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On metric properties of unconventional limit sets of contractive non-Archimedean dynamical systems

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ABSTRACT
In this paper, we define the limit set Λ^ξ of an unconventional set of contractive functions $\{f_k\}$ on the unit ball of non-Archimedean algebra. Then, we prove that Λ^ξ is compact, perfect and uniformly disconnected. It is shown that there is a new collection of contractive mappings $\{\tilde{F}_k\}$ defined on Λ^ξ. Moreover, we establish that the set Λ^ξ coincides with the limit set generated by the semi-group of $\{\tilde{F}_k\}$. This result allows us to further investigate the structure of Λ^ξ by means of this limiting set. As an application, we demonstrate the existence of invariant measures on Λ^ξ. We should stress that the non-Archimedeanity of the space is essentially used in the paper. Therefore, the methods applied in this paper are not longer valid in the Archimedean setting (i.e. in case of real or complex numbers).

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1. Introduction

In this paper, we deal with metric properties of unconventional sets of discrete contractive dynamical systems defined over non-Archimedean algebras. The field of non-Archimedean dynamical systems is one of the most popular areas of the modern mathematics. There are many works devoted to p-adic and non-Archimedean dynamics.[1–11] We stress that applications of p-adic numbers in p-adic mathematical physics,[12–14] quantum mechanics and many others [15–18] stimulated an increasing interest in the study of p-adic and non-Archimedean dynamical systems.

On the other hand, the metric properties of limit sets in the Euclidean spaces have been studied in investigations of random dynamical systems (see, for example [19–22]). These investigations have found their applications in the fractal geometry.[23,24] This naturally motivates to consider the metric properties of limit sets in a non-Archimedean setting. In this direction, very recently, in [25] it has been considered a semi-group G generated by a finite set $\{f_i\}_{i=1}^N$ of contractive functions on $\mathcal{O}$ (here $\mathcal{O} = \{x \in K : |x| \leq 1\}$ is the closed unit ball of the non-Archimedean algebra K). Namely, $G = \bigcup_{k\geq 1} G_k$, where $G_k = \{f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_k}, 1 \leq i_j \leq N, 1 \leq j \leq k\}$. Furthermore, it was studied metric properties of the limit set Λ of G which is a complement of the set of all points $x \in \mathcal{O}$ for which there exist open neighbourhoods U_x of x such that $g(U_x) \cap U_x = \varnothing$ for all but finitely many $g \in G$.

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Note that the limit set Λ of G is a very important object in the study of random dynamical systems (see, for example [5,11,26]).

One can easily see that the composition of two contractive mappings is again a contraction. But, in general, the product or the sum of such kind of contractions is not a contraction, but in a non-Archimedean situation, they are also contractions. Using that fact in [27,28], the uniqueness limiting sets of unconventional iterates of contractive mappings have been studied. Note that these results play an important role in the theory of p-adic Gibbs measures.[9,29–32]

In this paper, we define the limit set Λ^{ξ} of an unconventional set (see, for the definition Section 3) of contractive functions $\{f_k\}$ on $\mathcal{O}$, and a family of mappings ξ. Note that this set can be considered as a non-Archimedean fractal. In Section 4, we prove that Λ^{ξ} is compact, perfect and uniformly disconnected. Moreover, we show that the set is self-similar with respect to some functions $\{\tilde{F}_k\}$. Note that the results of this section can be considered as an extension of some results of [25] to more general setting. Based on results of Section 4, in Section 5 it is introduced a new metric for which the functions $\{\tilde{F}_k\}$ become contractions of Λ^{ξ}. Moreover, we establish that the set Λ^{ξ} coincides with the limit set generated by $\{\tilde{F}_k\}$. This result allows us to further investigate the structure of Λ^{ξ} by means of this limiting set. As an application, we demonstrate the existence of invariant measures on Λ^{ξ}. We should stress that the non-Archimedeanity of the algebra is essentially used in the paper. Therefore, the methods applied in this paper are not longer valid in the Archimedean setting (i.e. in case of real or complex numbers).

2. Definitions and preliminary results

Let K be a field with a non-Archimedean norm $|\cdot|$, i.e. for all $x, y \in K$ one has

(1) $|x| \geq 0$ and $|x| = 0$ implies $x = 0$;
(2) $|xy| = |x| \cdot |y|$; and
(3) $|x + y| \leq \max\{|x|, |y|\}$.

Standard examples of such fields are fields of p-adic numbers $\mathbb{Q}_p$. Let p be a prime, the set $\mathbb{Q}_p$ is defined as a completion of the rational numbers $\mathbb{Q}$ with respect to the norm $|\cdot|_p : \mathbb{Q} \to \mathbb{R}$ given by

$$|x|_p = \begin{cases} p^{-r} & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{2.1}$$

here, $x = p^r \frac{m}{n}$ with $r, m \in \mathbb{Z}$, $n \in \mathbb{N}$, $(m, p) = (n, p) = 1$. The absolute value $|\cdot|_p$ is non-Archimedean. There are also many other examples of non-Archimedean fields (see, for example [33]).

Now let $(\mathcal{A}, \|\cdot\|)$ be a non-Archimedean Banach algebra over K. This means that the norm $\|\cdot\|$ of the algebra satisfies the non-Archimedean property, i.e. $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for any $x, y \in \mathcal{A}$. We recall a nice property of the norm, i.e. if $\|x\| > \|y\|$ then $\|x + y\| = \|x\|$. Note that this is a crucial property which is proper to the non-Archimedenity of the norm. There are many examples of such kind of spaces (see [34,35]).

Let us consider some basic examples of non-Archimedean Banach algebras.

(1) Let K be a non-Archimedean field and put

$$K^n = \{\mathbf{x} = (x_1, \dots, x_n) : \ x_k \in K, \ k = 1, \dots, n\}.$$

Then $(K^n, \|\cdot\|)$ with a norm $\|\mathbf{x}\| = \max|x_k|$ and usual pointwise summation and multiplication operations, is a non-Archimedean Banach algebra over K.

(2) Let K be as above and put

$$c_0 = \{\mathbf{x} = (x_n) : \ x_n \in K, \ x_n \to 0\}.$$

The defined set is endowed with usual pointwise summation and multiplication operations. Put $\|\mathbf{x}\| = \max|x_k|$, then $(c_0, \|\cdot\|)$ is a non-Archimedean Banach algebra over K.

In what follows, by $\mathcal{A}$ we denote a non-Archimedean Banach algebra.

Put

$$B^-(a, r) = \{x \in \mathcal{A} : \|x - a\| < r\}, \ \ B(a, r) = \{x \in \mathcal{A} : \|x - a\| \le r\},$$
$$S(a, r) = \{x \in \mathcal{A} : \|x - a\| = r\},$$

where $a \in \mathcal{A}$, $r > 0$.

In what follows, we will use the following lemma.

Lemma 2.1 ([36]): *Let* $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset \mathcal{A}$ *such that* $\|a_i\| \le 1$, $\|b_i\| \le 1$, $i = 1, \dots, n$, *then*

$$\left\| \prod_{i=1}^n a_i - \prod_{i=1}^n b_i \right\| \le \max_{i \le i \le n}\{\|a_i - b_i\|\}.$$

We should stress that a similar inequality does not exist in the Archimedean setting. We refer the reader, for the basics of non-Archimedean analysis, to [33,35].

