$,

then the solution of the investigated problem exists and unique.

**Proof:** Using the method separations of variables for the homogeneous equation we will get integer order differential equations

$$\frac{d^2}{dx^2} X^{(k)}(x) + \lambda^2 X^{(k)}(x) = 0, \quad \lambda \in R \setminus \{0\}, \quad k = 1, 2, 3$$  (8)
and fractional order differential equations

\[ D^{\alpha,\mu}_{0+}T(t) + \lambda^2 T(t) = 0, \quad 0 < \alpha < l, \quad 0 \leq \mu \leq 1, \]

moreover, from the conditions (3)-(5), we obtain

\[ X^{(1)}(0) = X^{(2)}(0) = X^{(3)}(0), \quad (9) \]

\[ \frac{d}{dx} X^{(1)}(0) + \frac{d}{dx} X^{(2)}(0) + \frac{d}{dx} X^{(3)}(0) = 0, \quad (10) \]

\[ X^{(k)}(M_k) = 0, k = 1, 2, 3. \quad (11) \]

By virtue conditions (9)-(11) from the general solution of equation (8), we can find eigenfunction and eigenvalues:

\[ X^{(k)}(x) = a_k \cos \lambda x + b_k \sin \lambda x; \quad x \in B_k. \quad (12) \]

We assume that \( f^{(k)}(x, t) \in L_2[0; M_k] \) and then we expand into the Fourier series in terms of eigenfunctions, i.e.

\[ f^{(k)}(x, t) = \sum_{n=0}^{\infty} f_n(t) X_n^{(k)}(x), \quad (13) \]

where \( f_n(t) \) is the coefficients of the Fourier series (13). Further, we search a solution of the equation (1) in the form

\[ u^{(k)}(x, t) = \sum_{n=0}^{\infty} x_n^{(k)}(x) W_n(t) \quad k = 1, 2, 3. \quad (14) \]

Substituting (14) into the equation (1), we obtain

\[ \sum_{n=0}^{\infty} \left( D^{\alpha,\mu}_{0+} W_n(t) + \lambda_n^2 W_n(t) - f_n(t) \right) X_n^{(k)}(x) = 0. \]

Consequently

\[ D^{\alpha,\mu}_{0+} W_n(t) + \lambda_n^2 W_n(t) = f_n(t). \quad (15) \]

General solution of the Eq. (15) has a form (see. [33], Lemma 2):

\[ W_n(t) = \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha} \left[ -\lambda_n^2 (t - z)^{\alpha} \right] f_n(\tau)d\tau + A_n t^{\gamma-1} E_{\alpha,\gamma} (-\lambda_n^2 t^\alpha), \quad 0 < \gamma < 1 \quad (16) \]

where \( \gamma = \alpha + \mu - \alpha \mu \). Considering (14) and (16) we can write the general solution of equation (1) in the following form:

\[ u^{(k)}(x, t) = \sum_{n=0}^{\infty} \left[ \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha} \left[ -\lambda_n^2 (t - z)^{\alpha} \right] f_n(\tau)d\tau + A_n t^{\gamma-1} E_{\alpha,\gamma} (-\lambda_n^2 t^\alpha) \right] X_n^{(k)}(x). \quad k = 1, 2, 3. \quad (17) \]

Applying the operator \( I^{1-\gamma}_{0+} \) to the (17) and considering Definition 1, we have

\[ I^{1-\gamma}_{0+} u^{(k)}(x, t) = \frac{1}{\Gamma(1 - \gamma)} \sum_{n=0}^{\infty} A_n X_n^{(k)}(x) \int_0^t (t - \tau)^{-\gamma} \tau^{\gamma-1} E_{\alpha,\gamma} (-\lambda_n^2 \tau^\alpha)d\tau + \]

