Chaos, Solitons and Fractals 87 (2016) 190–196

Contents lists available at ScienceDirect

Chaos, Solitons and Fractals

Nonlinear Science, and Nonequilibrium and Complex Phenomena

journal homepage: www.elsevier.com/locate/chaos

# Phase transition and chaos: *P*-adic Potts model on a Cayley tree

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ARTICLE INFO

*Article history:*
Received 16 February 2016
Accepted 5 April 2016

*MSC:*
46S10
82B26
12J12
39A70
47H10
60K35

*Keywords:*
*p*-adic numbers
Potts model
*p*-adic quasi Gibbs measure
Periodic
Shift

ABSTRACT

In our previous investigations, we have developed the renormalization group method to *p*-adic models on Cayley trees, this method is closely related to the investigation of dynamical system associated with a given model. In this paper, we are interested in the following question: how is the existence of the phase transition related to chaotic behavior of the associated dynamical system (this is one of the important question in physics)? To realize this question, we consider as a toy model the *p*-adic *q*-state Potts model on a Cayley tree, and show, in the phase transition regime, the associated dynamical system is chaotic, i.e. it is conjugate to the full shift. As an application of this result, we are able to show the existence of periodic (with any period) *p*-adic quasi Gibbs measures for the model. This allows us to know that how large is the class of *p*-adic quasi Gibbs measures. We point out that a similar kind of result is not known in the case of real numbers.



## 1. Introduction

Models of interacting systems have been intensively studied in the last years and new methodologies have been developed in the attempt to understanding their intriguing features. One of the most promising directions is the combination of statistical mechanics tools and methods adopted in dynamical systems. One of such tools is the renormalization group (RG) which has had a profound impact on modern statistical physics[1]. The renormalization method is then applied in statistical mechanics and yielded lots of interesting results. Since such investigations of phase transitions of spin models on hierarchical lattices showed that they make the exact calculation of various physical quantities [5,14]. One of the most simple hierarchical lattice is a Cayley tree or a Bethe lattice (see [39]). This lattice is not a realistic lattice, however, investigations of phase transitions of spin models on trees like the Cayley tree show that they make the exact calculation of various physical quantities [42].

On the other hand, there are many investigations that have been conducted to discuss and debate the question due to the assumption that *p*-adic numbers provide a more exact and more adequate description of microworld phenomena (see, for example [18,48,49]). Consequently, various models in physics described in the language of *p*-adic analysis (see [2,3,11,49,50]), and numerous applications of such an analysis to mathematical physics have been studied in [4,18,19,48]. These investigations proposed to study new probability models (namely *p*-adic probability), which cannot be described using ordinary Kolmogorov's probability theory (see [6,17,22,25,28]). Therefore, *p*-adic probability models were investigated in [21,24,25]. Using that, *p*-adic measure theory in [20,22,28], the theories of *p*-adic and non-Archimedean stochastic processes have been developed. In [13,23,29–38,44] it has been developed *p*-adic statistical mechanics within the scheme of the theory of *p*-adic probability and *p*-adic stochastic processes. For complete review of the *p*-adic mathematical physics we refer to [7].

In [34] we have developed the renormalization group method to *p*-adic $\lambda$-models on Cayley trees (which are generalizations of the Ising model [16,36]). Note that the renormalization method is closely related to the investigation of *p*-adic dynamical system associated with a given model (see [1,24,26]). In this paper, we are interested in the following question: how is the existence of

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[1] This method appeared after Wilsons seminal work in the early 1970s [51], based also on the ground breaking foundations laid by Fisher [10].

http://dx.doi.org/10.1016/j.chaos.2016.04.003

the phase transition related to chaotic behavior of the associated $p$-adic dynamical system (this is one of the important question in physics [15])? In the present paper, we consider as a toy model the $p$-adic $q$-state Potts model on a Cayley tree [37]. It is known [35,38,43] that for this model there exists a phase transition if $q$ is divisible by $p$. We will show that, in the phase transition regime, the associated $p$-adic dynamical system (which is rational dynamical system, a few investigates are devoted to such kind of dynamical systems [41,46], but most study is devoted to polynomial dynamical systems [8]) is chaotic, i.e. it is conjugate to the full shift. Note that some $p$-adic chaotic dynamical systems have been studied in [9,52]. As an application of this result, we are able to show the existence of periodic (with any period) $p$-adic quasi Gibbs measures for the model. We point out that similar kind of result is not known in the case of real numbers. A few attempts have been done to find out either 2-periodic or weakly periodic Gibbs measures on the Cayley tree (see [40,45]). The main result of this paper allows us to know that how large is the class of $p$-adic quasi Gibbs measures. If one considers $p$-adic Gibbs measures, then it was shown [38] that there is no periodic $p$-adic Gibbs measures except for translation-invariant ones for the Potts model. We stress that the set of $p$-adic quasi Gibbs measures is larger than the set of $p$-adic Gibbs measures. As is well known, $p$-adic spaces have the fractal (although very special) structure. Hence, our study opens a new perspective in rational $p$-adic dynamical systems on fractals.

## 2. Preliminaries

### 2.1. *p-adic numbers*

In what follows $p$ will be a fixed prime number. The set $\mathbb{Q}_p$ is defined as a completion of the rational numbers $\mathbb{Q}$ with respect to the norm $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ given by

$$|x|_p = \begin{cases} p^{-r} & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{2.1}$$

here, $x = p^r \frac{m}{n}$ with $r, m \in \mathbb{Z},\ n \in \mathbb{N},\ (m, p) = (n, p) = 1$. The absolute value $|\cdot|_p$ is non-Archimedean, meaning that it satisfies the strong triangle inequality $|x+y|_p \le \max\{|x|_p, |y|_p\}$. We recall a nice property of the norm, i.e. if $|x|_p > |y|_p$ then $|x+y|_p = |x|_p$. Note that this is a crucial property which is proper to the non-Archimedenity of the norm.