Recall that a metric space X is said to be *doubling* if there is a constant k such that every disk B in X can be covered with at most k disks of half the radius of B. A number of metric spaces have this property, e.g. the Euclidean space, a compact Riemann surface, etc. However, a non-Archimedean space is not necessarily a doubling space, e.g. $\mathbb{C}_p$ (complex p-adic field). Hence, it is very important to know whether a subspace of a non-Archimedean space is a doubling space. Note that $\mathbb{Q}_p$ is doubling. Let (X, d) be a complete metric space. Let $\mathcal{B}$ be the collection of all bounded subsets of X. For a set $E \in \mathcal{B}$, we denote the diameter of E by $\mathrm{diam}(E) = \sup_{z,w \in E} d(z, w)$. By definition, a set $E \in \mathcal{B}$ is called a *uniformly perfect set* if E contains at least two points and there exists a constant $c > 0$ such that for any point $x_0 \in E$ and $0 < r < \mathrm{diam}(E)$, the annulus $\{x \in X: cr \le d(x, x_0) \le r\}$ meets E. We say that a metric space (X, d) is *uniformly disconnected* if there is a constant $C > 1$ so that for each $x \in X$ and $r > 0$ we can find a closed subset A of X such that $B_{r/C}(x) \subset A \subset B_r(x)$, and $\mathrm{dist}(A, X \backslash A) \ge C^{-1} r$.

3. The unconventional limit set

Let $\mathcal{A}$ be a non-Archimedean Banach algebra over the field K, and $\mathcal{O} = \{x \in \mathcal{A} : \|x\| \leq 1\}$. A mapping $f : \mathcal{O} \to \mathcal{O}$ is called *contractive*, if there is a constant $\lambda_f \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \lambda_f \|x - y\|, \tag{3.1}$$

for all $x, y \in \mathcal{O}$.

Now assume that we are given a collection $\{f_i\}_{i=1}^N$ of contractive mappings defined on $\mathcal{O}$. In what follows, we denote $\lambda = \max\{\lambda_{f_i}\}$. It is clear that $\lambda \in (0, 1)$.

For convenience, instead of $\{1, 2, \dots, N\}$ we will write $[1, N]$. In what follows, we frequently use the denotation $\Sigma = [1, N]^{\mathbb{N}}$. The *shift* mapping $\sigma\colon \Sigma \to \Sigma$ is defined by usual way, i.e. $\sigma(\alpha)_k = \alpha_{k+1}$, $k \in \mathbb{N}$. Here $\alpha = (\alpha_1, \dots, \alpha_n, \dots)$.

Let M, L be fixed positive integers and consider a family $\xi := \{\xi_{ij}\colon [1, N] \to [1, N]\colon (i, j) \in [1, M] \times [1, L]\}$ of mappings.

For each $\alpha \in \Sigma$, $n \in \mathbb{N}$ we denote

$$F_{\alpha,n} = \sum_{i=1}^{M} \prod_{j=1}^{L} F_{\alpha,n}^{\xi_{ij}}, \tag{3.2}$$

where

$$F_{\alpha,n}^{\xi_{ij}} = f_{\xi_{ij}(\alpha_1)} \circ \cdots \circ f_{\xi_{ij}(\alpha_n)}, \quad \alpha = (\alpha_1, \dots, \alpha_n, \dots). \tag{3.3}$$

Put

$$\mathcal{F}_\xi = \bigcup_{n \geq 1} \mathcal{F}_{\xi,n}, \quad \mathcal{F}_{\xi,n} = \left\{F_{\alpha,n} : \alpha \in \Sigma\right\}.$$

The set $\mathcal{F}_\xi$ is called an *unconventional set* of the semi-group G generated by a finite set $\{f_i\}_{i=1}^N$.

Remark 3.1: *In the sequel, we always assume that the family ξ satisfies the following condition:*

$$\bigcup_{k=1}^{N} \bigcup_{i=1}^{M} \bigcup_{j=1}^{L} \{\xi_{ij}(k)\} = [1, N]. \tag{3.4}$$

Otherwise, the set $\mathcal{F}_\xi$ will be generated by a subset of $\{f_i\}_{i=1}^N$.

Example 3.1: Let us construct an example of a family of mappings $\xi = \{\xi_{ij}\colon [1, N] \to [1, N]\}$ which satisfies (3.4). For any integer number $\ell \in [1, N]$, we define an action on $[1, N]$ by

$$(\ell * k) = \begin{cases} (\ell + k)(\mathrm{mod}\, N) & \text{if } N \not| (\ell + k) \\ N & \text{if } N | (\ell + k). \end{cases} \quad k \in [1, N].$$

Then for any number $\xi_{ij} \in [1, N]$, we define $\xi_{ij}(k) := (\xi_{ij} {}^* k)$, $k \in [1, N]$. It is clear that ξ_{ij}: $[1, N] \to [1, N]$ and (3.4) holds.

In what follows, we need the following auxiliary fact.

Lemma 3.1: *Every element $F \in \mathcal{F}_\xi$ is a contractive mapping of $\mathcal{O}$.*

Proof: Let $F \in \mathcal{F}_\xi$. By construction of the set $\mathcal{F}_\xi$, there exist $\alpha \in \Sigma$ and an integer number $n \geq 1$ such that

$$F = \sum_{i=1}^{M} \prod_{j=1}^{L} F_{\alpha,n}^{\xi_{ij}}. \tag{3.5}$$

It is clear that $F_{\alpha,n}^{\xi_{ij}}$ be a mapping from $\mathcal{O}$ into $\mathcal{O}$ for any $(i, j) \in [1, M] \times [1, L]$. Consequently, for any $x \in \mathcal{O}$ using the strong triangle inequality from (3.5) one can find

$$\|F(x)\| \leq \max_{1\leq i\leq M} \left\| \prod_{j=1}^{L} F_{\alpha,n}^{\xi_{ij}}(x) \right\| \leq \max_{\substack{1\leq i\leq M\\ 1\leq j\leq L}} \left\| F_{\alpha,n}^{\xi_{ij}}(x) \right\| \leq 1.$$

This means that F is a mapping from $\mathcal{O}$ into $\mathcal{O}$.

Let $x, y \in \mathcal{O}$. Then by means of Lemma 2.1, we find

$$\left\| F(x) - F(y) \right\| \leq \max_{\substack{1\leq i\leq M\\ 1\leq j\leq L}} \left\| F_{\alpha,n}^{\xi_{ij}}(x) - F_{\alpha,n}^{\xi_{ij}}(y) \right\|. \tag{3.6}$$

By (3.3), one gets

$$\left\| F_{\alpha,n}^{\xi_{ij}}(x) - F_{\alpha,n}^{\xi_{ij}}(y) \right\| \leq \lambda \|x - y\| \quad \text{for any } (i,\ j) \in [1, M] \times [1, L].$$

Now substituting the last one into (3.6) we obtain the required assertion. □

The main aim of this paper is to study the limiting set of $\mathcal{F}_\xi$. First, one defines *the discontinuity set* $\Omega^\xi \subset \mathcal{O}$ of $\mathcal{F}_\xi$ as follows: $x \in \Omega^\xi$ if and only if there is a disk $B_r(x)$ such that there are only finitely many $F \in \mathcal{F}_\xi$ satisfying $F(B_r(x)) \cap B_r(x) = \emptyset$. The *limit set* of $\mathcal{F}_\xi$ is denoted by Λ^ξ, which is the compliment of the discontinuity set Ω^ξ, i.e. $\Lambda^\xi = \mathcal{O} \setminus \Omega^\xi$.

4. Some properties of the unconventional limit set

In this section, we study several metric properties of the set Λ^ξ. Namely, the followings are the main results of the paper.

It is well-known from Hutchinson [21] that the limit set of a semi-group generated by a finite set of contractive functions on a metric space is always a compact set. It turns out that we also have a similar kind of result.

Theorem 4.1: *One has $\Lambda^\xi = \widetilde{\Lambda}^\xi$, where $\widetilde{\Lambda}^\xi$ is defined by (4.9). Moreover, Λ^ξ is compact.*

The perfectness is a very important property for a metric space, since it has a number of applications.

Theorem 4.2: *If Λ^ξ contains at least two points then it is perfect.*

The doubling property of a metric space is also very important. Many metric spaces have the doubling property, e.g. the Euclidean space, a compact Riemann surface,etc. However, not all non-Archimedean spaces have the doubling property, e.g. the limit set can be viewed as a metric subspace of $\mathcal{A}$. Hence, it is natural to ask whether a limit set has the doubling property.