\[ + \sum_{n=0}^{\infty} A_n X_n^{(k)}(x) \frac{1}{\Gamma(1 - \gamma)} \int_0^t (t - \tau)^{-\gamma} \int_0^\tau (\tau - s)^{\alpha-1} E_{\alpha,\alpha} (-\lambda_n^2 (\tau - s)^\alpha)f_n(s)dsdr = \]

\[ = \sum_{n=0}^{\infty} A_n F_{1n}(t) X_n^{(k)}(x) + \sum_{n=0}^{\infty} F_{2n}(t) X_n^{(k)}(x). \quad (18) \]

\[ F_{1n}(t) = \frac{1}{\Gamma(1 - \gamma)} \int_0^t z^{-\gamma} (t - z)^{\gamma-1} E_{\alpha,\gamma} (-\lambda_n^2 (t - z)^\alpha)dz = E_{\alpha,1}(-\lambda_n^2 t^\alpha), \]
where, \( \varphi_n \) are the coefficients of the Fourier series (19).

By virtue (2) from (19) we find that

\[ A_n = \varphi_n. \]  

Further, integrating by parts two times the functions \( \varphi^{(k)}(x) \) and considering (6)-(7), we get:

\[ \sum_{n=0}^{\infty} \varphi_n = \frac{1}{\|X_n^{(k)}(x)\|^2} \sum_{n=0}^{\infty} \int_{0}^{M_k} \frac{d^2}{dx^2} \varphi^{(k)}(x) X_n^{(k)}(x) dx. \]  

It is required to prove the convergence of functions \( u^{(k)}(x, t), u_{xx}^{(k)}(x, t), D_0^{\alpha, \mu} u^{(k)}(x, t) \) in the domain \( B_k \times (0, T) \). We use Lemma 5 and the following inequalities

\[ |X_n^{(k)}(x)| = |a_k \cos \lambda_n x + b_k \sin \lambda_n x| \leq \sqrt{a_k^2 + b_k^2}, \]  

owing to (20)-(21), we find

\[ |A_n| = |\varphi_n| \leq \frac{C_1}{\lambda_n^2}. \]

where \( C_1 = \text{const} > 0 \). From (13) and based on the conditions of the Theorem,

\[ \sum_{n=0}^{\infty} f_n^{(k)}(t) = \frac{1}{|X_n(x)|^2} \sum_{n=0}^{\infty} \int_{0}^{L_k} \frac{d^2}{dx^2} f_n^{(k)}(x, t) X_n^{(k)}(x) dx, \]

taking into account (22),

\[ |f_n^{(k)}(t)| = \sqrt{a_k^2 + b_k^2} \frac{1}{\lambda_n^2} \sum_{k=1}^{3} \left| \int_{0}^{M_k} \frac{d}{dx} f_n^{(k)}(x, t) dx \right| \leq \frac{C_2}{\lambda_n^2}. \]  

where \( C_2 \) is positive const. From (17) and (21)-(23) we obtain

\[ |u^{(k)}(x, t)| = \left| \sum_{n=0}^{\infty} \left[ \int_{0}^{t} (t - \tau)^{\alpha - 1} E_{\alpha, \alpha} \left[ -\lambda_n^2 (t - z)^\alpha \right] f_n(\tau) d\tau + \right. \right.\]

\[ + A_n t^{\gamma - 1} E_{\alpha, \gamma} (-\lambda_n^2 t^{\alpha}) \right] X_n^{(k)}(x) \right| \leq \]

\[ \leq \sum_{n=0}^{\infty} \left| \int_{0}^{t} (t - \tau)^{\alpha - 1} E_{\alpha, \alpha} \left[ -\lambda_n^2 (t - z)^\alpha \right] f_n(\tau) d\tau + \right. \right.\]
+ \sum_{n=0}^{\infty} \frac{C_3}{\lambda_n^2 (1 + \lambda_n^2)} + \sum_{n=0}^{\infty} \frac{C_4}{\lambda_n^2 (1 + \lambda_n^2)} \leq \sum_{n=0}^{\infty} \frac{C_5}{\lambda_n^2}

where \( C_i = \text{const} > 0, i = 3, 5 \) and \( C_5 \geq C_3 + C_4 \). So \( u^{(k)}(x, t) \) are uniform convergent. Taking into account \( u_{xx}^{(k)}(x, t) = -\lambda_n^2 u^{(k)}(x, t) \), we can write

\[
\left| u_{xx}^{(k)}(x, t) \right| = \lambda_n^2 \left| u^{(k)}(x, t) \right| \leq \sum_{n=0}^{\infty} \frac{C_7}{\lambda_n^2}
\]