Any $p$-adic number $x \in \mathbb{Q}_p,\ x \neq 0$ can be uniquely represented in the form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + \cdots), \tag{2.2}$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and $x_j$ are integers, $0 \le x_j \le p-1$, $x_0 > 0$, $j = 0, 1, 2, \ldots$ In this case $|x|_p = p^{-\gamma(x)}$.

For each $a \in \mathbb{Q}_p$, $r > 0$ we denote

$$B_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}$$

and the set of all *p-adic integers*

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \ |x|_p \le 1\}.$$

The set $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$ is called a set of $p$-adic units.

Recall that the $p$-adic exponential is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for every $x \in B_{p^{-1/(p-1)}}(0)$.

Put

$$\mathcal{E}_p = \{x \in \mathbb{Q}_p : \quad |x - 1|_p < p^{-1/(p-1)}\}.$$

It is known [27] the following fact.

**Lemma 2.1.** *The set $\mathcal{E}_p$ has the following properties:*

(a) *$\mathcal{E}_p$ is a group under multiplication;*
(b) *$|a - b|_p < 1$ for all $a, b \in \mathcal{E}_p$;*
(c) *If $a, b \in \mathcal{E}_p$ then it holds*

$$|a+b|_p = \begin{cases} \frac{1}{2}, & \text{if} \quad p = 2 \\ 1, & \text{if} \quad p \neq 2. \end{cases}$$

(d) *If $a \in \mathcal{E}_p$, then there is an element $h \in B_{p^{-1/(p-1)}}(0)$ such that $a = \exp_p(h)$.*

Note that the basics of $p$-adic analysis, $p$-adic mathematical physics are explained in [27,48].

### 2.2. *p-adic sub-shift*

Let $f : X \to \mathbb{Q}_p$ be a mapping from a compact open set $X$ of $\mathbb{Q}_p$ into $\mathbb{Q}_p$. We assume that (i) $f^{-1}(X) \subset X$; (ii) $X = \bigcup_{j\in I} B_r(a_j)$ can be written as a finite disjoint union of balls of centers $a_j$ and of the same radius $r$ such that for each $j \in I$ there is an integer $\tau_j \in \mathbb{Z}$ such that

$$|f(x) - f(y)|_p = p^{-\tau_j}|x - y|_p, \qquad x, y \in B_r(a_j). \tag{2.3}$$

For such a map $f$, define its Julia set by

$$J_f = \bigcap_{n=0}^{\infty} f^{-n}(X). \tag{2.4}$$

It is clear that $f^{-1}(J_f) = J_f$ and then $f(J_f) \subset J_f$.

Following [9] the triple $(X, J_f, f)$ is called a $p$-adic *weak repeller* if all $\tau_j$ in (2.3) are nonnegative, but at least one is positive. We call it a $p$-adic *repeller* if all $\tau_j$ in (2.3) are positive. For any $i \in I$, we let

$$I_i := \left\{ j \in I : B_r(a_j) \cap f(B_r(a_i)) \neq \emptyset \right\} = \{ j \in I : B_r(a_j) \subset f(B_r(a_i)) \}$$

(the second equality holds because of the expansiveness and of the ultrametric property). Then define a matrix $A = (a_{ij})_{I\times I}$, called *incidence matrix* as follows

$$a_{ij} = \begin{cases} 1, \text{if} & j \in I_i; \\ 0, \text{if} & j \notin I_i. \end{cases}$$

If $A$ is irreducible, we say that $(X, J_f, f)$ is *transitive*. Here the irreducibility of $A$ means, for any pair $(i, j) \in I \times I$ there is positive integer $m$ such that $a_{ij}^{(m)} > 0$, where $a_{ij}^{(m)}$ is the entry of the matrix $A^m$.

Given $I$ and the irreducible incidence matrix $A$ as above. Denote

$$\Sigma_A = \{(x_k)_{k\ge 0} : \ x_k \in I, \ A_{x_k, x_{k+1}} = 1, \ k \ge 0\}$$

which is the corresponding subshift space, and let $\sigma$ be the shift transformation on $\Sigma_A$. We equip $\Sigma_A$ with a metric $d_f$ depending on the dynamics which is defined as follows. First for $i, j \in I, i \neq j$ let $\kappa(i, j)$ be the integer such that $|a_i - a_j|_p = p^{-\kappa(i,j)}$. It clear that $\kappa(i, j) < \tau$. By the ultra-metric inequality, we have

$$|x - y|_p = |a_i - a_j|_p \quad i \neq j, \ \forall x \in B_r(a_i), \forall y \in B_r(a_j)$$

For $x = (x_0, x_1, \ldots, x_n, \ldots) \in \Sigma_A$ and $y = (y_0, y_1, \ldots, y_n, \ldots) \in \Sigma$, define

$$d_f(x, y) = \begin{cases} p^{-\tau_{x_0} - \tau_{x_1} - \cdots - \tau_{x_{n-1}} - \kappa(x_n, y_n)} & , \ \text{if } n \neq 0 \\ p^{-\kappa(x_0, y_0)} & , \ \text{if } n = 0 \end{cases}$$

where $n = n(x, y) = \min\{i \ge 0 : x_i \neq y_i\}$. It is clear that $d_f$ defines the same topology as the classical metric which is defined by $d(x, y) = p^{-n(x,y)}$.