Theorem 4.3: *Let K have a doubling property. Assume that $\mathcal{A} = K^n$, then Λ^ξ is doubling and uniformly disconnected.*

To prove our main results, we need some auxiliary and preparatory results.

Let us denote

$$\Lambda_0^\xi = \left\{ \sum_{i=1}^{M} \prod_{j=1}^{L} x_{\xi_{ij},\alpha}^{(n)} : x_{\xi_{ij},\alpha}^{(n)} = F_{\alpha,n}^{\xi_{ij}}\left(x_{\xi_{ij},\alpha}^{(n)}\right) \text{ for some } \alpha \in \Sigma \text{ and } n \in \mathbb{N} \right\}. \tag{4.1}$$

Proposition 4.1: *The limit set Λ^ξ of $\mathcal{F}_\xi$ coincides with the closure of Λ_0^ξ.*

Proof: Let us first show that Λ^ξ is closed. It is enough to establish that Ω^ξ is open. Take any $x \in \Omega^\xi$. Then, there exist $r > 0$ and $\{F_k\}_{k=1}^m \subset \mathcal{F}_\xi$ such that $F_k(B_r(x)) \cap B_r(x) = \emptyset$, $k = \overline{1,m}$. Since for any $y \in B_r^-(x)$, one has $B_r(y) = B_r(x)$, hence we have $F_k(B_r(y)) \cap B_r(y) = \emptyset$. This implies that $B_r^-(x) \subset \Omega^\xi$, so Ω^ξ is open.

Let $x \in \Lambda_0^\xi$. Then, there exist $\alpha \in \Sigma$ and an integer number $n \geq 1$ such that

$$x = \sum_{i=1}^{M} \prod_{j=1}^{L} x_{\xi_{ij},\alpha}^{(n)}, \tag{4.2}$$

where $x_{\xi_{ij},\alpha}^{(n)}$ is a fixed point of $F_{\alpha,n}^{\xi_{ij}}$ (which due to Lemma 3.1 exists).

Consider a sequence $\{F_m\}_{m=1}^\infty$ defined by

$$F_m = \sum_{i=1}^{M} \prod_{j=1}^{L} \left(F_{\alpha,n}^{\xi_{ij}}\right)^m = \sum_{i=1}^{M} \prod_{j=1}^{L} (f_{\xi_{ij}(\alpha_1)} \circ \cdots \circ f_{\xi_{ij}(\alpha_n)})^m. \tag{4.3}$$

It is clear that for any $m \geq 1$, we have $F_m \in \mathcal{F}_{\xi,nm}$, hence $F_m \in \mathcal{F}_\xi$. Take any $r > 0$ and $y \in B_r(x)$. Then, from (4.2) and (4.3) one finds

$$\begin{aligned} F_m(y) - x &= \sum_{i=1}^{M} \prod_{j=1}^{L} \left(F_{\alpha,n}^{\xi_{ij}}\right)^m(y) - \sum_{i=1}^{M} \prod_{j=1}^{L} x_{\xi_{ij},\alpha}^{(n)} \\ &= \sum_{i=1}^{M} \prod_{j=1}^{L} \left(F_{\alpha,n}^{\xi_{ij}}\right)^m(y) - \sum_{i=1}^{M} \prod_{j=1}^{L} F_{\alpha,n}^{\xi_{ij}}\left(x_{\xi_{ij},\alpha}^{(n)}\right) \end{aligned}$$

$$= \sum_{i=1}^{M} \prod_{j=1}^{L} \left(F_{\alpha,n}^{\xi_{ij}}\right)^m (y) - \sum_{i=1}^{M} \prod_{j=1}^{L} \left(F_{\alpha,n}^{\xi_{ij}}\right)^m \left(x_{\xi_{ij},\alpha}^{(n)}\right)$$

$$= \sum_{i=1}^{M} \sum_{j=1}^{L} \left[\left(F_{\alpha,n}^{\xi_{ij}}\right)^m (y) - \left(F_{\alpha,n}^{\xi_{ij}}\right)^m \left(x_{\xi_{ij},\alpha}^{(n)}\right)\right] \prod_{k>j} \left(F_{\alpha,n}^{\xi_{ik}}\right)^m (y) \prod_{l<j} \left(F_{\alpha,n}^{\xi_{il}}\right)^m \left(x_{\xi_{il},\alpha}^{(n)}\right).$$

The last equality with the strong triangle inequality implies that

$$\left\|F_m(y) - x\right\| \leq \max_{i,j} \left\|\left(F_{\alpha,n}^{\xi_{ij}}\right)^m (y) - \left(F_{\alpha,n}^{\xi_{ij}}\right)^m \left(x_{\xi_{ij},\alpha}^{(n)}\right)\right\|. \tag{4.4}$$

The contractivity $F_{\alpha,n}^{\xi_{ij}}$ with (4.4) yields

$$\left\|F_m(y) - x\right\| \leq \lambda^m \max_{i,j} \left\|y - x_{\xi_{ij},\alpha}^{(n)}\right\| \leq \lambda^m.$$

Then, there exists a positive integer m_r such that $F_m(y) \in B_r(x)$ for all $m > m_r$. This means that $F_m(B_r(x)) \cap B_r(x) \neq \emptyset$. Consequently, $x \in \Lambda^\xi$. Since Λ^ξ is closed, we have $\overline{\Lambda_0^\xi} \subset \Lambda^\xi$.

Now suppose that $x_0 \notin \overline{\Lambda_0^\xi}$. Then, there exists $r > 0$ such that $B_r(x_0) \cap \overline{\Lambda_0^\xi} = \emptyset$. Choose a positive integer n_0 such that $\lambda^{n_0} < r$. Consider a function $F \in \mathcal{F}_\xi$ defined by

$$F = \sum_{i=1}^{M} \prod_{j=1}^{L} F_{\alpha,n_0}^{\xi_{ij}} \quad \text{for some } \alpha \in \Sigma.$$

It is easy to see that

$$\|F(x) - F(y)\| \leq \lambda^n \|x - y\| \leq \lambda^n \quad \text{for any } x, y \in \mathcal{O}. \tag{4.5}$$

Denote

$$x_F = \sum_{i=1}^{M} \prod_{j=1}^{L} x_{\xi_{ij},\alpha}^{(n_0)},$$

here as before $x_{\xi_{ij},\alpha}^{(n_0)}$ is a fixed point of $F_{\alpha,n_0}^{\xi_{ij}}$. Then $x_F \in \Lambda_0^\xi$. According to Lemma 3.1, the function F has a unique fixed point z_F on $\mathcal{O}$. Now from the strong triangle inequality and (4.5) we obtain

$$\begin{aligned}
\|z_F - x_F\| &= \|F(z_F) - x_F\| \\
&= \left\|\sum_{i=1}^{M} \prod_{j=1}^{L} F_{\alpha,n_0}^{\xi_{ij}}(z_F) - \sum_{i=1}^{M} \prod_{j=1}^{L} F_{\alpha,n_0}^{\xi_{ij}}\left(x_{\xi_{ij},\alpha}^{(n_0)}\right)\right\|
\end{aligned}$$

$$= \left\| \sum_{i=1}^{M} \sum_{j=1}^{L} \left[F_{\alpha,n_0}^{\xi_{ij}}(z_F) - F_{\alpha,n_0}^{\xi_{ij}}\left(x_{\xi_{ij},\alpha}^{(n_0)}\right)\right] \prod_{\substack{k>j\\ l<j}} F_{\alpha,n_0}^{\xi_{ik}}(z_F) F_{\alpha,n_0}^{\xi_{il}}\left(x_{\xi_{il},\alpha}^{(n_0)}\right) \right\|$$

$$\leq \max_{i,j} \left\| F_{\alpha,n_0}^{\xi_{ij}}(z_F) - F_{\alpha,n_0}^{\xi_{ij}}\left(x_{\xi_{ij},\alpha}^{(n_0)}\right)\right\| \prod_{\substack{k>j\\ l<j}} \left\| F_{\alpha,n_0}^{\xi_{ik}}(z_F) F_{\alpha,n_0}^{\xi_{il}}\left(x_{\xi_{il},\alpha}^{(n_0)}\right)\right\| \leq \lambda^{n_0} < r. \tag{4.6}$$