We infer, that the function \( u_{xx}^{(k)}(x, t) \) be uniformly convergent due to the Definition 4.

\[
D_{0+}^{\alpha, \mu} u^{(k)}(x, t) = I_{0+}^{\gamma - \alpha} \frac{d}{dt} I_{0+}^{\gamma - \mu} u^{(k)}(x, t) =
\]

\[
= \sum_{n=0}^{\infty} A_n \lambda_n^2 t^{\gamma - 1} E_{\alpha, \gamma} (-\lambda_n^2 \tau^\alpha) X_n^{(k)}(x) -
\]

\[- \sum_{n=0}^{\infty} f_n(0) E_{\alpha, 1} (-\lambda_n^2 \tau^\alpha) X_n^{(k)}(x) - \sum_{n=0}^{\infty} X_n^{(k)}(x) \int_0^t E_{\alpha, 1} (-\lambda_n^2 (t - \tau)^\alpha) f_n'(\tau) d\tau.
\]

Now consider \( D_{0+}^{\alpha, \mu} u^{(k)}(x, t) \) for convergence.

\[
\left| D_{0+}^{\alpha, \mu} u^{(k)}(x, t) \right| = \left| \sum_{n=0}^{\infty} A_n \lambda_n^2 t^{\gamma - 1} E_{\alpha, \gamma} (-\lambda_n^2 \tau^\alpha) X_n^{(k)}(x) -
\]

\[- \sum_{n=0}^{\infty} f_n(0) E_{\alpha, 1} (-\lambda_n^2 \tau^\alpha) X_n^{(k)}(x) - \sum_{n=0}^{\infty} X_n^{(k)}(x) \int_0^t E_{\alpha, 1} (-\lambda_n^2 (t - \tau)^\alpha) f_n'(\tau) d\tau \right| \leq
\]

\[
\leq \sum_{n=0}^{\infty} \left| A_n \lambda_n^2 t^{\gamma - 1} E_{\alpha, \gamma} (-\lambda_n^2 \tau^\alpha) \right| \left| X_n^{(k)}(x) \right| +
\]

\[\]
\[
+ \sum_{n=0}^{\infty} \left| f_n(0) E_{\alpha, 1} (-\lambda_n^2 \tau^\alpha) \right| \left| X_n^{(k)}(x) \right| + \sum_{n=0}^{\infty} \left| X_n^{(k)}(x) \right| \int_0^t \left| E_{\alpha, 1} (-\lambda_n^2 (t - \tau)^\alpha) f_n'(\tau) \right| d\tau \leq
\]

\[
\leq \sum_{n=0}^{\infty} \left| A_n \lambda_n^2 t^{\gamma - 1} E_{\alpha, \gamma} (-\lambda_n^2 \tau^\alpha) \right| \left| X_n^{(k)}(x) \right| +
\]

\[\]
\[
+ \sum_{n=0}^{\infty} \left| f_n(0) E_{\alpha, 1} (-\lambda_n^2 \tau^\alpha) \right| \left| X_n^{(k)}(x) \right| + \sum_{n=0}^{\infty} \left| X_n^{(k)}(x) \right| \int_0^t \left| E_{\alpha, 1} (-\lambda_n^2 (t - \tau)^\alpha) f_n'(\tau) \right| d\tau \leq
\]