**Theorem 2.2** [9]**.** *Let $(X, J_f, f)$ be a transitive $p$-adic weak repeller with incidence matrix $A$. Then the dynamics $(J_f, f, |\cdot|_p)$ is isometrically conjugate to the shift dynamics $(\Sigma_A, \sigma, d_f)$.*

### 2.3. p-adic measure

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets $X$. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a *p-adic measure* if for any $A_1, \dots, A_n \subset \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ $(i \neq j)$ the equality holds

$$\mu\Big(\bigcup_{j=1}^{n} A_j\Big) = \sum_{j=1}^{n} \mu(A_j).$$

A $p$-adic measure is called a *probability measure* if $\mu(X) = 1$. A $p$-adic probability measure $\mu$ is called *bounded* if $\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty$. For more detail information about $p$-adic measures we refer to [17,24].

### 2.4. Cayley tree

Let $\Gamma^k_+ = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root $x^0$ (whose each vertex has exactly $k+1$ edges, except for the root $x^0$, which has $k$ edges). Here $V$ is the set of vertices and $L$ is the set of edges. The vertices $x$ and $y$ are called *nearest neighbors* and they are denoted by $l = \langle x, y\rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x, x_1\rangle \cdots \langle x_{d-1}, y\rangle$ is called a *path* from the point $x$ to the point $y$. The distance $d(x, y)$, $x, y \in V$, on the Cayley tree, is the length of the shortest path from $x$ to $y$.

$$W_n = \big\{x \in V \mid d(x, x^0) = n\big\},$$

$$V_n = \bigcup_{m=0}^{n} W_m, \quad L_n = \{l =< x, y >\in L \mid x, y \in V_n\}.$$

The set of direct successors of $x$ is defined by

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, x \in W_n.$$

Observe that any vertex $x \neq x^0$ has $k$ direct successors and $x^0$ has $k+1$.

### 2.5. p-adic quasi Gibbs measure

In this section we recall the definition of $p$-adic quasi Gibbs measure (see [31]).

Let $\Phi = \{1, 2, \dots, q\}$, here $q \geq 2$, ($\Phi$ is called a *state space*) and is assigned to the vertices of the tree $\Gamma^k_+ = (V, \Lambda)$. A configuration $\sigma$ on $V$ is then defined as a function $x \in V \to \sigma(x) \in \Phi$; in a similar manner one defines configurations $\sigma_n$ and $\omega$ on $V_n$ and $W_n$, respectively. The set of all configurations on $V$ (resp. $V_n$, $W_n$) coincides with $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$). One can see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$. Using this, for given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$ we define their concatenations by

$$(\sigma_{n-1} \vee \omega)(x) = \begin{cases} \sigma_{n-1}(x), \text{ if } x \in V_{n-1}, \\ \omega(x), \quad \text{ if } \ x \in W_n. \end{cases}$$

It is clear that $\sigma_{n-1} \vee \omega \in \Omega_{V_n}$.

The (formal) Hamiltonian of $p$-adic Potts model is

$$H(\sigma) = J \sum_{\langle x,y\rangle \in L} \delta_{\sigma(x)\sigma(y)}, \tag{2.5}$$

where $J \in B(0, p^{-1/(p-1)})$ is a coupling constant, and $\delta_{ij}$ is the Kroneker's symbol.

A construct of a generalized $p$-adic quasi Gibbs measure corresponding to the model is given below.

Assume that $\mathbf{h} : V \setminus \{x^{(0)}\} \to \mathbb{Q}_p^{\Phi}$ is a mapping, i.e. $\mathbf{h}_x = (h_{1,x}, h_{1,x}, \dots, h_{q,x})$, where $h_{i,x} \in \mathbb{Q}_p$ $(i \in \Phi)$ and $x \in V \setminus \{x^{(0)}\}$. Given $n \in \mathbb{N}$, we consider a $p$-adic probability measure $\mu^{(n)}_{\mathbf{h},\rho}$ on $\Omega_{V_n}$ defined by

$$\mu^{(n)}_{\mathbf{h}}(\sigma) = \frac{1}{Z^{(\mathbf{h})}_n} \exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x} \tag{2.6}$$

Here, $\sigma \in \Omega_{V_n}$, and $Z^{(\mathbf{h})}_n$ is the corresponding normalizing factor

$$Z^{(\mathbf{h})}_n = \sum_{\sigma\in\Omega_{V_n}} \exp\{H_n(\sigma)\} \prod_{x\in W_n} h_{\sigma(x),x}. \tag{2.7}$$

In this paper, we are interested in a construction of an infinite volume distribution with given finite-dimensional distributions. More exactly, we would like to find a $p$-adic probability measure $\mu$ on $\Omega$ which is compatible with given ones $\mu^{(n)}_{\mathbf{h}}$, i.e.

$$\mu(\sigma \in \Omega : \sigma|_{V_n} = \sigma_n) = \mu^{(n)}_{\mathbf{h}}(\sigma_n), \quad \text{forall } \sigma_n \in \Omega_{V_n}, \ n \in \mathbb{N}. \tag{2.8}$$

We say that the $p$-adic probability distributions (2.6) are *compatible* if for all $n \geq 1$ and $\sigma \in \Phi^{V_{n-1}}$:

$$\sum_{\omega\in\Omega_{W_n}} \mu^{(n)}_{\mathbf{h}}(\sigma_{n-1} \vee \omega) = \mu^{(n-1)}_{\mathbf{h}}(\sigma_{n-1}). \tag{2.9}$$

This condition according to the Kolmogorov extension theorem (see [12,22]) implies the existence of a unique $p$-adic measure $\mu_{\mathbf{h}}$ defined on $\Omega$ with a required condition (2.8). Such a measure $\mu_{\mathbf{h}}$ is said to be *a p-adic quasi Gibbs measure* corresponding to the model [31,32]. If one has $h_x \in \mathcal{E}_p$ for all $x \in V \setminus \{x^{(0)}\}$, then the corresponding measure $\mu_{\mathbf{h}}$ is called *p-adic Gibbs measure* (see [37,38]).