For any $y \in B_r^-(x_0)$ due to contractivity of F, one gets

$$\|F(y) - z_F\| = \|F(y) - F(z_F)\| < \|y - z_F\|.$$

Again the strong triangle inequality implies

$$\|F(y) - y\| = \|F(y) - z_F + z_F - y\| = \|y - z_F\|.$$

Since $y \in B_r^-(x_0)$ and $B_r^-(x_0) \cap \overline{\Lambda_0^\xi} = \emptyset$, $x_F \in \Lambda_0^\xi$, one concludes that $\|y - x_F\| > r$. Therefore, from (4.6) it follows that

$$\|y - z_F\| = \|y - x_F + x_F - z_F\| = \|y - x_F\| > r.$$

Hence,

$$\|F(y) - y\| = \|y - x_F\| > r.$$

Consequently, with $\|x_0 - y\| < r$ one finds

$$\|F(y) - x_0\| = \|F(y) - y + y - x_0\| = \|F(y) - y\| > r.$$

This means that $F(y)$ does not belong to the disk $B_r(x_0)$. Hence, $F(B_r(x_0)) \cap B_r(x_0) = \emptyset$ which implies $x_0 \in \Omega^\xi$. It follows that $\Lambda^\xi \subset \overline{\Lambda_0^\xi}$. Hence $\Lambda^\xi = \overline{\Lambda_0^\xi}$. This completes the proof. □

Lemma 4.1: *Let $\alpha \in \Sigma$. Then a sequence defined by*

$$x_\alpha^{(n)} = \sum_{i=1}^{M} \prod_{j=1}^{L} x_{\xi_{ij},\alpha}^{(n)} \tag{4.7}$$

converges as $n \to \infty$. Here as before $x_{\xi_{ij},\alpha}^{(n)}$ is a fixed point of $F_{\alpha,n}^{\xi_{ij}}$.

Proof: Due to the closedness of $\mathcal{O}$ it is enough to show that the sequence (4.7) is Cauchy.

Take an arbitrary $\varepsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that $\lambda^{n_0} < \varepsilon$. Then for any $n, m \geq n_0$ $(n > m)$ from Lemma 2.1, we have

$$\begin{aligned}\left\|x_\alpha^{(n)} - x_\alpha^{(m)}\right\| &\leq \max_{i,j} \left\|x_{\xi_{ij},\alpha}^{(n)} - x_{\xi_{ij},\alpha}^{(m)}\right\| \\ &= \max_{i,j} \left\|F_{\alpha,n}^{\xi_{ij}}\left(x_{\xi_{ij},\alpha}^{(n)}\right) - F_{\alpha,m}^{\xi_{ij}}\left(x_{\xi_{ij},\alpha}^{(m)}\right)\right\| \\ &= \max_{i,j} \left\|F_{\alpha,m}^{\xi_{ij}} \circ F_{\sigma^m(\alpha),n-m}^{\xi_{ij}}\left(x_{\xi_{ij},\alpha}^{(n)}\right) - F_{\alpha,m}^{\xi_{ij}}\left(x_{\xi_{ij},\alpha}^{(m)}\right)\right\| \\ &\leq \lambda^m \max_{i,j} \left\|F_{\sigma^m(\alpha),n-m}^{\xi_{ij}}\left(x_{\xi_{ij},\alpha}^{(n)}\right) - x_{\xi_{ij},\alpha}^{(m)}\right\| \leq \lambda^m < \varepsilon.\end{aligned}$$

This means that $\{x_\alpha^{(n)}\}$ is a Cauchy sequence. □

For a given $\alpha \in \Sigma$ due to Lemma 4.1, we denote

$$x_\alpha = \lim_{n\to\infty} x_\alpha^{(n)}. \tag{4.8}$$

Put

$$\widetilde{\Lambda}^\xi = \{x_\alpha : \alpha \in \Sigma\}. \tag{4.9}$$

From the above given proof we infer the following

Corollary 4.1: *Let $\alpha \in \Sigma$. Then a sequence $\{x_{\xi_{ij},\alpha}^{(n)}\}$ converges as $n \to \infty$.*

Taking into account the last corollary, we denote

$$x_{\xi_{ij},\alpha} = \lim_{n\to\infty} x_{\xi_{ij},\alpha}^{(n)}. \tag{4.10}$$

Hence, (4.8) can be rewritten as follows:

$$x_\alpha = \sum_{i=1}^{M} \prod_{j=1}^{L} x_{\xi_{ij},\alpha}. \tag{4.11}$$

Lemma 4.2: *Let $F_{\alpha,n} \in \mathcal{F}_\xi$. For any $x = \sum_{i=1}^M \prod_{j=1}^L x_{\xi_{ij},\beta}^{(\ell)} \in \Lambda_0^\xi$ let us define a mapping by*

$$\widetilde{F}_{\alpha,n}[x] := \sum_{i=1}^{M} \prod_{j=1}^{L} F_{\alpha,n}^{\xi_{ij}}\big(x_{\xi_{ij},\beta}^{(\ell)}\big), \tag{4.12}$$

Then, one has $\widetilde{F}_{\alpha,n}[\Lambda_0^\xi] \subset \Lambda^\xi$.

Proof: Let us establish $\widetilde{F}_{\alpha,n}[x] \in \Lambda^\xi$. Take any $r > 0$ and $y \in B_r(\widetilde{F}_{\alpha,n}[x])$. Consider the following sequence:

$$F_m = \sum_{i=1}^{M} \prod_{j=1}^{L} F_{\alpha,n}^{\xi_{ij}} \circ \left(F_{\beta,l}^{\xi_{ij}}\right)^m.$$

It is clear that $F_m \in \mathcal{F}_\xi$ for all $m \geq 1$. Then by Lemma 2.1, one gets

$$\begin{aligned}\left\|F_m(y) - \widetilde{F}_{\alpha,n}[x]\right\| &= \left\|\sum_{i=1}^{M}\prod_{j=1}^{L} F_{\alpha,n}^{\xi_{ij}} \circ \left(F_{\beta,\ell}^{\xi_{ij}}\right)^m (y) - \sum_{i=1}^{M}\prod_{j=1}^{L} F_{\alpha,n}^{\xi_{ij}}\left(x_{\xi_{ij},\beta}^{(\ell)}\right)\right\| \\ &\leq \max_{i,j}\left\|F_{\alpha,n}^{\xi_{ij}} \circ \left(F_{\beta,\ell}^{\xi_{ij}}\right)^m (y) - F_{\alpha,n}^{\xi_{ij}}\left(x_{\xi_{ij},\beta}^{(\ell)}\right)\right\| \\ &\leq \lambda^n \max_{i,j}\left\|\left(F_{\beta,\ell}^{\xi_{ij}}\right)^m (y) - x_{\xi_{ij},\beta}^{(\ell)}\right\| \\ &< \lambda^n \max_{i,j}\left\|\left(F_{\beta,\ell}^{\xi_{ij}}\right)^m (y) - \left(F_{\beta,\ell}^{\xi_{ij}}\right)^m\left(x_{\xi_{ij},\beta}^{(\ell)}\right)\right\| \\ &\leq \lambda^{m+n} \max_{i,j}\left\|y - x_{\xi_{ij},\beta}^{(\ell)}\right\| \leq \lambda^m.\end{aligned}$$

This means that there exists $m_r \in \mathbb{N}$ such that

$$F_m(B_r(\widetilde{F}_{\alpha,n}[x])) \cap B_r(x) \neq \emptyset \quad \text{for all } m \geq m_r$$

which yields $\widetilde{F}_{\alpha,n}[x] \in \Lambda^\xi$. This completes the proof. □

Remark 4.1: *Since $\Lambda_0^\xi \subset \Lambda^\xi$ a natural question arises: how can we extend the function (4.12) to Λ^ξ?*

Given $\alpha, \beta \in \Sigma$ and $n \in \mathbb{N}$ we define an element of Σ by

$$\alpha^{[n]} \vee \beta = (\alpha_1, \ldots, \alpha_n, \beta_1, \beta_2, \ldots).$$

Lemma 4.3: *For each $F_{\alpha,n} \in \mathcal{F}_{\xi,n}$ and $\beta \in \Sigma$ the sequence*

$$\left\{\sum_{i=1}^{M}\prod_{j=1}^{L} F_{\alpha,n}^{\xi_{ij}}\left(x_{\xi_{ij},\beta}^{(m)}\right)\right\}_{m\in\mathbb{N}}$$

is Cauchy. Here, as before, $x_{\xi_{ij},\beta}^{(m)}$ is a fixed point of $F_{\beta,m}^{\xi_{ij}}$.

