O‘ZBEKISTON RESPUBLIKASI
OLIY VA O‘RTA MAXSUS TA’LIM VAZIRLIGI

SAMARQAND DAVLAT UNIVERSITETI

Algebra va sonlar nazariyasidam masala va masqlar

II qism

Samarqand – 2021

Algebra va sonlar nazariyasidan masala va mashqlar. II-qism. O'quv qo'llanma. – Samarqand, 2021. – 201 bet.

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Samarqand davlat universiteti Ilmiy kengashining 2021 yil **___** aprelidagi yig'lishida nashrga tavsiya etilgan.

Mundarija

So’z boshi

Ushbu o’quv qo‘llanma «Algebra va sonlar nazariyasi» fani bo‘yicha 5130100 – Matematika», 5110100 – Matematika va informatika, 5140300–Mexanika va matematik modellashtirish ta’lim yo‘nalishlari bakalavr talabalari uchun mo‘ljallangan. Qo’llanmadan Amaliy matematika va informatika, Fizika va astronomiya kabi ta’lim yo’nalishlari bakalavr talabalari ham foydalanishlari mumkin. Unda shu fanning namunaviy o‘quv dasturidan kelib chiqib, determinantlar nazariyasi, matrisalar algebrasi, kompleks sonlar maydoni va ko’phadlar halqasi mavzulariga oid qisqacha nazariy ma’lumotlar, namunaviy misollarni yechish uchun uslubiy tavsiylar, mustaqil ish topshiriqlari va boshqa tarqatma materiallar keltirilgan.

Qo’llanmadan 5A130101 – Matematika (yo’nalishlar bo’yicha) magistratura mutaxassisligi talabalari ham Abstrakt algebra fanining elementlarini o’rganishda foydalanishlari mumkin.

Qo’llanmada determinantlar nazariyasi, matrisalar algebrasi, kompleks sonlar maydoni va ko’phadlar halqasi mavzulariga oid jami 1000 ga yaqin misollar, mashqlar va nazariy masalalar jamlangan bo’lib, ulardan bir qanchasining yechimlari to’liq berilgan yoki yechish uchun ko’rsatmalar berilgan.

Raqamiga * belgisi qo’yilgan mashqlarning to’liq yechimlari keltirilgan.

Qo’llanma talabalarga Algebra va sonlar nazariyasi bo’yicha keltirilgan mavzularni yanada chuqurroq o‘zlashtirishga yaqindan yordam beradi degan umiddamiz.

Mualliflar

I-bob
Determinantlar nazariyasi

Tayanch iboralar: *o'rin almashtirishlar; tartib; juft va toq o'rin almashtirishlar; inversiya; transpozitsiya; dekrement; sikl; siklning uzunligi; matritsa; bosh diagonal; Vandermont determinanti; minor; Laplas teoremasi; algebraik to'ldiruvchi; zinapoyali determinant; o'zaro determinant.*

1-§. O'rin almashtirishlar va o'rniga qo'yishlar

n ta 1, 2, .., n *sonlar* (yoki n ta har xil $a_1, a_2, .., a_n$ *simvollar*) ning ma'lum tartibda mumkin bo'lgan ixtiyoriy joylashuviga shu sonlarning (yoki simvollarning) *o'rin almashtirishi* deyiladi. Berilgan n ta simvollarni 1, 2, ..., n, sonlar bilan tartiblash mumkin bo'lganligi sababli ixtiyoriy n ta simvollarning o'rin almashtirishlarini o'rganish 1, 2, ..., n larning o'rin almashtirishlarini o'rganishga keltiradi. n ta sonlarning barcha o'rin almashtirishlari soni $1\cdot 2\cdot 3\cdots n = n!$ («*n-faktorial*» deb o'qiladi) ga teng. Masalan, a_1, a_2, a_3 simvollarning barcha o'rin almashtirishlari quyidagilardir: $a_1\, a_2\, a_3$, $a_1\, a_3\, a_2$, $a_2\, a_1\, a_3$, $a_2\, a_3\, a_1$, $a_3\, a_1\, a_2$, $a_3\, a_2\, a_1$. Ularning soni 3! = 6 ta.

Agar o'rin almashtirishda ikki sondan kattasi kichigidan oldin kelsa bu sonlar *inversiyani* tashkil etadi, agar kichigi kattasidan oldin kelsa, *tartib* deyiladi.

Inversiyalar sonini hisoblash usuli: o'rin almashtirishdagi sonlarni yozilish tartibi bo'yicha (chapdan o'ngga) har bir son uchun undan o'ng tomonda turgan kichik sonlar sanaladi va hosil bo'lgan barcha sonlar qo'shiladi. Masalan, (613542) o'rin almashtirishda inversiyalar soni $5 + 1 + 2 + 1 = 9$ ga teng.

Inversiyalar sonining juft, toqligiga qarab o'rin almashtirish *juft* yoki *toq* deyiladi.

O'rin almashtirishdagi ikki sonni o'rnini almashtirish *transpozitsiya* deyiladi. i va j sonlarning transpozitsiyasi (i, j) bilan belgilanadi. n ta sonning har qanday o'rin almashtirishidan shu sonlarning istagan boshqa o'rin almashtirishiga bir nechta transpozitsiyalarni bajarish bilan kelish mumkin bo'lib, bunda n-1 tadan ko'p bo'lmagan transpozitsiyalar bilan chegaralanishi mumkin. Misol. (312546) o'rin almashtirishdan (631254) almashtirishga beshta: (3, 6), (3, 1), (1, 2), (2, 5), (5, 4) transpozitsiyalarni bajarish bilan kelish mumkin.

1, 2, ... , n sonlarning barcha $n!$ o'rin almashtirishlarni har keyingisi oldingisida bitta transpozitsiyani bajarishdan hosil bo'lgan qilib (tushirib qoldirmaydigan va takrorlanmaydigan), birin-ketin joylashtirish mumkin. Har bir transpozitsiya o'rin almashtirishning juft-toqligini o'zgartiradi. $n \geq 2$ son uchun n ta sondan tuzilgan o'rin almashtirishlardan juftlari soni toqlari soniga, ya'ni $\frac{1}{2}n!$ ga teng bo'ladi.

n ta 1, 2, ... , n sonlar to'plamining o'ziga o'zaro bir qiymatli akslantirishiga (biyeksiyasiga) bu sonlarning *o'rniga qo'yish* yoki *n-tartibli o'rniga qo'yish* deyiladi. Shunday qilib, o'rniga qo'yishda 1 dan n gacha bo'lgan har bir songa shu sonlardan qandaydir biri mos keltirilgan bo'lib, ikkita har xil songa ikkita har xil son mos keladi. O'rniga qo'yish umumiy qavsga olingan ikkita satr ko'rinishida: yuqori satrda turgan har bir sonning tagida unga mos keluvchi sonni yozish bilan ifodalanadi. Masalan, $\begin{pmatrix} 623451 \\ 436251 \end{pmatrix}$ o'rniga qo'yishda $1 \to 1$, $2 \to 3$, $3 \to 6$, $4 \to 2$, $5 \to 5$, $6 \to 4$ mos keltirilganligini bildiradi.

Sonlarning yuqori satrda joylashuviga qarab bitta o'rniga qo'yishni bir nechta ko'rinishda yozish mumkin. Masalan,

$$\begin{pmatrix} 1234 \\ 3412 \end{pmatrix}, \begin{pmatrix} 2134 \\ 4312 \end{pmatrix}, \begin{pmatrix} 3124 \\ 1342 \end{pmatrix}, \begin{pmatrix} 4123 \\ 2341 \end{pmatrix}$$

o'rniga qo'yishlarning barchasida 1 soni 3 ga, 2 soni 4 ga, 3 soni 1 ga, 4 soni 2 ga o'tganligi sababli, ular aynan bitta o'rniga qo'yishni ifodalaydi. n ta son yordamida tuzilgan har bir o'rniga qo'yishni $n!$ har xil ko'rinishlarda yozish mumkin. n ta sondan tuzilgan har xil o'rniga qo'yishlar soni ham $n!$ ga tengdir.

Agar o'rniga qo'yishning ikkala satridagi inversiyalar yig'indisi juft bo'lsa, o'rniga qo'yish *juft* deb, agar inversiyalar yig'indisi toq bo'lsa, *toq* deb aytiladi. Demak, agar ikkala satrdagi inversiyalar bir xilda juft, yoki ikkalasi ham toq bo'lsa, o'rniga qo'yish juft, agar har xil bo'lsa o'rniga qo'yish toq bo'ladi. O'rniga qo'yishning juft-toqligi uning ikkita satr yordamida ko'rinishidan bog'liq emas, ya'ni bitta o'rniga qo'yishning har xil ko'rinishida inversiyalar juft-toqligi bir xildir. Masalan, $\begin{pmatrix} 3214 \\ 1324 \end{pmatrix} = \begin{pmatrix} 1234 \\ 2314 \end{pmatrix}$ o'rniga qo'yishning birinchi yozuvida to'rtta, ikkinchisida ikkita inversiya bor, ya'ni juft.

n elementdan tuzilgan juft o'rniga qo'yishlar soni toq o'rniga qo'yishlar soniga va demak, $\frac{1}{2}n!$ ga tengdir ($n \geq 2$).

O'rniga qo'yishning juft-toqligini aniqlashning boshqa usuli ham bor. Bir nechta sonlar ketma-ketligida berilgan o'rniga qo'yishda birinchi son - ikkinchisiga, ikinchisi - uchinchisiga va h.k oxirgisi - birinchisiga o'tsa, bu sonlar *sikl* deb ataladi. Siklni undagi sonlarni umumiy qavslarga olib yozish bilan belgilanadi. Agarda son yana o'ziga o'tsa, u ham bitta siklni tashkil etadi. Umumiy sonlarga ega bo'lmagan sikllar, *o'zaro bog'liq bo'lmagan sikllar* deyiladi. Har qanday o'rniga qo'yishni o'zaro bog'liq bo'lmagan sikllarga ajratish mumkin (yoki yoyish mumkin). Masalan, $\begin{pmatrix} 123456 \\ 613542 \end{pmatrix} = (162)(45)(3)$.

O'rniga qo'yishdagi elementlar soni n va uning yoyilmasidagi sikllar soni k ning ayirmasi bo'lgan d soniga, ya'ni $d = n - k$ ga *o'rniga qo'yishning dekrementi* deyiladi. O'rniga qo'yishning juft-toqligi uning dekrementining juft-toqligi bilan bir xildir. Masalan: $n = 6$, $k = 3$, $d = 3$ bo'lib, o'rniga qo'yish toq.

n-tartibli ikkita o'rniga qo'yishni ketma-ket bajarishdan hosil bo'lgan o'rniga qo'yishga ularning *ko'paytmasi* deyiladi. Masalan, agar $a = \begin{pmatrix} 12345 \\ 31254 \end{pmatrix}$, $b = \begin{pmatrix} 12345 \\ 25314 \end{pmatrix}$, bo'lsa, u holda $ab = \begin{pmatrix} 12345 \\ 32541 \end{pmatrix}$ bo'ladi.

Agar siklni o'rniga qo'yish deb tushunsak, u holda o'rniga qo'yishni o'zaro bog'liq bo'lmagan sikllarga yoyilmasiga uning shu sikllarning ko'paytmasi ko'rinishidagi ifodasi deb qarash mumkin. Agar 1, 2, ..., *n*, sonlarning o'rniga qo'yishda i_1 son i_2 *ga*, $i_2 \to i_3$ ga ,... $i_{k-1} \to i_k$ ga ($k \le n$),
$i_k \to i_1$ ga o'tib, qolgan sonlar o'ziga o'tsa, bunday o'rniga qo'yishga *sikl* yoki *siklik o'rniga qo'yish* deyiladi va $(i_1, i_2, ..., i_k)$ ko'rinishida belgilanadi. $(i_1, i_2, ..., i_k)$ va masalan, $(i_2, i_3, ..., i_k, i_1)$ sikllar o'zaro tengdir. *k* songa *siklning uzunligi* deyiladi.

Uzunligi 1 ga teng sikl ko'paytmada yozilmaydi. Masalan, $\begin{pmatrix} 12345678 \\ 82157463 \end{pmatrix} = (183)(4576)$. Uzunligi ikkiga teng sikl *transpozitsiya* deyiladi. Har qanday o'rniga qo'yishni transpozitsiyalar ko'paytmasi shaklida ifodalash mumkin. Masalan, $(i_1, i_2, ..., i_k) = (i_1, i_2)(i_1, i_3) ... (i_1, i_k)$. Bu ifodalanish yagona emas, har qanday juft o'rniga qo'yishni juft sondagi transpozitsiyalar, toq o'rniga qo'yishni toq sondagi transpozitsiyalar ko'paytmasi ko'rinishida ifodalash mumkin.

1-m i s o l. Agar RPOKUTME harfli o'rin almashtirishni tartib deb qarab, unga nisbatan KOMPUTER o'rin almashtirishining juft yoki toqligini aniqlang.

Yechish. K harfi O, P, R harflari bilan 3 inversiyani tashkil qiladi. O harfi P, R bilan 2 inversiyani, M harfi P, U, T, R harflari bilan 4 ta inversiyani, P harfi R harfi bilan 1 inversiyani, U harfi R bilan 1 inversiyani, U harfi R bilan 1 inversiyani, T harfi R bilan 1 inversiyani, E harfi R bilan 1 inversiyanm hosil qiladi. Hammasi bo'lib KOMPUTER o'rin almashtirishida 14 ta inversiya bor. Demak, bu o'rin almashtirish juft. ■

2-m i s o l. (2*n*, 2*n*-2, ..., 6, 4, 2, 2*n*-1, 2*n*-3, ..., 5, 3, 1) (1)
o'rin almashtirishida inversiyalar sonini toping. O'rin almashtirishi juft bo'ladigan *n* larning, va toq bo'ladigan *n* larning umumiy ko'rinishini ko'rsating.

Yechish. Inversiyalar sonini hisoblaymiz:

$$(2n-1)+(2n-3)+...+5+3+1+(n-1)+(n-2)+...+2+1=$$
$$\frac{1+(2n-1)}{2}\cdot n+\frac{1+(n-1)}{2}\cdot(n-1)=n^2+\frac{n(n-1)}{2}=\frac{1}{2}n(3n-1)$$

bundan $n = 4k$ va $n = 4k + 3$ bo'lgandagina (1) o'rin almashtirish juft bo'lishini ko'ramiz.■

3-m i s o l. (9, 5, 1, 8, 3, 7, 4, 6, 2) o'rin almashtirishdan (9, 8, 7, 6, 5, 4, 3, 2, 1) o'rin almashtirishga o'tish mumkin bo'lgan transpozitsiyalarni ko'rsating.

Yechish. Bu transpozitsiyalar quyidagilardan iboratligini ko'rish qiyin emas. (5, 8), (1, 7), (5, 6), (3, 5), (1, 4), (1, 3), (2, 1). ■

4-m i s o l. n ta 1, 2, ..., n sonlarning (1 2 ... n) o'rin almashtirishidan farqli har qanday o'rin almashtirishida ma'lum bir transpozitsiya bajarish bilan undagi inversiyalar sonini bittaga kamaytirish mumkinligini ko'rsating.

Yechish. Qaraladigan o'rin almashtirishda kamida bitta α_k, α_{k+1}, ($\alpha_k > \alpha_{k+1}$) juftlik topiladi. (α_k, α_{k+1}) transpozitsiya inversiyalar sonini bittaga kamaytiradi. ■

5-m i s o l. Quyidagi o'rin almashtirishni sikllar ko'paytmasiga yoying va dekrement orqali juft-toqligini aniqlang

$$\begin{pmatrix} 1234\ldots.2n-1 & 2n \\ 2143\ldots.2n & 2n-1 \end{pmatrix}.$$

Yechish. Berilgan o'rniga qo'yishni o'zaro bog'liq bo'lmagan (1 2) (3 4) .... (2n-1, 2n) sikllarning ko'paytmasi ko'rinishida yoyish mumkin. Demak, uning dekrementi $2n - n = n$ ga teng bo'lib, o'rniga qo'yishning juft-toqligi n ning juft-toqligi bilan bir xildir. ■

6-m i s o l. (3 2 1) (6 5 4) .... (3n, 3n-1, 3n-2) o'rniga qo'yishda sikllardagi yozuvdan ikki satrlardagi yozuvga o'ting.

Yechish. Birinchi sikldan 1 ning 3 ga, 3 ning 2 ga, 2 ning 1 ga o'tishini ko'ramiz. Ikkinchi siklda 4 – 6 ga, 6 – 5 ga, 5 – 4 ga o'tadi. Oxirgi siklda $3n \to 3n$-1 ga, $3n$-1 $\to 3n$-2 ga, $3n$-2 esa $3n$ *ga* o'tadi. Natijada, biz quyidagi $\begin{pmatrix} 123456\ldots.3n-2 & 3n-1 & 3n \\ 312645\ldots.3n & 3n-2 & 3n-1 \end{pmatrix}$ o'rniga qo'yishni hosil qilamiz. ■

7-m i s o l. Hisoblang:

$$\begin{pmatrix} \alpha_1\alpha_2..\alpha_n \\ \beta_1\beta_2...\beta_n \end{pmatrix}^{-2} (\alpha_1\alpha_2..\alpha_n) \begin{pmatrix} \alpha_1\alpha_2..\alpha_n \\ \beta_1\beta_2...\beta_n \end{pmatrix}.$$

Yechish.

$$\begin{pmatrix} \beta_1\beta_2...\beta_n \\ \alpha_1\alpha_2..\alpha_n \end{pmatrix}\begin{pmatrix} \alpha_1\alpha_2..\alpha_n \\ \alpha_2\alpha_3..\alpha_1 \end{pmatrix}\begin{pmatrix} \alpha_1\alpha_2..\alpha_n \\ \beta_1\beta_2...\beta_n \end{pmatrix} = \begin{pmatrix} \beta_1\beta_2...\beta_n \\ \alpha_2\alpha_3..\alpha_1 \end{pmatrix}\begin{pmatrix} \alpha_1\alpha_2..\alpha_n \\ \beta_1\beta_2...\beta_n \end{pmatrix} =$$

$$\begin{pmatrix} \beta_1\beta_2...\beta_n \\ \beta_2\beta_3...\beta_1 \end{pmatrix} = (\beta_1\beta_2...\beta_n).■$$

8-m i s o l. Agar

$$A = \begin{pmatrix} 12345 \\ 31254 \end{pmatrix},\ B = \begin{pmatrix} 12345 \\ 42135 \end{pmatrix},\ C = \begin{pmatrix} 12345 \\ 53124 \end{pmatrix}$$

bo'lsa, $A^{-1}XB = C$ tenglikdan X o'rniga qo'yishni toping.

Yechish. $A^{-1}XB = C$ tenglikni chapdan A ga, o'ngdan B^{-1} ga ko'paytirsak, $X=ACB^{-1}$ ni topgan bo'lamiz.

$B^{-1} = \begin{pmatrix} 12345 \\ 32415 \end{pmatrix}$ bo'lganligi sababli,

$X = \begin{pmatrix} 12345 \\ 31254 \end{pmatrix}\begin{pmatrix} 12345 \\ 53124 \end{pmatrix}\begin{pmatrix} 12345 \\ 32415 \end{pmatrix} = \begin{pmatrix} 12345 \\ 35412 \end{pmatrix}$ ni hosil qilamiz. ■

M A S H Q L A R

1.1. Berilgan o’rin almashtirishlarning biridan ikkinchisiga o’tish mumkin bo’lgan transpozitsiyalarni ko’rsating.

a) (10, 1, 2, 8, 7, 4, 3, 6, 9, 5) o’rin almashtirishdan (8, 9, 5, 1, 10, 7, 2, 3, 6, 4) o’rin almashtirishda;

v) (2, 4, 6, ..., 2n, 1, 3, 5, ..., 2n-1) o’rin almashtirishdan (2n, 2n-1, ..., 4, 3, 2, 1) o’rin almashtirishda.

1.2. i va k larning (1, 2, 7, 4, i, 5, 6, k, 9) o’rin almashtirish juft bo’ladigan qiymatlarini toping.

1.3. Agar ASQMUI o’rin almashtirishni tartib deb qarab, MUSIQA o’rin almashtirishidagi inversiyalar sonini toping.

1.4. Quyidagi o’rin almashtirishlardagi inversiyalar sonini toping. Qanday n larda o’rin almashtirishlar juft, qanday n larda toq bo’lishini aniqlang va shu n larning umumiy ko’rinishini bering.

a) (2, 4, 6, ..., 2n, 1, 3, 5, ..., 2n - 1);

v) (1, 3, 5, ..., 2n - 1, 2, 4, 6, ..., 2n);

c) (2n, 1, 2n - 1, 2, 2n - 2, 3, ..., n + 1, n);

d) (1, 4, 7, ..., 3n - 2, 2, 5, ..., 3n - 1, 3, 6, ..., 3n);

e) (3, 6, ..., 3n, 1, 4, ..., 3n - 2, 2, 5, ..., 3n - 1);

f) (1, 5, ..., 4n - 3, 2, 6, ..., 4n - 2, 3, 7, ..., 4n - 1, 4, 8, ..., 4n);

g) (4n, 4n - 4, ..., 8, 4, 4n - 1, 4n-5, ..., 7, 3, 4n-2, 4n - 6, ..., 6, 2, 4n - 3, 4n-7, ..., 5, 1).

1.5. (1, 2,.., n) o’rin almashtirishning k-chi o’rnida turgan 1 soni nechta inversiyani hosil qiladi?

1.6. 1, 2, 3, ..., n o’rin almashtirishning k-chi o’rnida turgan n soni nechta inversiyani hosil qiladi?

1.7. 1, 2, ..., n sonlarning ixtiyoriy o’rin almashtirishidagi inversiyalar soni va tartiblar soni yig’indisini toping.

1.8. Qanday n lar uchun 1, 2 , ..., n sonlarning istalgan o’rin almashtirishidagi inversiyalar soni va tartiblar sonlarining juft-toqligi bir xil, qanday n lar uchun har xil bo’ladi?

1.9. 1, 2, ..., n, sonlarining biror o’rin almashtirishida (12) transpozitsiya bajarsak, inversiyalar soni qanday o’zgaradi?

1.10. $a_1, a_2, ..., a_{n-1}, a_n$ o’rin almashtirishdagi inversiyalar soni k ekanligi berilgan. U holda $a_n, a_{n-1}, ..., a_2, a_1$ o’rin almashtirishdagi inversiyalar soni nechta bo’ladi?

1.11. n ta elementning barcha o’rin almashtirishlaridagi inversiyalarning hammasi nechta?

1.12. 1, 2, ..., n sonlarning inversiyalari soni k ga teng bo’lgan ixtiyoriy o’rin almashtirishidan boshlang’ich holatiga k ta transpozitsiyalarni bajarish natijasida o’tish mumkin bo’lib, undan kam sondagi transpozitsiyalar orqali o’tish mumkin emasligini isbotlang.

1.13. Ixtiyoriy butun k ($0 \le k \le C_n^2$) soni uchun 1, 2, 3, ..., n, sonlarning o'rin almashtirishlari ichida inversiyalari soni k ga teng bo'lgani borligini ko'rsating.

1.14. Qanday umumiy va yetarli shartda $(\alpha_1\ \alpha_2\ \alpha_3 \ldots \alpha_n)$ o'rin almashtirishida yonma-yon turmagan ikki sonning transpozitsiyasi undagi inversiyalar sonini bittaga oshiradi? Qanday shartda bittaga kamaytiradi?

1.15. Quyidagi binar munosabatlardan qaysilari o'rniga qo'yish bo'ladi:

$$a)\begin{pmatrix} 2134 \\ 2321 \end{pmatrix}; \quad b)\begin{pmatrix} 4213 \\ 4312 \end{pmatrix}; \quad c)\begin{pmatrix} 1234 \\ 1324 \end{pmatrix};$$

$$d)\begin{pmatrix} 21324 \\ 23421 \end{pmatrix}; \quad e)\begin{pmatrix} 13423 \\ 43123 \end{pmatrix}; \quad f)\begin{pmatrix} 123453 \\ 43213 \end{pmatrix}.$$

1.16. Quyidagi o'rniga qo'yishlarni o'zaro bog'liq bo'lmagan sikllar ko'paytmasi ko'rinishida ifodalang va dekrement bo'yicha uning juft-toqligini aniqlang.

$$a)\begin{pmatrix} 12345678 \\ 17453682 \end{pmatrix}; \; b)\begin{pmatrix} 123456789 \\ 876543219 \end{pmatrix};$$

$$c)\begin{pmatrix} 123456\ldots 3n-2 & 3n-1 & 3n \\ 321654\ldots 3n & 3n-1 & 3n-2 \end{pmatrix};$$

$$d)\begin{pmatrix} 123456\ldots 3n-2 & 3n-1 & 3n \\ 321654\ldots 3n-1 & 3n & 3n-2 \end{pmatrix};$$

$$e)\begin{pmatrix} 1 & 2 & 3 & \cdots & k & \cdots & nk-k+1 & nk-k+2 & \cdots & nk \\ k+1 & k+2 & k+3 & \cdots & 2k & \cdots & 1 & 2 & \cdots & k \end{pmatrix}.$$

1.17. O'rniga qo'yishlarni ko'paytiring:

$$a)\begin{pmatrix} 123456 \\ 234561 \end{pmatrix}\begin{pmatrix} 123456 \\ 124356 \end{pmatrix}; \; b)\begin{pmatrix} 123456 \\ 231564 \end{pmatrix}\begin{pmatrix} 123456 \\ 124356 \end{pmatrix};$$

$$c)\begin{pmatrix} 1234 \\ 2341 \end{pmatrix}^2; \; d)\begin{pmatrix} 12345 \\ 34512 \end{pmatrix}^3.$$

1.18. Quyidagi o'rniga qo'yishlarda sikllar bo'yicha yozuvlardan 2 ta satrlar bo'yicha yozuvga o'ting:

$$a)(15)(234); \quad b)(13)(25)(4);$$

$$c)(7531)(246)(8)(9); \; d)(12)(34)\ldots(2n-1,2n);$$

$$e)(1,2,3,4,\ldots,2n-1,2n).$$

1.19. Hisoblang:

$$\begin{pmatrix} \alpha_1 \ldots \alpha_n \\ \beta_1 \ldots \beta_n \end{pmatrix}(\alpha_1 \ldots \alpha_n)\begin{pmatrix} \alpha_1 \ldots \alpha_n \\ \beta_1 \ldots \beta_n \end{pmatrix}^{-1}.$$

1.20. i va k larning $\begin{pmatrix} 123456789 \\ 731i46k29 \end{pmatrix}$ o'rniga qo'yish: a) juft; b) toq bo'ladigan qiymatlarini toping.

1.21. Agar siklning biror darajasi birga teng bo’lsa, u holda daraja ko’rsatgichini siklning uzunligiga bo’linishini isbotlang.

1.22. Birga teng bo’lgan o’rniga qo’yishlarning barcha darajalari ichida eng kichik ko’rsatgich o’rniga qo’yishning sikllarga yoyilmasidagi sikllarning uzunliklarining eng kichik umumiy karralisiga tengligini ko’rsating.

1.23. Agar $A=\begin{pmatrix}1&2&3&4&5&6&7\\2&4&7&5&6&1&3\end{pmatrix}$ bo’lsa, A^{101} ni toping.

1.24. Agar $A=\begin{pmatrix}1&2&3&4&5&6&7\\7&6&5&4&3&2&1\end{pmatrix}$ bo’lsa, A^{n} ni toping.

1.25. Agar

$A=\begin{pmatrix}1&2&3&4&5&6&7&8\\5&1&3&2&7&6&8&4\end{pmatrix}$, $B=\begin{pmatrix}1&2&3&4&5&6&7&8\\3&1&2&7&4&5&6&8\end{pmatrix}$,

$C=\begin{pmatrix}1&2&3&4&5&6&7&8\\8&7&6&5&4&3&2&1\end{pmatrix}$ bo’lsa, $AXB = C$ tenglikdan X o’rniga qo’yishni toping.

1.26. O’rniga qo’yishni (α,β) transpozitsiyaga chapdan ko’paytirish shu o’rniga qo’yishning yuqori satrida α va β transpozitsiyani bajarishga, o’ngdan ko’paytirish esa quyi satrda α va β transpozitsiyani bajarishga teng kuchli ekanligini isbotlang.

1.27. Agar α va β lar o’rniga qo’yishning biror sikliga kirsa, u holda bu o’rniga qo’yishni (α, β) transpozitsiyaga ko’paytirganda (o’ngdan yoki chapdan) berilgan ikkita siklga ajraladi, agar α va β lar har xil sikllarga kirsa, u holda yuqorida aytilgan ko’paytirishda bu sikllar bittaga birlashadi. Shuni isbotlang.