By $Q\mathcal{G}(H)$ we denote the set of all $p$-adic quasi Gibbs measures associated with functions $\mathbf{h} = \{\mathbf{h}_x, \ x \in V\}$. If there are at least two distinct generalized $p$-adic quasi Gibbs measures such that at least one of them is unbounded, then we say that *a phase transition* occurs.

The following statement describes conditions on $h_x$ guaranteeing compatibility of $\mu^{(n)}_{\mathbf{h}}(\sigma)$.

**Theorem 2.3** *[31]. The measures $\mu^{(n)}_{\mathbf{h}}$, $n = 1, 2, \dots$ (see (2.6)) associated with q-state Potts model (2.5) satisfy the compatibility condition (2.9) if and only if for any $n \in \mathbb{N}$ the following equation holds:*

$$\hat{h}_x = \prod_{y\in S(x)} \mathbf{F}(\hat{\mathbf{h}}_y, \theta), \tag{2.10}$$

*here and below a vector $\hat{\mathbf{h}} = (\hat{h}_1, \dots, \hat{h}_{q-1}) \in \mathbb{Q}_p^{q-1}$ is defined by a vector $\mathbf{h} = (h_1, h_1, \dots, h_q) \in \mathbb{Q}_p^q$ as follows*

$$\hat{h}_i = \frac{h_i}{h_q}, \quad i = 1, 2, \dots, q-1 \tag{2.11}$$

*and mapping $\mathbf{F} : \mathbb{Q}_p^{q-1} \times \mathbb{Q}_p \to \mathbb{Q}_p^{q-1}$ is defined by $\mathbf{F}(\mathbf{x}; \theta) = (F_1(\mathbf{x}; \theta), \dots, F_{q-1}(\mathbf{x}; \theta))$ with*

$$F_i(\mathbf{x}; \theta) = \frac{(\theta - 1)x_i + \sum_{j=1}^{q-1} x_j + 1}{\sum_{j=1}^{q-1} x_j + \theta},$$
$$\mathbf{x} = \{x_i\} \in \mathbb{Q}_p^{q-1}, \quad i = 1, 2, \dots, q-1. \tag{2.12}$$

**Remark 2.1.** In what follows, without loss of generality, we may assume that $h_q = 1$. Otherwise, in (2.6) we multiply and divide the expression on the right hand side by $\prod_{x\in W_n} h_{q,x}$, and after replacing $h_i$ by $h_i/h_q$, we get the desired equality.

Let us first observe that the set $(\underbrace{1, \dots, 1, h}_{m}, 1, \dots, 1)$ $(m = 1, \dots, q-1)$ is invariant for the Eq. (2.10). Therefore, in what follows, we restrict ourselves to one of such lines, let us say $(h, 1, \dots, 1)$.

In [38] to establish the phase transition, we considered translation-invariant (i.e. $\mathbf{h} = \{\mathbf{h}_x\}_{x\in V\setminus\{x^0\}}$ such that $\mathbf{h}_x = \mathbf{h}_y$ for all

$x, y$) solutions of (2.10). Then the Eq. (2.10) reduced to the following one

$$h = f_\theta(h), \tag{2.13}$$

where

$$f_\theta(x) = \left(\frac{\theta x + q - 1}{x + \theta + q - 2}\right)^k. \tag{2.14}$$

Hence, to establish the existence of the phase transition we showed [37] that (5.5) has three nontrivial solutions if $q$ is divisible by $p$. Note that full description of all solutions of the last equation has been carried out in [43,47]. In the section, we will show that the function (2.14) is chaotic.

## 3. The fixed points of *p*-adic dynamical system (2.14)

In this section, we study behavior of the fixed points of the function (2.14).

In what follows, for the sake of simplicity, we assume that $p \geq 3$, $k = 2$, $0 < |\theta - 1|_p < |q|_p < 1$. It is known [37,43] that, in this case, there exist three translation invariant $p$-adic Gibbs measures $\mu_0$, $\mu_1$, $\mu_2$ (note that they are not bounded) which correspond to the fixed points of $f_\theta$. Namely, the fixed points are $x_0 = 1$ and

$$x_1 = \frac{-2(q-1) + (\theta-1)^2 + (\theta-1)\sqrt{-4(q-1) + (\theta-1)^2}}{2} \tag{3.1}$$

$$x_2 = \frac{-2(q-1) + (\theta-1)^2 - (\theta-1)\sqrt{-4(q-1) + (\theta-1)^2}}{2} \tag{3.2}$$

Let $x^{(0)}$ be a fixed point of an analytic function $f(x)$. Let

$$\lambda = \frac{d}{dx} f(x^{(0)}).$$

The fixed point $x^{(0)}$ is called *attractive* if $0 \leq |\lambda|_p < 1$, *indifferent* if $|\lambda|_p = 1$, and *repelling* if $|\lambda|_p > 1$.

**Theorem 3.1.** *For the fixed points $x_0, x_1, x_2$ of $f_\theta$ the following statements hold:*

(i) *$x_0$ is an attracting fixed point;*
(ii) *$x_1$ and $x_2$ are repelling fixed points.*

**Proof.** Let $x_i$ be a fixed point of (2.14). Then we have

$$f'_\theta(x_i) = \frac{2(\theta-1)(\theta-1+q)x_i}{(x_i+\theta+q-2)(\theta x_i+q-1)} \tag{3.3}$$

(i) From (3.3) we get

$$f'_\theta(x_0) = \frac{2(\theta-1)}{\theta-1+q}$$

Since $|\theta - 1|_p < |q|_p$ using non-Archimedean norm's property we obtain

$$|f'_\theta(x_0)|_p = \frac{|\theta-1|_p}{|q|_p} < 1.$$

which means that $x_0$ is attracting.