Proof: The proof immediately follows from

$$\begin{aligned}\left\|\sum_i\prod_j F_{\alpha,n}^{\xi_{ij}}\left(x_{\xi_{ij},\beta}^{(m)}\right) - \sum_i\prod_j F_{\alpha,n}^{\xi_{ij}}\left(x_{\xi_{ij},\beta}^{(\ell)}\right)\right\| &\leq \max_{i,j}\left\|F_{\alpha,n}^{\xi_{ij}}\left(x_{\xi_{ij},\beta}^{(m)}\right) - F_{\alpha,n}^{\xi_{ij}}\left(x_{\xi_{ij},\beta}^{(\ell)}\right)\right\| \\ &\leq \lambda^n \max_{i,j}\left\|x_{\xi_{ij},\beta}^{(m)} - x_{\xi_{ij},\beta}^{(\ell)}\right\| \\ &\leq \lambda^{n+\min\{m,\ell\}} \to 0\end{aligned}$$

□

For any $F_{\alpha,n} \in \mathcal{F}_\xi$, we define

$$\widetilde{F}_{\alpha,n}[x_\beta] := \lim_{m\to\infty}\sum_{i=1}^{M}\prod_{j=1}^{L} F_{\alpha,n}^{\xi_{ij}}\left(x_{\xi_{ij},\beta}^{(m)}\right). \tag{4.13}$$

Proposition 4.2: *For any $F_{\alpha,n} \in \mathcal{F}_\xi$, one has*

$$\widetilde{F}_{\alpha,n}[x_\beta] = x_{\alpha^{[n]}\vee\beta}.$$

Moreover, $\widetilde{F}_{\alpha,n}[\Lambda^\xi] \subset \Lambda^\xi$.

Proof: By definition, we have

$$x_{\alpha^{[n]}\vee\beta} = \lim_{m\to\infty} \sum_{i=1}^{M} \prod_{j=1}^{L} x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta}, \tag{4.14}$$

where $x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta}$ is a fixed point of

$$\underbrace{f_{\xi_{ij}(\alpha_1)} \circ \cdots \circ f_{\xi_{ij}(\alpha_n)}}_{F^{\xi_{ij}}_{\alpha,n}} \circ \underbrace{f_{\xi_{ij}(\beta_1)} \circ \cdots \circ f_{\xi_{ij}(\beta_{n-m})}}_{F^{\xi_{ij}}_{\beta,m-n}}.$$

On the other hand, one has

$$\widetilde{F}_{\alpha,n}[x_\beta] = \lim_{m\to\infty} \sum_{i=1}^{M} \prod_{j=1}^{L} F^{\xi_{ij}}_{\alpha,n}\big(x^{(m)}_{\xi_{ij},\beta}\big). \tag{4.15}$$

Thus, one gets

$$\begin{aligned}
\left\| \sum_i \prod_j F^{\xi_{ij}}_{\alpha,n}\big(x^{(m)}_{\xi_{ij},\beta}\big) - \sum_i \prod_j x^{(m)}_{\xi_{ij},\gamma} \right\| &\le \max_{i,j} \left\| F^{\xi_{ij}}_{\alpha,n}\big(x^{(m)}_{\xi_{ij},\beta}\big) - x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta} \right\| \\
&= \max_{i,j} \left\| F^{\xi_{ij}}_{\alpha,n}\big(x^{(m)}_{\xi_{ij},\beta}\big) - F^{\xi_{ij}}_{\alpha,n} \circ F^{\xi_{ij}}_{\beta,m-n}\big(x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta}\big) \right\| \\
&\le \lambda^n \max_{i,j} \left\| F^{\xi_{ij}}_{\beta,m}\big(x^{(m)}_{\xi_{ij},\beta}\big) - F^{\xi_{ij}}_{\beta,m-n}\big(x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta}\big) \right\| \\
&\le \lambda^m \max_{i,j} \left\| F^{\xi_{ij}}_{\sigma^{m-n}(\beta),n}\big(x^{(m)}_{\xi_{ij},\beta}\big) - x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta} \right\| \\
&\le \lambda^m.
\end{aligned}$$

Consequently, from (4.14) and (4.15) we find the desired equality. This completes the proof. □

Let $k \in [1, N]$ and $\beta \in \Sigma$. For any $\alpha \in \Sigma$ with $\alpha_1 = k$, we denote $F^{\xi_{ij}}_{k,1} := F^{\xi_{ij}}_{\alpha,1}$ and $k\vee\beta := \alpha^{[1]}\vee\beta$.

Corollary 4.2: *Let $F_{k,1} = \sum_{i=1}^{M} \prod_{j=1}^{L} F^{\xi_{ij}}_{k,1}$. Then, one has*

$$\bigcup_{k=1}^{N} \widetilde{F}_{k,1}(\Lambda^\xi) = \Lambda^\xi. \tag{4.16}$$

Now we are ready to turn to the proofs of main results.

Proof of Theorem 4.1: Let us first show that $\widetilde{\Lambda}^{\xi}$ is compact. To do so, we define a mapping $\pi : \Sigma \to \widetilde{\Lambda}^{\xi}$ as follows:

$$\pi(\alpha) = x_{\alpha}.$$

It is known that Σ is compact, and to establish compactness of $\widetilde{\Lambda}^{\xi}$ it is enough to show that π is continuous. Suppose that $\alpha = (\alpha_1, \dots, \alpha_k, \dots) \in \Sigma$ and $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $\lambda^{n_0} < \varepsilon$. Then, for any $\beta \in \Sigma$ with $\beta_k = \alpha_k, k \le n_0$, we find

$$\begin{aligned} \left\| \sum_{i=1}^{M} \prod_{j=1}^{L} F^{\xi_{ij}}_{\beta,n_0}\big(x^{(n_0)}_{\xi_{ij},\beta}\big) - \sum_{i=1}^{M} \prod_{j=1}^{L} F^{\xi_{ij}}_{\alpha,n}\big(x^{(n)}_{\xi_{ij},\alpha}\big) \right\| &\le \max_{i,j} \left\| F^{\xi_{ij}}_{\beta,n_0}\big(x^{(n_0)}_{\xi_{ij},\beta}\big) - F^{\xi_{ij}}_{\alpha,n}\big(x^{(n)}_{\xi_{ij},\alpha}\big) \right\| \\ &= \max_{i,j} \left\| F^{\xi_{ij}}_{\alpha,n_0}\big(x^{(n_0)}_{\xi_{ij},\beta}\big) - F^{\xi_{ij}}_{\alpha,n_0} \circ F^{\xi_{ij}}_{\sigma^{n_0}(\alpha),n-n_0}\big(x^{(n)}_{\xi_{ij},\alpha}\big) \right\| \\ &\le \lambda^{n_0} \max_{i,j} \left\| x^{(n_0)}_{\xi_{ij},\beta} - F^{\xi_{ij}}_{\sigma^{n_0}(\alpha),n-n_0}\big(x^{(n)}_{\xi_{ij},\alpha}\big) \right\| \\ &\le \lambda^{n_0} < \varepsilon. \end{aligned} \tag{4.17}$$

From (4.17) as $n \to \infty$ one gets $\|x^{(n_0)}_{\beta} - x_{\alpha}\| < \varepsilon$. It follows that π is continuous. Therefore, $\widetilde{\Lambda}^{\xi}$ is compact. It is clear that $\Lambda^{\xi}_0 \subset \widetilde{\Lambda}^{\xi} \subset \Lambda^{\xi}$. As a result, due to closedness of $\widetilde{\Lambda}^{\xi}$ and $\overline{\Lambda^{\xi}_0} = \Lambda^{\xi}$ (see Proposition 4.1) we immediately find $\Lambda^{\xi} = \widetilde{\Lambda}^{\xi}$. Consequently, Λ^{ξ} is compact. The proof is complete. □

From the last proof, we immediately find the following.