1.28. Oldingi ikki masaladan foydalanib har qanday o’rniga qo’yishning inversiyalar soni va dekrementining juft-toqligi bir xildir.

1.29. Berilgan o’rniga qo’yish transpozitsiyalar ko’paytmasi ko’rinishida ifodalashda eng kam miqdordagi transpozitsiyalar soni o’rniga qo’yishning dekrementiga tengligini isbotlang.

1.30. 1, 2, 3, 4 sonlarining o’rniga qo’yishlari ichida $S=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$ o’rniga qo’yish bilan o’rin almashinuvchi bo’lgan o’rniga qo’yishlarni toping.

1.31. 1, 2, 3, 4, 5 larning o’rniga qo’yishlari ichida $\begin{pmatrix}1&2&3&4&5\\3&4&5&2&1\end{pmatrix}$ o’rniga qo’yish bilan o’rin almashinuvchi bo’ladiganini toping.

2-§. Determinantning ta'rifi va asosiy xossalari.

Determinantni satr yoki ustun bo'yicha yoyish

To'g'ri to'rtburchak ko'rinishidagi sonlar jadvaliga *matritsa* deyiladi.

Matritsani belgilashda qavslardan foydalanamiz, masalan,

$$\begin{pmatrix} -1 & 2 & 3 \\ 5 & 4 & -6 \end{pmatrix}.$$

Matritsani tashkil etuvchi sonlarni uning *elementlari* deyiladi. Matritsa elementlarining gorizontal qatoriga uning *satrlari*, vertikal qatoriga *ustunlari* deyiladi. Agar matritsadagi satrlar soni ustunlar soniga teng bo'lsa, undagi satrlar soni – *matritsa tartibi* deb ataladi. Umumiy ko'rinishda yozilganda matritsaning elementlari ikkita indeksli bitta harf orqali belgilanib, birinchi indeks satrning tartib raqamini (nomerini), ikkinchi indeks ustun tartib raqamini ifodalaydi. Masalan, n-tartibli A matritsaning umumiy ko'rinishi quyidagicha yoziladi:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} = \left(a_{ij}\right)_1^n .$$

Kvadrat matritsaning yuqori chap burchagini quyi o'ng burchagi bilan tutushtiruvchi kesmada yotuvchi elementlar qatori matritsaning *bosh dioganali*, yuqori o'ng burchagini quyi chap burchagi bilan tutashtiruvchi kesmadagi elementlar qatori *yordamchi diagonali* deyiladi.

n-tartibli determinant yoki n>1 *da A matritsaning determinanti* deb, shu matritsaning elementlaridan quyidagi formula yordamida hosil qilingan songa aytiladi:

$$|A| = \det A = \left|a_{ij}\right|_1^n = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} =$$

$$= \sum (-1)^{s+t} a_{i_1 j_1} a_{i_2 j_2} \dots a_{i_n j_n} = \sum (-1)^k a_{1k_1} a_{2k_2} \dots a_{nk_n}$$

Bunda birinchi to'rtta ifoda determinantning belgtlanishi; birinchi summa o'zaro teng bo'lmagan barcha

$$\begin{pmatrix} i_1 & i_2 & \dots & i_n \\ j_1 & j_2 & \dots & j_n \end{pmatrix}, \qquad (*)$$

o'rniga qo'yishlar bo'yicha olinib, bunda s – yuqori satrdagi inversiyalar soni, t – quyi satrdagi inversiyalar soni, ikkinchi summa barcha $(k_1, k_2, \dots, k_n)$ o'rin almashtirishlar bo'yicha olinib, k -bu o'rin almashtirishdagi inversiyalar soni. Bu ikki summa aynan tengdir. Summalardagi qo'shiluvchilar *determinantning hadlari* deyiladi; determinantning har bir hadi – matritsaning har bir satridan bittadan, har bir ustunidan bittadan olingan n ta elementlar ko'paytmasiga teng bo'lib, agar (*) o'rniga qo'yish juft bo'lsa, bu ko'paytma o'z ishorasi bilan, agar o'rniga qo'yish toq bo'lsa, teskari ishora bilan olinadi. Birinchi tartibli determinant o'zining yagona elementiga teng. n-tartibli determinantning barcha elementlari soni $n!$ ga teng. A matritsaning elementlari, satrlari, ustunlari mos determinantning *elementlari, satrlari, ustunlari* deb ataladi.

1-m i s o l. Ikkinchi tartibli determinant:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}. \blacksquare$$

2-m i s o l. Uchinchi tartibli determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \blacksquare$$

Bu ifoda uchburchaklar qoidasi (*Sarryus qoidasi*) bo'yicha topiladi. Uni quyidagi jadvallar orqali tasvirlash mumkin bo'lib, bir xil ishora bilan bitta ko'paytmada ishtirok etuvchi elementlar kesmalar bilan birlashtirilib ko'rsatilgandir:

Matritsa (yoki determinantlar) ning barcha satrlarini mos ustunlar bilan almashtirishga *transponirlash* deyiladi. Demak, berilgan matritsaning satrlari transponirlangan matritsaning o'sha tartibda yozilgan ustunlaridan iborat, va aksincha.

Kvadrat matritsa (yoki determinant) bo'lgan holda transponirlash matritsani (yoki determinantni) bosh dioganal atrofida 180^0 burishdan iborat bo'ladi.

Bir nechta bir xil uzunlikdagi satrlar yig'indisi deganda, har bir elementi berilgan satrlardan mos elemantlar yig'indisidan iborat satrga aytiladi. *Satrni songa ko'paytirish* deganda quyidagi satr tushuniladiki, uning har bir elementi shu songa ko'paytirishdan hosil bo'ladi.

Bir xil uzunlikdagi satrlarning *chiziqli kombinasiyasi* deb, berilgan satrlarni *chiziqli kombinasiya koeffisiyentlari* deb ataluvchi sonlarga ko'paytmalarining yig'indisiga aytiladi.

Agar biror satr boshqalarining chiziqli kombinasiyasidan iborat bo'lsa, u holda berilgan satr bu satrlar orqali *chiziqli bog'langan* deyiladi. Agar bir xil uzunlikdagi satrlarning hyech biri qolganlari orqali chiziqli bog'lanishda bo'lmasa, bunday satrlar *chiziqli bog'lanmagan* deyiladi.

Masalan, (-1, -7, 5, -3)=2(1, -1, -2, -3)-3 (1, 2, -3, -1) tenglik birinchi satr qoligan ikki satrning chiziqli kombinasiyasidan iborat ekanligini ko'rsatadi.

D e t e r m i n a n t l a r n i n g   a s o s i y   x o s s a l a r i

1. Determinantda hamma satrlar mos ustunlar qilib yozilsa, ya’ni transponirlanganda, determinatning qiymati o’zgarmaydi.
2. Determinantning biror satridagi (yoki biror ustunidagi) barcha elementlar nolga teng bo’lsa, bunday determinant nolga teng bo’ladi.
3. Determinantda istalgan ikki satrni (yoki ikki ustunni) o’zaro almashtirsak, determinantning faqat ishorasi o’zgaradi.
4. Ikki satri (yoki ikki ustuni) teng bo’lgan determinant nolga tengdir.
5. Determinantning biror satridagi (yoki ustunidagi) barcha elementlarni aynan bitta songa ko’paytirilsa, u holda determinant ham shu songa ko’paytiriladi. Boshqacha aytganda, satrdagi (yoki ustundagi) barcha elementlarning umumiy ko’paytuvchisini determinant belgisi ostidan chiqarish mumkin.
6. Biror satridagi barcha elementlari boshqa bir satrining mos elementlariga proporsional bo’lgan determinant nolga tengdir. Xuddi shunday ustunlar uchun ham o’rinli.
7. Agar determinantni *i*-chi satridagi barcha elementlar *k* ta qo’shiluvchidan iborat bo’lsa, u holda bu determinantni *k* ta determinantlarning yig’indisi ko’rinishida ifodalash mumkin bo’lib, bunda ularning *i*-chidan farqli barcha satrlari berilgan deteminantdagidek, *i*-satri esa birinchi determinantda birinchi qo’shiluvchilardan ikkinchisida -ikkinchilaridan va h.k. tuzilgandir. Xuddi shunday, ustunlar uchun ham o’rinlidir. Xususiy holda bitta satrga boshqa bir satrni (ustunni) qo’shish (yoki undan ayirish) mumkin.
8. Agar determinantning hyech bo’lmaganda bitta satri boshqa satrlari orqali chiziqli bog’langan bo’lsa, bu determinant nolga tengdir. Aksincha, agar *n*-tartibli $(n \geq 2)$ determinant nolga teng bo’lsa, u holda uning hyech bo’lmaganda bitta satri boshqa satrlari orqali chiziqli ifodalangan bo’ladi. Xuddi shunday ustunlar uchun ham o’rinlidir.

3-m i s o l. Quyidagi ko’paytmalardan qaysi birlari mos tartibli determinantga kiradi:

a) $a_{33}\, a_{16}\, a_{72}\, a_{27}\, a_{55}\, a_{61}\, a_{44}$; v) $a_{27}\, a_{36}\, a_{51}\, a_{74}\, a_{25}\, a_{43}\, a_{62}$.

Yechish. a) bu ko’paytma yettinchi tartibli determinantga kiradi, chunki u har bir satr va har bir ustundan bittadan olib tuzilgan yettita elementning ko’paytmasidan iborat. Uning ishorasini aniqlash uchun berilgan ko’paytmadagi indekslardan o’rniga qo’yishni tuzib uning juft-toqligini aniqlaymiz:

$$\begin{pmatrix} 3 & 1 & 7 & 2 & 5 & 6 & 4 \\ 3 & 6 & 2 & 7 & 5 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 3 & 4 & 5 & 1 & 2 \end{pmatrix} = (16)(27)(3)(4)(5).$$

Dekrement 2 ga teng bo’lganligi sababli, bu o’rniga qo’yish juft, demak, ko’paytma plyus ishora bilan kiradi.

v) bu ko’paytma birinchi satrdagi elementni saqlamagani uchun determinantga kirmaydi. ■

4-m i s o l. i va k larni $a_{47}\ a_{63} a_{1i}\ a_{55}\ a_{7k}\ a_{24}\ a_{31}$ ko'paytma 7-tartibli determinantga plyus ishorasi bilan kiradigan qilib tanlang.

Yechish. Berilgan ko'paytmaning 7-tartibli determinantga plyus ishorasi bilan qatnashishi uchun ko'paytuvchilarning indekslaridan tuzilgan $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ i & 4 & 1 & 7 & 5 & 3 & k \end{pmatrix}$ o'rniga qo'yish juft bo'lishi zarur. Bu o'rniga qo'yish $i=6$, $k=2$ bo'lganda juft bo'ladi. Darhaqiqat, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 4 & 1 & 7 & 5 & 3 & 2 \end{pmatrix} = (163)(247)(5)$ o'rniga qo'yishning dekrementi 4 ga teng bo'lganligi sababli juftdir. ■

5-m i s o l. n tartibli determinantning birinchi ustunini oxiriga qo'yib, qolgan ustunlarni esa joylashish tartibini saqlagan holda chap tomonga siljitsak, determinant qanday o'zgaradi?

Yechish. Agar determinantda birinchi va ikkinchi ustunlarni, so'ng ikkinchi va uchinchi ustunlarni, va h.k. (n-1)-chi va n-chi ustunlar o'rinlarini almashtirib qo'ysak, natijada masala shartidagi almashtirishni hosil qilamiz. Hammasi bo'lib determinant ustunlarining (n-1) ta almashtirishini bajargan bo'lamiz. Demak, determinant $(-1)^{n-1}$ ga ko'paytirilgan bo'ladi. ■

6-m i s o l. Bosh dioganalga nisbatan simmetrik joylashgan elementlari qo'shma kompleks sonlardan iborat determinant haqiqiy sondan iboratligini ko'rsating.

Yechish. Faraz etaylik, determinant $d = a + b_i$ bo'lsin. d da transpozitsiya bajarib, mos elementlari d determinantning elementlariga qo'shma bo'lgan d^1 determinantni hosil qilamiz. Determinant o'zining elementlari ko'paytmalarining (ma'lum ishoralar bilan olingan) yig'indisiga teng bo'lganligi sababli, va bir nechta kompleks sonlarning yig'indisi va ko'paytmasiga qo'shma bo'lgan kompleks sonlarning xossasiga ko'ra: $d^1 = a - b_i$ bo'ladi. $d = d^1$ bo'lganligi sababli, $a + b_i = a - b_i$ tenglik bajariladi.

Bundan $b = 0$ ni va $d = a$ - haqiqiy son ekanligini ko'ramiz. ■

7-m i s o l. Bosh dioganalga nisbatan simmetrik joylashgan elementlari faqat ishora bilangina farq qilsa, ya'ni barcha i va k larda $a_{ik}=-a_{ki}$ shart bajarilsa, u holda bunday determinant *kososimmetrik* deyiladi. Toq tartibli kososimmetrik determinant nolga teng bo'lishini ko'rsating.

Yechish. d determinantning har bir satridan (-1) ko'paytuvchini chiqarsak, transpozitsiyalashgan d ga teng bo'lgan determinantni hosil qilamiz, ya'ni $d = (-1)^n d$. n toq bo'lganligi sababli $d = 0$ ni hosil qilamiz. ■

8-m i s o l. 24026, 40262, 02624, 26240, 62402 41 ga bo'linadi.

$$\begin{vmatrix} 2 & 4 & 0 & 2 & 6 \\ 4 & 0 & 2 & 6 & 2 \\ 0 & 2 & 6 & 2 & 4 \\ 2 & 6 & 2 & 4 & 0 \\ 6 & 2 & 4 & 0 & 2 \end{vmatrix}$$ determinant 41 ga bo'linishini isbotlang.

Yechish. Oxirgi ustunga 10000 ga ko'paytirilgan birinchi ustunni, 1000 ga ko'paytirilgan ikkinchi, 100 ga ko'paytirilgan uchinchi, 10 ga ko'paytirilgan to'rtinchi ustunni 41 soniga bo'linadigan 24026, 40262, 02624, 26240, 62402 sonlardan tuzilagan determinantni hosil qilamiz. Demak, berilgan determinant 41 ga bo'linadi. ■

n-tartibli $(n\geq 2)$ d determinantning a_{ij} elementining M_{ij} minori deb, *d* determinantning a_{ij} elementi turgan satr va ustunni o'chirishdan keyin qolgan *n-1* tartibli determinantga aytiladi.

a_{ij} elementning algebraik to'ldiruvchisi deb $A_{ij}=(-1)^{i+j}M_{ij}$ ga aytiladi.

Agar determinantning biror satr (ustun) elementlarini ularning algebraik to'ldiruvchilariga ko'paytirib yig'sak, shu determinantga teng bo'ladi. Xususiy holda, agar satrda (yoki ustunda) bitta elementdan boshqa barchasi nolga teng bo'lsa, u holda determinant shu elementning uni algebraik to'ldiruvchisiga ko'paytmasiga teng bo'ladi. Masalan,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

determinantda a_{23} element minori

$$M_{23}=\begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}.$$

ga teng bo'lib, uning algebraik to'ldiruvchisi $A_{23} = -M_{23}$.■

9-m i s o l. Determinantni uchinchi satr bo'yicha yoyib, hisoblang.

$$A=\begin{vmatrix} 2 & -3 & 4 & 1 \\ 4 & -2 & 3 & 2 \\ a & b & c & d \\ 3 & -1 & 4 & 3 \end{vmatrix}.$$

Yechish.

$$A=a(-1)^{3+1}\begin{vmatrix} -3 & 4 & 1 \\ -2 & 3 & 2 \\ -1 & 4 & 3 \end{vmatrix}+b(-1)^{3+2}\begin{vmatrix} 2 & 4 & 1 \\ 4 & 3 & 2 \\ 3 & 4 & 3 \end{vmatrix}+$$

$$c(-1)^{3+3}\begin{vmatrix} 2 & -3 & 1 \\ 4 & -2 & 2 \\ 3 & -1 & 3 \end{vmatrix}+d(-1)^{3+4}\begin{vmatrix} 2 & -3 & 4 \\ 4 & -2 & 3 \\ 3 & -1 & 4 \end{vmatrix}=8a+15b+12c-19d.■$$

10-m i s o l. Determinantni yoymasdan quyidagi ayniyatni isbotlang.

$$\begin{vmatrix} o & x & y & z \\ x & o & z & y \\ y & z & o & x \\ z & y & x & o \end{vmatrix} = x^2y^2z^2 \begin{vmatrix} o & 1 & 1 & 1 \\ 1 & o & z^2 & y^2 \\ 1 & z^2 & o & x^2 \\ 1 & y^2 & x^2 & o \end{vmatrix}.$$

Yechish. Chap tomondagi determinantning ikkinchi ustunini yz ga, uchinchi ustunini xz ga, to'rtinchi ustunini xy ga ko'paytirib, quyidagini hosil qilamiz:

$$\begin{vmatrix} o & xyz & xyz & xyz \\ x & o & xz^2 & xy^2 \\ y & yz^2 & o & x^2y \\ z & y^2z & x^2z & o \end{vmatrix}.$$

Bu determinantning birinchi satridan xyz ni, ikkinchi satridan x ni, uchinchidan u ni, 4-dan z ni chiqarsak, o'ng tomondagi determinant hosil bo'ladi. ■

11-m i s o l. Determinant xossalaridan satr yoki ustun bo'yicha yoyishdan foydalanib, quyidagi ayniyatlarni isbotlang:

$$\begin{vmatrix} \cos\frac{\alpha-\beta}{2} & \sin\frac{\alpha+\beta}{2} & \cos\frac{\alpha+\beta}{2} \\ \cos\frac{\beta-\gamma}{2} & \sin\frac{\beta+\gamma}{2} & \cos\frac{\beta+\gamma}{2} \\ \cos\frac{\gamma-\alpha}{2} & \sin\frac{\gamma+\alpha}{2} & \cos\frac{\gamma+\alpha}{2} \end{vmatrix} = \frac{1}{2}\left[\sin(\beta-\alpha)+\sin(\gamma-\beta)+\sin(\alpha-\lambda)\right].$$

Yechish. Determinantni birinchi ustun bo'yicha yoysak, quyidagilar hosil bo'ladi:

$$\cos\frac{\alpha-\beta}{2}\begin{vmatrix} \sin\frac{\beta+\gamma}{2} & \cos\frac{\beta+\gamma}{2} \\ \sin\frac{\gamma+\alpha}{2} & \cos\frac{\gamma+\alpha}{2} \end{vmatrix} - \cos\frac{\beta-\gamma}{2}\begin{vmatrix} \sin\frac{\alpha+\beta}{2} & \cos\frac{\alpha+\beta}{2} \\ \sin\frac{\gamma+\alpha}{2} & \cos\frac{\gamma+\alpha}{2} \end{vmatrix} +$$

$$+\cos\frac{\gamma-\alpha}{2}\begin{vmatrix} \sin\frac{\alpha+\beta}{2} & \cos\frac{\alpha+\beta}{2} \\ \sin\frac{\beta+\gamma}{2} & \cos\frac{\beta+\gamma}{2} \end{vmatrix} =$$

$$=\cos\frac{\alpha-\beta}{2}\left(\sin\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}-\sin\frac{\gamma+\alpha}{2}\cos\frac{\beta+\gamma}{2}\right)-$$

$$-\cos\frac{\beta-\gamma}{2}\left(\sin\frac{\alpha+\beta}{2}\cos\frac{\gamma+\alpha}{2}-\sin\frac{\gamma+\alpha}{2}\cos\frac{\alpha+\beta}{2}\right)+$$

$$=\cos\frac{\gamma-\alpha}{2}\left(\sin\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}-\sin\frac{\beta+\gamma}{2}\cos\frac{\alpha+\beta}{2}\right)=$$

$$=\cos\frac{\alpha-\beta}{2}\sin\frac{\beta-\alpha}{2}-\cos\frac{\beta-\gamma}{2}\sin\frac{\beta-\gamma}{2}+\cos\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\gamma}{2}=$$

$$=\frac{1}{2}\left[\sin(\beta-\alpha)+\sin(\gamma-\beta)+\sin(\alpha-\gamma)\right].\blacksquare$$

M A S H Q L A R

1.32 Determinantlarni hisoblang:

$$a)\begin{vmatrix}3 & 5\\ 5 & 8\end{vmatrix};\quad b)\begin{vmatrix}ab & ac\\ bd & cd\end{vmatrix};\quad c)\begin{vmatrix}\sin\alpha & \sin\beta\\ \cos\alpha & \cos\beta\end{vmatrix};\quad d)\begin{vmatrix}\ell og_b a & 1\\ 1 & \ell og_a b\end{vmatrix};$$

$$e)\begin{vmatrix}\cos\alpha+i\sin\alpha & 1\\ 1 & \cos\alpha-i\sin\alpha\end{vmatrix};\quad f)\begin{vmatrix}a+bi & c+di\\ -c+di & a-bi\end{vmatrix}.$$

1.33. Determinantlarni hisoblang:

$$a)\begin{vmatrix}-1 & 5 & 4\\ 3 & -2 & 0\\ -1 & 3 & 6\end{vmatrix};\quad b)\begin{vmatrix}0 & 2 & 2\\ 2 & 0 & 2\\ 2 & 2 & 0\end{vmatrix};\quad c)\begin{vmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{vmatrix};$$

$$d)\begin{vmatrix}a & b & c\\ b & c & a\\ c & a & b\end{vmatrix};\quad e)\begin{vmatrix}0 & a & 0\\ b & c & d\\ 0 & e & 0\end{vmatrix};\quad f)\begin{vmatrix}\sin\alpha & \cos\alpha & 1\\ \sin\beta & \cos\beta & 1\\ \sin\gamma & \cos\gamma & 1\end{vmatrix};$$

$$g)\begin{vmatrix}1 & 0 & 1+i\\ 0 & 1 & i\\ 1-i & -i & 1\end{vmatrix};\quad h)\begin{vmatrix}1 & \varepsilon & \varepsilon^2\\ \varepsilon^2 & 1 & \varepsilon\\ \varepsilon & \varepsilon^2 & 1\end{vmatrix}\left(\varepsilon=-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right);$$

$$i)\begin{vmatrix}1 & 1 & 1\\ 1 & \varepsilon & \varepsilon^2\\ 1 & \varepsilon^2 & \varepsilon\end{vmatrix}\left(\varepsilon=\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi\right).$$

1.34. Determinantlarni yoymasdan turib, quyidagi ayniyatlarni isbotlang:

$$a)\begin{vmatrix}1 & a & bc\\ 1 & b & ca\\ 1 & c & ab\end{vmatrix}=(b-a)(c-a)(c-b);$$

$$b)\begin{vmatrix}1 & 1 & 1\\ a^2 & b^2 & c^2\\ a^3 & b^3 & c^3\end{vmatrix}=(ab+ac+bc)(b-a)(c-a)(c-b);$$

$$c)\begin{vmatrix}1 & a & bc\\ 1 & b & ca\\ 1 & c & ab\end{vmatrix}=\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix};\quad d)\begin{vmatrix}1 & a & a^3\\ 1 & b & b^3\\ 1 & c & c^3\end{vmatrix}=(a+b+c)\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}.$$

1.35. Quyidagi ko'paytmalardan qaysi birlari mos tartibli determinantlar yoyilmasiga kiradi:

a) $a_{43}\, a_{21}\, a_{35}\, a_{12}\, a_{54}$; b) $a_{61}\, a_{23}\, a_{45}\, a_{36}\, a_{12}\, a_{54}$;

c) $a_{12}\, a_{23}\, a_{34} \ldots a_{n-1,\, n}\, a_{kk}\ (1\le k\le n)$; d) $a_{12}\, a_{23} \ldots a_{n-1,\, n}\, a_{n1}$;

e) $a_{11}\, a_{2,n}\, a_{3,n-1} \ldots a_{n,2}$;

f) $a_{13}\, a_{22}\, a_{31}\, a_{46}\, a_{55}\, a_{64} \ldots a_{3n-2,\, 3n}\, a_{3n-1,\, 3n-1} a_{3n,\, 3n-2}$.

1.36. i va k ni shunday tanlangki, $a_{62}\, a_{i5}\, a_{33}\, a_{k4}\, a_{46}\, a_{21}$ ko'paytma 6-tartibli determinantga minus ishorasi bilan qatnashsin.

1.37. *a*) $\begin{vmatrix} 5x & 1 & 2 & 3 \\ x & x & 1 & 2 \\ 1 & 2 & x & 3 \\ x & 1 & 2 & 2x \end{vmatrix}$ determinantning x^4 va x^3 ni saqlovchi hadlarni toping.

b) $\begin{vmatrix} x & 1 & 2 & 3 \\ 1 & x & 3 & 2 \\ 3 & 1 & x & 2 \\ 5 & 3 & 1 & x \end{vmatrix}$, determinantning x^4, x^3 va x^2 ni saqlovchi hadlarni toping.

1.38. *a*) $\begin{vmatrix} 1 & 1 & 1 & a \\ 2 & 2 & 1 & b \\ 3 & 2 & 1 & c \\ 1 & 2 & 3 & d \end{vmatrix}$ determinantni 4-ustun elementlari bo'yicha yoying;

b) $\begin{vmatrix} a & 1 & 1 & 1 \\ b & 0 & 1 & 1 \\ c & 1 & 0 & 1 \\ d & 1 & 1 & 0 \end{vmatrix}$ determinantni 1-ustun elementlari bo'yicha yoying;

c) $\begin{vmatrix} 1 & 2 & -1 & -2 \\ 2 & -1 & -2 & 1 \\ a & b & c & d \\ -2 & -1 & 1 & 2 \end{vmatrix}$ determinantni 3-satr elementlari bo'yicha yoying.

1.39. Determinantni uni yoymasdan turib hisoblang:

$$\begin{vmatrix} a & b & c & 1 \\ b & c & a & 1 \\ c & a & b & 1 \\ \frac{b+c}{2} & \frac{c+a}{2} & \frac{a+b}{2} & 1 \end{vmatrix}.$$

1.40. Determinantning xossalaridan foydalanib (satr va ustun bo'yicha yoy-ishni ham hisobga olganda) ayniyatlarni isbotlang:

$$a)\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix} = 2abc(a+b+c)^3;$$

$$b)\begin{vmatrix} \frac{1}{a+x} & \frac{1}{a+y} & 1 \\ \frac{1}{b+x} & \frac{1}{b+y} & 1 \\ \frac{1}{c+x} & \frac{1}{c+y} & 1 \end{vmatrix} = \frac{(a-b)(a-c)(b-c)(x-y)}{(a+x)(b+x)(c+x)(a+y)(b+y)(c+y)};$$

$$c)\begin{vmatrix} a^2+(1-a^2)\cos\varphi & ab(1-\cos\varphi) & ac(1-\cos\varphi) \\ ba(1-\cos\varphi) & b^2+(1-b^2)\cos\varphi & bc(1-\cos\varphi) \\ ca(1-\cos\varphi) & cb(1-\cos\varphi) & c^2+(1-c^2)\cos\varphi \end{vmatrix} = \cos^2\varphi;$$

$$d)\begin{vmatrix} a & b & c & d \\ -b & a & d & -c \\ -c & -d & a & b \\ -d & c & -b & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2;$$

$$e)\begin{vmatrix} 0 & 1 & 1 & a \\ 1 & 0 & 1 & b \\ 1 & 1 & 0 & c \\ a & b & c & d \end{vmatrix} = a^2+b^2+c^2-2ab-2bc-2ac+2d.$$

1.41. Agar n-tartibli determinantda:

a) har bir elementi ishorasi o’zgartirilsa;

v) har bir elementni yordamchi dioganaliga nisbatan simmetrik bo’lgan element bilan almashrirsak;

s) har bir elementni «markaz»ga nisbatan simmetrik bo’lgan element bilan almashtirsak, u holda determinant qanday o’zgaradi?

1.42. Agar determinantning

a) ikkinchisidan boshlab har bir ustuniga oldingi ustunni qo’shib borilsa;

v) ikkinchi ustunidan boshlab har bir ustunga oldingi barcha ustunlar qo’shib borilsa;

s) oxirgisidan boshqa har bir satrdan keyingi satr ayrilib borilsa, oxirgisidan esa oldingi birinchi satri ayrilsa, determinant qanday o’zgaradi?