(ii) Using (3.1) and (3.2) one can calculate that

$$x_{1,2} + \theta + q - 2 = \frac{(\theta-1)\left(2 \pm \sqrt{4-4q+(\theta-1)^2} + \theta - 1\right)}{2}$$
$$\theta x_{1,2} + q - 1 = \frac{(\theta-1)\left(2 \pm \theta\sqrt{4-4q+(\theta-1)^2} - 2q + \theta(\theta-1)\right)}{2}$$

Due to $|\theta - 1|_p < |q|_p < 1$ and using strong triangle inequality one gets

$$|x_{1,2}|_p = 1,$$
$$|x_{1,2} + \theta + q - 2|_p \leq |\theta - 1|_p,$$
$$|\theta x_{1,2} + q - 1|_p \leq |\theta - 1|_p.$$

Putting these into (3.3) we can easily get

$$\left|f'_\theta(x_{1,2})\right|_p \geq \frac{|q|_p}{|\theta-1|_p} > 1.$$

This yields the assumption. □

Now, we are going to describe basin of attraction

$$A(x_0) = \left\{x \in \mathbb{Q}_p : \ f^n_\theta(x) \to x_0\right\}$$

of the fixed point $x_0 = 1$.

Let us denote

$$K_1 = \{x \in \mathbb{Q}_p : |x - x_0|_p < |q|_p\}$$
$$K_2 = \{x \in \mathbb{Q}_p : |x - x_0|_p > |q|_p\}$$

It is easy to check that $x_{1,2} \in \mathbb{Q}_p \setminus (K_1 \cup K_2)$. We show that $f_\theta(x) \in K_1$ for any $x \in K_1 \cup K_2$.

Due to $x_0 = 1$ we have

$$f_\theta(x) - x_0 = \frac{(\theta-1)[(\theta+1)(x-x_0) + 2(\theta-1) + q]}{(x - x_0 + \theta - 1 + q)^2}(x - x_0) \tag{3.4}$$

The non-Archimedean norm's property implies that

$$|(\theta+1)(x-x_0) + 2(\theta-1) + q|_p = \begin{cases} |q|_p, & \text{if} \quad x \in K_1, \\ |x - x_0|_p, & \text{if} \quad x \in K_2, \end{cases} \tag{3.5}$$

and

$$|x - x_0 + \theta - 1 + q|_p = \begin{cases} |q|_p, & \text{if} \quad x \in K_1, , \\ |x - x_0|_p, & \text{if} \quad x \in K_2. \end{cases} \tag{3.6}$$

Inserting (3.5), (3.6) into (3.4) we find

$$|f_\theta(x) - x_0|_p = \begin{cases} \frac{|(\theta-1)(x-x_0)|_p}{|q|_p}, & \text{if} \quad x \in K_1 \\ |\theta - 1|_p, & \text{if} \quad x \in K_2 \end{cases} \tag{3.7}$$

According to $|\theta - 1|_p < |q|_p$, $|x - x_0|_p < |q|_p$ one gets $|f_\theta(x) - x_0|_p < |q|_p$ for any $x \in K_1$. It yields that $f_\theta(x) \in K_1$ for any $x \in K_1 \cup K_2$.

Moreover, from (3.7) one can see that $f_\theta$ is a contraction on $K_1$, which means that $K_1 \subset A(x_0)$. Besides, we also infer that $K_2 \subset f_\theta^{-1}(K_1)$, so $K_2 \subset A(x_0)$. Therefore, we conclude that the set

$$B = \left(\bigcup_{n \geq 0} f_\theta^{-n}(K_1)\right)$$

also belongs to $A(x_0)$.

Thus, we have proven the following result

**Theorem 3.2.** *Let $k = 2$ and $p \geq 3$. If $0 < |\theta - 1|_p < |q|_p$ then*

$$\left(\bigcup_{n \geq 0} f_\theta^{-n}(K_1)\right) = A(x_0).$$

## 4. Chaotic behavior of (2.14)

In this section, we study the dynamics of the function $f_\theta$. In the sequel, we assume that $p \geq 3$, $k = 2$, $q = mp^n$ and $0 < |\theta - 1|_p \leq p^{-2n-1}$, for some $m, n \in \mathbb{N}$ and $(m, p) = 1$.

Now, we are ready to formulate the main result of this section.

**Theorem 4.1.** *Let $|\theta-1|_p\le p^{-2n-1}$, $r=|p^n(\theta-1)|_p$ and $X=B_r(x_1)\cup B_r(x_2)$. If $f_\theta: X\to\mathbb{Q}_p$ be a function defined by (2.14) then the dynamics $(J_{f_\theta}, f_\theta, |\cdot|_p)$ is isometrically conjugate to the shift dynamics $(\Sigma_A,\sigma,d_f)$.*

To proof this thereom we need several auxiliary facts.