Corollary 4.3: *One has $\Lambda^{\xi} = \widetilde{\Lambda}^{\xi}$. Moreover, the mapping $\pi\colon \Sigma \to \Lambda^{\xi}$ given by $\pi(\alpha) = x_{\alpha}$ is well-defined* (*see* (4.8)).

Proof of Theorem 4.2: Let Λ^{ξ} contain at least two points. Since $\Lambda^{\xi} = \widetilde{\Lambda}^{\xi}$ and $\overline{\Lambda^{\xi}_0} = \widetilde{\Lambda}^{\xi}$ it is enough to show that each $x \in \Lambda^{\xi}_0$ is not isolated point of $\widetilde{\Lambda}^{\xi}$.

Let $x \in \Lambda^{\xi}_0$. Then, there exist $\alpha \in \Sigma$ and a positive integer n such that

$$x = \sum_{i=1}^{M} \prod_{j=1}^{L} x^{(n)}_{\xi_{ij},\alpha},$$

where $x^{(n)}_{\xi_{ij},\alpha}$ is a fixed point of $F^{\xi_{ij}}_{\alpha,n}$. It is clear that

$$x = \sum_{i=1}^{M} \prod_{j=1}^{L} \big(F^{\xi_{ij}}_{\alpha,n}\big)^m \big(x^{(n)}_{\xi_{ij},\alpha}\big) \quad \text{for all } m \ge 1.$$

Take any $r > 0$ and $y_{\beta} \in \widetilde{\Lambda}^{\xi} \setminus B_r(x)$. Choose a positive integer m such that $\lambda^m < \frac{r}{2}$. Take $\gamma \in \Sigma$ such that $\gamma_{jn+i} = \alpha_i$ for all $i = \overline{1, n}$ and $j = \overline{0, m-1}$.

Note that

$$F^{\xi_{ij}}_{\gamma,mn} = \underbrace{\big(f_{\xi_{ij}(\alpha_1)} \circ \cdots \circ f_{\xi_{ij}(\alpha_n)}\big) \circ \cdots \circ \big(f_{\xi_{ij}(\alpha_1)} \circ \cdots \circ f_{\xi_{ij}(\alpha_n)}\big)}_{m} = \big(F^{\xi_{ij}}_{\alpha,n}\big)^m.$$

So,

$$F_{\gamma,mn} = \sum_{i=1}^{M} \prod_{j=1}^{L} F_{\gamma,mn}^{\xi_{ij}} \in \mathcal{F}_{\xi}.$$

Due to Proposition 4.2, we have

$$\widetilde{F}_{\gamma,mn}[y_\beta] = \lim_{k\to\infty} \sum_{i=1}^{M} \prod_{j=1}^{L} F_{\gamma,mn}^{\xi_{ij}}\big(y_{\xi_{ij},\beta}^{(k)}\big)$$

which belongs to $\widetilde{\Lambda}^{\xi}$. Now choose $k \geq 1$ such that

$$\left\| \widetilde{F}_{\gamma,mn}[y_\beta] - \sum_{i=1}^{M} \prod_{j=1}^{L} F_{\gamma,mn}^{\xi_{ij}}\big(y_{\xi_{ij},\beta}^{(k)}\big) \right\| < \frac{r}{2}. \tag{4.18}$$

One can see that

$$\begin{aligned} \widetilde{F}_{\gamma,mn}[y_\beta] - x = {} & \widetilde{F}_{\gamma,mn}[y_\beta] - \sum_{i=1}^{M} \prod_{j=1}^{L} F_{\gamma,mn}^{\xi_{ij}}\big(y_{\xi_{ij},\beta}^{(k)}\big) + \sum_{i=1}^{M} \prod_{j=1}^{L} F_{\gamma,mn}^{\xi_{ij}}\big(y_{\xi_{ij},\beta}^{(k)}\big) \\ & - \sum_{i=1}^{M} \prod_{j=1}^{L} \big(F_{\alpha,n}^{\xi_{ij}}\big)^m \big(x_{\xi_{ij},\alpha}^{(n)}\big) \\ = {} & \widetilde{F}_{\gamma,mn}[y_\beta] - \sum_{i} \prod_{j} F_{\gamma,mn}^{\xi_{ij}}\left(y_{\xi_{ij},\beta}^{(k)}\right) \\ & + \sum_{i} \sum_{j} \left[F_{\gamma,mn}^{\xi_{ij}}\left(y_{\xi_{ij},\beta}^{(k)}\right) - F_{\gamma,mn}^{\xi_{ij}}\left(x_{\xi_{ij},\alpha}^{(n)}\right)\right] \prod_{\substack{l>j \\ u<j}} F_{\gamma,mn}^{\xi_{il}}\left(y_{\xi_{il},\beta}^{(k)}\right) F_{\gamma,mn}^{\xi_{iu}}\left(x_{\xi_{iu},\alpha}^{(n)}\right). \end{aligned} \tag{4.19}$$

Noting

$$\left\| F_{\gamma,mn}^{\xi_{ij}}\big(y_{\xi_{ij},\beta}^{(k)}\big) - F_{\gamma,mn}^{\xi_{ij}}\big(x_{\xi_{ij},\alpha}^{(n)}\big) \right\| \leq \lambda^{mn} < \lambda^m < \frac{r}{2}$$

and using (4.18) and the strong triangle inequality from (4.19) we obtain $\|\widetilde{F}_{\gamma,mn}[y_\beta] - x\| < r$. Hence, $\widetilde{F}_{\gamma,mn}[y_\beta] \in \widetilde{\Lambda}^{\xi} \cap B_r(x)$. This means that x is not isolated point of $\widetilde{\Lambda}^{\xi}$. Consequently, Λ^{ξ} is perfect. □

Proof of Theorem 4.3: Let K be doubling. Now, we show that $\mathcal{A}$ (here $\mathcal{A} = K^n$) has the same property. Take any ball $B_r(\mathbf{a})$ in $\mathcal{A}$ (where $\mathbf{a} = (a_1, \ldots, a_n)$). Then, one can see that $B_r(\mathbf{a}) = B_r(a_1) \times \cdots \times B_r(a_n)$. Due to the doubling of K, there is k such that

$$B_r(a_i) \subset \bigcup_{j_i=1}^{k} B_{r/2}(a_{i,j_i}).$$

Hence, one finds

$$B_r(\mathbf{a}) \subset \bigcup_{j_1,\dots,j_n=1}^{k} B_{r/2}(a_{1,j_1}) \times \cdots B_{r/2}(a_{n,j_n})$$
$$= \bigcup_{j_1,\dots,j_n=1}^{k} B_{r/2}(\mathbf{a}_{j_1,\dots,j_n}),\quad \mathbf{a}_{j_1,\dots,j_n} = (a_{1,j_1},\dots,a_{n,j_n}).$$

This means that $\mathcal{A}$ is doubling. Now since $\Lambda^\xi \subset \mathcal{O} \subset \mathcal{A}$, the limiting set also has doubling property. For every $a \in \Lambda^\xi$ and $r > 0$, let $A = B_r(a)$. Then $B_{r/2}(a) \subset A \subset B_r(a)$. For any $y \in \Lambda^\xi \setminus B_r(a)$ and $x \in B_r(a)$, by the strong triangle equality, we have $\|x - y\| = \|y - a\| \geq r > r/2$, namely $\mathrm{dist}(B_r(a), \Lambda^\xi \setminus B_r(a)) \geq r/2$, which shows that Λ^ξ is uniformly disconnected. □

5. Invariant measures

In this section, we will show that there is a measure on Λ^ξ which is invariant with respect to the mappings $\{\widetilde{F}_k\}$. This is an analogue of the famous Hutchinson's result about the existence of the invariant measure for $\{f_i\}_{i=1}^N$.