1.43. 185, 518, 851 sonlari 37 ga bo’linadi. $\begin{vmatrix} 1 & 8 & 5 \\ 5 & 1 & 8 \\ 8 & 5 & 1 \end{vmatrix}$ determinant 37 ga bo’linishini isbotlang.

1.44. 20604, 53227, 25755, 20927, 289 sonlari 17 ga bo'linadi. $\begin{vmatrix} 2 & 0 & 6 & 0 & 4 \\ 5 & 3 & 2 & 2 & 7 \\ 2 & 5 & 7 & 5 & 5 \\ 2 & 0 & 9 & 2 & 7 \\ 0 & 0 & 2 & 8 & 9 \end{vmatrix}$ determinant 17 ga bo'linishini isbotlang.

1.45. Determinantni yoymasidan foydalanib, quyidagi determinantlarni hisoblang:

$$a) \begin{vmatrix} a_{12} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{vmatrix}; \qquad b) \begin{vmatrix} 0\dots & 0 & 0 & a_{1n} \\ 0\dots & 0 & a_{2,n-1} & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}\dots & a_{n,n-2} & a_{n,n-1} & a_{nn} \end{vmatrix}$$

$$c) \begin{vmatrix} a & 3 & 0 & 5 \\ 0 & b & 0 & 2 \\ 1 & 2 & c & 3 \\ 0 & 0 & 0 & d \end{vmatrix}; \quad d) \begin{vmatrix} 1 & 0 & 2 & a \\ 2 & 0 & b & 0 \\ 3 & c & 4 & 5 \\ d & 0 & 0 & 0 \end{vmatrix}; \quad e) \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & 0 & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 & 0 \\ a_{51} & a_{52} & 0 & 0 & 0 \end{vmatrix}.$$

1.46. Determinantlarni hisoblamasdan tenglamalarni yeching:

$$a) \begin{vmatrix} 1 & a_1 & a_2 & \dots & a_{n-1} \\ 1 & x & a_2 & \dots & a_{n-1} \\ 1 & a_2 & x & \dots & a_{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & a_2 & a_3 & \dots & x \end{vmatrix} = 0; \quad b) \begin{vmatrix} 1 & a & a_2\dots & a_{n-1} \\ 1 & a_1 + x + 1 & a_2\dots & a_{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & a_1 & a_2\dots & a_{n-1} + x + n - 1 \end{vmatrix} = 0;$$

$$c) \begin{vmatrix} 1 & x & x^2\dots & x^{n-1} \\ 1 & a_1 & a_1^2\dots & a_1^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & a_{n-1} & a_{n-1}^{\;2} & a_{n-1}^{n-1} \end{vmatrix} = 0.$$

3-§. Elementar almashtirishlar yordamida determinantlarni hisoblash

n-tartibli determinantni hisoblashni *(n-1)-tartibli* bitta determinantni hisoblashga olib kelish mumkin.

$$d=\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Determinantni hisoblash kerak bo'lsin. Agar birinchi ustunning barcha elementlari nolga teng bo'lsa, u holda $d = 0$ (2-xossa) bo'ladi; agar $a_{11} = 0$, lekin $a_{k1} \neq 0$ bo'lsa, u holda 1- va k- satrlar o'rnini almashtirib, yuqori chap burchakda noldan farqli elementni hosil qilamiz. Demak, umumiylikni saqlagan holda $a_{11}\neq 0$ deb hisoblash mumkin. 2-satrga $\left(-\frac{a_{21}}{a_{11}}\right)$ga ko'paytirilgan 1-satrni, 3-satrga $\left(-\frac{a_{31}}{a_{11}}\right)$ga ko'paytirilgan 1-satrni, ..., n-satrga $\left(-\frac{a_{n1}}{a_{11}}\right)$ga ko'payti-rilgan 1-ustunni qo'shamiz. Determinantning qiymati o'zgarmaydi (8-xossa) va quyidagi ko'rinishni oladi:

$$d=\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & b_{22} & \dots & b_{2n} \\ \dots & \dots & \dots & \dots \\ 0 & b_{n2} & \dots & b_{nn} \end{vmatrix} = a_{11}\left|b_{ij}\right|_1^{n-1}.$$

1-m i s o l. $d=\begin{vmatrix} 1 & -1 & 2 & 1 \\ 3 & 4 & -1 & 0 \\ 2 & 1 & 3 & 1 \\ -5 & 0 & 1 & -2 \end{vmatrix}$ determinantni hisoblang.

Yechish.. Determinantda quyidagi almashtirishlarni bajaramiz:

a) 1-ustun elementlarini 2-ustunning mos elementlariga qo'shamiz;

v) (-2) ga ko'paytirilgan 1-ustun elementlarini 3-ustunning mos elementlariga qo'shamiщ;

s) (-1) ga ko'paytirilgan 1-ustun elementlarini 4-ustunning mos elementlariga qo'shamiz.

Natijada determinantning qiymati o'zgarmaydi:

$$\begin{vmatrix} 1 & -1 & 2 & 1 \\ 3 & 4 & -1 & 0 \\ 2 & 1 & 3 & 1 \\ -5 & 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 3 & 7 & -7 & -3 \\ 2 & 3 & -1 & -1 \\ -5 & -5 & 11 & 3 \end{vmatrix}.$$

Birinchi satrda uchta nol bo'lganligi sababli bu determinantni 1-satr elementlari bo'yicha yoyish qulaydir. Shunday qilib,

$$d = 1\cdot(-1)^{1+1}\begin{vmatrix} 7 & -7 & -3 \\ 3 & -1 & -1 \\ -5 & 11 & 3 \end{vmatrix} = -21-99-35+15+77+63=0\,.\blacksquare$$

2-m i s o l. $\Delta=\begin{vmatrix}1&1&1&1\\1&-1&1&1\\1&1&-1&1\\1&1&1&-1\end{vmatrix}$ determinantni hisoblang.

Yechish. Har bir keyingi ustundan 1-ustunni ayiramiz.

$$\Delta=\begin{vmatrix}1&0&0&0\\1&-2&0&0\\1&0&-2&0\\1&0&0&-2\end{vmatrix}=1\cdot(-1)^{1+1}\begin{vmatrix}-2&0&0\\0&-2&0\\0&0&-2\end{vmatrix}=-8\,.\blacksquare$$

M A S H Q L A R

1.47. Determinantlarni hisoblang:

$a)\begin{vmatrix}-1&2&7&5\\1&3&-1&2\\2&1&2&3\\-5&2&-1&3\end{vmatrix}$; $b)\begin{vmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{vmatrix}$; $c)\begin{vmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{vmatrix}$; $d)\begin{vmatrix}3&-1&2&5\\2&1&0&3\\4&-2&1&6\\-1&3&-2&4\end{vmatrix}$;

$e)\begin{vmatrix}10&6&8&3\\1&1&3&-4\\4&-2&-1&11\\9&7&6&-13\end{vmatrix}$; $f)\begin{vmatrix}1&3&3&1\\3&4&3&2\\3&2&1&4\\2&4&2&3\end{vmatrix}$; $g)\begin{vmatrix}1&2&3&4\\-2&1&-4&3\\3&-4&-1&2\\4&3&-2&-1\end{vmatrix}$; $h)\begin{vmatrix}4&7&9&2\\2&3&4&-6\\5&9&6&9\\3&6&8&5\end{vmatrix}$;

$i)\begin{vmatrix}1&2&3&4\\2&2&1&4\\3&1&3&3\\4&4&3&4\end{vmatrix}$; $j)\begin{vmatrix}2&3&0&0&0\\4&7&0&0&0\\-8&2&1&0&0\\25&17&0&1&0\\17&8&0&0&1\end{vmatrix}$; $k)\begin{vmatrix}5&-1&3&-3&1\\8&4&1&0&1\\13&3&4&-3&2\\3&2&-2&4&-5\\7&-6&0&8&1\end{vmatrix}$;

$e)\begin{vmatrix}6&1&1&1&1\\1&5&1&1&1\\1&1&4&1&1\\1&1&1&3&1\\1&1&1&1&2\end{vmatrix}$; $m)\begin{vmatrix}1&2&3&4\\-2&1&5&6\\-3&-5&1&7\\-4&-6&-7&1\end{vmatrix}$; $n)\begin{vmatrix}1&2&3&4&5\\5&1&2&3&4\\4&5&1&2&3\\3&4&5&1&2\\2&3&4&5&1\end{vmatrix}$;

$$o)\begin{vmatrix} 23 & 11 & 48 & 106 \\ 19 & 32 & 45 & 116 \\ 7 & 25 & 43 & 83 \\ 67 & 73 & 81 & 289 \end{vmatrix}; \quad p)\begin{vmatrix} \frac{1}{2} & -1 & -2 & \frac{1}{3} \\ -1 & \frac{1}{2} & 0 & \frac{1}{5} \\ 1 & 1 & -\frac{1}{2} & -\frac{1}{4} \\ 2 & 3 & 1 & \frac{1}{5} \end{vmatrix}; \quad q)\begin{vmatrix} \frac{2}{3} & \frac{4}{8} & \frac{6}{9} & 1 \\ 1 & \frac{1}{2} & \frac{1}{3} & -1 \\ \frac{5}{6} & \frac{1}{3} & \frac{3}{5} & 2 \\ \frac{1}{7} & \frac{1}{3} & \frac{1}{2} & -\frac{1}{4} \end{vmatrix};$$

$$r)\begin{vmatrix} \frac{1}{3} & -\frac{5}{12} & \frac{2}{5} & \frac{3}{2} \\ 3 & -12 & \frac{21}{5} & 15 \\ \frac{2}{3} & -\frac{9}{2} & \frac{4}{5} & \frac{5}{2} \\ -\frac{1}{7} & \frac{2}{7} & -\frac{1}{7} & \frac{3}{7} \end{vmatrix}; \quad s)\begin{vmatrix} 0 & \sqrt{2} & \sqrt{3} & \sqrt{5} \\ -\sqrt{2} & 0 & \sqrt{3} & \sqrt{5} \\ -\sqrt{5} & -\sqrt{3} & 0 & \sqrt{7} \\ -\sqrt{5} & -\sqrt{5} & -\sqrt{7} & 0 \end{vmatrix};$$

$$t)\begin{vmatrix} 0 & \sqrt{2} & \sqrt{6} & \sqrt{10} \\ -\sqrt{2} & 0 & \sqrt{3} & \sqrt{7} \\ -\sqrt{6} & -\sqrt{3} & 0 & \sqrt{5} \\ -\sqrt{10} & -\sqrt{7} & -\sqrt{5} & 0 \end{vmatrix}.$$

§ 4. *n*-tartibli determinantlarni hisoblash usullari

Sonli determinantlarni hisoblashda qo'llaniladigan ma'lum usullar juda ko'p hisoblashlarni bajarishni talab qiladi. Harfiy va sonli determinantlarning ma'lum bir ko'rinishlari uchun ularni hisoblashning ba'zi bir usullari ishlab chiqilgan.

1. *Determinantni uchburchak ko'rinishiga olib kelish usuli*

Bu usulning asosiy g'oyasi – dioganaldan bir tomonda turgan barcha elementlar elementar almashtirishlarni bajarib nolga keltiriladi. Agar bosh dioganaldan bir tomonda yotgan elementlar nolga teng bo'lsa, bunday determinant bosh diagonaldagi barcha elementlar ko'paytmasiga teng bo'ladi. Agar determinantning yordamchi diagonalidan bir tomonda yotgan barcha elementlar nolga teng bo'lsa, u holda bunday determinant $(-1)^{\frac{n(n-1)}{2}}$ ishora bilan olingan diagonaldagi barcha elementlar ko'paytmasiga teng.

1-m i s o l. *n*-tartibli determinantni hisoblang:

$$d=\begin{vmatrix} 1 & 2 & 3 & \dots & n \\ -1 & 0 & 3 & \dots & n \\ -1 & -2 & 0 & \dots & n \\ \dots & \dots & \dots & \dots & \dots \\ -1 & -2 & -3 & \dots & n \end{vmatrix}.$$

Yechish. Birinchi satrni qolgan satrlariga qo'shib chiqamiz. Natijada bosh diagonalning pastida turgan barcha elementlari nolga teng bo'lgan determinant hosil bo'ladi:

$$d=\begin{vmatrix} 1 & 2 & 3 & \dots & n \\ 0 & 2 & 6 & \dots & 2n \\ 0 & 0 & 3 & \dots & 2n \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & n \end{vmatrix}.$$

Demak, $d=1\cdot 2\cdot 3\cdots n=n!$ ■

2-m i s o l. n-tartibli $d=\begin{vmatrix} a & a\dots & a & a+x \\ a & a\dots & a+x & a \\ \dots & \dots & \dots & \dots \\ a+x & a\dots & a & a \end{vmatrix}$ determinantni hisoblang.

Yechish. Oxirgi ustunga oldingi barcha ustunlarni qo'shamiz:

$$d=\begin{vmatrix} a & a\dots & a & na+x \\ a & a\dots & a+x & na+x \\ \dots & \dots & \dots & \dots \\ a+x & a\dots & a & na+x \end{vmatrix}.$$

Determinant belgisi ostidan oxirgi ustundagi umumiy ko'paytuvchi - $na + x$ ni chiqaramiz. Oldingi ustunlarning har biridan a ga ko'paytirilgan oxirgi ustunni ayiramiz. Natijada yordamchi diagonalidan yuqorida turgan barcha elementlari nollardan iboratbo'lgan uchburchak ko'rinishidagi determinantga kelamiz:

$$d=(na+x)\begin{vmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & x & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & x & \dots & 0 & 1 \\ x & 0 & \dots & 0 & 1 \end{vmatrix}.$$

Demak,

$$d=(-1)^{\frac{n(n-1)}{2}}(x+na)x^{n-1}.■$$

2. *Chiziqli ko'paytuvchilarni ajratish usuli*

Bu usulning asosiy g'oyasi n-tartibli determinantga bir yoki bir necha o'zgaruvchilarning m-tartibli ko'phadi sifatida qaraydi. Bevosita yoki ma'lum almashtirishlarni bajarib determinant bo'linadigan m ta o'zaro tub bo'lgan chiziqli ko'paytuvchilar topiladi. U holda determinant o'zgarmas ko'paytuvchi S aniqligida shu chiziqli ko'paytuvchilarning ko'paytmasiga teng bo'ladi. O'zgarmas S soni mos ravishda determinantning hadi va chiziqli ko'paytuvchilar ko'paytmasidagi hadini solishtirish natijasida topiladi.

3-m i s o l. n-tartibli determinantni hisoblang.

$$d = \begin{vmatrix} 1 & 2 & 3 & \dots & n \\ 1 & x+a & 3 & \dots & n \\ 1 & 2 & x+a & \dots & n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & \dots & x+a \end{vmatrix}.$$

Yechish. Determinantning diagonalidagi elementlari ko'paytmasi x ni eng katta - $(n - 1)$-darajada saqlaydi. Demak, bu determinant x ning $(n - 1)$ darajali ko'phadidir. x ning $x = 2 - a, x = 3 - a, ..., x = n - a$ qiymatlarida bu determinantning mos holda 1- va 2-, 1- va 3-, .. , 1- va n-satrlari bir xil bo'ladi va natijada determinant nolga teng bo'ladi. Shunday qilib, d determinant $x + a - 2$, $x + a - 3, ..., x + a - n$ ga bo'linadi va demak,

$$d = c (x + a -2)(x + a -3)...(x + a - n) \qquad (*)$$

c sonni topish uchun bosh diagonaldagi elementlarini ko'paytirishda hosil bo'lgan x^{n-1} hadni (*) ning o'ng tomonidagi $c\, x^{n-1}$ had bilan solishtiramiz. Bu hadlar teng bo'lishi shartidan $s = 1$ ni va natijada

$d = (x + a - 2) (x + a - 3)...(x + a - n)$ ni hosil qilamiz. ■

4-m i s o l. $\begin{vmatrix} -x & a & b & c \\ a & -x & c & b \\ b & c & -x & a \\ c & b & a & -x \end{vmatrix}$ determinantni hisoblang.

Yechish. 1-ustunga qolgan ustunlarni qo'shamiz; natijada birinchi ustunning barcha elementlari $(x - a - b - c)$ ga teng bo'ladi. Demak, d determinant $(x - a - b - c)$ ga bo'linadi. Agar 1-ustundan 2-ustunni ayirib, 3-ustunni qo'shib va 4-ustunni ayirsak, u holda 1-ustunning barcha elementlari ±1 aniqligida $x + a - b + c$ ga teng bo'ladi. Demak, d determinant $x + a - b + c$ ga bo'linadi. Agar 1-ustundan 2- va 3-ustunlarni ayirsak, va 4-us-tunni qo'shsak, u holda ±1 aniqligida 1-ustun elementlari $x + a + b - c$ ga teng bo'ladi, demak, d determinant $x + a + b - c$ ga bo'linadi. Natijada

$d = m (x - a - b - c) (x + a - b + c) (x - a + b + c) (x + a + b - c)$ ga ega bo'lamiz. Bu ifodada m soni x, a, b, c sonlardan bog'liq emas. m sonini aniqlash uchun d determinantning bosh diagonali elementlarni ko'paytirishdan hosil bo'ladigan x^4 hadni o'ng tomonda hosil bo'ladigan mx^4 bilan solishtirib, $m = 1$ ni hosil qilamiz. Shunday qilib,

$d = (x - a - b - c)(x + a - b + c)(x - a + b + c)(x + a + b - c)$. ■

3. *Rekurrent munosabatlar usuli*

Bu usulda berilgan determinant xuddi shu ko'rinishdagi tartibi kichik bo'lgan bitta yoki bir nechta determinantlarga keltiriladi. Buning uchun determinant biror satr yoki ustun bo'yicha yoyiladi. Ba'zi hollarda ma'lum almashtirishlar bajarib determinant qulay ko'rinishga keltiriladi va so'ng satr yoki ustun bo'yicha yoyiladi. Determinantni xuddi shu ko'rinishla pastki tartibli bitta yoki bir nechta determinant orqali ifodalovchi tenglikka rekurrent yoki qaytish munosibatlari deyiladi. Rekurrent munosibatdan matematik induksiya usulidan foydalanib berilgan determinantning umumiy ifodasi keltirib chiqariladi.

Bu usul quyidagi o'zgartirilgan shaklda ham qo'llanilishi mumkin: n-tartibli determinantlar orqali ifodalovchi rekurrent munosibatda, shu rekurrent munosibatdagi n ni $(n - 1)$ bilan almashtirgandagi ifodasi keltirib qo'yiladi; xuddi shunday $(n - 2)$-tartibli ifodasi va h.k. qo'yib chiqiladi. Natijada n-tartibli determinantning umumiy ko'rinishi hosil bo'ladi. Bu ifodalashning to'g'riligi matematik induksiya usuli yordamida tekshirib ko'riladi.

5-m i s o l. $(n + 1)$-tartibli

$$d_{n+1} = \begin{vmatrix} a_0 & -1 & 0 & 0 & \dots & 0 & 0 \\ a_1 & x & -1 & 0 & \dots & 0 & 0 \\ a_2 & 0 & x & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & 0 & 0 & 0 & \dots & x & -1 \\ a_n & 0 & 0 & 0 & \dots & 0 & x \end{vmatrix}$$ determinantni hisoblang.

Yechish. d_{n+1} ni oxirgi satr bo'yicha yoyib chiqamiz.

$$d_{n+1} = a_n(-1)^n \begin{vmatrix} -1 & 0 & \dots & 0 \\ x & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -1 \end{vmatrix} + x \begin{vmatrix} a_0 & -1 & 0 & \dots & 0 \\ a_1 & x & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & 0 & 0 & \dots & x \end{vmatrix}.$$

O'ng tomondagi 1-determinant uchburchak shakliga ega. 2-determinant berilgan d_{n+1} determinant ko'rinishiga ega bo'lib, tartibi n-ga teng va a_n ni saqlamaydi. Natijada $d_{n+1} = a_n + x\, d_n$ (1) rekurrent munosabat hosil bo'ladi.

d_{n+1} determinantning umumiy ko'rinishini keltirib chiqarish uchun d_1 va d_2 larni qaraymiz:

$$d_1 = a_0, \quad d_2 = \begin{vmatrix} a_0 & -1 \\ a_1 & x \end{vmatrix} = a_0 x + a_1.$$

d_1 - x ga nisbatan nolinchi darajali, koeffisiyenti a_0 ga teng bo'lgan ko'phad, d_2 - birinchi darajali, koeffisiyenti a_0 va a_1 ga teng bo'lgan ko'phad.

d_{n+1} uchun xuddi shu kabi: $d_{n+1} = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$ munosabat o'rinli ekanligini ko'rsatamiz.

Faraz qilaylik, $d_n = a_0 x^{n-1} + a_1 x^{n-2} + \ldots + a_{n-1}$ o'rinli bo'lsin. d_n ning bu ifodasini (1) ga qo'ysak, $d_{n+1} = a_0 x^n + a_1 x^{n-1} + \ldots + a_n$ hosil bo'ladi. ■

6-m i s o l. n-tartibli

$$d_n = \begin{vmatrix} 2\cos^2\frac{\theta}{2} & \cos\theta & 0 & \ldots & 0 & 0 \\ 1 & 2\cos^2\frac{\theta}{2} & \cos\theta & \ldots & 0 & 0 \\ 0 & 1 & 2\cos^2\frac{\theta}{2} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 2\cos^2\frac{\theta}{2} & \cos\theta \\ 0 & 0 & 0 & \ldots & 1 & 2\cos^2\frac{\theta}{2} \end{vmatrix}$$

determinantni hisoblang.

Yechish. d_n ni 1-ustun bo'yicha yoyamiz.

$$d_n = 2\cos^2\frac{\theta}{2} \cdot d_{n-1} - \begin{vmatrix} \cos\theta & 0 & 0 & \ldots & 0 & 0 \\ 1 & 2\cos^2\frac{\theta}{2} & \cos\theta & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 2\cos^2\frac{\theta}{2} \end{vmatrix}$$

O'ng tomondagi ikkinchi determinantni birinchi satri bo'yicha yoyamiz. Natijada $d_n = 2\cos^2\frac{\theta}{2} \cdot d_{n-1} - \cos\theta \cdot d_{n-1}$ rekurrent munosabat hosil bo'ladi. Bunda $2\cos^2\frac{\theta}{2}$ ni 1 + $\cos\theta$ bilan almashtirib,

$$d_n = (1 + \cos\theta) d_{n-1} - \cos\theta \cdot d_{n-2} \qquad (2)$$

ga kelamiz. (2) rekurrent munosabatda n ni n-1 bilan almashtiramiz. U holda $d_{n-1} = (1 + \cos\theta) d_{n-2} - \cos\theta \cdot d_{n-3}$ ga teng bo'ladi. d_{n-1} ning bu ifodasini (2) ga qo'ysak,

$$d_n = (1 + \cos\theta + \cos^2\theta) d_{n-2} - (1 + \cos\theta)\cos\theta \cdot d_{n-3} \qquad (3)$$

ga ega bo'lamiz. (2) da n ni n-2 bilan almashtirib,

$d_{n-2} = (1 + \cos\theta) d_{n-3} - \cos\theta \cdot d_{n-4}$ va uni (3) ga qo'yib,

$d_n = (1 + \cos\theta + \cos^2\theta + \cos^3\theta) d_{n-3} - (1 + \cos\theta + \cos^2\theta)\cos\theta \cdot d_{n-4}$ ifodani topamiz va h.k.

$$d_n = (1+\cos\theta+...+\cos^{n-2}\theta)d_2 - (1+\cos\theta+...+\cos^{n-3}\theta)\cos\theta\cdot d_1,$$

$$d_2 = \begin{vmatrix} 2\cos^2\frac{\theta}{2} & \cos\theta \\ 1 & 2\cos^2\frac{\theta}{2} \end{vmatrix} = 1+\cos\theta+\cos^2\theta, \quad d_1 \ -1+\cos\theta$$

ifodasini va natijada $d_n = 1+\cos\theta+...+\cos^n\theta$ ni hosil qilamiz. d_n ning bu ifodasini matematik induksiya usuli yordamida tekshiramiz.

Rekurrent munosabatlar usuli determinantning umumiy ko'rinishidagi qonuniyatni topish talab etadi. Bu usulning qiyinchiligi ham shunda.

$$d_n = pd_{n-1} + qd_{n-2}, \quad n > 2, \qquad (4)$$

ko'rinishdagi rekurrent munosibatni qanoatlantiruvchi determinantlarni o'rganish bilan chegaralanamiz. (4) dagi p va q lar n dan bog'liq bo'lmagan o'zgarmas kattaliklar.

$q = 0$ bo'lganda, d_n geometrik progressiyasining hadini hisoblagan kabi hisoblanadi: $d_n = p^{n-1}d_1$, bunda d_1 - 1-tartibli determinant, ya'ni d_n ning chap yuqori burchagida turgan element.

$q \neq 0$ bo'lsin. α, β-lar $x^2 - px - q = 0$ kvadrat tenglamaning ildizlari bo'lsin. U holda $p = \alpha+\beta, \ q = -\alpha\beta$ bo'lib, (4) tenglikni quyidagi ikki xil ko'rinishda yozish mumkin:

$$d_n - \beta d_{n-1} = \alpha(d_{n-1} - \beta d_{n-2}), \qquad (5)$$

$$d_n - \alpha d_{n-1} = \beta(d_{n-1} - \alpha d_{n-2}), \qquad (6)$$

$\alpha \neq \beta$ bo'lsin. Geometrik progressiyaning (n-1)-hadini topish formulasidan

(5) va (6) larga ko'ra $d_n - \beta d_{n-1} = \alpha^{m-2}(d_2 - \beta d_1)$ va

$d_n - \alpha d_{n-1} = \beta^{m-2}(d_2 - \alpha d_1)$, bundan esa,

$d_n = \dfrac{\alpha^{n-1}(d_2 - \beta d_1) - \beta^{n-1}(d_2 - \alpha d_1)}{\alpha - \beta}$ yoki $d_n = c_1\alpha^n + c_2\beta^n$, bunda

$$\alpha \neq \beta \quad c_1 = \frac{d_2 - \beta d_1}{\alpha(\alpha-\beta)}, \ c_2 = \frac{d_2 - \alpha d_1}{\beta(\alpha-\beta)} \qquad (7)$$

ni hosil qilamiz. d_n ning bu ifodasi $n > 2$ uchun hosil qilingan bo'lib, n=1 va n=2 lar uchun bevosita tekshiriladi. c_1 va c_2 larning (7) munosibatdan emas, balki boshlang'ich $d_1 = c_1\alpha + c_2\beta$, $d_2 = c_1\alpha^2 + c_2\beta^2$ shartlardan topish mumkin.

$\alpha = \beta$ bo'lsin. (5) va (6) tengliklar bitta $d_n - \alpha d_{n-1} = \alpha(d_{n-1} - \alpha d_{n-2})$, tenglikka aylanadi. Bundan

$$d_n - \alpha d_{n-1} = A\alpha^{n-2}, \qquad (8)$$

hosil qilamiz. Bunda $A = d_2 - \alpha d_1$.

(8) da n ni n-1 bilan almashtirib, $d_{n-1} - \alpha d_{n-2} = A\alpha^{n-3}$ ni, va undan esa $d_{n-1} = \alpha d_{n-2} + A\alpha^{n-2}$ ni hosil qilamiz. Bu ifodani (5) tenglikka qo'yib: $d_n = \alpha^2 d_{n-2} + 2A\alpha^{n-2}$ ni topamiz. Bu usulni bir necha marta qo'llab, $d_n = \alpha^{n-1}d_1 + (n-1)A\alpha^{n-2}$ yoki $d_n = \alpha^n[(n-1)c_1 + c_2]$, bunda $c_1 = \frac{A}{\alpha^2}$, $c_2 = \frac{d_1}{\alpha}$ ($q \neq 0$ bo'lganligi uchun $\alpha \neq 0$) ni hosil qilamiz. ■

7-m i s o l. n-tartibli $d_n = \begin{vmatrix} 7 & 4 & 0 & 0 & \dots & 0 & 0 \\ 3 & 7 & 4 & 0 & \dots & 0 & 0 \\ 0 & 3 & 7 & 4 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 3 & 7 \end{vmatrix}$ determinantni hisoblang.