**Lemma 4.2.** *Let $r=|p^n(\theta-1)|_p$. Then for any $x\in B_r(x_1)$ and $y\in B_r(x_2)$ there exist p-adic integers $\alpha_x$ and $\beta_y$ such that*

$$\begin{aligned} x&=1-mp^n+(1+m_0p^n+p^{n+1}\alpha_x)(\theta-1)\\ y&=1-mp^n-(1+m_0p^n+p^{n+1}\beta_y)(\theta-1)\end{aligned}\tag{4.1}$$

*where $2m_0\equiv -m(\mathrm{mod}\, p)$.*

**Proof.** It is enough to show that

$$\begin{aligned} x_1&=1-mp^n+(1+m_0p^n)(\theta-1)+p^{n+1}(\theta-1)\alpha\\ x_2&=1-mp^n-(1+m_0p^n)(\theta-1)+p^{n+1}(\theta-1)\beta\end{aligned}\tag{4.2}$$

where $\alpha,\beta\in\mathbb{Z}_p$. Since $|\theta-1|_p\le p^{-2n-1}$ there exists $p$-adic integer $\gamma$ such that

$$-4(q-1)+(\theta-1)^2=4\big(1-mp^n+p^{n+1}\gamma\big).$$

It follows that

$$\sqrt{-4(p-1)+(\theta-1)^2}=2\big(1+m_0\cdot p^n+p^{n+1}\gamma'\big),$$

where $\gamma'\in\mathbb{Z}_p$. Put the last one into (3.1) one gets

$$x_1=1-mp^n+(1+m_0p^n)(\theta-1)+p^{n+1}(\theta-1)\alpha,\quad \alpha\in\mathbb{Z}_p.$$

Similarly, we have

$$x_2=1-mp^n-(1+m_0p^n)(\theta-1)+p^{n+1}(\theta-1)\beta,\quad \beta\in\mathbb{Z}_p.$$

Thus, we have shown that (4.2) holds. Using non-Archimedean norm's property from (4.2) one finds (4.1). □

**Corollary 4.3.** *Let $r=|p^n(\theta-1)|_p$. Then one has $B_r(x_1)\cap B_r(x_2)=\varnothing$.*

**Proof.** It is enough to show that $x_1\notin B_r(x_2)$. From (4.2), using non-Archimedean norm's property we have

$$|x_1-x_2|_p=|\theta-1|_p>r,$$

which yields $x_1\notin B_r(x_2)$. □

**Lemma 4.4.** *Let $r=|p^n(\theta-1)|_p$ and $X=B_r(x_1)\cup B_2(x_2)$. Then $f_\theta^{-1}(X)\subset X$.*

**Proof.** We know that $f_\theta$ has two inverse branches on $X$, which are

$$f_{1,\theta}(x)=-\frac{(\theta+q-2)\sqrt{x}+q-1}{\theta+\sqrt{x}},$$

$$f_{2,\theta}(x)=\frac{(\theta+q-2)\sqrt{x}-q+1}{\theta-\sqrt{x}}.$$

Let us show that $f_{1,\theta}(x)\in B_r(x_2)$ for any $x\in X$. We have

$$\begin{aligned} &f_{1,\theta}(x)+mp^n-1+m_0p^n(\theta-1)\\ &=\frac{(\theta-1)\big[m_0p^n(\sqrt{x}+1)+mp^n+(1+m_0p^n)(\theta-1)\big]}{\theta+\sqrt{x}}\\ &=\frac{(\theta-1)\big[m_0p^n(\sqrt{x}-1)+(2m_0+m)p^n+(1+m_0p^n)(\theta-1)\big]}{\theta+\sqrt{x}}\end{aligned}\tag{4.3}$$

Since $\theta,\sqrt{x}\in\mathcal{E}_p$ by Lemma 2.1 we get

$$|\theta+\sqrt{x}|_p=1,\quad |\sqrt{x}-1|_p\le\frac{1}{p}\tag{4.4}$$

From $2m_0\equiv -m(\mathrm{p})$ we have

$$|2m_0+m|_p\le\frac{1}{p}\tag{4.5}$$

Inserting (4.4), (4.5) and $|\theta-1|_p\le\frac{1}{p^{n+2}}$ into (4.3) and using strong triangle inequality, we obtain

$$|f_{1,\theta}(x)+mp^n-1+(1+m_0p^n)(\theta-1)|_p\le\left|p^{n+1}(\theta-1)\right|_p$$

which is equivalent to

$$f_{1,\theta}(x)=1-mp^n-(1+m_0p^n)(\theta-1)+p^{n+1}(\theta-1)\beta,$$
for some $\beta\in\mathbb{Z}_p$.

According to Lemma 4.2 one has $f_{1,\theta}(x)\in B_r(x_2)$.

Now, we show that $f_{2,\theta}(x)\in B_r(x_1)$, for any $x\in X$. Let $x\in X$. Then we have

$$\begin{aligned} &f_{2,\theta}(x)+mp^n-1-(1+m_0p^n)(\theta-1)\\ &=\frac{(\theta-1)\big[\sqrt{x}-1+mp^n-(\theta-\sqrt{x})(1+m_0p^n)\big]}{\theta-\sqrt{x}}\\ &=\frac{(\theta-1)\big[(\sqrt{x}-1)(2+m_0p^n)+mp^n-(\theta-1)(1+m_0p^n)\big]}{\theta-\sqrt{x}}\end{aligned}\tag{4.6}$$

Let us establish

$$\left|(\sqrt{x}-1)(2+m_0p^n)+mp^n\right|_p\le\frac{1}{p^{2n+1}}.$$

Indeed, using $x=1-mp^n+(\theta-1)\alpha$, $|\alpha|_p=1$ we have

$$\begin{aligned} &(\sqrt{x}+1)\big[(\sqrt{x}-1)(2+m_0p^n)+mp^n\big]\\ &=(x-1)(2+m_0p^n)+mp^n(\sqrt{x}+1)\\ &=mp^n(\sqrt{x}-1-m_0p^n)+(\theta-1)(2+m_0p^n)\alpha\end{aligned}$$

Since $\left|\sqrt{x}-1+m_0p^n\right|_p<p^{-n}$ and $|\sqrt{x}+1|_p=1$, the strong triangle inequality with (4.7) implies

$$\left|(\sqrt{x}-1)(2+m_0p^n)+mp^n\right|_p\le\frac{1}{p^{2n+1}}.$$

Plugging the last one into (4.6) one gets

$$|f_{2,\theta}(x)+mp^n-1-(1+m_0p^n)(\theta-1)|_p<|p^n(\theta-1)|_p.$$

By Lemma 4.2 we find $f_{2,\theta}(x)\in B_r(x_1)$.