Consider the set Λ^ξ. Due to Corollary 4.3, one has $\Lambda^\xi = \widetilde{\Lambda}^\xi$. Therefore, in what follows, we deal with $\widetilde{\Lambda}^\xi$. From (4.10), we infer that

$$\widetilde{\Lambda}^\xi = \left\{ x_\alpha = \sum_{i=1}^{M} \prod_{j=1}^{L} x_{\xi_{ij},\alpha} \ :\ \alpha \in \Sigma \right\}. \tag{5.1}$$

Now we define a mapping $d : \widetilde{\Lambda}^\xi \times \widetilde{\Lambda}^\xi \to \mathbb{R}_+$ by

$$d(x_\alpha, x_\beta) = \max_{i,j} \|x_{\xi_{ij},\alpha} - x_{\xi_{ij},\beta}\|. \tag{5.2}$$

It is clear that d is a metric on $\widetilde{\Lambda}^\xi$.

Lemma 5.1: *The set $\widetilde{\Lambda}^\xi$ is close with respect to the metric d.*

Proof: It is enough to establish that the mapping $\pi : \Sigma \to \widetilde{\Lambda}^\xi$ is continuous w.r.t. the metric d. Take any $\alpha = (\alpha_1, \dots, \alpha_k, \dots) \in \Sigma$ and $\varepsilon > 0$. Due to (4.10), there is $n_{0,1} \in \mathbb{N}$ such that

$$\left\| x_{\xi_{ij},\alpha} - x^{(n)}_{\xi_{ij},\alpha} \right\| < \varepsilon \tag{5.3}$$

for all $n \geq n_{0,1}$.

Now we choose $n_0 \in \mathbb{N}$ with $n_0 > n_{0,1}$ such that $\lambda^{n_0} < \varepsilon$. Then, for any $\beta \in \Sigma$ with $\beta_k = \alpha_k, k \leq n_0$, we have

$$\begin{aligned}\left\| x^{(n_0)}_{\xi_{ij},\beta} - x^{(n)}_{\xi_{ij},\beta} \right\| &= \left\| F^{\xi_{ij}}_{\beta,n_0}\left(x^{(n_0)}_{\xi_{ij},\beta}\right) - F^{\xi_{ij}}_{\beta,n}\left(x^{(n)}_{\xi_{ij},\beta}\right) \right\| \\ &\leq \lambda^{n_0} \left\| x^{(n_0)}_{\xi_{ij},\beta} - F^{\xi_{ij}}_{\sigma^{n_0}(\beta),n-n_0}\left(x^{(n)}_{\xi_{ij},\beta}\right) \right\| \\ &\leq \lambda^{n_0} < \varepsilon, \quad \text{for all } n \geq n_0.\end{aligned}$$

From this due to (4.10), we have

$$\left\| x^{(n_0)}_{\xi_{ij},\beta} - x_{\xi_{ij},\beta} \right\| < \varepsilon. \tag{5.4}$$

Noting $x^{(n_0)}_{\xi_{ij},\alpha} = x^{(n_0)}_{\xi_{ij},\beta}$ from (5.3),(5.4) using the strong triangle inequality one gets

$$\|x_{\xi_{ij},\alpha} - x_{\xi_{ij},\beta}\| < \varepsilon.$$

Consequently, by definition of d we get

$$d(x_\alpha, x_\beta) < \varepsilon.$$

The last one yields the continuity of π. This completes the proof. □

Proposition 5.1: *Let $\alpha \in \Sigma$. Then for any $n \geq 1$ the mapping $\widetilde{F}_{\alpha,n}$ defined as (4.13) is a contraction on $(\widetilde{\Lambda}^\xi, d)$.*

Proof: Take any $x_\beta, x_\gamma \in \widetilde{\Lambda}^\xi$. Then from Proposition 4.2, one finds

$$\widetilde{F}_{\alpha,n}[x_\beta] = x_{\alpha^{[n]}\vee\beta}, \quad \widetilde{F}_{\alpha,n}[x_\gamma] = x_{\alpha^{[n]}\vee\gamma}.$$

Therefore, we have

$$d(\widetilde{F}_{\alpha,n}[x_\beta], \widetilde{F}_{\alpha,n}[x_\gamma]) = \max_{i,j} \|x_{\xi_{ij},\alpha^{[n]}\vee\beta} - x_{\xi_{ij},\alpha^{[n]}\vee\gamma}\|.$$

Due to (4.10) and by non-Archimedean norm's property one can find $m_0 \geq n$ such that for all $m \geq m_0$ one has

$$d(\widetilde{F}_{\alpha,n}[x_\beta], \widetilde{F}_{\alpha,n}[x_\gamma]) = \max_{i,j} \left\| x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta} - x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\gamma} \right\|. \tag{5.5}$$

Hence, we obtain

$$\begin{aligned} \left\| x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta} - x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\gamma} \right\| &= \left\| F^{\xi_{ij}}_{\alpha,n} \circ F^{\xi_{ij}}_{\beta,m-n}\left(x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta}\right) - F^{\xi_{ij}}_{\alpha,n} \circ F^{\xi_{ij}}_{\gamma,m-n}\left(x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\gamma}\right) \right\| \\ &\leq \lambda^n \left\| F^{\xi_{ij}}_{\beta,m-n}\left(x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta}\right) - F^{\xi_{ij}}_{\gamma,m-n}\left(x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\gamma}\right) \right\|. \end{aligned} \tag{5.6}$$

On the other hand, one has

$$\begin{aligned} \left\| F^{\xi_{ij}}_{\beta,m-n}\left(x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta}\right) - x^{(m)}_{\xi_{ij},\beta} \right\| &= \left\| F^{\xi_{ij}}_{\beta,m-n}\left(x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta}\right) - F^{\xi_{ij}}_{\beta,m}\left(x^{(m)}_{\xi_{ij},\beta}\right) \right\| \\ &\leq \lambda^{m-n} \left\| x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta} - F^{\xi_{ij}}_{\sigma^{m-n}(\beta),n}\left(x^{(m)}_{\xi_{ij},\beta}\right) \right\| \\ &\leq \lambda^{m-n} \to 0 \text{ as } m \to \infty. \end{aligned}$$

Similarly, we get

$$\left\| F^{\xi_{ij}}_{\beta,m-n}\left(x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta}\right) - x^{(m)}_{\xi_{ij},\beta} \right\| \leq \lambda^{m-n} \to 0 \text{ as } m \to \infty.$$

Accordingly, the last inequalities with (5.6) imply that

$$\left\| x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\beta} - x^{(m)}_{\xi_{ij},\alpha^{[n]}\vee\gamma} \right\| \le \lambda^n \left\| x^{(m)}_{\xi_{ij},\beta} - x^{(m)}_{\xi_{ij},\gamma} \right\|.$$

It then follows from (5.5) that

$$\begin{aligned} d(\widetilde{F}_{\alpha,n}(x_\beta), \widetilde{F}_{\alpha,n}(x_\gamma)) &\le \lambda^n \max_{i,j} \left\| x^{(m)}_{\xi_{ij},\beta} - x^{(m)}_{\xi_{ij},\gamma} \right\| \\ &= \lambda^n d(x_\beta, x_\gamma). \end{aligned}$$

This completes the proof. □

Corollary 5.1: *For any $k \in [1, N]$, the mapping $\widetilde{F}_{k,1}$ is a contraction on $(\widetilde{\Lambda}^\xi, d)$.*

The last corollary yields that we can consider a collection $\{\widetilde{F}_{k,1}\}_{k=1}^N$ of contractions on $(\widetilde{\Lambda}^\xi, d)$ for which one can ask the following question: Does the set Λ^ξ coincide with the limiting set Λ generated by the collection of contractions $\{\widetilde{F}_{k,1}\}_{k=1}^N$?