Yechish. Birinchi satr bo'yicha yoyib, $d_n = 7d_{n-1} - 12d_{n-2}$ ni hosil qilamiz. Bu rekurrent munosabatga muvofiq keluvchi $x^2 - 7x + 12 = 0$ kvadrat tenglama $\alpha = 3$, $\beta = 4$ $(\alpha \neq \beta)$ ildizlarga ega. Demak, $d_n = c_1 3^n + c_2 4^n$. c_1 va c_2 koeffisiyentlarni $c_1 = \frac{d_2 - \beta d_1}{\alpha(\alpha - \beta)}$, $c_2 = -\frac{d_2 - \alpha d_1}{\beta(\alpha - \beta)}$ formulalardan topamiz. $d_2 = \begin{vmatrix} 7 & 4 \\ 3 & 7 \end{vmatrix} = 37$, $d_1 = 7$, bo'lganligidan c_1=-3, c_2=4 bo'ladi. Demak, $d_n = 4^{n+1} - 3^{n+1}$ bo'ladi. ■

4. *Determinantni determinantlar yig'indisiga yoyish usuli*

Ba'zida n-tartibli harfli determinantni ikki yoki bir nechta determinantlarning yig'indisi ko'rinishida ifodalash orqali oson yo'l bilan hisoblash mumkin.

8-m i s o l. n-tartibli $d = \begin{vmatrix} a & b & 0 & 0 & \dots & 0 & 0 \\ 0 & a & b & 0 & \dots & 0 & 0 \\ 0 & 0 & a & b & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a & b \\ 0 & 0 & 0 & 0 & \dots & 0 & a \end{vmatrix}$ determinantni hisoblang.

Yechish. Determinantni birinchi ustun bo'yicha yoyamiz:

$$d = a\begin{vmatrix} a & b & 0 & \dots & 0 \\ 0 & a & b & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & a \end{vmatrix} + (-1)^{n-1}b\begin{vmatrix} b & 0 & 0 & \dots & 0 \\ a & b & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b \end{vmatrix} =$$

$$= a \cdot a^{n-1} + (-1)^{n+1} b \cdot b^{n-1} = a^n + (-1)^{n+1} b^n .$$

Ikkita determinant ham uchburchak ko'rinishiga ega. ■

9-m i s o l. n-tartibli $d_n = \begin{vmatrix} x+a_1 & a_2 & a_3 & \dots & a_n \\ a_1 & x+a_2 & a_3 & \dots & a_n \\ a_1 & a_2 & x+a_3 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & x+a_n \end{vmatrix}$ determinantni hisoblang.

Yechish. Berilgan determinantni:

$$d = \begin{vmatrix} x+a_1 & 0+a_2 & 0+a_3 & \dots & 0+a_n \\ 0+a_1 & x+a_2 & 0+a_3 & \dots & 0+a_n \\ 0+a_1 & 0+a_2 & x+a_3 & \dots & 0+a_n \\ \dots & \dots & \dots & \dots & \dots \\ 0+a_1 & 0+a_2 & 0+a_3 & \dots & x+a_n \end{vmatrix}$$ ko'rinishda yozamiz.

Determinantning har bir ustun elementini ikkita qo'shiluvchining yig'indisi ko'rinishida ifodalab, determinantning ma'lum xossasiga ko'ra d ni n-tartibli 2^n ta determinantlar yig'indisiga yoyish mumkin. Bunda hosil bo'lgan determinantlarning ba'zilarida bir xil ustunlar hosil bo'ladi va bunday determinantlarlar qiymati nolga teng bo'lganligi sababli ularni tashlab, qolganlarini quyidagi ko'rinishda yozish mumkin:

$$d = \begin{vmatrix} x & 0 & 0 & \dots & 0 \\ 0 & x & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x \end{vmatrix} + \sum_{i=1}^{n} \begin{vmatrix} x & 0 & \dots & a_i & \dots & 0 \\ 0 & x & \dots & a_i & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_i & \dots & x \end{vmatrix}$$

(bunda a_i lar i-ustunda).

$\begin{vmatrix} x & 0 & \dots & a_i & \dots & 0 \\ 0 & x & \dots & a_i & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_i & \dots & x \end{vmatrix}$ determinantni oxirgi ustun bo'yicha yoyib, hosil bo'lgan (n-1)–tartibli determinant yana oxirgi ustun bo'yicha yoyib va h.k. oxirgi ustuni a_i larga teng bo'lgan uchburchak ko'rinishidagi i-tartibli determinant hosil bo'lguncha qadar davom ettiramiz. Natijada

$$d_i = x^{n-i} \begin{vmatrix} x & 0 & \dots & a_i \\ 0 & x & \dots & a_i \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_i \end{vmatrix} = x^{n-i} \cdot x^{i-1} \cdot a_i = x^{n-1} a_i .$$

n-tartibli $\begin{vmatrix} x & 0 & 0 & \dots & 0 \\ 0 & x & 0 & \dots & 0 \\ 0 & 0 & x & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x \end{vmatrix}$ x^n ga teng bo'lganligi sababli quyidagi ifodani hosil qilamiz: $d = x^n + \sum_{i=1}^{n} x^{n-1} a_i = x^{n-1}(x + a_1 + \dots + a_n)$.■

5. *Determinantning elementlarini o'zgartirish usuli*

Bu usulda determinantning barcha elementlarini bitta songa o'zgartirish yo'li bilan barcha elementlarning algebraik to'ldiruv-chilarini hisoblash qulay bo'lgan holga keltiriladi. Bu usul quyidagi xossaga asoslangandir: agar *d* determinantning barcha elementlariga aynan bitta *x* sonini qo'shsak, u holda, determinant *x* sonini d.

$$d=\begin{vmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}, \qquad d'=\begin{vmatrix} a_{11}+x & \dots & a_{1n}+x \\ \dots & \dots & \dots \\ a_{n1}+x & \dots & a_{nn}+x \end{vmatrix} \text{ bo'lsin.}$$

d' ni 1-satrga nisbatan ikkita determinantga, ularning har birini esa 2-satrga nisbatan ikkitadan determinantlarga va h.k. yozamiz. Barcha elementlari *x* ga teng bo'lgan satrlari bittadan ko'p bo'lgan determinantlar nolga teng, elementlari *x* ga teng bo'lgan satrlar soni bitta bo'lgan determinantlarni shu satr bo'yicha yoyamiz. U holda isbot qilinishi kerak bo'lgan $d' = d + x\sum_{i,j=1}^{n} A_{ij}$ tenglikni hosil qilamiz. Shunday qilib, d' determinantni hisoblash *d* determinantni va uning algebraik to'ldiruvchilari yig'indisini hisoblashga olib kelinadi.

10-m i s o l. *n*-tartibli $d=\begin{vmatrix} a_1-x & 0 & \dots & 0 \\ 0 & a_2-x & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_n-x \end{vmatrix}$ determinantni hisoblang.

Yechish. Bosh diagonalda yotmagan elementlarning algebraik to'ldiruvchilari nolga teng, bosh diagonaldagi elementlarniki esa bosh diagonaldagi boshqa elementlarning ko'paytmasiga teng. Shuning uchun,

$$d_n=(a_1-x)\dots(a_n-x)+x\sum_{i=1}^{n}(a_1-x)\dots(a_{i-1}-x)(a_{i+1}-x)\dots(a_n-x)=$$

$$=x(a_1-x)(a_2-x)\dots(a_n-x)\cdot\left(\frac{1}{x}+\frac{1}{a_i-x}+\dots+\frac{1}{a_n-x}\right)$$

6. *n-tartibli determinantni Vandermond determinantiga olib kelib hisoblash*

Vandermond determinanti deb,

$$V_n=\begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & x_2^3 & \dots & x_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & x_n^3 & \dots & x_n^{n-1} \end{vmatrix}$$

ko'rinishidagi determinantga aytiladi.

U quyidagi formula yordamida hisoblanadi:

$$V_n=(x_2-x_1)(x_3-x_1)\ldots(x_n-x_1)(x_3-x_2)(x_4-x_2)\ldots(x_n-x_2)\ldots(x_n-x_{n-1})=$$
$$=\prod_{n\ge i\ge k\ge 1}(x_i-x_k).$$

Ba'zi determinantlarni Vandermond determinantiga olib kelish yo'li bilan hisoblash mumkin.

11-m i s o l. Vandermond determinantiga keltirish yo'li bilan determinantni hisoblang.

$$d=\begin{vmatrix}\alpha_1^n & \alpha_1^{n-1} & \beta_1 & \ldots & \beta_1^n\\ \alpha_2^n & \alpha_2^{n-1} & \beta_1 & \ldots & \beta_2^n\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ \alpha_{n+1}^n & \alpha_{n+1}^{n-1} & \beta_{n+1} & \ldots & \beta_{n+1}^n\end{vmatrix}.$$

Yechish. Determinant belgisi ostidan $\alpha_1^n,\alpha_2^n,\ldots,\alpha_{n+1}^n$ ko'paytuvchilarni mos ravishda birinchi, ... , (n+1)-satrlardan chiqaramiz. Natijada Vandermond determinantini hosil qilamiz:

$$d=\alpha_1^n\alpha_2^n\ldots\alpha_{n+1}^n\begin{vmatrix}1 & \dfrac{\beta_1}{\alpha_1} & \ldots & \left(\dfrac{\beta_1}{\alpha_1}\right)^n\\ 1 & \dfrac{\beta_2}{\alpha_2} & \ldots & \left(\dfrac{\beta_2}{\alpha_2}\right)^n\\ \ldots & \ldots & \ldots & \ldots\\ 1 & \dfrac{\beta_{n+1}}{\alpha_{n+1}} & \ldots & \left(\dfrac{\beta_{n+1}}{\alpha_{n+1}}\right)^n\end{vmatrix}=\alpha_1^n\alpha_2^n\ldots\alpha_{n+1}^n\cdot\prod_{i>j}\left[\left(\frac{\beta_i}{\alpha_i}\right)-\left(\frac{\beta_j}{\alpha_j}\right)\right]=$$

$$=\alpha_1^n\alpha_2^n\ldots\alpha_{n+1}^n\prod_{i>j}\frac{\alpha_j\beta_i-\alpha_i\beta_j}{\alpha_i\alpha_j}=\prod_{i>j}(\alpha_j\beta_i-\alpha_i\beta_j).\blacksquare$$

12-m i s o l. Vandermond determinantiga

$$d=\begin{vmatrix}1 & x_1 & x_1^2 & \ldots & x_1^{p-1} & x_1^{p+1} & \ldots & x_1^n\\ 1 & x_2 & x_2^2 & \ldots & x_2^{p-1} & x_2^{p+1} & \ldots & x_2^n\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 1 & x_n & x_n^2 & \ldots & x_n^{p-1} & x_n^{p+1} & \ldots & x_n^n\end{vmatrix}\quad(1\le p\le n-1)$$

ko'rinishdagi determinantni ham keltirish mumkin.

$x_1, x_2, ..., x_n$ elementlarning *n-p* tadan olingan barcha ko'paytmalaridan tuzilgan yig'indini s_{n-p} bilan, d_n bilan shu elementlardan tuzilgan Vandermond determinantini belgilasak, $d = s_{n-p} d_n$ tenglik o'rinlidir.

M A S H Q L A R

Determinantinng tartibi aniq bo'lmagan holda, uning tartibi *n*-ga teng deb qaraladi.

1.48. Quyidagi determinantlarni uchburchak ko'rinishiga keltirib yeching:

a) $\begin{vmatrix} 1 & 2 & 3 & ... & n-2 & n-1 & n \\ 2 & 3 & 4 & ... & n-1 & n & n \\ 3 & 4 & 5 & ... & n & n & n \\ ... & ... & ... & ... & ... & ... & ... \\ n & n & n & ... & n & n & n \end{vmatrix}$; b) $\begin{vmatrix} x_1 & a_{12} & a_{13} & ... & a_{1n} \\ x_1 & x_2 & a_{23} & ... & a_{2n} \\ x_1 & x_2 & x_3 & ... & a_{3n} \\ ... & ... & ... & ... & ... \\ x_1 & x_2 & x_3 & ... & x_4 \end{vmatrix}$;

c) $\begin{vmatrix} 1 & ... & 1 & 1 & 1 \\ a_1 & ... & a_1 & a_1+b_1 & a_1 \\ a_2 & ... & a_2-b_2 & a_2 & a_2 \\ ... & ... & ... & ... & ... \\ a_n-b_n & ... & a_n & a_n & a_n \end{vmatrix}$; d) $\begin{vmatrix} 3 & 2 & 2 & ... & 2 \\ 2 & 3 & 2 & ... & 2 \\ 2 & 2 & 3 & ... & 2 \\ ... & ... & ... & ... & ... \\ 2 & 2 & 2 & ... & 3 \end{vmatrix}$;

e) $\begin{vmatrix} a_0 & a_1 & a_2 & ... & a_n \\ -x & x & 0 & ... & 0 \\ 0 & -x_2 & x_3 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & x_n \end{vmatrix}$; f) $\begin{vmatrix} a_1 & a_2 & a_3 & ... & a_n \\ -x_1 & x_2 & 0 & ... & 0 \\ 0 & -x_2 & x_3 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & x_n \end{vmatrix}$.

1.49. Elementlari $a_{ij} = \min(i, j)$ shart bilan aniqlangan *n*-tartibli determinantni hisoblang.

1.50. Elementlari $a_{ij} = \max(i, j)$ shart bilan aniqlangan n-tartibli determinantni hisoblang.

1.51*. Elementlari $a_{ij} = |i - j|$ shart bilan aniqlangan n-tartibli determinantni hisoblang.

1.52. Determinantlarni chiziqli ko'paytuvchilarni ajratish usuli bilan hisoblang:

a) $\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2-x & 1 & \dots & 1 \\ 1 & 1 & 3-x & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & n+1-x \end{vmatrix}$; b) $\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n \\ a_0 & x & a_2 & \dots & a_n \\ a_0 & a_1 & x & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_0 & a_1 & a_2 & \dots & x \end{vmatrix}$;

c) $\begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-x^2 \end{vmatrix}$; d)* $\begin{vmatrix} 1+x & 1 & 1 & 1 \\ 1 & 1-x & 1 & 1 \\ 1 & 1 & 1+z & 1 \\ 1 & 1 & 1 & 1-z \end{vmatrix}$.

53. Rekurrent munosibatlar usuli bo'yicha determinantlarni hisoblang.

a) $\begin{vmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \dots & 0 \\ 0 & 1 & 2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{vmatrix}$; b) $\begin{vmatrix} 3 & 2 & 0 & \dots & 0 \\ 1 & 3 & 2 & \dots & 0 \\ 0 & 1 & 3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 3 \end{vmatrix}$; c) $\begin{bmatrix} 7 & 5 & 0 & \dots & 0 \\ 2 & 7 & 5 & \dots & 0 \\ 0 & 2 & 7 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 7 \end{bmatrix}$;

d) $\begin{vmatrix} 5 & 6 & 0 & 0 & 0 & \dots & 0 & 0 \\ 4 & 5 & 2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 3 & 2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 3 \end{vmatrix}$; ye) $\begin{vmatrix} 1 & 2 & 0 & 0 & 0 & \dots & 0 & 0 \\ 3 & 4 & 3 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2 & 5 & 3 & 0 & \dots & 0 & 0 \\ 0 & 0 & 2 & 5 & 3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots & 5 & 3 \\ 0 & 0 & 0 & 0 & 0 & \dots & 2 & 5 \end{vmatrix}$;

f) $\begin{vmatrix} \alpha+\beta & \alpha\beta & 0 & 0 & \dots & 0 \\ 1 & \alpha+\beta & \alpha\beta & 0 & \dots & 0 \\ 0 & 1 & \alpha+\beta & \alpha\beta & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \alpha+\beta \end{vmatrix}$;

g) $\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 2 & 2^2 & \dots & 2^n \\ 1 & 3 & 3^2 & \dots & 3^n \\ \dots & \dots & \dots & \dots & \dots \\ 1 & n+1 & (n+1)^2 & \dots & (n+1)^2 \end{vmatrix}$.

1.54. Determinantlarni determinantlar summasi ko'rinishida ifodalab, hisoblang:

a)* $\begin{vmatrix} x_1 & a_2 & \dots & a_n \\ a_1 & x_2 & \dots & a_n \\ \dots & \dots & \dots & \dots \\ a_1 & a_2 & \dots & x_n \end{vmatrix}$; b)* $\begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ x_1 & a_1 & 0 & 0 & \dots & 0 & 0 \\ x_2 & x_2 & a_2 & 0 & \dots & 0 & 0 \\ x_3 & x_3 & x_3 & a_3 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_n & x_n & x_n & x_n & \dots & x_n & a_n \end{vmatrix}$;

c) $\begin{vmatrix} x_1 & a_1b_2 & a_1b_3 & \dots & a_1b_n \\ a_2b_1 & x_2 & a_2b_3 & \dots & a_2b_n \\ a_3b_1 & a_3b_2 & x_3 & \dots & a_3b_n \\ \dots & \dots & \dots & \dots & \dots \\ a_nb_1 & a_nb_2 & a_nb_3 & \dots & x_n \end{vmatrix}$.

1.55. Determinantlarni Vandermond determinantiga keltirib hisoblang:

a) $\begin{vmatrix} x_1 & x_1^2 & \dots & x_1^n \\ x_2 & x_2^2 & \dots & x_2^n \\ \dots & \dots & \dots & \dots \\ x_n & x_n^2 & \dots & x_n^n \end{vmatrix}$; b) $\begin{vmatrix} \sin^{n-1}\varphi_1 & \sin^{n-2}\varphi_1 & \dots & \sin\varphi_1 & 1 \\ \sin^{n-1}\varphi_2 & \sin^{n-2}\varphi_2 & \dots & \sin\varphi_2 & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \sin^{n-1}\varphi_n & \sin^{n-1}\varphi_n & \dots & \sin\varphi_n & 1 \end{vmatrix}$;

$$
\text{c})^{*}\ \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_1+1 & x_2+1 & \dots & x_n+1 \\ x_1^2+1 & x_2^2+x_2 & \dots & x_n^2+x_n \\ \dots & \dots & \dots & \dots \\ x_1^{n-1}+x_1^{n-2} & x_2^{n-1}+x_2^{n-2} & \dots & x_n^{n-1}+x_n^{n-2} \end{vmatrix};
$$

$$
\text{d})^{*}\ \begin{vmatrix} x_1 & 1 & 1 & \dots & 1 \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix}; \qquad \text{e})\ \begin{vmatrix} a^n & (a-1)^n & \dots & (a-n)^n \\ a^{n-1} & (a-1)^{n-1} & \dots & (a-n)^{n-1} \\ \dots & \dots & \dots & \dots \\ a & a-1 & \dots & a-n \\ 1 & 1 & \dots & 1 \end{vmatrix}.
$$

§5. Minorlar va ularning algebraik to’ldiruvchilari. Laplas teoremasi

n-tartibli d determinantning *k-tartibli* $(1 \le k \le n)$ *minori* deb d determinantning k ta ixtiyoriy satrlari va k ta ixtiyoriy ustunlarning kesishgan joyida turgan elementlardan tuzilgan M determinantga aytiladi. Xususiy holda d ning n-tartibli minori d ning o’ziga teng. Nolinchi tartibli minor ta’rifiga ko’ra 1 ga teng deb qabul qilinadi. d determinantning M minorida turgan k ta satr va k ta ustunlarni o’chirish natijasida hosil bo’lgan M' minorga d determinantning M minorining *to’ldiruvchi minori* deyiladi. d determinantning M minorining *algebraik to’ldiruvchisi* deb, M minor turgan satr va ustun nomerlari yig’indisi juft bo’lganda plyus ishora bilan toq bo’lganda minus ishora bilan olinadigan to’ldiruvchi M' minorga aytiladi.

Laplas teoremasi. d determinantning qandaydir k-ta satri (yoki qandaydir k ta ustuni) tanlangan bo’lsin. n $(1 \le k \le n-1)$. Agar shu satrlarda (yoki ustunlarda) joylashgan barcha k-tartibli minorlarni ularning algebraik to’ldiruvchilariga mos ravishda ko’paytirib, bu ko’paytmalar qo’shilsa, d determinant hosil bo’ladi.

1-m i s o l. Determinantni hisoblang: $d = \begin{vmatrix} 2 & 1 & 4 & 3 & 5 \\ 3 & 4 & 0 & 5 & 0 \\ 3 & 4 & 5 & 2 & 1 \\ 1 & 5 & 2 & 4 & 3 \\ 4 & 6 & 0 & 7 & 0 \end{vmatrix}$.

Yechish. Ikkinchi va beshinchi satrlardagi ikkinchi tartibli o'nta minorlardan faqat uchtasi noldan farqli. Shu satrlar bo'yicha yoyamiz:

$$d = (-1)^{10}\begin{vmatrix} 3 & 4 \\ 4 & 6 \end{vmatrix}\begin{vmatrix} 4 & 3 & 5 \\ 5 & 2 & 1 \\ 2 & 4 & 3 \end{vmatrix} + (-1)^{12}\begin{vmatrix} 3 & 5 \\ 4 & 7 \end{vmatrix}\cdot\begin{vmatrix} 1 & 4 & 5 \\ 4 & 5 & 1 \\ 5 & 2 & 1 \end{vmatrix} + (-1)^{18}\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix}\begin{vmatrix} 2 & 4 & 5 \\ 3 & 5 & 1 \\ 1 & 2 & 3 \end{vmatrix} =$$

$$= 2\cdot 49 + 1\cdot(-100) + 2(-1) = -4. \blacksquare$$

Xususiy holda, agar d determinantning bosh diagonalini umumiy elementlarga ega bo'lmagan kvadrat matritsalar yordamida qoplash mumkin bo'lib, ularning determinantlari d_1 va d_2 bo'lsa, va ularning bir tomonida hamma elementlar nolga teng bo'lsa, u holda $d = d_1 d_2$ bo'ladi.

2-m i s o l. $\begin{vmatrix} 5 & 3 & 0 & 0 \\ 3 & 7 & 0 & 0 \\ 3 & 2 & 4 & 8 \\ 3 & 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 5 & 3 \\ 3 & 7 \end{vmatrix}\cdot\begin{vmatrix} 4 & 8 \\ 8 & 9 \end{vmatrix} = 26\cdot(-28) = -728. \blacksquare$

Zinapoyali determinantlar deb ataluvchi umumiy holda, ya'ni d determinantning bosh diagonalida determinantlari $d_1, d_2, ..., d_k$, ga teng bo'lgan kvadrat matritsalar ketma-ket turgan bo'lib, bu ketma-ket matritsalar bir tomonidagi barcha elementlar nolga teng bo'lsa, u holda $d = d_1 \cdot d_2 \cdot \cdot \cdot d_k$ bo'ladi.

3-m i s o l.

$$\begin{vmatrix} 7 & 5 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 7 & 1 & 8 & 0 & 0 & 0 & 0 \\ -2 & 3 & 1 & 4 & -3 & 0 & 0 \\ 4 & 5 & 7 & 2 & 3 & 0 & 0 \\ 1 & -1 & 2 & 5 & -2 & 3 & 5 \\ 7 & 3 & 1 & 4 & 3 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 7 & 5 \\ 1 & 2 \end{vmatrix}\cdot 8\cdot\begin{vmatrix} 4 & -3 \\ 2 & 3 \end{vmatrix}\begin{vmatrix} 3 & 5 \\ 2 & -1 \end{vmatrix} = 9\cdot 8\cdot 18\cdot(-13) = -14848. \blacksquare$$

Ba'zi hollarda avval determinantda almashtirishlar bajarib, so'ng Laplas teoremasini qo'llash qulaydir.

4-m i s o l. Determinantni hisoblang:

$$d=\begin{vmatrix} 2 & -1 & 3 & 4 & -5 \\ 4 & -2 & 7 & 8 & -7 \\ -6 & 4 & -9 & -2 & 3 \\ 3 & -2 & 4 & 1 & -2 \\ -2 & 6 & 5 & 4 & -3 \end{vmatrix}.$$

Yechish. Ikkinchi satrdan ikkilangan birinchi satrni ayiramiz, uchinchi satrga ikkilangan to'rtinchi satrni qo'shamiz. Natijada,

$$d=\begin{vmatrix} 2 & -1 & 3 & 4 & -5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 0 & -1 \\ 3 & -2 & 4 & 1 & -2 \\ -2 & 6 & 5 & 4 & -3 \end{vmatrix} \text{ ni hosil qilamiz.}$$

Determinantni ikkinchi va uchinchi satrlar bo'yicha yoyib, quyidagini hosil qilamiz:

$$d=\begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix}(-1)^{2+3+3+5}\begin{vmatrix} 2 & -1 & 4 \\ 3 & -2 & 1 \\ -2 & 6 & 4 \end{vmatrix}=2\cdot(-1)\cdot 42=-84. \blacksquare$$

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1.56. Laplas teoremasidan foydalanib, quyidagi determinantlarni hisoblang:

a) $\begin{vmatrix} 1 & 0 & 0 & -1 \\ 2 & 3 & 4 & 7 \\ -3 & 4 & 5 & 9 \\ -4 & -5 & 6 & 1 \end{vmatrix}$; b) $\begin{vmatrix} 5 & 62 & -79 & 4 \\ 0 & 2 & 3 & 0 \\ 6 & 183 & 201 & 5 \\ 0 & 3 & 4 & 0 \end{vmatrix}$; c) $\begin{vmatrix} 3 & -1 & 5 & 2 \\ 2 & 0 & 7 & 0 \\ -3 & 1 & 2 & 0 \\ 5 & -4 & 1 & 2 \end{vmatrix}$;

d) $\begin{vmatrix} 9 & 7 & 6 & 8 & 5 \\ 3 & 0 & 0 & 2 & 0 \\ 5 & 3 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 7 & 5 & 4 & 6 & 0 \end{vmatrix}$; e) $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 2 & 3 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{vmatrix}$; f) $\begin{vmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 2 & 0 & 3 & 0 \\ 2 & 0 & 3 & 0 & 4 \\ 0 & 3 & 0 & 4 & 0 \\ 3 & 0 & 4 & 0 & 3 \end{vmatrix}$;

g) $\begin{vmatrix} 2 & 3 & 1 & 2 & 9 & 8 \\ 3 & 4 & 2 & 7 & 5 & 3 \\ 0 & 0 & 5 & 3 & 3 & 1 \\ 0 & 0 & 8 & 5 & 7 & 5 \\ 0 & 0 & 0 & 0 & 9 & 7 \\ 0 & 0 & 0 & 0 & 4 & 3 \end{vmatrix}$; h) $\begin{vmatrix} 2 & -3 & 7 & 1 & 9 & 11 \\ 1 & 0 & 3 & 0 & -4 & 0 \\ 7 & 4 & 9 & -1 & 11 & -5 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 9 & -4 & 11 & 1 & 13 & 2 \\ 4 & 0 & 1 & 0 & -1 & 0 \end{vmatrix}$; i) $\begin{vmatrix} 0 & a & b & c \\ 1 & x & 0 & 0 \\ 1 & 0 & y & 0 \\ 1 & 0 & 0 & z \end{vmatrix}$;

j) $\begin{vmatrix} 0 & a & b & c \\ a' & 1 & 0 & 0 \\ b' & 0 & 1 & 0 \\ c' & 0 & 0 & 1 \end{vmatrix}$; k) $\begin{vmatrix} 1 & x & x & x \\ 1 & a & 0 & 0 \\ 1 & 0 & b & 0 \\ 1 & 0 & 0 & c \end{vmatrix}$; l) $\begin{vmatrix} 1 & 1 & 0 & 0 \\ x_1 & x_2 & \cos\alpha & \sin\alpha \\ y_1 & y_2 & \cos\beta & \sin\beta \\ z_1 & z_2 & \cos\gamma & \sin\gamma \end{vmatrix}$;

m) $\begin{vmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & x_1 & x_2 & x_3 & x_4 \\ 0 & x_1^2 & x_2^2 & x_3^2 & x_4^2 \end{vmatrix}$; n) $\begin{vmatrix} a_{11} & 1 & a_{12} & 1 & \dots & a_{1n} & 1 \\ 1 & 0 & 1 & 0 & \dots & 1 & 0 \\ a_{21} & x_1 & a_{22} & x_2 & \dots & a_{2n} & x_n \\ x_1 & 0 & x_2 & 0 & \dots & x_n & 0 \\ a_{31} & x_1^2 & a_{32} & x_2^2 & \dots & a_{3n} & x_n^2 \\ x_1^2 & 0 & x_2^2 & 0 & \dots & x_n^2 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n1} & x_1^{n-1} & a_{n2} & x_2^{n-1} & \dots & a_{nn} & x_n^{n-1} \\ x_1^{n-1} & 0 & x_2^{n-1} & 0 & \dots & x_n^{n-1} & 0 \end{vmatrix}$.