Since $x$ is an arbitrary, we conclude that $f_\theta^{-1}(X)\subset X$. This completes the proof. □

**Lemma 4.5.** *Let $r=|p^n(\theta-1)|_p$. Then one has*

$$|f_\theta(x)-f_\theta(y)|_p=\frac{|x-y|_p}{p^2r},\quad\text{for any } x,y\in B_r(x_1)$$

*and*

$$|f_\theta(x)-f_\theta(y)|_p=\frac{|x-y|_p}{r},\quad\text{for any } x,y\in B_r(x_2)$$

**Proof.** Let $x\ne y$. Then we have

$$\frac{f_\theta(x)-f_\theta(y)}{x-y}=\frac{(\theta-1)(\theta+mp^n-1)R(x,y)}{Q^2(x)Q^2(y)}\tag{4.7}$$

where

$$\begin{aligned} R(x,y)&=2\theta xy+\big[\theta^2+\theta(mp^n-2)+mp^n-1\big](x+y)\\ &\quad+2(mp^n-1)(\theta+mp^n-2)\end{aligned}\tag{4.8}$$

$$Q(x)=x+\theta+mp^n-2$$

It is easy to check that

$$|Q(x)|_p=\begin{cases}|\theta-1|_p, & \text{if } x\in B_r(x_1)\\ |p^n(\theta-1)|_p, & \text{if } x\in B_r(x_2)\end{cases}\tag{4.9}$$

Let $x,\ y\in B_r(x_1)$. Then by Lemma 4.2 we have

$$\begin{aligned} x&=1-mp^n+(1+m_0p^n)(\theta-1)+p^{n+1}(\theta-1)\alpha_x,\\ y&=1-mp^n+(1+m_0p^n)(\theta-1)+p^{n+1}(\theta-1)\alpha_y.\end{aligned}$$

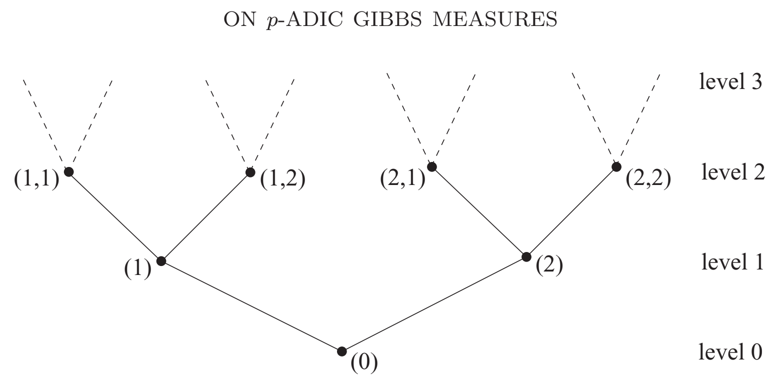


**Fig. 1.** The first levels of $\Gamma^2_+$.

Plugging last ones into (4.8) one gets

$$R(x,y) = (\theta-1)^2\Big[4(\theta-1) + p^{n+1}\Big[3(\theta-1) + p^n(2m_0\theta - m) + 4\Big] (\alpha_x+\alpha_y) + 2p^n\Big[(\theta-1)(2m_0+1) + \theta p^{n+2}\alpha_x\alpha_y + m_0p^n(m_0\theta - m) - 2m\Big] + 8\Big] \tag{4.10}$$

Similarly, for $x, y \in B_r(x_2)$ we obtain

$$R(x,y) = p^n(\theta-1)^2\Big[2m_0p^n\Big[m_0(\theta-1) + m_0 + m\Big] + 2\theta p^{n+2}\alpha_x\alpha_y - p^{n+1}\Big[p(\theta-1) - 2m_0\theta + m\Big]\Big(\alpha_x+\alpha_y\Big) + 2(\theta-1)m_0\Big] \tag{4.11}$$

Keeping into account the following relations

$$\begin{aligned} &|\theta-1|_p \le \tfrac{1}{p^{2n+1}},\\ &|3(\theta-1) + p^n(2m_0\theta-m) + 4|_p = 1,\\ &\Big|(\theta-1)(2m_0+1) + \theta p^{n+2}\alpha_x\alpha_y + m_0p^n(m_0\theta-m) - 2m\Big|_p = 1,\\ &\Big|2m_0p^n\Big[m_0(\theta-1)+m_0+m\Big|_p = 1,\\ &|p(\theta-1) - 2m_0\theta + m|_p < 1, \end{aligned}$$

and using the non-Archimedean norm's property, from (4.10) and (4.11), we find

$$|R(x,y)|_p = \begin{cases} \Big|(\theta-1)^2\Big|_p & , \text{ if } x,y \in B_r(x_1)\\ \Big|p^{2n}(\theta-1)^2\Big|_p & , \text{ if } x,y\in B_r(x_2)\end{cases} \tag{4.12}$$

Hence, by means of (4.9), (4.12), from (4.7) one gets

$$\frac{|f_\theta(x) - f_\theta(y)|_p}{|x-y|_p} = \begin{cases} \frac{1}{p^n|\theta-1|_p} & , \text{ if } x,y\in B_r(x_1)\\ \frac{p^n}{|\theta-1|_p} & , \text{ if } x,y\in B_r(x_2)\end{cases}$$