To get an affirmative answer to this question we recall some notions. Following to Hutchinson,[21] the limit set Λ of the collection $\{\widetilde{F}_{k,1}\}_{k=1}^N$ is the closure (see [25])

$$\Lambda_0 = \left\{ x \in \widetilde{\Lambda}^\xi : x \text{ is a fixed point of } \widetilde{F}_{\alpha_1,1} \circ \cdots \circ \widetilde{F}_{\alpha_n,1} \text{ for some } \alpha_1, \dots, \alpha_n \in [1, N] \right\}.$$

For a given collection $\{\alpha_1, \dots, \alpha_n\} \subset [1, N]$ by $(\alpha_1, \dots, \alpha_n)$ we denote an element of Σ defined by

$$(\alpha_1, \dots, \alpha_n) = (\alpha_1, \dots, \alpha_n, \alpha_1, \dots, \alpha_n, \dots).$$

Namely, $(\alpha_1, \dots, \alpha_n)$ is a n-periodic point of the shift σ. Then, due to Proposition 4.2 a fixed point of $\widetilde{F}_{\alpha_1,1} \circ \cdots \circ \widetilde{F}_{\alpha_n,1}$ is the element $x_{(\alpha_1,\dots,\alpha_n)}$.

Consequently, one finds

$$\Lambda_0 = \left\{ x \in \widetilde{\Lambda}^\xi : x = x_\alpha,\ \alpha \text{ is } n\text{-periodic point of } \sigma \text{ for some } n \in \mathbb{N} \right\}.$$

We want to show that $\overline{\Lambda_0}^d = \widetilde{\Lambda}^\xi$. Indeed, take any $x_\alpha \in \widetilde{\Lambda}^\xi$ and $\varepsilon > 0$. Choose $n_0 \in \mathbb{N}$ such that $\lambda^{n_0} < \varepsilon$. Then, using the argument as in Lemma 5.1 for any $\beta = (\beta_1, \dots, \beta_{n_0})$ we obtain $d(x_\alpha, x_\beta) < \varepsilon$. This means that $\overline{\Lambda_0}^d = \widetilde{\Lambda}^\xi$. Thus, we have proven the following result

Theorem 5.1: *Let Λ be a limit set of the collection of contractions $\{\widetilde{F}_{k,1}\}_{k=1}^N$ on $(\widetilde{\Lambda}^\xi, d)$. Then one has $\Lambda = \widetilde{\Lambda}^\xi$.*

This theorem allows us to further investigate the structure of $\widetilde{\Lambda}^\xi$ by means of the limiting set of semi-group generated by $\{\widetilde{F}_{k,1}\}_{k=1}^N$ on $\widetilde{\Lambda}^\xi$. Furthermore, one can study the Hausdorff dimension of the set $\widetilde{\Lambda}^\xi$. Now we are going to demonstrate the existence of invariant measures on $\widetilde{\Lambda}^\xi$.

Let $\mathcal{B}_\xi$ be the Borel σ-algebra of subsets of $\widetilde{\Lambda}^\xi$. By $\mathcal{M}^1(\widetilde{\Lambda}^\xi)$ we denote the set of all probability measures defined on the measurable space $(\Lambda^\xi, \mathcal{B}_\xi)$.

Assume that $\rho = \{\rho_i\}_{i=1}^N$ is a collection of positive numbers such that $\rho_i \in (0, 1)$ and $\sum_{i=1}^n \rho_i = 1$. Now we define a mapping $\mathcal{S}_\rho : \mathcal{M}^1(\widetilde{\Lambda}^\xi) \to \mathcal{M}^1(\widetilde{\Lambda}^\xi)$ by

$$(\mathcal{S}_\rho \mu)(A) = \sum_{i=1}^N \rho_i \mu \left(\widetilde{F}_{i,1}^{-1}(A)\right), \quad A \in \mathcal{B}_\xi. \tag{5.7}$$

A measure $\nu \in \mathcal{M}^1(\widetilde{\Lambda}^\xi)$ is called *invariant* w.r.t. $\mathcal{S}_\rho$ if one has $\mathcal{S}_\rho \nu = \nu$.

Following a general scenario, we consider the *Hutchinson metric* [21] on $\mathcal{M}^1(\widetilde{\Lambda}^\xi)$ which is defined as follows:

$$d_H(\mu, \nu) = \sup \left\{ \int \varphi d\mu - \int \varphi d\nu \ : \ \varphi \in C^0(\widetilde{\Lambda}^\xi, \mathbb{R}) : \ |\varphi(x) - \varphi(y)| \le d(x, y) \ \ \forall x, y \in \widetilde{\Lambda}^\xi \right\}.$$

Due to Proposition 5.1 and following [21] one can prove the following.

Theorem 5.2: *The map $\mathcal{S}_\rho$ is a contractive mapping w.r.t. the d_H metric. Moreover, there is a unique invariant measure $\nu_\xi \in \mathcal{M}^1(\widetilde{\Lambda}^\xi)$.*

Now we want to exactly construct the invariant measure ν_ξ. First, recall that $\mathcal{F}$ denotes a σ-algebra $\mathcal{F}$ generated by cylindrical subsets of Σ. Let μ_ρ be the product measure on $(\Sigma, \mathcal{F})$ induced by the measure $\rho(i) = \rho_i$ on each factor $\{1, \dots, N\}$. In what follows, by σ_i we denote i^{th}shift operator σ_i: $\Sigma \to \Sigma$ defined by $\sigma_i(\alpha) = i \vee \alpha$, i.e. $\sigma_i(\alpha_1, \dots, \alpha_n, \dots) = (i, \alpha_1, \dots, \alpha_n, \dots)$.

From Proposition 4.2 and 4.8, one has

$$\pi \circ \sigma_i = \widetilde{F}_{i,1} \circ \pi \tag{5.8}$$

for every $i \in [1, N]$.

Now on the measurable space $(\widetilde{\Lambda}^\xi, \mathcal{B}_\xi)$, we define a measure ν_ξ as follows:

$$\tilde{\nu}_\xi(A) = \mu_\rho(\pi^{-1}(A)), \quad A \in \mathcal{B}_\xi. \tag{5.9}$$

Theorem 5.3: *The measure $\tilde{\nu}_\xi$ is invariant with respect to $\mathcal{S}_\rho$.*

Proof: First, we note that the measure μ_ρ is invariant with respect to $\{\sigma_1, \dots, \sigma_N\}$ and ρ, i.e. one has

$$\mu_\rho(B) = \sum_{i=1}^n \rho_i \mu_\rho(\sigma_i^{-1}(B)), \quad B \in \mathcal{F}.$$

Therefore, the last equality with (5.8),(5.9) yields

$$\begin{aligned}\tilde{\nu}_\xi(A) &= \mu_\rho(\pi^{-1}(A)) \\ &= \sum_{i=1}^{n} \rho_i \mu_\rho\big(\sigma_i^{-1}(\pi^{-1}(A))\big) \\ &= \sum_{i=1}^{n} \rho_i \mu_\rho\big(\pi^{-1}(\widetilde{F}_{i,1}^{-1}(A))\big) \\ &= \sum_{i=1}^{n} \rho_i \tilde{\nu}_\xi\big(\widetilde{F}_{i,1}^{-1}(A)\big), \quad A \in \mathcal{B}_\xi.\end{aligned}$$

This completes the proof. □

From Theorems 5.2 and 5.3, we infer that the measure $\tilde{\nu}_\xi$ is the unique invariant measure for $\mathcal{S}_\rho$. Our result shows that even in a general setting rather than [21] one can find unique invariant measure on the set $\widetilde{\Lambda}^\xi$.

Disclosure statement

No potential conflict of interest was reported by the authors.

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