1.57. Quyidagi determinantlarni avval almashtirishlar bajarib soddalashtiring, so’ng Laplas teoremasidan foydalanib hisoblang:

a) $\begin{vmatrix} 3 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ -8 & 5 & 9 & 5 \\ -11 & 7 & 7 & 4 \end{vmatrix}$; b) $\begin{vmatrix} 9 & 7 & 9 & 7 \\ 8 & 6 & 8 & 6 \\ -9 & -7 & 9 & 7 \\ -8 & -6 & 8 & 6 \end{vmatrix}$; c) $\begin{vmatrix} 6 & 8 & -9 & -12 \\ 4 & 6 & -6 & -9 \\ -3 & -4 & 6 & 8 \\ -2 & -3 & 4 & 6 \end{vmatrix}$;

d) $\begin{vmatrix} 213 & 186 & 162 & 137 \\ 344 & 157 & 295 & 106 \\ 419 & 418 & 419 & 418 \\ 417 & 416 & 417 & 416 \end{vmatrix}$; e) $\begin{vmatrix} 8 & 10 & 3 & 1 & 4 \\ 7 & 9 & 4 & 1 & 6 \\ 1 & -2 & 2 & 1 & 3 \\ 2 & 5 & -4 & -2 & -6 \\ -1 & 2 & 6 & 3 & 9 \end{vmatrix}$;

f)* $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 7 & 6 & 7 & 8 \\ 2 & 5 & 9 & 10 & 11 \\ 5 & 9 & 1 & 1 & 1 \\ 9 & 1 & 2 & 3 & 4 \end{vmatrix}$; g) $\begin{vmatrix} 5 & -5 & -3 & 4 & 2 \\ -4 & 4 & 3 & 6 & 3 \\ 3 & -1 & 5 & -9 & -5 \\ -7 & 7 & 6 & 8 & 4 \\ 5 & -3 & 2 & -1 & -2 \end{vmatrix}$;

h) $\begin{vmatrix} 5 & 9 & -2 & -4 & 5 \\ 2 & -3 & 4 & -3 & 3 \\ -5 & -7 & 2 & 4 & -2 \\ 4 & -5 & 8 & -6 & 8 \\ 6 & -5 & 3 & -3 & 7 \end{vmatrix}$; i) $\begin{vmatrix} 3 & 4 & -3 & -1 & 2 \\ -5 & 6 & 5 & 2 & 3 \\ 4 & -9 & -3 & 7 & -5 \\ -1 & -4 & 1 & 1 & -2 \\ -3 & 7 & 5 & 2 & 3 \end{vmatrix}$;

j) $$\begin{vmatrix} 1+x & x & \dots & x & x & \dots & x & 1+x \\ x & 1+x & \dots & x & x & \dots & 1+x & x \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x & x & \dots & 1+x & 1+x & \dots & x & x \\ x & x & \dots & 1+2x & 1+x & \dots & x & x \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x & 1+2x & \dots & x & x & \dots & 1+x & x \\ 1+2x & x & \dots & x & x & \dots & x & 1+x \end{vmatrix}.$$

(determinant tartibi $2n$ ga teng).

1.58. $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$ matritsani mos ravishda birinchi, ikkinchi, uchinchi va to’rtinchi ustunlarni o’chirish natijasida hosil bo’lgan uchinchi tartibli determinantlarni A, B, C, D-lar bilan belgilaymiz. U holda

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & 0 & 0 \\ a_2 & b_2 & c_2 & d_2 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 & 0 & 0 \\ 0 & 0 & a_1 & b_1 & c_1 & d_1 \\ 0 & 0 & a_2 & b_2 & c_2 & d_2 \\ 0 & 0 & a_3 & b_3 & c_3 & d_3 \end{vmatrix} = AD - BC$$ ekanligini isbotlang.

§ 6. Determinantlarni ko'paytirish

Bir xil n-tartibli $\det A = \det(a_{ij})_1^n$ va $\det B = \det(b_{ij})_1^n$ determinantlarning ko'paytmasi deb, xuddi shu tartibli va barcha elementlari quyidagi to'rtta formulalarning

1) $c_{ij} = a_{i1}b_{ij} + a_{i2}b_{j2} + \dots\dots + a_{in}b_{jn}$;

2) $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots\dots + a_{in}b_{nj}$;;

3) $c_{ij} = a_{i1}b_{j1} + a_{2j}b_{j2} + \dots\dots + a_{ni}b_{jn}$;

4) $c_{ij} = a_{i1}b_{ij} + a_{2j}b_{2j} + \dots\dots + a_{ni}b_{nj}$;

biri orqali hisoblanadigan $\det C = \det(c_{ij})_1^n$ determinantga yatiladi. Birinchi holda c_{ij} element *detA* determinantning *i*-satri elementlarini mos ravishda *detB* determinantning *j*-satri elementlariga ko'paytirib, hosil bo'lgan ko'paytmalarni qo'shish natijasida hosil qilinadi. Bu holda ko'paytma birinchi determinant satrlarini ikkinchi determinant satrlariga ko'paytirishdan, ikkinchi holda satrlarni ustunlarga ko'paytirishdan, uchinchi holda ustunlarni satrlarga, to'rtinchisida ustunlarni ustunlarga ko'paytirishdan hosil qilingan deyiladi. Bu to'rt holda *detC=detA·detB* ning c_{ij} elementlari har xil bo'lgani bilan *detC* determinantning qiymati bir xildir.

1-m i s o l. $d_1 = \begin{vmatrix} 1 & 0 & 4 \\ -1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix}$ va $d_2 = \begin{vmatrix} 2 & -5 & 2 \\ 1 & -4 & 3 \\ 1 & -3 & 1 \end{vmatrix}$

determinantlarni to'rt xil usulda ko'apytiring va barcha xollarda hosil bo'lgan qiymatlar berilgan determinantlar qiymatlarining ko'paytmasiga teng ekanligiga ishonch hosil qiling.

Yechish.

$$1)\ d_1d_2=\begin{vmatrix}10 & 13 & 5\\ -6 & 0 & -4\\ -2 & -2 & -2\end{vmatrix}=-72;\quad 2)\ d_1d_2=\begin{vmatrix}6 & -17 & 6\\ 3 & -12 & 7\\ 9 & -26 & 13\end{vmatrix}=-72;$$

$$3)\ d_1d_2=\begin{vmatrix}13 & 14 & 7\\ -6 & -2 & -4\\ -5 & -5 & -4\end{vmatrix}=-72;\quad 4)\ d_1d_2=\begin{vmatrix}4 & -10 & 2\\ 4 & -14 & 8\\ 12 & -35 & 18\end{vmatrix}=-72;$$

d_1 = - 36, d_2 =2 Demak, d_1 d_2 = - 72.. ■

2-m i s o l.

$$\Delta=\begin{vmatrix}a & b & c & d\\ -b & a & d & -c\\ -c & -d & a & b\\ -d & c & -b & a\end{vmatrix}$$ determinantni hisoblang.

Yechish. Δ determinantni satrni satrga ko'paytirish yo'li bilan kvadratga ko'taramiz. Natijada, bosh diagonalda bir xil ifoda $a^2+b^2+c^2+d^2$, bosh diagonaldan tashqarida esa nollar hosil bo'lishini ko'ramiz. Shu sababli $\Delta^2=$ $\left(a^2+b^2+c^2+d^2\right)^2$ bo'ladi. Δ ning bosh diagonali a^4 ga teng ko'paytmani saqlagani sababli, oxirgi tenglikning har ikkala tomonidan plyus ishorali ildiz chiqarish mumkin, shu sababli $\Delta=\left(a^2+b^2+c^2+d^2\right)^2$.■

3-m i s o l.

$$\Delta=\begin{vmatrix}s_0 & s_1 & s_2 & \dots & s_{n-1}\\ s_1 & s_2 & s_3 & \dots & s_n\\ s_2 & s_3 & s_4 & \dots. & s_{n+1}\\ \dots & \dots & \dots & \dots & \dots\\ s_{n-1} & s_n & s_{n+1} & \dots & s_{2n-2}\end{vmatrix}$$ determinantni hisoblang.

Bunda $s_k=x_1+x_2^k+\dots.+x_n^k\quad (k=0,1,2,\dots)$ − x_1 x_2 ..., x_4 (xususiy holda, $s_0=n$) o'zgaruvchilarning darajali yig'indisidir.

Yechish. Vandermond determinanti

$$V_n=\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix}$$

Ni o'zini o'ziga ustunlarni ustunlarga ko'paytirish yo'li bilan ko'paytirib va V_n (4-§, 6-bandini qarang) ning ifodasidan foydalanib, $\Delta = V_n^2 = \prod_{n\geq i>j\geq 1}(x_i - x_j)^2$.ni hosil qilamiz. ¼¼■

4-m i s o l. $\Delta = \begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & -3 & -8 \\ -1 & 1 & 0 & -13 \\ 2 & 3 & 5 & 15 \end{vmatrix}$, $\delta = \begin{vmatrix} 1 & -2 & -3 & -11 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$ berilgan.

Δ determinantni δ determinantga ko'paytirish orqali Δ determinantni hisoblang.

Yechish. Δ determinantni δ determinantga satrlarni satrga ko'paytirish yo'li bilan ko'paytiramiz

$$\Delta\delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 3 & 3 & 0 \\ 2 & -1 & -1 & 4 \end{vmatrix} = 24.$$

δ =1 bo'lganligi sababli Δ=24 ekanligini hosil qilamiz. ■

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1.59. *Δ* determinantni δ determinantga ko'paytirish orqali toping:

a) $\Delta = \begin{vmatrix} -1 & -9 & -2 & 3 \\ -5 & 5 & 3 & -2 \\ -12 & -6 & 1 & 1 \\ 9 & 0 & -2 & 1 \end{vmatrix}$, $\delta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ -3 & 4 & 2 & 1 \end{vmatrix}$;

b) $\Delta=\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}, \quad \delta=\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix}.$

1.60. Determinantlarning kvadratlarini toping:

a) $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{vmatrix};$ b) $\begin{vmatrix} 1 & -1 & 1 & -1 \\ 2 & 2 & 1 & 1 \\ 2 & 0 & -3 & -1 \\ 3 & -7 & -1 & 9 \end{vmatrix}.$

1.61. Determinantni kvadratga ko'tarish yo'li bilan hisoblang:

$$\begin{vmatrix} a & b & c & d & e & f & g & h \\ -b & a & d & -c & f & -e & -h & g \\ -c & -d & a & b & g & h & -e & -f \\ -d & c & -b & a & h & -g & f & -e \\ e & -f & -g & -h & a & b & c & d \\ -f & e & -h & g & -b & a & -d & c \\ -g & h & e & -f & -c & d & a & -b \\ -h & -g & f & e & -d & -c & b & a \end{vmatrix}.$$

1.62. Quyidagi determinantlarni determinantlar ko'paytmasi ko'rinishida ifodalash yo'li bilan hisoblang:

a)* $\begin{vmatrix} 1+x_1y_1 & 1+x_1y_2 & \ldots & 1+x_1y_n \\ 1+x_2y_1 & 1+x_2y_2 & \ldots & 1+x_2y_n \\ \ldots\ldots\ldots & \ldots\ldots\ldots & \ldots\ldots & \ldots\ldots \\ 1+x_ny_1 & 1+x_ny_2 & \ldots & 1+x_ny_n \end{vmatrix};$

b) $\begin{vmatrix} \cos(\alpha_1-\beta_1) & \cos(\alpha_1-\beta_2) & \ldots & \cos(\alpha_1-\beta_n) \\ \cos(\alpha_2-\beta_1) & \cos(\alpha_2-\beta_2) & \ldots & \cos(\alpha_2-\beta_n) \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots\ldots \\ \cos(\alpha_n-\beta_1) & \cos(\alpha_n-\beta_2) & \ldots & \cos(\alpha_n-\beta_n) \end{vmatrix};$

c) $\begin{vmatrix} 1 & \cos(\alpha_1-\alpha_2) & \cos(\alpha_1-\alpha_3) & \dots & \cos(\alpha_1-\alpha_n) \\ \cos(\alpha_1-\alpha_2) & 1 & \cos(\alpha_2-\alpha_3) & \dots & \cos(\alpha_2-\alpha_n) \\ \cos(\alpha_1-\alpha_3) & \cos(\alpha_2-\alpha_3) & 1 & \dots & \cos(\alpha_3-\alpha_n) \\ \dots & \dots & \dots & \dots & \dots \\ \cos(\alpha_1-\alpha_n) & \cos(\alpha_2-\alpha_n) & \cos(\alpha_3-\alpha_n) & \dots & 1 \end{vmatrix}$;

d) $\begin{vmatrix} \sin 2\alpha_1 & \sin(\alpha_1+\alpha_2) & \dots & \sin(\alpha_1+\alpha_n) \\ \sin(\alpha_2+\alpha_1) & \sin 2\alpha_2 & \dots & \sin(\alpha_2+\alpha_n) \\ \dots & \dots & \dots & \dots \\ \sin(\alpha_n+\alpha_1) & \sin(\alpha_n+\alpha_2) & \dots & \sin 2\alpha_n \end{vmatrix}$;

e)* $\begin{vmatrix} 1^{n-1} & 2^{n-1} & \dots & n^{n-1} \\ 2^{n-1} & 3^{n-1} & \dots & (n+1)^{n-1} \\ \dots & \dots & \dots & \dots \\ n^{n-1} & (n+1)^{n-1} & \dots & (2n-1)^{n-1} \end{vmatrix}$;

f)* $\begin{vmatrix} s_0 & s_1 & s_2 & \dots & s_{n-1} & 1 \\ s_1 & s_2 & s_3 & \dots & s_n & x \\ s_2 & s_3 & s_4 & \dots & s_{n+1} & x^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ s_n & s_{n+1} & s_{n+2} & \dots & s_{2n-1} & x^n \end{vmatrix}$, bunda $s_k = x_1^k + x_2^k + \dots + x_n^k$

1.63*. Agar $f(x)=a_1 + a_2 x + a_3 x^2 + \dots + a_n x^n$, $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ - lar birning n-darajali ildizlarining barchasi bo'lsa, u holda *sirkulyantning* qiymatlari:

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_n & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_2 & a_3 & a_4 & \dots & a_1 \end{vmatrix} = f(\varepsilon_1)\ f(\varepsilon_2) \dots f(\varepsilon_n)$$ tenglik yordamida

aniqlanishini isbotlang.

1.64. Oldingi masaladagi belgilashlarda

$$\begin{vmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_2 & a_3 & a_4 & \dots & a_1 \\ a_3 & a_4 & a_5 & \dots & a_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_n & a_1 & a_2 & \dots & a_{n-1} \end{vmatrix} = (-1)^{\frac{(n-1)(n-2)}{2}} f(\varepsilon_1) f(\varepsilon_2) \dots f(\varepsilon_n)$$ tenglik o'rinli

ekanligini isbotlang.

1.65[*]. Quyidagi ikki determinantni

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_2 & -x_1 & -x_4 & x_3 \\ x_3 & x_4 & -x_1 & -x_2 \\ x_4 & -x_3 & x_2 & -x_1 \end{vmatrix} \cdot \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & -y_1 & -y_4 & y_3 \\ y_3 & y_4 & -y_1 & -y_2 \\ y_n & -y_3 & y_2 & -y_1 \end{vmatrix}$$

ko'paytirib, Eyler ayniyati:

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 +$$
$$+ (x_1y_2 - x_2y_1 - x_3y_4 + x_4y_3)^2 + (x_1y_3 + x_2y_4 - x_3y_1 - x_4y_2)^2$$
$$+ +(x_1y_4 - x_2y_4 + x_3y_2 - x_4y_1)^2.$$

ni isbotlang. Butun sonlarning qanday xossasi bu tenglikdan kelib chiqadi?

1.66[*]. Quyidagi ayniyatni determinantlarning ko'paytmasi yordamida isbotlang: $(a^3 + b^3 + c^3 - 3abc)(a'^3 + b'^3 + c'^3 - 3a'b'c') = A^3 + B^3 + C^3 - 3ABC$,

bunda $A = aa' + bc' + cb'$, $B = ac' + bb' + ca'$, $C = ab' + ba' + cc'$.

Butun sonlarning qanday xossasi bu tenglikdan kelib chiqadi?

1.67[*]. Oldingi masala belgilashlaridan foydalanib, quyidagi ayniyatni isbotlang:

$$(a^2 + b^2 + c^2 - ab - ac - bc)(a'^2 + b'^2 + c'^2 - a'b' - a'c' - b'c') = A^2 + B^2 + C^2 - AB - AC - BC.$$

1.68[*]. $n > 1$ tartibli d determinantning barcha elementlarini ularning algebraik to'ldiruvchilari bilan almashtirishdan hosil bo'lgan d' determinantga *o'zaro determinant* deyiladi. $d' = d^{n-1}$ ekanligini isbtolang.

1.69[*]. M – d determinantning m-tartibli minori, A – M ning algebraik to'ldiruvchisi, M' - M ga mos keluvchi d' o'zaro determinantning minori

(ya'ni d determinantning M minorga kirgan elementlarini ularning algebraik to'ldiruvchilari bilan almashtirishdan hosil bo'lgan) bo'lsin.

U holda $M' = d^{m-1}A$ tenglikni isbotlang.

1.70*. d determinantning i-chi va j-chi satrlarini, k-chi va l-chi ustunlarini o'chirishdan keyin hosil bo'lgan $(n-2)$-tartibli minorini C bilan, (bunda $i<j$ va $k<l$), a_{pq} elementning algebraik to'ldiruvchisini A_{pq} bilan belgilaymiz.

$$\begin{vmatrix} A_{ik} & A_{il} \\ A_{ik} & A_{jl} \end{vmatrix} = (-1)^{i+j+k+l} dC$$ ekanligini isbotlang.

1.71*. Agar d determinant nolga teng bo'lsa, u holda o'zaro determinantning barcha satrlari (hamda barcha ustunlari) o'zaro proporsional ekanligini isbotlang..

1.72*. Agar a_{ij} – n- tartibli d determinantning elementi, A'_{ij} - d determinantga o'zaro bo'lgan d' determinantning A_{ij} elementiga mos keluvchi algebraik to'ldiruvchisi bo'lsa, u holda $A'_{ij} = d^{n-2} a_{ij}$ ekanligini isbotlang.

1.73*. Agar M – n-tartibli d determinatning m-tartibli минори, M' - d' o'zaro determinantning M ga mos keluvchi minori, A' - M' minorning algebraik to'ldiruvchisi bo'lsa, u holda $A' = d^{n-m-1}M$ ekanligini isbotlang.

1.74*. Noldan farqli d determinantning barcha elementlarining minorlarini bilgan holda, uning elementlarini toping.

1.75*. Determinantlarni ko'paytirish yordamida satrlar (yoki ustunlar) o'rnini almashtirganda determinantning ishorasi o'zga-rishini isbotlang.

1.76*. Determinantlarni ko'paytrish yordamida biror satrga (ustunga) c songa ko'paytirilgan boshqa satrni (ustunni) qo'shganda o'zgarmasligini isbotlang.

II - bob

MATRITSALAR ALGEBRASI

Tayanch iboralar: *matritsa; satr; ustun; matritsa elementlari; kvadrat matritsa; matritsalar yig'indisi; matritsalarni transponirlash; qo'shma kompleks matritsa; qo'shma ermit matritsa; nol matritsa; matritsaning izi; diagonal matritsa; birlik matritsa; matritsaviy ko'phad; matritsa kommutatori; matritsalarning Yordan ko'paytmasi; teskari matritsa; matritsaning elementar almashtirishlari; teskarilanuvchi matritsa; xos (maxsus) matritsa; xosmas (maxsusmas) matritsa; matritsaviy tenglama; skalyar matritsa; unimodulyar matritsa; o'rin almashtirish matritsasi; elementar matritsa; yuqori uchburchakli matritsa; pastki uchburchakli matritsa; simmetrik matritsa; kososimmetrik matritsa; ermit matritsasi; kosoermit matritsasi; ortogonal matritsa; unitar matritsa; manfiymas matritsa; stoxastik (markov) matritsa; nilpotent matritsa; davriy ; blokli (katakli) matritsa; matritsalarning (o'ng) Kroneker ko'paytmasi (yoki o'ng to'g'ri ko'paytmasi).*

1-§. Matritsalar ustida amallar

Sonlardan tuzilgan quyidagi to'g'ri burchakli jadvalga (tablisaga) *matritsa* deb aytiladi:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}.$$

Matritsaning gorizontal qatoridagi sonlari uning *satrlari*, vertikal qatoridagi sonlari uning *ustunlari* deb aytiladi. a_{ij} sonlar *matritsaning elementlari* deb aytiladi. Matritsa *m* ta satrlarga va *n* ta ustunlarga ega bo'lsa, uni $m \times n$ *matritsa* deb aytiladi. Agar $m = n$ bo'lsa, bunday matritsa *n-tartibli kvadrat matritsa* deb aytiladi.

B matritsa *A* matritsa bilan α sonning ko'paytmasidan iborat deb aytiladi, agar ularning hamma elementlari uchun $b_{ij} = \alpha a_{ij}$ tenglik bajarilsa (*A* va *B* matritsalarning o'lchovlari bir xil) va $B = \alpha A$ deb belgilanadi.

Uchta *A, B, C* – matritsalar bir xil o'lchovli bo'lsin. *C* matritsa *A* va *B* matritsalarning yig'indisi deb aytiladi va $C = A + B$ deb belgilanadi, agar *i* va *j* indekslarning hamma qiymatlari uchun $c_{ij} = a_{ij} + b_{ij}$ tenglik bajarilsa.

Faraz qilaylik, $m \times n$-o'lchovli $A = (a_{ij})$ va $n \times p$-o'lchovli $B = (b_{ij})$ matritsalar berilgan bo'lsin. Bu matritsalarning ko'paytmasi deb shunday $C = AB = (c_{ik})$ matritsaga aytiladiki, uning elementlari quyidagi formula bilan beriladi:

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \dots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}, \quad i = 1,2,\dots,m; \quad k = 1,2,\dots,p.$$

B matritsa *A* matritsaga nisbatan *transponirlangan matritsa* deb aytiladi va $B = A^T$ deb belgilanadi, agar *B* matritsaning ustunlari *A* matritsaning mos satrlari bo'lsa, ya'ni hamma i, j indekslar uchun $b_{ij} = a_{ji}$. *A* matritsadan A^T matritsaga o'tish amali *A* matritsani *transponirlash* deb aytiladi. Agar *A* matritsa $m \times n$ o'lchovli bo'lsa, A^T matritsa $n \times m$ o'lchovli bo'ladi.

B matritsa *A* kompleks matritsaga nisbatan *qo'shma kompleks matritsa* deb ataladi va $B = \overline{A}$ deb belgilanadi, agar hamma *i, j* indekslar uchun $b_{ij} = \overline{a}_{ij}$ tenglik bajarilsa. *B* matritsa *A* matritsaga nisbatan *qo'shma ermit matritsa* deb aytiladi va $B = A^H$ deb belgilanadi, agar hamma i, j lar uchun $b_{ij} = \overline{a}_{ji}$ tenglik bajarilsa.

A matritsa *nol matritsa* deb aytiladi, agar uning hamma elementlari 0 ga teng bo'lsa va *A=0* deb belgalanadi. *A* matritsa i_0, j_0 *indeksli birlik matritsa* deb aytiladi, agar $a_{i_0 j_0} = 1$ bo'lib, qolgan elementlari nolga teng bo'lsa.

$a_{11}, a_{22}, \dots, a_{nn}$ elementlar *n* tartibli $A = (a_{ij})$ kvadrat matritsaning bosh dioganalini tashkil qiladi va uning *diagonal elementlari* deb aytiladi. Matritsaning diagonal elementlari yig'indisi *A* matritsaning *izi* deb aytiladi va trA deb belgilanadi. Shunday qilib, $trA = \sum_{i=1}^{n} a_{ii}$.

Kvadrat matritsa diagonal matritsa deb aytiladi, agar uning diagonalida bo'lmagan elementlari 0 ga teng bo'lsa, ya'ni $a_{ij} = 0$, $i \neq j$. *n*-tartibli diagonal matritsa $diag(a_{11},...,a_{nn})$ deb belgilanadi. Diagonal elementlari 1 ga teng bo'lgan n-tartibli diagonal matritsa birlik matritsa deb aytiladi va E yoki E_n deb belgilanadi. Birlik matritsaning elementlari δ_{ij} deb belgilanadi: $E=(\delta_{ij})$,

$$\delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$$

Bizga $f(x)=a_0+a_1x+...+a_kx^k$ - ko'phad berilgan bo'lsin. $B=a_0E+a_1A+...+a_kA^k$ matritsa A matritsadan *ko'phad* deb aytiladi va $B=f(A)$ deb belgilanadi.

1-m i s o l. Matritsalarning chiziqli kombinasiyasi topilsin:

$$2\begin{pmatrix}2&7\\-1&3\end{pmatrix}-\begin{pmatrix}5&4\\-2&1\end{pmatrix}-5\begin{pmatrix}1&0\\0&8\end{pmatrix}=$$

$$\begin{pmatrix}2\cdot2-5-5\cdot1 & 2\cdot7-4-5\cdot0\\ 2\cdot(-1)-2(-2)-5\cdot0 & 2\cdot3-1-5\cdot8\end{pmatrix}=\begin{pmatrix}-6&10\\0&-35\end{pmatrix}$$ ■.

2-m i s o l. Matritsalarning ko'paytmasi topilsin:

$$A=\begin{pmatrix}5&8&-4\\6&9&-5\\4&7&-3\end{pmatrix},\qquad B=\begin{pmatrix}3&2&5\\4&-1&3\\9&6&5\end{pmatrix}.$$

Yechish. Matritsalarni ko'paytmasi formulasiga asosan quyidagi tenglik kelib chiqadi:

$$AB=\begin{pmatrix}5\cdot3+8\cdot4+(-4)\cdot9 & 5\cdot2+8\cdot(-1)+(-4)\cdot6 & 5\cdot5+8\cdot3+(-4)\cdot5\\ 6\cdot3+9\cdot4+(-5)\cdot9 & 6\cdot2+9\cdot(-1)+(-5)\cdot6 & 6\cdot5+9\cdot3+(-5)\cdot5\\ 4\cdot3+7\cdot4+(-3)\cdot9 & 4\cdot2+7\cdot(-1)+(-3)\cdot6 & 4\cdot5+7\cdot3+(-3)\cdot5\end{pmatrix}=$$

$$=\begin{pmatrix}11&-22&29\\9&-27&32\\13&-17&26\end{pmatrix}.$$ ■

3-m i s o l. Quyidagi matritsa bilan o'rin almashinuvchi hamma matritsalar topilsin.

$$A=\begin{pmatrix}7&-3\\5&-2\end{pmatrix}.$$

Yechish. Shunday X matritsa topishimiz $AX=XA$ bo'lsin. $X=\begin{pmatrix}x_1&x_2\\x_3&x_4\end{pmatrix}$ deb belgilaymiz. U holda

$$\begin{pmatrix}7&-3\\5&-2\end{pmatrix}\begin{pmatrix}x_1&x_2\\x_3&x_4\end{pmatrix}=\begin{pmatrix}x_1&x_2\\x_3&x_4\end{pmatrix}\begin{pmatrix}7&-3\\5&-2\end{pmatrix}.$$

Bundan

$$\begin{pmatrix} 7x_1-3x_3 & 7x_2-3x_4 \\ 5x_1-2x_3 & 5x_2-2x_4 \end{pmatrix} = \begin{pmatrix} 7x_1+5x_2 & -3x_1-2x_2 \\ 7x_3+5x_4 & -3x_3-2x_4 \end{pmatrix}.$$

Shunday qilib,

$$\begin{cases} 7x_1-3x_3=7x_1+5x_2 \\ 7x_2-3x_4=-3x_1-2x_2 \\ 5x_1-2x_3=7x_3+5x_4 \\ 5x_2-2x_4=-3x_3-2x_4 \end{cases} \Rightarrow \begin{cases} 5x_2+3x_3=0 \\ 3x_1+9x_2-3x_4=0 \\ 5x_1-9x_2-5x_4=0 \\ 5x_2+3x_3=0 \end{cases}$$

Bu sistemani yechib quyidagi tengliklarni hosil qilamiz:
$x_1=\alpha;\ x_2=3\beta;\ x_3=-5\beta;\ x_4=\alpha+9\beta$, bu yerda α va β -ixtiyoriy sonlar.
Izlanayotgan matritsa quyidagi ko'rinishga ega bo'ladi:

$$\begin{pmatrix} \alpha & 3\beta \\ -5\beta & \alpha+9\beta \end{pmatrix},\ \alpha,\beta\in C.\blacksquare$$

4-m i s o l. $f(A)$ ni hisoblang, agar:

$$f(x)=x^2-2x+1,\ A=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Yechish. Matritsaviy ko'phad ta'rifiga asosan quyidagi tenglikka ega bo'lamiz:

$$f(A)=\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 - 2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

M A S H Q L A R

2.1. Matritsalarning chiziqli kombinasiyasi topilsin:

a) $3\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} - 4\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$; b) $2\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$;

c) $2\begin{pmatrix} 1 & 8 & 7 & -15 \\ 1 & -5 & -6 & 11 \end{pmatrix} - \begin{pmatrix} 5 & 24 & -7 & -1 \\ -1 & 2 & 7 & 3 \end{pmatrix}$;

d) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$; ye) $\begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix} - \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$;

f) $2\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$.