This completes the proof. □

**Corollary 4.6.** *Let $r = |p^n(\theta-1)|_p$. Then $B_r(x_i) \subset f_\theta(B_r(x_j)), i, j \in \{1, 2\}$.*

**Proof.** By Lemma 4.4 we have $B_r(x_j) \subset f_\theta(B_r(x_j)), \quad j = 1,2$. Since $|x_1 - x_2|_p = p^n r \le \frac{1}{p^{2n+1}}$ by Lemma 4.4 one gets $x_1 \in B_1(x_2)$, $x_2 \in B_{\frac{1}{p^{2n}}}(x_1)$. This yields that $B_r(x_1) \subset f_\theta(B_r(x_2))$ and $B_r(x_2) \subset f(B_r(x_1))$. □

**Proof of Thorem 4.1.** According to Lemma 4.4 we have $f_\theta^{-1}(X) \subset X$. By Lemma 4.5 the triple $(X, J_{f_\theta}, f_\theta)$ is a $p$-adic repeller. Finally, by Corollary 4.6 an incidence matrix $A$ has the following form:

$$A = \begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}$$

So, the triple $(X, J_{f_\theta}, f_\theta)$ be a transitive. According to Theorem 2.2 we conclude that the dynamics $(J_{f_\theta}, f_\theta, |\cdot|_p)$ is isometrically conjugate to the shift dynamics $(\Sigma_A, \sigma, d_f)$. □

## 5. Periodic *p*-adic quasi Gibbs measures

In this section, we will given a consequence of Theorem 4.1, i.e. it allows us to show that how the set of the periodic $p$-adic quasi Gibbs measures is huge.

First, we recall a coordinate structure in $\Gamma^k_+$: every vertex $x$ (except for $x^0$) of $\Gamma^k_+$ has coordinates $(i_1, \dots, i_n)$, here $i_m \in \{1, \dots, k\}$, $1 \le m \le n$ and for the vertex $x^0$ we put $(0)$. Namely, the symbol $(0)$ constitutes level 0, and the sites $(i_1, \dots, i_n)$ form level $n$ (i.e. $d(x^0, x) = n$) of the lattice.

Let us define on $\Gamma^k_+$ a binary operation $\circ : \Gamma^k_+ \times \Gamma^k_+ \to \Gamma^k_+$ as follows: for any two elements $x = (i_1, \dots, i_n)$ and $y = (j_1, \dots, j_m)$ put

$$x \circ y = (i_1, \dots, i_n) \circ (j_1, \dots, j_m) = (i_1, \dots, i_n, j_1, \dots, j_m) \tag{5.1}$$

and

$$x \circ x^0 = x^0 \circ x = (i_1, \dots, i_n) \circ (0) = (i_1, \dots, i_n). \tag{5.2}$$

By means of the defined operation $\Gamma^k_+$ becomes a noncommutative semigroup with a unit. Using this semigroup structure one defines translations $\tau_g : \Gamma^k_+ \to \Gamma^k_+$, $g \in \Gamma^k_+$ by

$$\tau_g(x) = g \circ x. \tag{5.3}$$

It is clear that $\tau_{(0)} = id$.

Let $H \subset \Gamma^k_+$ be a sub-semigroup of $\Gamma^k_+$ and $h : \Gamma^k_+ \to Y$ be a $Y$-valued function defined on $\Gamma^k$. We say that $h$ is *H-periodic* if $h(\tau_g(x)) = h(x)$ for all $g \in H$ and $x \in \Gamma^k_+$. Any $\Gamma^k_+$-periodic function is called *translation invariant*. A $p$-adic quasi Gibbs measure $\mu_{\mathbf{h}}$ is called *H-periodic*, if $\mathbf{h}$ is $H$-periodic function (see Fig. 1).

Now for each $m \ge 2$ we put

$$H_m = \{x \in \Gamma^k_+ : \ d(x, x^0) \equiv 0 (\mathrm{mod}\ m)\}. \tag{5.4}$$

One can check that $H_m$ is a sub-semigroup.

**Remark 5.1.** We stress that in [32] we have established the existence of the phase transition for the considered model. To do so, we found only translation-invariant solutions of (2.10). In [38] it was shown that $H_2$-periodic solutions of (2.10) belonging to $\mathcal{E}_p$ coincides with translation-invariant ones. Therefore, it is natural to find periodic solutions of (2.10) in a general setting, which allows to find periodic quasi Gibbs measures.

Let us consider a $H_m$-periodic function $\mathbf{h} = \{\mathbf{h}_x\}_{x \in V\setminus\{x^0\}}$ on the invariant line $(h, 1, \dots, 1)$ of the Eq. (2.10). From the $H_m$-periodicity we infer that there is a $m$-collection of vectors $\{\mathbf{h}_0, \dots, \mathbf{h}_{m-1}\}$, such that $\mathbf{h}_x = \mathbf{h}_i$, if $d(x, x^0) \equiv i(\mathrm{mod}\, m)$, $i = 0, \dots, m-1$. On the invariant line, we have $\mathbf{h}_i = (h_i, 1, \dots, 1)$ $(i = 0, \dots, m-1)$.

Then the Eq. (2.10), for the $H_m$-periodic functions, reduces to the following system

$$h_i = f_\theta(h_{i+1}), \quad h_m = f_\theta(h_0), \ i = 1, \dots, m-1. \tag{5.5}$$

where $f_\theta$ is defined as (2.14).

It is clear that the Eq. (5.5) is equivalent to finding $m$-periodic points of the function $f_\theta$. Hence, the existence of periodic orbits of the function implies the existence of $H_m$-periodic $p$-adic quasi Gibbs measures. It is well-known that the shift operator has infinitely many periodic points, therefore, Theorem 4.1 implies that the function $f_\theta$ also has infinitely many periodic points. Hence, there are many $H_m$-periodic $p$-adic quasi Gibbs measures.

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