\[
\leq \sum_{n=0}^{\infty} \left| A_n \lambda_n^2 t^{\gamma - 1} E_{\alpha, \gamma} (-\lambda_n^2 \tau^\alpha) \right| \left| X_n^{(k)}(x) \right| +
\]

\[\]
\[
+ \sum_{n=0}^{\infty} \left| f_n(0) E_{\alpha, 1} (-\lambda_n^2 \tau^\alpha) \right| \left| X_n^{(k)}(x) \right| + \sum_{n=0}^{\infty} \left| X_n^{(k)}(x) \right| \int_0^t \left| E_{\alpha, 1} (-\lambda_n^2 (t - \tau)^\alpha) f_n'(\tau) \right| d\tau \leq
\]

\[
\leq \sum_{n=0}^{\infty} \frac{m_1}{1 + \lambda^2} + \sum_{n=0}^{\infty} \frac{m_2}{1 + \lambda^2} + \sum_{n=0}^{\infty} \frac{m_3}{\lambda^2 (1 + \lambda^2)} \leq \sum_{n=0}^{\infty} \frac{m_4}{\lambda^2},
\]
where \( m_i = \text{const} > 0 \), \((i = 1, 4)\) and \( m_1 + m_2 + m_3 \leq m_4 \). According to the asymptotes of \( \lambda_n \sim cn \ (c = \text{const}) \) (see [11]) we can conclude that series of \( D_{0+}^{(\alpha,n)}u^{(k)}(x,t) \) is uniformly convergent.

The operator \( D_{0+}^{(\alpha,n)}u \) which is defined in definition 4 can be written as
\[
I_0^{\mu(1-\alpha)}D_0^{1-\mu(1-\alpha)}u^{(k)}(x,t) - u^{(k)}_{xx}(x,t) = f^{(k)}(x,t),
\]
\[
I_0^{1-\alpha}D_0^{1-\gamma}u^{(k)}(x,t) = u^{(k)}_{xx}(x,t) + f^{(k)}(x,t), \gamma = \alpha + \mu - \alpha\mu.
\]

Introducing notation
\[
I_0^{1-\gamma}u^{(k)}(x,t) = v^{(k)}(x,t), k = 1, 2, 3
\]
equation [1] we will rewrite as follows:
\[
I_0^{1-\gamma}Dv^{(k)}(x,t) = u^{(k)}_{xx}(x,t) + f^{(k)}(x,t)
\]
Further applying \( I_0^{1-\gamma} \) and considering Lemma 2, we deduce
\[
I_0^{1-\alpha}Dv^{(k)}(x,t) = v^{(k)}_{xx}(x,t) + f^{(k)}_1(x,t).
\]

Using Definition 3 we obtain
\[
cDv^{(k)}(x,t) - v^{(k)}_{xx}(x,t) = f^{(k)}_1(x,t)
\]
where \( f^{(k)}_1(x,t) = I_0^{1-\gamma}f^{(k)}(x,t) \). Considering [2-4], and from [24] we deduce
\[
v^{(k)}(x,0) = \varphi^{(k)}(x), k = 1, 2, 3, \ x \in B_k,
\]
and vertex conditions
\[
v^{(1)}(0,t) = v^{(2)}(0,t) = v^{(3)}(0,t), t \in [0,T],
\]
\[
v^{(1)}_x(0,t) + v^{(2)}_x(0,t) + v^{(3)}_x(0,t) = 0, \ t \in [0,T], \ k = 1, 2, 3
\]
boundary conditions
\[
v^{(k)}(L_k,t) = 0, \ t \in [0,T], \ k = 1, 2, 3.
\]

Using standart scheme, a solution the problem [25]-[29] (see [7,10,11]) is unique. If \( f^{(k)}_1(x,t) = 0 \) and \( \varphi^{(k)}(x) = 0 \), then \( v^{(k)}(x,t) \equiv 0 \). Based on [24] we get \( u^{(k)}(x,t) \equiv 0 \). Therefore, we can say that the solution to problem [11]-[15] is unique.

References


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