2.2. Qanday shartlarda quyidagi ayniyatlar o'rinli bo'ladi:

a) $A+B=B+A$; b) $A+(B+C)=(A+B)+C$;

c) $\alpha(\beta A)=(\alpha\beta)A$; d) $\alpha(A+B)=\alpha A+\alpha B$;

e) $(\alpha+\beta)A=\alpha A+\beta A$.

2.3. Matritsalarning ko'paytamsi hisoblansin:

a) $\begin{pmatrix}2 & -3 & 0\end{pmatrix}\begin{pmatrix}4\\3\\1\end{pmatrix}$; b) $\begin{pmatrix}4\\3\\1\end{pmatrix}\begin{pmatrix}2 & -3 & 0\end{pmatrix}$; c) $\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}3 & 5\\5 & 9\end{pmatrix}$;

d) $\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$; e) $\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}\begin{pmatrix}1 & 4 & 3\\0 & 3 & 2\\0 & 1 & 0\\0 & 2 & 1\end{pmatrix}$;

f) $\begin{pmatrix}3 & 3 & -4 & -3\\0 & 6 & 1 & 1\\5 & 4 & 2 & 1\\2 & 3 & 3 & 2\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$; g) $\begin{pmatrix}3 & 1\\2 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$

h) $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 3 & 4\\3 & 4 & 5\end{pmatrix}$; i) $\begin{pmatrix}1 & 1 & 1 & 1\\1 & 0 & 1 & 1\\0 & 0 & 1 & 0\\0 & 0 & 1 & 1\end{pmatrix}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$;

j) $\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}\begin{pmatrix}\frac{1}{\lambda_1} & & 0\\ & \ddots & \\ 0 & & \frac{1}{\lambda_n}\end{pmatrix}$; k) $\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}\begin{pmatrix}0 & & \lambda_1\\ & \iddots & \\ \lambda_n & & 0\end{pmatrix}$;

l) $\begin{pmatrix}0 & & \lambda_1\\ & \iddots & \\ \lambda_n & & 0\end{pmatrix}\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$; m) $\begin{pmatrix}-1 & 1 & 1\\-5 & 21 & 17\\6 & -26 & -21\end{pmatrix}^2$;

n) $\begin{pmatrix}1 & 1 & 1 & \dots & 1\\0 & 1 & 1 & \dots & 1\\ \dots & \dots & \dots & \dots & \dots\\0 & 0 & 0 & \dots & 1\end{pmatrix}^2$; o) $\begin{pmatrix}1 & 1 & 1 & \dots & 1\\1 & \varepsilon & \varepsilon^2 & \dots & \varepsilon^{n-1}\\1 & \varepsilon^2 & \varepsilon^4 & \dots & \varepsilon^{2(n-1)}\\ \dots & \dots & \dots & \dots & \dots\\1 & \varepsilon^{n-1} & \varepsilon^{2(n-1)} & \dots & \varepsilon^{(n-1)^2}\end{pmatrix}^2$,

bu yerda $\varepsilon = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}$.

2.4. *A* va *B* matritsalar qanday shartlarni qanoatlantirganda quyidagilar o'rili bo'ladi:

a) *AB* ko'patma mavjud bo'ladi;
b) *BA* ko'paytma mavjud bo'ladi;
e) *AB* va *BA* ko'paytmalar mavjud bo'ladi.

2.5. Ayniyatlarning to'g'riligini tekshiring (*A*, *B*, *C* – matritsalar, α – son):

a) $\alpha(AB) = (\alpha A)B$; b) $(AB)C = A(BC)$;

s) $A(B+C)=AB+AC$; d) $(A+B)C=AC+BC$;

E) $A(B+C+D)=AB+AC+AD$.

2.6. Ko'paytmaning mavjudligini tekshiring va mavjud bo'lganda hisoblang:

a) $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}(1\ \ 2)\begin{pmatrix}2\\ 1\end{pmatrix}$; b) $\begin{pmatrix}2\\ 4\end{pmatrix}(1\ \ 2)\begin{pmatrix}2\\ 1\end{pmatrix}$; c) $(1\ \ 2)\begin{pmatrix}2\\ 1\end{pmatrix}(2\ \ 4)$;

d) $(-12\ \ 13)\begin{pmatrix}13547 & 13647\\ 28423 & 28523\end{pmatrix}\begin{pmatrix}-1\\ 1\end{pmatrix}(-12\ \ 13)$.

2.7. Hisoblang:

a) $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}^n$; b) $\begin{pmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}^n$; c) $\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0\end{pmatrix}^3$;

d) $\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}^n$; e) $\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^n$; f) $\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}^n$;

g) $\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}^n$; h) $\begin{pmatrix}0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1\\ 1 & 0 & 0 & \dots & 0\end{pmatrix}^n$.

2.8. Ayniyatlarning to'g'riligini tekshiring:

a) $(\alpha A)^T=\alpha A^T$; b) $(AB)^T=B^TA^T$;

c) $(ABC)^T=C^TB^TA^T$; d) $(A+B)^T=A^T+B^T$.

2.9. $A\times B=AB-BA$ (A va B matritsalarning kommutatori) matritsani hisoblang, agar:

a) $A=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, $B=\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}$;

b) $A=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, $B=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$.

2.10. Ayniyatlarning to'g'riligini tekshiring (9 misolga qarang):

a) $A\times B=-B\times A$; b) $A\times A=0$;

c) $A\times E=E\times A=0$; d) $A\times(B+C)=A\times B+A\times C$.

2.11. $A*B=\frac{1}{2}(AB+BA)$ matritsani hisoblang (A va B matritsalarning Yordan ko'paytmasi), agar:

a) $A=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$, $B=\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}$;

b) $A=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad B=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}.$

2.12. Ayniyatlarning to'g'riligini tekshiring:

a) $A*B=B*A$; b) $A*A=A^2$;

c) $A*E=A$; d) $A*(B+C)=A*B+A*C$.

2.13. *f(A)* ni hisoblang, agar

a) $f(x)=x^2-2x+1,\ A=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$;

b) $f(x)=x^2-2x+1, \quad A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$;

c) $f(x)=x^2-3x+2, \quad A=\begin{pmatrix}0 & -2\\ 1 & 3\end{pmatrix}$;

d) $f(x)=(x-\varepsilon)^2, \quad A=\begin{pmatrix}\varepsilon & 1\\ -1 & \varepsilon\end{pmatrix}$;

e) $f(x)=x^2+x+1, \quad A=\begin{pmatrix}-1 & 1 & 1\\ -5 & 21 & 17\\ 6 & -26 & -21\end{pmatrix}$;

f) $f(x)=x^3-5x^2+2x+4, \quad A=\begin{pmatrix}2 & 0 & 0\\ 0 & -3 & 0\\ 0 & 0 & 1\end{pmatrix}$;

g) $f(x)=x^2-4x+3, \quad A=\begin{pmatrix}2 & 1 & 0 & 0\\ 1 & 2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$.

2.14. Agar *AB*=*BA* shart bajarilsa, quyidagi tenglamalarning to'g'riligini isbotlang:

a) $(A+B)^2=A^2+2AB+B^2$; b) $(A+B)(A-B)=A^2-B^2$;

c) $A^n-B^n=(A-B)(A^{n-1}+A^{n-2}B+\ldots+AB^{n-2}+B^{n-1})$;

d) $(A+B)^n=A^n+nA^{n-1}+\dfrac{n(n-1)}{2!}A^{n-2}B^2+\ldots+B^n$.

Agar $AB\neq BA$ bo'lsa, yuqoridagi tengliklar to'g'ri bo'ladimi?

2.15. *A* matritsa bilan o'rin almashinuvchi bo'lgan hamma matritsalar topilsin, agar:

a) $A=\begin{pmatrix}-1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 5\end{pmatrix}$; b) $A=\begin{pmatrix}2 & -1\\ 3 & -1\end{pmatrix}$; c) $A=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$;

d) $A=\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$; e) $A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

f) $A=\begin{pmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_n \end{pmatrix}$, bunda $i \neq j$ bo'lsa, $a_i \neq a_j$.

2.16. *p*-tartibli skalyar matritsalar va faqat ular ixtiyoriy *p*-tartibli kvadrat matritsa bilan o'rin almashinuvchi ekanligini isbotlang.

2.17. Isbot qilingki, $trAB = trBA$.

2.18. $\begin{pmatrix} 17 & -6 \\ 35 & -12 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix}$ tenglikdan foydalanib, $\begin{pmatrix} 17 & -6 \\ 35 & -12 \end{pmatrix}^5$ ni hisoblang.

2.19. $\begin{pmatrix} 4 & 3 & -3 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 & -1 \\ 1 & 1 & -1 \\ -2 & -5 & 4 \end{pmatrix}$ tenglikdan foydalanib, $\begin{pmatrix} 4 & 3 & -3 \\ 2 & 3 & -2 \\ 4 & 4 & -3 \end{pmatrix}^6$ ni hisoblang.

2.20. $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ matritsa $x^2-(a+d)x+ad-bc=0$ tenglamani qanoatlantirishini isbotlang.

2.21. Kvadrati birlik matritsaga teng bo'lgan ikkinchi tartibli matritsalar topilsin.

2.22. Kvadrati nol matritsaga teng bo'lgan ikkinchi tartibli matritsalar topilsin.

2-§. Matritsalarni ko'paytirish bilan elementar almashtirishlar orasidagi munosabat

Quyidagi uchta almashtirishlar *matritsaning elementar almashtirishlari* deb aytiladi:

1) matritsaning biror satriga noldan farqli songa ko'paytirish;
2) matritsaning biror satrini songa ko'paytirib boshqa satriga qo'shish;
3) ikkita satrlarning o'rinlarini almashtirish.

Xuddi shunday uchta almashtirishlarni matritsaning ustunlari uchun ham ta'riflash mumkin.

M A S H Q L A R

2.23. Isbot qilingki, *AB* matritsaning *k*- ustuni *A* matritsa bilan *B* matritsaning *k*- ustuni ko'paytmasiga tengdir.

2.24. Matritsaning satrlari uchun 23-masalaga o'xshash masalani ifodalang va uni isbot qiling.

2.25. *AB* matritsaning *k*-ustuni koeffisiyentlari *B* matritsaning *k*-ustuni elementlaridan iborat bo'lgan *A* matritsa ustunlarining chiziqli kombinasiyasidan iborat ekanligini isbotlang.

2.26. *AB* matritsaning satrlari uchun 25-masalaga o'xshash masalani ifodalang va uni isbotlang.

2.27*. Isbot qiling:

a) agar *A* matritsaning ikkita satrlarining o'rinlari almashtirilsa, *AB* matritsaning mos satrlarining ham o'rinlari almashadi;

b) agar *A* matritsaning *k*-ustuni λ soniga ko'paytirilsa, *AB* matritsaning *k*-satri ham λ ga ko'paytiriladi;

c) agar *A* matritsaning *k*-satriga uning *j*-satrini qo'shsak, *AB* matritsa bilan ham xuddi shunday elementar almashtirish bajariladi.

2.28. matritsaning ustunlari uchun 27-masala hga o'xshash jumlalarni ifodalab isbot qiling.

2.29*. Isbot qilingki, matritsa ikkita satrlarining o'zaro o'rin almashtirishi uning satrlari ustida ketma-ket boshqa elementar almashtirishlar bilan bajariladi: satrni noldan farqli songa ko'paytirish va biror satrini boshqa satrga qo'shish.

2.30. Ixtiyoriy *A* matritsa uchun $e_i A e_j^T$ ko'paytmani hisoblang (e_i bilan mos tartibli birlik matritsaning *i*-satrini belgilang).

2.31. Ixtiyoriy *A* matritsa va unga mos o'lchovli E_{ij} matritsaviy birlik uchun quyidagi ko'paytmalarni hisoblang:

a) $E_{ij}A$; b) AE_{ij}.

2.32*. *A* va *B* matritsalar shunday matritsalarki, ixtiyoriy *a* va *b* ustunlar uchun (mos birlikka ega bo'lgan) $a^T Ab = a^T Bb$ tenglik o'rinlidir. *A=B* tenglik o'rinli ekanligini isbotlang

2.33. *A* – $m \times n$-o'lchovli matritsa , E_m va E_n - lar mos ravishda *m* va *n* tartibli birlik matritsalar bo'lsa, $E_m A = AE_n = A$ tenglikni isbotlang.

2.34. *A* – matritsani qanday matritsa ko'paytrish natijasida quyidagilar hosil bo'ladi:

a) *A* matritsaning birinchi ustuni;

b) *A* matritsaning birinchi satri?

2.35. Elementar matritsa *K* ni shunday tanlangki, *A* matritsadan *KA* matritsa quyidagicha almashtirish natijasida hosil bo'lsin:

a) *A* matritsaning birinchi ikkita satrlari o'rin almashtirishdan;

b) birinchi satrni ikkinchi satrga qo'shishdan;

c) *A* ning birinsi satrini $\lambda \neq 0$ songa ko'paytirishdan.

2.36. Elementar matritsa K shunday tanlab olinsinki, AK ko'paytma A matritsadan ustunlarning berilgan elementar almashtirishi natijasida hosil bo'lsin.

3-§. Teskari matritsa

A – matritsa n-chi tartibli kvadrat matritsa bo'lsin. A matritsa uchun $AB=BA=E$ tenglikni qanoatlantiruvchi B matritsa A ga *teskari matritsa* deyiladi va u $B=A^{-1}$ ko'rinishda belgilanadi. Teskari matritsa elementlarini

$$b_{ij}=\frac{A_{ji}}{\det A}$$

formula yordamida topish mumkin, bunda A_{ji} lar a_{ji} elementlarning algebraik to'ldiruvchisidir. A matritsa *teskarilanuvchi* deb aytiladi, agar $\det A\neq 0$, ya'ni A matritsa xosmas bo'lsa.

Har qanday xosmas A matritsani faqat satrlar (yoki faqat ustunlar) elementar almashtirishlari yordamida birlik matritsaga keltirish mumkin. Elementar almashtirishlarni xuddi shunday ketma-ketlikda E birlik matritsaga tadbiq qilsak, teskari matritsa A^{-1} ni hosil qilamiz. A va E matritsalarni chiziq yordamida qo'shni yozib ular ustida elementar almashtirishlarni bir vaqtda bijarish juda qulaydir.

Teskari matritsani hisoblash $AX=B,\ \ YA=B$ matritsaviy tenglamalarni yechish bir-biri bilan bog'langandir, bunda A, B – berilgan matritsalar, X, Y – izlanayotgan noma'lum matritsalar. Agar A matritsa to'g'ri burchakli matritsa yoki xosmas matritsa bo'lsa, matritsaviy tenglamalarni yechish X matritsaning har bir ustuni yoki Y matritsaning har bir satri elementlari uchun hosil bo'ladigan chiziqli tenglamalar sistemasini yechishga keltiriladi. Bu tenglamalarni hosil qilish uchun tenglamaning har ikkala tomonidagi matritsalarning mos elementlarini bir-biriga tenglashtirish lozim. Agar A matritsa xosmas bo'lsa, matritsaviy tenglamalarning yechimlari qyidagi formulalar yordamida topiladi:

$$X=A^{-1}B,\ \ Y=BA^{-1}.$$

1-m i s o l. Berilgan matritsa uchun teskari matritsa topilsin

$$A=\begin{pmatrix}3 & -4 & 5\\ 2 & -3 & 1\\ 3 & -5 & -1\end{pmatrix}.$$

Yechish. $\det A=-1$ bo'lganligi uchun teskari matritsa A^{-1} mavjud. Matritsa elementlarining algebraik to'ldiruvchilarini topamiz:

$A_{11}=8$, $A_{21}=-29$, $A_{31}=11$,

A_{12}=5, A_{22}=-18, A_{32}=7,
A_{13}=-1, A_{33}=3, A_{33}=-1.

Teskari matritsani topish formulasiga asosan bu algebraik to'ldiruvchilarni (-1) ga bo'lib, teskari matritsani hosil qilamiz:

$$A^{-1} = \begin{pmatrix} -8 & 29 & -11 \\ -5 & 18 & -7 \\ 1 & -3 & 1 \end{pmatrix}. \blacksquare$$

2-m i s o l. Satrlarning elementar almashtirishlari yordamida teskari matritsa A^{-1} ni toping

$$A = \begin{pmatrix} 3 & -4 & 5 \\ 2 & -3 & 1 \\ 3 & -5 & -1 \end{pmatrix}.$$

Yechish. Quyidagilarni hosil qilamiz:

$$\left(\begin{array}{ccc|ccc} 3 & -4 & 5 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ 3 & -5 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & -1 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ 3 & -5 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 2R_1}$$

$$\xrightarrow{R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & -1 & 0 \\ 0 & -1 & -7 & -2 & 3 & 0 \\ 3 & -5 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & -1 & 0 \\ 0 & -1 & -7 & -2 & 3 & 0 \\ 0 & -2 & -13 & -3 & 3 & 1 \end{array}\right) \xrightarrow{R_3 - 2R_2}$$

$$\xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 4 & 1 & -1 & 0 \\ 0 & -1 & -7 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right) \xrightarrow{R_1 - 4R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 3 & 11 & -4 \\ 0 & -1 & -7 & -2 & 3 & 0 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right) \xrightarrow{R_2 + 7R_1}$$

$$\xrightarrow{R_2 + 7R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -3 & 11 & -4 \\ 0 & -1 & 0 & 5 & -18 & 7 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -8 & 29 & -11 \\ 0 & -1 & 0 & 5 & -18 & 7 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right) \xrightarrow{(-1)R_2}$$

$$\xrightarrow{(-1)R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -8 & 29 & -11 \\ 0 & 1 & 0 & -5 & 18 & -7 \\ 0 & 0 & 1 & 1 & -3 & 1 \end{array}\right).$$

Bunda R_i matritsaning *i*-satri.

Shunday qilib, teskari matritsa quyidagi ko'riniga ega bo'ladi:

$$A^{-1} = \begin{pmatrix} -8 & 29 & -11 \\ -5 & 18 & -7 \\ 1 & -3 & 1 \end{pmatrix}. \blacksquare$$

3-m i s o l. Quyidagi tenglamalardan *X* matritsani toping:

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} X \begin{pmatrix} -4 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Yechish. $\det A = -1 \neq 0$ va $\det B = -4 \neq 0$ bo'lgani uchun A^{-1} va B^{-1} teskari matritsalar mavjud. Tenglamaning chap va o'ng tomonlarini chapdan A^{-1} ga, o'ngdan B^{-1} ga ko'paytirsak, quyidagi tenglikka ega bo'lamiz:

$$A^{-1}(AXB)B^{-1} = A^{-1}CB^{-1}.$$

$$A^{-1}(AXB) = (A^{-1}A)X(BB^{-1}).$$

$A^{-1}A = E, BB^{-1} = E$ va $EXE = X$ bo'lganligi uchun $X = A^{-1}CB^{-1}$ kelib chiqadi.

Bu tenglikka asosan X matritsani hisoblaymiz:

$$A^{-1} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} -0,25 & 0 \\ 0,5 & 1 \end{pmatrix},$$

$$X = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -0,25 & 0 \\ 0,5 & 1 \end{pmatrix} = \begin{pmatrix} -0,5 & 1 \\ 0,5 & 1 \end{pmatrix}. \ \blacksquare$$

4-m i s o l. Har qanday qanday elementar almashtirishlar xosmas matritsalarning ko'paytmasidan iborat ekanligini isbotlang.

Yechish. A matritsa satrlarining har qanday elementar almashtirishlari A matritsani chapdan elementar matritsaga ko'paytirishdan iborat bo'lib, bu elementar matritsani birlik matritsadan o'shanday elementar almashtirish yordamida hosil qilish mumkin. Xosmas matritsani elementar almashtirishlar yordamida birlk matritsaga keltirish mumkin.
Shuning uchun quyidagiga $S_k \cdots S_1 A = E,$
ega bo'lamiz:

bundan esa $S_k \cdots S_1 = A^{-1}, \quad A = S_1^{-1} \cdots S_k^{-1}$. $S_1^{-1},...,S_k^{-1}$ matritsalar hamda $S_1,..., S_k$ matritsalar ham elementar matritsalardir, ularni birlik matritsadan satrlarning «teskari» elementar almashtirishi yordamida hosil qilish mumkin. ■

5-m i s o l. Berilgan matritsani elementar matritsalarning ko'paytmasiga yoying: $\begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$.

Yechish. To'rtinchi misolning yechimiga asosan bunda $A = S_1^{-1} \cdots S_k^{-1}$ matritsalar A matritsa satrlarining elementar almashtirishlariga mos keladiki, uni birlik matritsaga keltiradi. $S_1,...,S_k$ matritsalarni tanlab keyin $S_1^{-1},...,S_k^{-1}$ larni topamiz. Bu jarayonni bitta qadamga kamaytirish mumkinligini ko'rsatamiz. $A = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$ matritsani soddalashtiramiz. Matritsaning ikkinchi satrini $\left(-\frac{1}{2}\right)$ ga ko'paytirish A ni chap tomondan $\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ matritsaga ko'paytirish bilan teng kuchlidir. Quyidagi tenglik hosil bo'ladi:

$$\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = B \qquad (*)$$

B matritsa elementar matritsadir. Hisoblaymiz:

$$\begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = S .$$

(*) tenglikning har ikkala tomonini chapdan *S* ga ko'paytirsak, izlanayotgan yoyilma kelib chiqadi:

$$\begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix} = SB = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \blacksquare$$

6-m i s o l. *A* va *B* matritsalar satrlarining elementar almashtirishlari yordamida $A^{-1}B$ ko'paytmani hosil qilish usulini keltiring.

Yechish. *A* va *B* matritsalarni ketma-ket yozamiz. (*AB*) matritsaning satrlari bilan elementar almashtirishlarni bajaramiz. Bu almashtirishlar *A* matritsani *E* matritsaga keltirsin. U holda bu almashtirishlar natijasida *A* matritsa joyida *E* matritsa, *B* matritsa joyida A^{-1} matritsa hosil bo'ladi.

Haqiqatdan ham, $S_1 \cdots S_k A = E$ bo'lgani uchun $S_1 \cdots S_k = A^{-1}$ bo'ladi. U holda $S_1 \cdots S_k B = A^{-1}B$ (4 misolga qarang).■

2.37. To'g'ri burchakli matritsa teskarilanuvchi bo'ladimi?

2.38*. *A* –matritsa ikkinchi tartibli xos matritsa, *m* – natural son bo'lsin. Shunday λ son mavjud bo'lib, $A^m = \lambda^{m-1}$ tenglik hamma *m* lar uchun bajarilishini isbotlang.

2.39. $A = (a_{ij})$ kvadrat matritsa berilgan. A^{-1} matritsa *j*-chi ustun elementlari qanoatlantiruvchi tenglamalar sistemasini yozing.

2.40. Teskari matritsani topish formulasidan foydalanib quyidagi matritsalar uchun teskari matritsani toping:

a) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; b) $\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$; c) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; d) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$;

e) $\begin{pmatrix} 2 & 7 & 3 \\ 3 & 9 & 4 \\ 1 & 5 & 3 \end{pmatrix}$; f) $\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$; g) $\begin{pmatrix} 2 & -2 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 2 \end{pmatrix}$;

h) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$; i) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & -1 \\ 1 & 0 & -2 & -6 \end{pmatrix}$; j) $\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$; k) $\begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$; l) $\begin{pmatrix} 1 & a & a^2 & a^3 & \dots & a^n \\ 0 & 1 & a & a^2 & \dots & a^{n-1} \\ 0 & 0 & 1 & a & \dots & a^{n-2} \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$;

m) $\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 2 \end{pmatrix}$; n) $\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}$;

o) $\begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}$; p) $\begin{pmatrix} 1+a & 1 & 1 & \dots & 1 \\ 1 & 1+a & 1 & \dots & 1 \\ 1 & 1 & 1+a & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1+a \end{pmatrix}$.

2.41. Elementar matritsaga teskari matritsa ham elementar matritsa bo'lishini isbotlang.

2.42. Quyidagi ayniyatlar o'rinli bo'ladimi?

a) $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$; b) $(\alpha A)^{-1} = \alpha^{-1}A^{-1}$;

c) $(AB)^{-1} = B^{-1}A^{-1}$; d) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$;

e) $\left(A^{-1}\right)^k = \left(A^k\right)^{-1}$; f) $(A+B)^{-1} = A^{-1} + B^{-1}$.

2.43. Berilgan matritsani elementar matritsalar ko'paytmasiga yoying:

a) $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$; b) $\begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$; c) $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$.

2.44. A matritsa uchun teskari matritsani toping, agar:

a) $A^2 - 4A + E = 0$; b) $A^3 + 5A^2 - 3A - E = 0$; c) $A^2 - A = 0$.

2.45. Matritsaviy tenglamalar sistemasini yeching:

a) $\begin{cases} X+Y=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} \\ 2X+3Y=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\end{cases}$; b) $\begin{cases} 2X-Y=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} \\ -4X-2Y=\begin{pmatrix}0 & -2\\ 2 & 0\end{pmatrix}\end{cases}$.

2.46. $A^2+A+E=0$ bo'lsin. A matritsa xosmas matritsa ekanligini isbotlang va A^{-1} ni hisoblashning eng sodda usulini ko'rsating.

2.47. Elementar almashtirishlar yordamida berilgan matritsa uchun teskari matritsani toping:

a) $\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{pmatrix}$; b) $\begin{pmatrix}0 & 0 & 1 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\end{pmatrix}$; c) $\begin{pmatrix}2 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 2 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}$;

d) $\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & 2 & 0\\ 1 & 0 & 0 & 0\\ 0 & 3 & 0 & 0\end{pmatrix}$; e) $\begin{pmatrix}1 & 2 & 2 & 2\\ 2 & 1 & 2 & 2\\ 2 & 2 & 1 & 2\\ 2 & 2 & 2 & 1\end{pmatrix}$; f) $\begin{pmatrix}0 & 1 & 1 & 1\\ -1 & 0 & 1 & 1\\ -1 & -1 & 0 & 1\\ -1 & -1 & -1 & 0\end{pmatrix}$;

g) $\begin{pmatrix}1 & 1 & 1 & \dots & 1\\ 0 & 1 & 1 & \dots & 1\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1\end{pmatrix}$; h) $\begin{pmatrix}1 & 0 & 0 & \dots & 0 & 0\\ 1 & 1 & 0 & \dots & 0 & 0\\ 0 & 1 & 1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1 & 1\end{pmatrix}$;

i) $\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$; j) $\begin{pmatrix}1 & -1 & 0 & \dots & 0 & 0\\ 0 & 1 & -1 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1 & -1\\ 0 & 0 & 0 & \dots & 0 & 1\end{pmatrix}$;

k) $\begin{pmatrix}0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1\\ 1 & 0 & 0 & \dots & 0\end{pmatrix}$; l) $\begin{pmatrix}1 & 2 & 3 & \dots & n\\ 0 & 1 & 2 & \dots & n-1\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & 0 & \dots & 1\end{pmatrix}$.

2.48. *A, B* matritsalar ustunlarining elementar almashtirishlari yordamida AV^{-1} ko'paytmani hisoblash metodini asoslab ifodalang.

2.49. $A^m=0$ bo'lsin. $(E-A)^{-1}E+A+\dots+A^{m-1}$ tenglikni isbotlang.

2.50. A matritsa B matritsa bilan o'rin almashinuvchidir. Isbot qilish lozimki, A^{-1} matritsa B^{-1} matritsa Bilan o'rinalmashinuvchi bo'ladi (A va B matritsalar teskarilanuvchidir).

2.51. Quyidagi formula tekshirilsin: $\left(S^{-1}AS\right)^m = S^{-1}A^m S$.

2.52. $S^{-1}AS = B$ va $f(t)$ – ko'phad bo'lsin. $f(B) = S^{-1}f(A)S$ tenglikni isbotlang.

2.53. A va C matritsalar xosmas matritsalar bo'lsin. Matritsaviy tenglamalarni yeching:

a) $AX=0$; b) $A(X+C)=B$.

2.54. Quyidagi tenglamalardan X matritsani toping:

a) $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$; b) $X\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$;

c) $\begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & -2 \\ 0 & -2 & 5 \end{pmatrix} X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$; d) $X\begin{pmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 5 & 2 \\ 5 & 8 & -1 \end{pmatrix}$;

e) $X\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$; f) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} X = X\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$;

g) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 2 \end{pmatrix} X = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 4 \\ 2 & 4 & 8 \end{pmatrix}$; h) $\begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{pmatrix}$;

i) $\begin{pmatrix} 3 & 2 \\ 3 & 1 \\ -2 & 0 \end{pmatrix} X = \begin{pmatrix} -1 & 4 & 3 \\ -2 & 5 & 3 \\ 2 & -4 & -2 \end{pmatrix}$; j) $\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} X = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{pmatrix}$;

k) $\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix} X = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 0 & 1 & 2 & \dots & n-1 \\ 0 & 0 & 1 & \dots & n-2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$.

2.55. Teskari A^{-1} matritsa qanday o'zgaradi, agar berilgan A matritsada:

a) i-chi va j-chi satrlar joylari almashtirilsa?

b) i-chi satrni noldan farqli s soniga ko'paytirilsa?

c) j-chi satrni s soniga ko'paytirib, i-chi satriga qo'shsa yoki shunday almashtirishni ustunlar ustida bajarsa?

4-§. Matritsalar bilan boshqa amallar. Maxsus ko'rinishdagi matritsalar

Endi *n*–tartibli matritsaning ba'zi bir maxsus ko'rinishlarini qarab chiqamiz: $A=(a_{ij})$

skalyar matritsa: $A=diag(\lambda,...,\lambda)$, λ – son;

unimodulyar matritsa: detA=1;

xos (maxsus)matritsa: detA=0;

xosmas (maxsusmas) matritsa: detA ≠ 0;

o'rinmalmashtirish matritsasi: A matritsa birlik matritsadan satrlarning o'rinlarini almashtirishdan hosil bo'ladi;

elementar matritsa: A matritsa birlik matritsadan elemetar almashtirishlar orqali hosil bo'ladi*;*

yuqori uchburchakli matritsa: $a_{ij}=0$ agar $i>j$;

pastki uchburchakli matritsa: $a_{ij}=0$ agar $i<j$;

simmetrik matritsa: $A^T=A$;

kososimmetrik matritsa: $A^T=-A$;

ermit matritsasi: $A^N=A$;

kosoermit matritsasi: $A^N=-A$;

ortogonal matritsa: $A^T=A^{-1}$;

unitar matritsa: $A^N=A^{-1}$;

manfiymas matritsa: $a_{ij}\geq 0$ hamma *i, j* lar uchun;

Markov (stoxastik) matritsasi: $a_{ij}\geq 0$ hamma *i, j* lar uchun va $\sum_{k=1}^{n} a_{ik}=1$ *i=1, 2,…,n*;

nilpotent matritsa: *k natural sonning qandaydir qiymatida* $A^k=0$ (bunday *k* ning eng kichik qiymatiga *A matritsaning nilpotentlik ko'rsatkichi* deb aytiladi);

davriy matritsa: *k natural sonning qandaydir qiymatida* $A^k=E$ (bunday *k* son *A* matritsaning *davri* deb aytiladi).

B matritsa katakli matritsa deb aytiladi, agar uning elementlari $m_i \times n_j$ o'lchovli B_{ij} matritsalardan iborat bo'lsa. bunda *B* matritsaning bitta satriga qarashli bo'lgan hamma B_{ij} matritsalar bir xil balandlikka ega bo'ladi, *B* matritsaning bitta ustuniga qarashli bo'lgan hamma B_{ij} matritsalar bir xil enga ega bo'ladi. katakli matritsalar ustida bajariladigan amallar oddiy sonli matritsalar ustida bajariladigan amallardan iborat. Agar *A* sonli matritsa gorizontal va vertikal to'g'ri chiziqlar bilan B_{ij} kataklarga ajratilgan bo'lib, tabiiy holda nomerlangan bo'lsa va bu kataklardan $B=(B_{ij})$ katakli matritsa tuzilgan bo'lsa, *V* matritsa *A* matritsadan kataklarga bo'lish natijasida hosil bo'lgan deb aytiladi. B_{ij} matritsalarning elementlaridan tabiiy ravishda $\sum_i m_i \times \sum_j n_j$ o'lchovli sonli matritsani tuzish mumkin. bu holda *A* matritsa *B* matritsa kataklarining birlashmasidan hosil bo'lgan deb aytiladi va $A=B^{\square}$ deb yoziladi. Agar tushunmovchilik uchun asos bo'lmasa, $^{\square}$ belgisini qoldirib sonli va katakli matritsalarni bir xil harflar bilan belgidash mumkin.

A=(a_{ij}) va *B* – matritsalar, $S=(s_{ij})$ – katakli matritsa $c_{ij}=a_{ij}B$ tenglik bilan hamma *i, j* lar uchun aniqlangan bo'lsin. *S* matritsa kataklarining birlashmasidan hosil bo’lgan sonli matritsa *A* va *B* matritsalarning *o’ng kroneker ko’paytmasi* deb aytiladi (yoki *o’ng to’g’ri ko’paytmasi deyiladi*) va $A\otimes B$ deb belgilanadi.

1-m i s o l. *A* diagonal matritsa bo’lib, uning hamma diaganal elementlari har xil bo’lsin va *AB=BA*. U holda *B* matritsa ham diaganal ekanligini isbotlang.

Yechish. Bu tasdiqning to’g’riligini ikkinchi tartibli matritsa uchun isbotlaymiz.

$A=\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix},\ \ d_1\neq d_2,\ \ B=\begin{pmatrix} x & y \\ z & t \end{pmatrix}$ bo’lsin. U holda

$$AB=\begin{pmatrix} d_1x & d_1y \\ d_2z & d_2t \end{pmatrix},\ BA=\begin{pmatrix} d_1x & d_2y \\ d_1z & d_2t \end{pmatrix}.$$

AB=BA bo’lganligi uchun $d_1y=d_2y$ va $d_2z=d_1z$ bo’ladi. bundan esa $(d_1-d_2)y=0,(d_2-d_1)z=0$, ya’ni $y=0,\ z=0$ bo’ladi, chunki $d_1-d_2\neq 0$.

Xuddi shunday bu tasdiqni *n*-chi tartibli matritsalar uchun isbotlash mumkin. ■

2-m i s o l. Har qanday *n* chi tartibli maxsusmas matritsa bilan o’rin almashinuvchi matritsalarni toping.

Yechish. Dioganal matritsa *diag(1, 2, ..., n)* maxsusmasdir. Bu matritsadan foydalanib birinchi misolni qo’llasak, *A* matritsaning dioganalligi kelib chiqadi. endi faqat *A* matritsaning hamma dioganal elementlari teng ekanligini isbotlash qoladi. Agar *A* matritsa ikkinchi tartibli bo’lsa, $A=\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, uni chapdan va o’ngdan $S=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ matritsaga ko’paytiramiz. *AS* va *SA* matritsalarni tenglashtirib $\lambda_1=\lambda_2$ tenglikni hosil qilamiz. Xuddi shunday ixtiyoriy tartibli *A* dioganal matritsa uchun *S* ni tanlab *A* matritsaning har qanday ikkita dioganal elementlarining tengligini tekshiramiz. ■

3-m i s o l. $A=\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ matritsa ermit matritsasidir, chunki

$$A^H=\overline{A}^T=\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.\ ■$$

4-m i s o l. $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ matritsa o’rinalmashtirishmatritsasidir, chunki bu matritsa birlik matritsadan 1-,2- chi va 3-, 4- chi satrlarning o’rinlarini almashtirishdan hosil bo’ladi. ■

5-m i s o l. $\frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$ matritsa unitar matritsadir, chunki

$$A^{-1}=A^{H}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}. \blacksquare$$

6-m i s o l. Agar A matritsa yuqori uchburchakli bo'lsa va uning hamma dioganal elamentlari noldan farqli bo'lsa, A^{-1} matritsa mavjudligini va yuqori uchburchakli matritsa ekanligini isbotlang.

Yechish. Teskari matritsani ($A|E$) matritsadan satrlarning elementar almashtirishlari yordamida izlaymiz. soddalashtirish prosessini oxirgi satrdan boshlaymiz. bunda A va E matritsalarning bosh dioganaldan pastda joylashgan elementlari o'zgarmaydi. natijada birlik matritsadan yuqori uchburchakli matritsa hosil bo'ladi. ■

7-m i s o l. $A=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$ matritsa ortogonal matritsadir, chunki $A^{T}=\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}=A^{-1}$. ■

8-m i s o l. $\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}$ matritsa nilpotentdir va uning nilpotentlik ko'rsatkichi 2 ga teng, chunki bu matritsaning kvadrati nol matritsadir. ■

9-m i s o l. $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ matritsa davriy matritsadir va uning davri 2 ga teng, chunki uning kvadrati birlik matritsaga tengdir.

10-m i s o l. $\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}$ matritsa stoxastik matritsadir. ■

11-m i s o l. Œrinalmashtirish matritsasi davriy matritsa ekanligini isbotlang.

Yechish. Faraz qilaylik, A - o'rin almashtirish matritsasi bo'lsin. Mumkin bo'lgan barcha A^k matritsalarni qaraymiz. Tekshirish mumkinki, o'rinalmashtirishlar matritsalarning ko'paytmasi yana o'rin almashtirish matritsasi bo'ladi. Har xil o'rinalmashtirishlar matritsalarning bir xil tartibga ega bo'lgan matritsalari soni cheklidir. Shuning uchun shunday p, q natural sonlar mavjudki, $q>p$ va $A^q=A^p$ bo'ladi. Bundan esa $A^{q-p}=E$ kelib chiqadi. ■

12-m i s o l. Berilgan A va B matritsalarni bloklarga ajratib AB ko'paytmani toping.

$$A=\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix},\quad B=\begin{pmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}.$$

Yechish. Berilgan A va B matritslarani bloklarga quyidagicha ajratamiz. Bu matritsalarni blokli matritsalarni ko'paytirish qoidasiga asosan ko'paytirsak:

$$A = \left(\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right), \quad B = \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right).$$

quyidagi matritsa hosil bo'ladi $AB = \left(\begin{array}{cc|cc} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right)$. ■

13-m i s o l. Agar $N = \begin{pmatrix} E & A \\ 0 & E \end{pmatrix}$ blokli matritsa bo'lsa, $(N^{□})^{-1}$ matritsa topilsin.

Yechish. $\det H = E \neq 0$ bo'lganligi uchun teskari matritsa mavjud. algebraik to'ldiruvchilarni hisoblamaymiz.

$$A_{11} = E, A_{12} = 0, A_{21} = -A, A_{22} = E.$$

bundan esa $(N^{□})^{-1} = \begin{pmatrix} E & -A \\ 0 & E \end{pmatrix}$ kelib chiqadi. ■

14-m i s o l. A va B matritsalarning kroneker ko'paytmasini hisoblang.

$$A = \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Yechish. Kroneker ko'paytmasining ta'rifiga asosan quyidagiga ega bo'lamiz:

$$A \otimes B = \begin{pmatrix} 3\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & 5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \\ 5\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} & 9\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 3 & 6 & 5 & 10 \\ 9 & 12 & 15 & 20 \\ 5 & 10 & 9 & 18 \\ 15 & 20 & 27 & 36 \end{pmatrix}. \ ■$$

M A S H Q L A R

2.56. A, B – bir xil tartibli diagonal matritsalar, α – son bo'lsin. isbot qilish lozimki, $\alpha A, A+B, AB, BA$ matritsalar ham diagonal matritsalardir va $AB=BA$.

2.57. $A = diag(\lambda_1,..., \lambda_n)$ bo'lsin. Isbot qilish lozimki:

1) BA matritsaning ustunlari V matritsaning ustunlarini $\lambda_1,..., \lambda_n$ sonlarga ko'paytirishdan hosil bo'ladi;

2) AB matritsaning satrlari B matritsaning satrlarini $\lambda_1,..., \lambda_n$ sonlarga ko'paytrishdan hosil bo'ladi.

2.58. A – diagonal matritsa, $f(t)$ – ko'phad bo'lsin. u holda $f(A)$ matritsa ham diagonal matritsa ekanligi isbotlang.

2.59. A matritsa diagonal matritsa bo'lib, uning hamma dioganal elementlari har xil bo'lsin. va $AB=BA$. U holda B matritsa ham diagonal matritsa ekanligi isbotlang.

2.60. A matritsa ixtiyoriy n-tartibli dioganal matritsa bilan o'rinalmashinuvchi bo'lsin. A matritsa n-tartibli dioganal matritsa ekanligini isbotlang.

2.61*. A matritsa hamma n-tartibli matritsaviy birliklar bilan o'rinalmashinuvchi bo'lsin. A matritsa skalyar matritsa ekanligini isbotlang.

2.62*. A matritsa n-tartibli har qanday matritsa bilan o'rinalmashinuvchi bo'lsin. A matritsa skalyar matritsa ekanligini isbotlang.

2.63. Berilgan matritsalarga qo'shma ermit matritsalar topilsin:

a) $\begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}$; b) $\begin{pmatrix} 5 & i \\ -i & 1 \end{pmatrix}$; c) $\begin{pmatrix} 1 & -i \\ 2+i & 1-2i \end{pmatrix}$; d) $\begin{pmatrix} 1+i\sqrt{2} & 3 \\ 1 & 1-i\sqrt{2} \end{pmatrix}$.

2.64. Ayniyatlarning to'g'riligini tekshiring:

a) $(A+B)^H = A^H + B^H$; b) $(\alpha A)^H = \overline{\alpha} A^H$;

c) $(A^H)^H = A$; d) $(AB)^H = B^H A^H$; e) $(A^H)^{-1} = (A^{-1})^H$.

2.65. Berilgan ikkinchi tartibli matritsalar dioganal, skalyar, uchburchakli, simmetrik, kososimmetrik, ermit, kosoermit, unitar, ortogonal yoki o'rinalmashtirish matritsalari ekanligini aniqlang:

a) $\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$; b) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; c) $\begin{pmatrix} 1 & 1+i \\ 1-i & 3 \end{pmatrix}$; d) $\begin{pmatrix} 5 & i \\ -i & 1 \end{pmatrix}$;

e) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$; f) $\begin{pmatrix} -4 & 0 \\ 1 & 4 \end{pmatrix}$; g) $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$; h) $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

2.66. Isbot qilingki:

a) kososimmetrik matritslarning hamma dioganal elementlari nolga teng;

b) ermit matritsasining dioganal elementlari haqiqiydir;

c) kosoermit matritsaning dioganal elementlari mavhum sondir;

2.67. Isbot qilish lozimki:

a) agar A ermit matritsasi bo'lsa, iA – kosoermit matritsasidir;

b) agar A kosoermit matritsasi bo'lsa, iA ermit matritsasidir.

2.68. a) ikkinchi tartibli ermit matritsalarning umumiy ko'rinishi topilsin;

b) ikkinchi tartibli kosoermit matritsalarning umumiy ko'rinishi topilsin;

c) hamma ikkinchi tartibli o'rinalmashtirishlarning matritsalarini ko'rsating.

2.69. Agar A diagonal matritsa bo'lib, uning hamma dioganal elementlari noldan farqli bo'lsa, teskari A^{-1} matritsa mavjud va dioganal matritsa ekanligini isbotlang.

2.70. Agar A – maxsusmas simmetrik matritsa bo'lsa, A^{-1} ham simmetrik matritsadir.

2.71. Agar A – maxsusmas kososimmetrik matritsa bo'lsa, A^{-1} ham kososimmetrik matritsadir.

2.72. Agar A – ortogona matritsa bo'lsa, A^{-1} matritsa mavjud va ortogonaldir.

2.73. Agar A – unitar matritsa bo'lsa, A^{-1} mavjud va unitardir.

2.74. Agar A – o'rinalmashtirish matritsasi bo'lsa, A^{-1} matritsa mavjud va u ham o'rinalmashtirish matritsasi bo'ladi.

2.75. Quyidagi matritsalar ortogonal ekanligini isbotlang va ularga teskari matritsani toping:

a) $\begin{pmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{1}{3\sqrt{2}} \\ \frac{1}{3} & 0 & \frac{4}{3\sqrt{2}} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} \end{pmatrix}$; b) $\begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{1}{2} & \frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{1}{\sqrt{2}} & \frac{2}{\sqrt{10}} \\ -\frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$;

c) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$; d) $\begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & -\frac{1}{2} & -\frac{1}{2} & \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{10} \end{pmatrix}$.

2.76. Quyidagi matritsalar unitar ekanlini isbotlang va ularga teskari matritsani toping:

a) $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$; b) $\frac{1}{\sqrt{2}}\begin{pmatrix} i & 0 & -i & 0 \\ 0 & i & 0 & -i \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$.

2.77. A va B – matritsalar yuqori uchburchakli matritsalar bo'lsin. AB matritsaning elementlarini A va B matritsalarning elementlari orqali ifodalang.

2.78. A va B – matritsalar yuqori uchburchakli matritsalar bo'lsin. U holda $A+B$ va AB – matritsalar ham yuqori uchburchakli matritsalar ekanligini isbotlang.

2.79. A va B matritsalar simmetrik matritsalar bo'lsin. Isbot qilish lozimki:

a) $A+B$ – simmetrik matritsadir;

b) har qanday k natural son uchun A^k – simmetrik matritsadir $k \in \mathbf{N}$;

c) AB matritsa simmetrik matritsa bo'lishi uchun A va B matritsalar o'rin almashinuvchi bo'lishi zarur va yetarlidir.

2.80. A va B – matritsalar kososimmetrik matritsalar bo'lsin. isbot qilish lozimki:

a) $A+B$ – kososimmetrik matritsa;

b) k toq son bo'lganda A^k – kososimmetrik matritsadir va k juft son bo'lganda

s) *AB* matritsa simmetrik bo'lishi uchun *A* va *B* matriyalar o'rinalmashuvchi bo'lishi zarur va yetarlidir;

d) *A* va *B* matritsalarning ko'paytmasi kososimmtrik matritsa bo'lishining yetarli va zarur shartlarini ifodalang va ularni isbotlang.

2.81. *A* – ixtiyoriy kvadrat matritsa bo'lsin. $A + A^T$ va AA^T matritsalar simmetrik, $A - A^T$ matritsa kososimmetrik ekanligini isbotlang.

2.82. Ixtiyoriy kvadrat matritsani simmetrik va kososimmetrik matritsalar yig'indisi ko'rinishida ifodalash mumkinligini isbotlang. Bunday yoyilma yagona bo'ladimi?

2.83. Berilgan matritsalarni simmetrik va kososimmetrik matritsalar yig'indisiga yoying:

a) $\begin{pmatrix} 4 & -3 \\ 12 & -8 \end{pmatrix}$; b) $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$; c) $\begin{pmatrix} 0 & 2 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & -2 \end{pmatrix}$.

2.84. Faraz qilaylik, *S* – xosmas matriya bo'lsin va $S^T AS = B$ bo'lsin. isbot qiling simmetriklik va kososimmetrik xossalarning har biri *A* va *B* matritsalar uchun bir vaqtda bajariladi (ya'ni biror xossa A matritsa uchun bajarilsa, *B* matritsa uchun ham bajariladi va aksincha).

2.85. Har qanday haqiqiy ermit matritsasi simmetrik matritsa ekanligini isbotlang.

2.86. *A* va *B* matritsalar ermit matritsalari bo'lsin. Isbot qilingki:

a) *A+B* matritsa ermit matritsasidir;

b) *AB* matritsa ermit matritsasi bo'lishi uchun *A* va *B* matritsalar o'rinalmashinuvchi bo'lishi zarur va yetarlidir.

2.87. *A* – ermit matritsasi va $A+B=iC$ bo'lsin, bunda *B* va *C* matritsalar haqiqiydir. Isbot qilingki, *B* – simmetrik matritsa, *C* – kososimmetrik matritsadir.

2.88. Har qanday kvadrat matritsani ermit va kosoermit matritsalar yig'indisiga yoyish mumkinligini isbotlang. Bunday yoyilma yagonami?

2.89. Haqiqiy unitar matritsa ortogonal matritsa ekanligini isbotlang.

2.90. Agar *A* va *B* matritsalar ortogonal bo'lsa, *AB* ortogonal ekanligini isbotlang.

2.91. Agar *A* va *B* matritsalar unitar bo'lsa, *AB* unitar ekanligini isbotlang.

2.92. Agar *A* – ortogonal matritsa bo'lsa, uning ixtiyoriy satr elementlari kvadratlarining yig'indisi 1 ga teng, ikkita har xil satrlarining mos elementlari ko'paytmalarining yig'indisi 0 ga teng ekanligini isbotlang. Bu xossalar aniqlovchi bo'ladimi?

2.93. Ortogonal matritsaning ustunlari uchun 92 masalaga o'xshash xossalarini ifodalang va uni isbot qiling.

2.94. Unitar matritsa uchun 92 va 93 masalalar xossalariga o'xshash xossalarni ifodalang va ularni isbot qiling.

2.95*. O'rinalmashtirish matritsasi ortogonal ekanligini isbotlang.

2.96. *A* va *B* – o'rinalmashtirishlar matritsasi bo'lsa, *AB* ham o'rinalmashtirish matritsasi ekanligini isbotlang.

2.97. A matritsa dioganal va ortogonaldir. uning dioganal elementlari λ_i haqida nima atish mumkin?

2.98. Berilgan matritslar davriy, nilpotent yoki stoxastik ekanligini tekshiring, ularning davri, nilpotentlik ko'rsatkichini toping:

a) $\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; b) $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$; c) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$;

d) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 4 \\ -1 & -2 & -3 \end{pmatrix}$; e) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 4 \\ -1 & -2 & -3 \end{pmatrix}$; f) $\begin{pmatrix} -1 & 1 & -2 \\ 3 & -3 & 6 \\ 2 & -2 & 4 \end{pmatrix}$;

g) $\begin{pmatrix} -2 & 5 & 3 \\ -2 & 5 & 3 \\ 2 & -5 & -3 \end{pmatrix}$; h) $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$; i) $\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$;

j) $\begin{pmatrix} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$; k) $\frac{1}{7}\begin{pmatrix} 1 & 2 & 2 & 2 \\ 2 & 1 & 2 & 2 \\ 2 & 2 & 1 & 2 \\ 2 & 2 & 2 & 1 \end{pmatrix}$; l) $\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$.

2.98–106 masalalarda ifodalangan kvadrat matritsalarning xossalarini tekshiring.

2.99. Nilpotent matritsa doimo xos matritsadir, davriy matritsa xosmasdir.

2.100*. Agar A – ikkinchi tartibli nilpotent matritsa bo'lsa, $A^2 = 0$ bo'ladi.

2.101. Uchburchakli matritsa nilpotent bo'lishi uchun uning hamma dioganal elementlari nolga teng bo'lishi zarur va yetarlidir.

2.102*. Agar A va B matritsalar nilpotent va o'rinalmashuvchi bo'lsa, $A+B$ va AB matritsalar nilpotentdir.

2.103. Agar A va B matritsalar davriy va o'rinalmashtiruvchi bo'lsa, AB – davriy matritsadir. uning birorta davrini A va B matritsalarning davrlari orqali ifodalang.

2.104*. $A^m + A^{m-1} + \dots + E = 0$ bo'lsin. A – matritsa davriy ekanligini isbotlang.

2.105. A matritsa bir vaqtda unitar va ermit matritsasi bo'lsin. A matritsa davriy ekanligini isbotlang.

2.106. S – xosmas matritsa va $S^{-1}AS = B$ bo'lsin. U holda A va B matritsalar uchun davriylik va nilpotentlik xossalarning har biri bir vaqtda bajariladi (ya'ni agar bu xossalar A matritsa uchun bajarilsa, B matritsa uchun ham bajariladi va aksincha).

2.107. Agar A va B matritsalar nomanfiy bo'lsalar $A+B$, AB – matritsalar ham nomanfiydir.

2.108. I – ustun birlardan iborat bo'lsin va A matritsa nomanfiy bo'lsin. $AI=I$ – shart A matritsaning stoxastikligining yetarli va zaruriy sharti ekanligini isbotlang.

2.109. Agar A va B matritsalar stoxastik bo'lsa, AB ham stoxastik matritsa ekanligini isbotlang.

2.110. A matritsa stoxastik bo'lsin. A^{-1} matritsa mavjudmi? Agar A^{-1} mavjud bo'lsa. stoxastik bo'ladimi?

2.111. Qaysi holda stoxastik matritsa ortogonal bo'ladi?

2.112. Ayniyatlarni isbotlang:

a) $tr(A+B) = trA + trB$;   b) $trAB = trBA$.

2.113*. A – uchburchakli matritsa, m – natural son bo'lsin. A^m matritsaning izi topilsin.

2.114. A – ixtiyoriy matritsa bo'lsin. Hisoblang:

a) $tr(A^T A)$; b) $tr(A^H A)$; c) agar $tr(A^H A) = 0$ bo'lsa, $A = 0$ ekanligini isbotlang.

2.115. Agar A – ikkinchi tartibli nilpotent matritsa bo'lsa, $trA = 0$ bo'lishini isbotlang.

2.116. $AB - BA = E$ tenglikni qanoatlantiruvchi A va B matritsalar mavjud emasligini isbotlang.

2.117. A va B – matritsalar ikkinchi tartibli katakli matritsalar bo'lsin. bu matritsalarni ko'paytirish mumkinligi shartlarini ifodalang. Agar AB ko'paytma mavjud bo'lsa, $(AB)^{\square} = A^{\square}B^{\square}$ ekanligini isbotlang.

2.118. A va B – matritsalar ikkinchi tartibli yuqori katakli uchburchakli matritsalar bo'lsin va AB ko'paytma mavjud bo'lsin. $A^{\square}B^{\square}$ matritsani hisoblash formulasini keltirib chiqaring.

2.119. A – ikkinchi tartibli katakli matritsa bo'lsin, B – katakli matritsa ikkita katakdan iborat bo'lgan ustun bo'lsin.

a) AB ko'paytma mavjud bo'lishi shartlarini ifodalang.

b) agar AV mavjud bo'lsa, $(AB)^{\square} = A^{\square}B^{\square}$ ekanligini isbotlang.

c) $A^{\square}B^{\square}$ ko'paytmani hisoblash formulasini keltirib chiqaring.

2.120. A va B – matritsalar katakli dioganal matritsalar bo'lsin. shunday shartlarni ifodalangki, ular bajarilganda:

a) AB ko'paytma aniqlangan bo'lsin;

b) $(AB)^{\square} = A^{\square}B^{\square}$;

c) AB va BA ko'paytmalar aniqlangan bo'lsin;

d) $AB=BA$.

2.121. Ayniyatlarning to'g'riligini tekshiring:

a) $(A+B)^{\square} = A^{\square}+B^{\square}$;   b) $(AB)^{\square} = A^{\square}B^{\square}$,

bunda A va B matritsalar ixtiyoriy katakli matritsalar.

2.122. Berilgan matritsalarni kataklarga ajratib matritsalar ko'paytmasini hisoblang:

a) $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix};$

b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix};$

c) $\begin{pmatrix} 2 & -3 & 1 & 0 \\ -2 & 0 & 2 & 0 \\ 6 & 3 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix};$

d) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 & -2 \\ -2 & -1 & 0 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix};$

e) $\begin{pmatrix} 1 & 2 & 5 & 5 & 5 \\ 3 & 4 & 6 & 6 & 6 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{pmatrix}.$

2.123. $(N□)^{-1}$ matritsani toping, agar N – quyidagi ko'rinishdagi katakli matritsa bo'lsa: $H = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, bunda A va C matritsalar teskarilanuvchidir.

2.124. E – r-tartibli birlik matritsa, D – ixtiyoriy $r \times s$ o'lchovli matritsa, o, b, x – ustunlar bo'lsin. Tenglamani yeching:

a) $(ED)□x=0$; b) $(ED)□x=b$.

2.125. Matritsalarning kroneker ko'paytmasi hisoblansin:

a) $\begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix} \otimes \begin{pmatrix} 4 \\ -2 \end{pmatrix}$; b) $\begin{pmatrix} 4 \\ -2 \end{pmatrix} \otimes \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}$; c) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} -1 \\ 1 \end{pmatrix}$;

d) $\begin{pmatrix} -1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$; e) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 3 & 5 \\ 5 & 9 \end{pmatrix}$; f) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$.

2.126. $a=(a_1 \ldots a_n)$, $b=(b_1 \ldots b_m)^T$ bo'lsin. $a \otimes b$, $b \otimes a$ larni hisoblang va ba bilan taqqoslang.

2.127. Quyidagi ayniyatlarning to'g'riligini tekshiring:

a) $(\alpha A) \otimes B = \alpha (A \otimes B)$;

b) $(A + B) \otimes C = A \otimes C + B \otimes C$;

c) $A \otimes (B + C) = A \otimes B + A \otimes C$;

d) $A \otimes (B \otimes C) = (A \otimes B) \otimes C$;

e) $AB \otimes CD = (A \otimes C)(B \otimes D)$;

f) $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

III - BOB

Kompleks sonlar nazariyasi

Tayanch iboralar: *kompleks son; mavhum birlik; kompleks sonning haqiqiy va mavhum qismi; kompleks-qo'shma son; kompleks tekislik; haqiqiy va mavhum o'q; kompleks sonning absolyut qiymati va argumenti; kompleks sonning trigonometrik shakli; yig'indining absolyut qiymati haqidagi teorema; Muavr formulasi; kompleks sondan n-dara-jali ildiz chiqarish formulasi; birning n-darajali ildizlari; birning n-darajali boshlang'ich ildizlari; doiraviy ko'phad; Eyler formulasi; kompleks sonning ko'rsatkichli shakli.*

1-§. Algebraik shakldagi kompleks sonlar

Kompleks son deb haqiqiy sonlarning tartiblangan juftligiga aytiladi. (a, o) kompleks sonni haqiqiy sondan farqlamaydilar. Barcha kompleks sonlar to'plamini S orqali belgilanadi. (a,b) va (c,d) juftliklar ularning mos koordintalari teng bo'lgandagina *teng* deyiladi, ya'ni

$$(a,\ b)=(c,\ d) <=> \begin{cases} a=c \\ b=d. \end{cases}$$

Kompleks sonlarni qo'shish va ko'paytirish amallari quyidagi tengliklar yordamida kiritiladi

$$(a,\ v)+(c,\ d) = (a+c,\ b+d),$$
$$(a,\ b)\cdot(c,\ d) = (ac-bd,\ ad+ bc)$$

(0,1) kompleks soni i harfi orqali belgilash va uni *mavhum bir* deb atash qabul qilingan. $i^2 + 1 = 0$ bo'lishini ko'rsatish qiyin emas, ya'ni i soni $x^2 + 1 = 0$ tenglamaning ildizi bo'ladi.

Har qanday z kompleks sonni $a + bi$ algebraik shaklda yozish mumkin. Agar $z = a + bi$ bo'lsa, a son z kompleks sonning *haqiqiy qismi* dyiladi va $Re\ z$ orqali belgilanadi, b son esa z kompleks sonning *mavhum qismi* deyiladi va $Im\ z$ orqali belgilanadi. $z = a - bi$ kompleks son, $z = a + bi$ kompleks sonning *kompleks qo'shma*si deyiladi.

Agar $a = c$, $b = d$ bo'lsa $a + bi$ va $c + di$ kompleks sonlar teng deyiladi.

Algebraik shakldagi kompleks sonlar ustida arifmetik amallar quyidagi tengliklar yordamida aniqlanadi:

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$
$$(a + bi) - (c + di) = (a - c) + (b - d)i,$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i,$$
$$\frac{a+bi}{c+di}=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i \ \ (c+di \neq 0, \text{ ya'ni } s^2+d^2 \neq 0).$$

Boshqacha aytganda, agar $i^2 = -1$ ekanligini hisobga olinsa, kompleks sonlar ustida barcha arifmetik amallar haqiqiy sonlar ustidagi xuddi shunday amalar kabi bajaradi.

Agar kompleks sonlarning yig'indisi, ayirmasi, ko'paytmasi va bo'linmasidagi barcha sonlarni ularning kompleks-qo'shmasiga almashtirilsa, natija ham o'zining qo'shmasiga almashadi:

$$\bar{z}_1 + \bar{z}_2 = \overline{z_1 + z_2}, \qquad \bar{z}_1 - \bar{z}_2 = \overline{z_1 - z_2}, \qquad \bar{z}_1 \cdot \bar{z}_2 = \overline{z_1 z_2}, \qquad \frac{\bar{z}_1}{\bar{z}_2} = \overline{\left(\frac{z_1}{z_2}\right)}$$

Kompleks sonni *darajaga ko'tarish amali* quyidagicha aniqlanadi:

$$z^n = \begin{cases} \underbrace{z \cdot z \cdot \ldots z}_{n\ марта}, & агар\ n \geq 2 \\ z, & агар\ n = 1,\ \ n \in \boldsymbol{N} \end{cases}.$$

Agar $z \neq 0$ bo'lsa: $z^0 = 1, \quad z^{-n} = \dfrac{1}{z^n}$ deb qabul qilinadi.

Kompleks sonning butun ko'rsatkichli darajasi quyidagi xossalarga ega:

$$z^p \cdot z^q = z^{p+q}, \quad (z^p)^q = z^{pq}, \quad (z_1 \cdot z_2)^p = z_1^p \cdot z_2^p,$$

$$\frac{z^p}{z^q} = z^{p-q}, \quad \left(\frac{z_1}{z_2}\right)^p = \frac{z_1^p}{z_2^p}, \quad бунда\ \ p,\ q \in \boldsymbol{Z}.$$

Kompleks son z ning n-darajali ildizi deb shunday ω, $\omega = \sqrt[n]{z}$, kompleks songa aytiladiki, $\omega^n = z \quad (n \geq 2,\ n \in \boldsymbol{N})$.

1-m i s o l. Quyidagi tenglamadan x va y haqiqiy sonlarni toping:

$$(5x - 3y) + (x - 2y)I = 6 + (8 - x + y)\,i.$$

Yechish. Kompleks sonlarning tenglik shartidan foydalanib,

$$\begin{cases} 5x - 3y = 6 \\ x - 2y = 8 - x + y \end{cases}$$

sistemani hosil qilamiz. Bu sistemadan x va y noma'lumlarni topamiz: $x = -\dfrac{2}{3},\ \ y = -\dfrac{28}{9}$. ■

2-m i s o l. i ning darajalarini toping.

Yechish. Ta'rifga ko'ra $i^0 = 1$, $i^1 = i$ va $i^2 = -1$. Shuning uchun $i^3 = i^2 i = -i$, $i^4 = i^3\ i = 1$, $i^5 = i^4 \cdot i = i$.

Umuman olganda: $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$, $n \in N$. ■

3-m i s o l. Darajaga ko'taring: $(1+i)^{20}$, $(1-i)^{21}$.

Yechish. Bu masalani Nyuton binomi formulasidan foydalanib hal qilsa bo'ladi, lekin uni quyidagicha yozish qulayroq:

$(1+i)^2 = 2i$, $(1-i)^2 = -2i$. U holda

$$(1+i)^{20} = \left[(1+i)^2\right]^{10} = (2i)^{10} = -2^{10},$$

$$(1-i)^{21} = \left[(1-i)^2\right]^{10}(1-i) = (-2i)^{10}(1-i) = -2^{10}(1-i). ■$$

Kompleks koeffisiyentli istagan kvadrat tenglamani yechish uchun, avvalo kompleks sonning kvadrat ildizini topa olish kerak. Ta’rifga ko’ra $x+yi$ son $a+bi$ sonning kvadrat ildizi bo’lishi:

$$(x + yi)^2 = a + bi \qquad (*)$$

tenglikning bajarilishiga teng kuchli.

(*) tenglik quyidagi formulalar yordamida topiladigan ikkita har xil yechimlarga ega bo’ladi:

$$x = \pm\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}; \quad y = \pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}},$$

bu yerda radikal arifmetik ildizni bildiradi, agar $b> 0$ bo’lsa, x va y larning ishoralari bir xil qilib, $b < 0$ bo’lganda esa har xil qilib tanlanadi.

4-m i s o l. $\sqrt{24-10i}$ ildizning qiymatlari $5 - i$ va $-5 + i$ bo’ladi.■

Kvadrat ildizni to’g’ridan to’g’ri topish ham mumkin.

5-m i s o l. Ildizdan chiqaring: $\sqrt{5+12i}$

Yechish. $\sqrt{5+12i} = x + yi$ bo’lsin. Ildizning ta’rifiga ko’ra

$$(x + yi)^2 = 5 + 12i \text{ yoki } (x^2 - y^2) + 2xyi = 5 + 12i,$$

bundan

$$\begin{cases} x^2 - y^2 = 5 \\ 2xy = 12 \end{cases}$$

sistemani hosil qilamiz.

Bu sistemadagi ikala tenglikni kvadratga ko’tarib va ularni qo’shib, $(x^2 + y^2)^2 = 25 + 144$ va $x^2 + y^2 = 13$ larni hosil qilamiz.

U holda $\begin{cases} x^2 + y^2 = 13 \\ x^2 - y^2 = 5 \end{cases}$ sistemadan x va y noma’lumlarni topamiz:

$$x = \pm 3,\ y = \pm 2.$$

Oldingi sistemaning ikkinchi tenglamasidan x va y larning bir xil ishorali bo’lishi kelib chiqadi. Shuning uchun $x_1 = 3$, $y_1 = 2$; $x_2 =-3$, $y_2 =-2$. Shunday qilib, $\sqrt{5+12i}$ ildiz ikkita $3 + 2i$ va $-3 - 2i$ qiymatlarga ega.■

Endi kompleks sonning kvadrat ildizini topishni bilgan holda aynan maktab matematika kursidekdagi kompleks koeffisiyentli

$$ax^2 + bx + c = 0$$

tenglamaning ildizlari

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

formula yordamida topilishini ko’rsatish mumkin.

6-m i s o l. $(3 - i)x^2 - 2(2 - 3i)x - 4i = 0$ kvadrat tenglamaning ildizlari $x_1 = 0{,}4 - 0{,}8i$ va $x_2 = 0{,}2 - 1{,}4i$ sonlardan iborat. ■

7-m i s o l. Sistemani yeching:

$$\begin{cases} (1+i)z_1 + (1-i)z_2 = 1+i \\ (1-i)z_1 + (1+i)z_2 = 1+3i \end{cases}.$$

Yechish. Sistemadagi birinchi tenglamaning ikkala tomonini $(1-i)$ ga, ikkinchi tenglamaning ikkala tomonini esa $(1 + i)$ ga ko'paytirib

$$\begin{cases} 2z_1 - 2iz_2 = 2 \\ 2z_1 + 2iz_2 = -2 + 4i \end{cases}$$

ni hosil qilamiz.

Bu tenglamalarni qo'shib, $4z_1 = 4i$ ga kelamiz. Bundan $z_1 = i$.

Birinchi tenglamadan ikkalasini ayirib $-4z_2 i = 4 - 4i$ ni hosil qilamiz. Bundan $z_2 = \dfrac{-1+i}{i} = 1+i$. ■

8-m i s o l. a ning qanday haqiqiy qiymatlarida

$$4i^4 - 3ai^3 + (2-a)i - 5 + a$$

son haqiqiy bo'ladi?

Yechish. $i^4 = 1$, $i^3 = -i$ bo'lganligi sababli

$$4i^4 - 3a^3 + (2-a)i - 5 + a = (2a+2)i + a - 1.$$

Shuning uchun $2a+2=0$ bo'lganda bu son haqiqiy bo'ladi, ya'ni $a = -1$. ■

9-m i s o l. $z\bar{z} + 2\bar{z} = 3 + 2i$ tenglamani yeching.

Yechish. $z = x + yi$ bo'lsin. U holda $x^2 + y^2 + 2x - 2yi = 3 + 2i$. Haqiqiy va mavhum qismlarini tenglashtirib

$$\begin{cases} x^2 + y^2 + 2x = 3 \\ -2y = 2 \end{cases}$$

sistemani hosil qilamiz. Bundan $y = -1,\quad x = -1 \pm \sqrt{3}$. Natijada,

$$z_1 = (-1+\sqrt{3}) - i,\quad z_2 = (-1-\sqrt{3}) - i. \ ■$$

M A S H Q L A R

3.1. Berilgan z_1 va z_2 kompleks sonlarning yig'indisi va ko'paytmasini toping:

a) $z_1 = 5+4i$, $z_2 = -2+3i$;   b) $z_1 = -8-7i$, $z_2 = -3i$;

c) $z_1 = 5 + \sqrt{3}\,i$, $z_2 = 5 - \sqrt{3}\,i$.

3.2. $z_2 - z_1$ ayirmani va $\dfrac{z_2}{z_1}$ bo'linmani toping:

a) $z_1 = 1+2i$, $z_2 = 5$;   b) $z_1 = -1 + \sqrt{3}i$, $z_2 = -\sqrt{2} + \sqrt{6}i$;

c) $z_1 = a - \sqrt{b}i$, $z_2 = a + \sqrt{b}i$.

3.3. Hisoblang:

$a)\quad (4+i)(5+3i)-(3+i)(3-i);\quad b)\ \dfrac{(5+i)(7-6i)}{3+i};\quad c)\ \dfrac{(5+i)(3+5i)}{2i};$

$d)\ \dfrac{(1+3i)(8-i)}{(2+i)^2};\quad e)\ \dfrac{(2+i)(4+i)}{1+i};\ f)\ \dfrac{(3\text{-}i)(1\text{-}4i)}{z\text{-}i};\ g)\ (2+i)^3+(2-i)^3;$

$h)\ (3+i)^3-(3-i)^3;\quad i)\ \dfrac{(1+i)^5}{(1-i)^3};\quad j)\ \left(-\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i\right)^3.$

3.4. Kompleks sonning haqiqiy qismini toping:

a) $z=\dfrac{(1+2i)^3}{i}+i^{19};$ b) $z=\dfrac{5+2i}{2-5i}-\dfrac{3-4i}{4+3i}+\dfrac{1}{i}.$

3.5. Kompleks sonning mavhum qismini toping:

a) $z=(2-i)^3(2+11i)$ b) $z=\dfrac{2-3i}{1+4i}+i^6.$

3.6. Tenglikni isbotlang:

a) $(1+i)^{8n}=2^{4n}\quad (n\in \boldsymbol{Z});$ $b)\quad (1+i)^{4n}=(-1)^n 2^{2n}\quad (n\in \boldsymbol{Z}).$

3.7. Tenglamalar sistemasini yeching:

a) $\begin{cases} iz_1+(1+i)z_2=2+2i \\ 2iz_1+(3+2i)z_2=5+3i \end{cases}$; b) $\begin{cases} (1-i)z_1-3z_2=-i \\ 2z_2-(3+3i)z_2=3-i \end{cases}$;

c) $\begin{cases} 2z_1-(2+i)z_2=-i \\ (4-2i)z_1-5z_2=-1-2i \end{cases}$

3.8. Hisoblang:

a) $i^4+i^{14}+i^{24}+i^{34}+i^{44}$; b) $i+i^2+i^3+...+i^n,\ n>4$;

c) $i\cdot i^2\cdot i^3\cdot i^4...i^{50}$.

3.9. $\omega=-\dfrac{1}{2}+\dfrac{i\sqrt{3}}{2}$ bo’lganda quyidagilarni hisoblang:

a) $(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$; b) $(a+b)(a+b\omega)(a+b\omega^2)$;

c) $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3$.

3.10. Tenglamani yeching:

a) $(i-z)(1+2i)+(1-iz)(3-4i)=1+7i$; b) $z^2+\bar{z}=0$;

c) $(1-i)\bar{z}-3iz=2-i$; d) $z\bar{z}+3(z-\bar{z})=4+3i$;

e) $z\bar{z}+3(z+\bar{z})=7$; f) $z\bar{z}+3(z+\bar{z})=3i$.

3.11. Hisoblang:

a) $\sqrt{2i}$; b) $\sqrt{-8i}$; c) $\sqrt{3-4i}$; d) $\sqrt{-15+8i}$; e) $\sqrt{-11+60i}$;

f) $\sqrt{-8-6i}$; q) $\sqrt{2-3i}$; h) $\sqrt{1-i\sqrt{3}}$; i) $\sqrt[4]{2-i\sqrt{12}}$; j) $\sqrt[4]{-1}$.

3.12. Tenglamani yeching:

a) $x^2-(2+i)x+(-1+7i)=0$; b) $x^2-(3-2i)x+(5-5i)=0$;

c) $(2+i)x^2-(5-i)x+(2-2i)=0$; d) $x^4-6x^2+25=0$;

e) $x^4+34x^2+289=0$; f) $x^2-(4+3i)x+1+5i=0$;

g) $x^2+5x+9=0$; h) $x^2+x+1+i=0$.

2-§. Kompleks sonning geometrik tasviri va trigonometrik shakli

Tekislikdagi nuqtalar bilan kompleks sonlar o'rtasida o'zaro bir qiymatli moslik o'rnatamiz. Buning uchun tekislikda biror to'g'ri burchakga Dekart koordinatalar sistemasi kiritamiz. Natijada, tekislikdagi har bir nuqtaga haqiqiy sonlarning (a,b) tartiblangan juftligi, ya'ni $(a,b)\in \boldsymbol{C}$ element mos qo'yiladi va aksincha har bir (a,b) kompleks songa tekislikdagi koordinatalari *a* va *b* ga teng nuqta mos keladi. Shu munosabat bilan tekislikning o'zini kompleks deb ataladi. Bunda haqiqiy sonlarga (a,o) ko'rinishdagi, ya'ni abssissalar o'qida yotuvchi nuqtalar mos keladi. Ordinatalar o'qidagi nuqtalarga esa (o,b) mavhum sonlar mos keladi. Shuning uchun kompleks tekislikning abssissalar o'qini *haqiqiy o'q*, ordinatalar o'qini esa *mavhum o'q* deb ataladi.

Odatda, *z* = *(a, b)* kompleks son tekislikdagi koordinatalari *a* va *b* sonlardan iborat nuqta orqali yoki abssissa va ordinatalar o'qidagi proyeksiyalari mos ravishda *a* va *b* ga teng bo'lgan vektor orqali tasvirlanadi. Ko'pincha *z* nuqta yoki *z* vektor ham deb aytiladi. Kompleks son *z* = *a* + *bi* ning absolyut qiymati deb $z=|z|=\sqrt{a^2+b^2}$ haqiqiy songa aytiladi. Absissalar o'qining musbat yo'nalishi va *z* vektorning yo'nalishi orasidagi φ burchak *z* kompleks sonning argumenti deyiladi va *arp z* orqali belgilanadi. 0 sonning argumenti aniqlanmagan.

Ixtiyoriy noldan farqli *z* = (*a*,*b*) kompleks sonni va tekislikda unga mos keluvchi vektorni qaraymiz. *z* nuqtaning tekislikdagi holatini uning qutb koordinatalari: koordinatalar boshidan *z* nuqtagacha bo'lgan masofa *r*, ya'ni $|z|$ va absissa o'qining musbat yo'nalishi bilan z vektor yo'nalishi orasidagi φ = *arg z* burchaklar to'liq aniqlaydi.

Agar *z* = *a* + *bi* bo'lsa, u holda $cos\varphi=\frac{a}{r}$, $sin\varphi=\frac{b}{r}$. Bundan har bir *z* kompleks son uchun $z=r(cos\varphi+i\,sin\varphi)$ kelib chiqadi. Kompleks sonning bunday ko'rinishi uning *trigonometrik shakli* deyiladi.

Trigonometrik shakldagi kompleks sonlarni ko'paytirish va bo'lish amallari quyidagicha amalga oshiriladi:

$$z_1=r_1(cos\varphi_1+i\,sin\varphi_1),\quad z_2=r_2(cos\varphi_2+i\,sin\varphi_2)$$

bo'lsin. U holda

$$z_1z_2=r_1r_2(cos(\varphi_1+\varphi_2)+i\,sin(\varphi_1+\varphi_2)),$$

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}(cos(\varphi_1-\varphi_2)+i\,sin(\varphi_1-\varphi_2)).\quad z_2\neq 0.$$

1-m i s o l. $|z|$ – son tekislikda *z* ni tasvirlovchi nuqtadan koordinatalar boshigacha bo'lgan masofadan iborat. Boshqacha aytganda $|z|$ son *z* kompleks sonni ifodalovchi vektorning uzunligidir. ■

2-m i s o l. Kompleks tekislikda $|z|=3$ shartni qanoatlantiruvchi z nuqtalar to’plami markazi koordinatalar boshida va radiusi 3 ga teng bo’lgan aylanadan iborat. ■

3-m i s o l. Kompleks tekislikda $|z|\le 3$ shartni qanoatlantiruvchi z nuqtalar to’plami markazi koordinatalar boshida va radiusi 3 ga teng bo’lgan yopiq dioradan iborat. ■

4-m i s o l. Kompleks tekislikda $|z-1+2i|=3$shartni qanoatlantiruvchi z nuqtalar to’plami markazi (1,–2) nuqtada va radiusi 3 ga teng bo’lgan aylanadan iborat. ■

5-m i s o l. Kompleks tekislikda $|z-1+2i|\le 3$ shartni qanoatlantiruvchi z nuqtalar to’plami markazi (1,–2) nuqtada va radiusi 3 ga teng bo’lgan aylanadan iborat. ■

6-m i s o l. $z_1, z_2 \in \boldsymbol{C}$ bo’lsin. Tekislikda z_1 va z_2 sonlarni ifodalovchi nuqtalar orasidagi masofa $|z_1-z_2|$ ga teng (*ayirmaning absolyut qiymati haqidagi teorema*). ■

7-m i s o l. Uchlari *o*, z_1, z_2 nuqtalarda joylashgan uchburchakning tomonlarini taqqoslab, *yig’indining absolyut qiymati* haqidagi teoremaga ega bo’lamiz:

$$|z_1|-|z_2|\le|z_1+z_2|\le|z_1|+|z_2|.\ ■$$

Bu teoremadan quyidagi tasdiq kelib chiqadi.

8-m i s o l. Barcha $z_1, z_2, ..., z_n \in \boldsymbol{C}$ sonlar uchun

$$|z_1|-|z_2|-...-|z_n|\le|z_1+z_2+....+z_n|\le|z_1|+|z_2|+...+|z_n|.\ ■$$

9-m i s o l. Tekislikda $|z-i|+|z+i|<4$ tengsizlikni qanoatlantiruvchi kompleks sonlarni tasvirlaydigan nuqtalar to’plamini aniqlang.

Yechish. $z = x + iy$, $x, y \in R$ bo’lsin. U holda $|x+yi-i|+|x+yi+i|<4$. Bundan $\sqrt{x^2+(y-1)^2}+\sqrt{x^2+(y+1)^2}<4$.

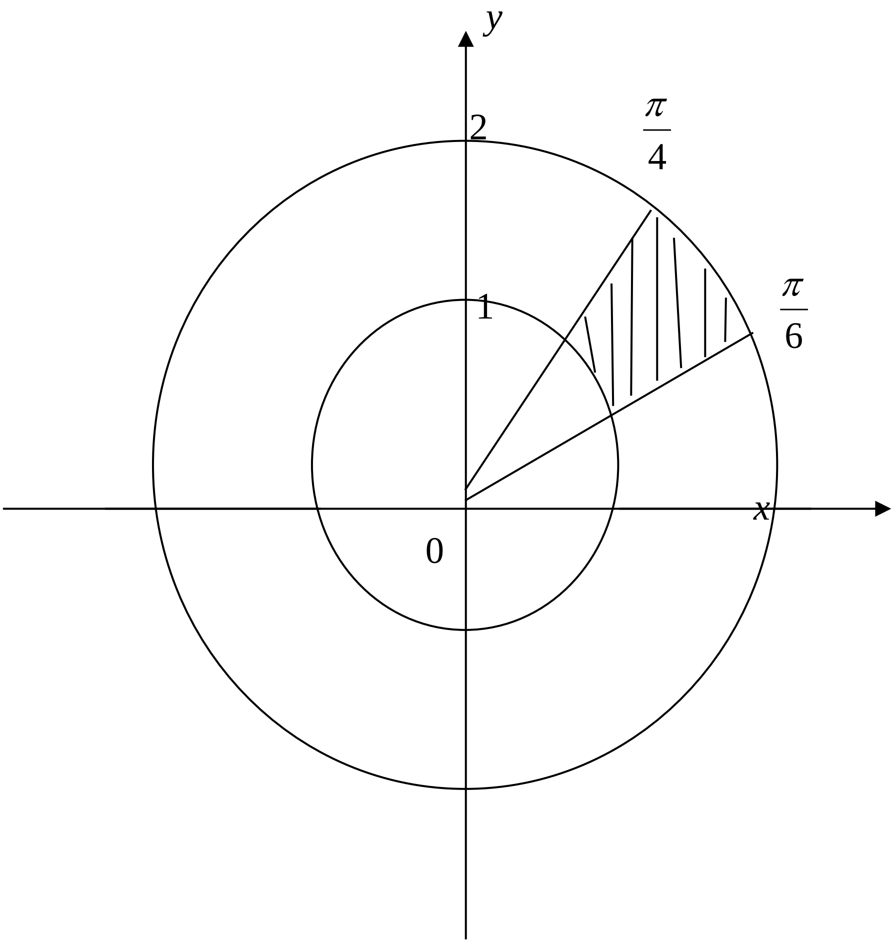
Bu tengsizlikni soddalashtirib, unga teng kuchili bo’lgan

$$4x^2+3y^2<12 \quad \text{yoki} \quad \frac{x^2}{3}+\frac{y^2}{4}<1$$

tengsizlikni hosil qilamiz. Shunday qilib, izlanayotgan to’plam tekislikning $\frac{x^2}{3}+\frac{y^2}{4}<1$ ellips bilan chegaralangan qismidan iborat. ■

10-m i s o l. Kompleks sonlar $1<|z|<2$, $\frac{\pi}{6}<arg\ z<\frac{\pi}{4}$ shartlarni qanoatlantiradi. Bunday kospleks sonlarni ifodalovchi nuqtalar qayerda joylashgan?

Yechish. $1<|z|<2$ bo’lganligi uchun, bu nuqtalar markazi *O* nuqtada va radiuslari 1 va 2 ga teng bo’lgan aylanalar bilan chegaralangan halqada yotadi.

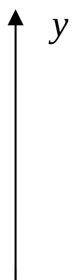


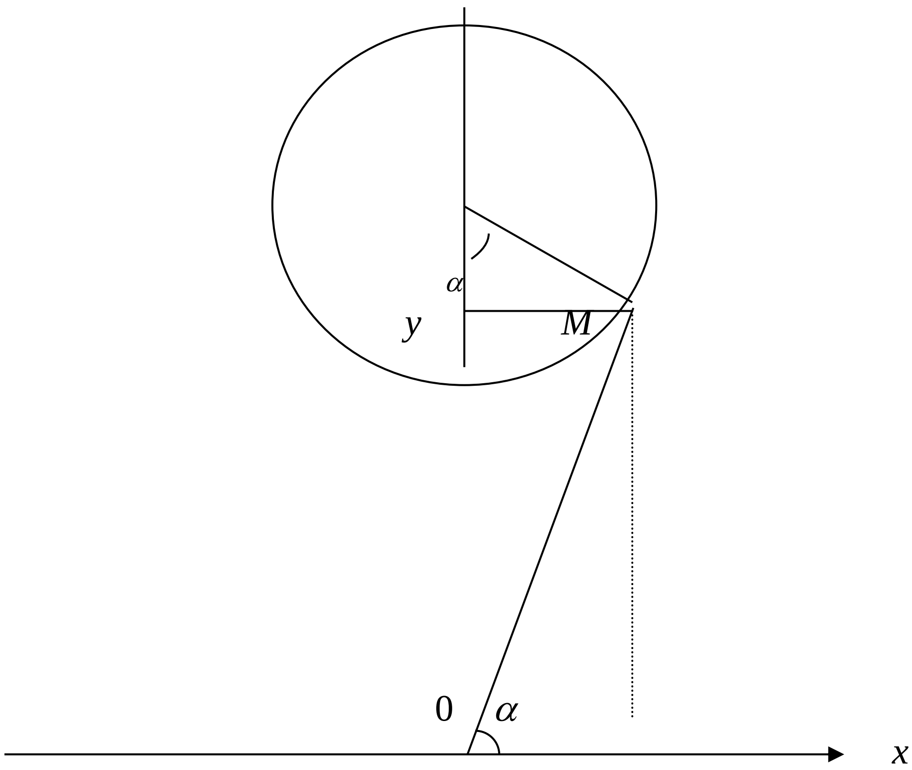
1-rasm

$\frac{\pi}{6} < arg\ z < \frac{\pi}{4}$ bo'lganligi uchun masala shartini qanoatlantiruvchi nuqtalar 1-rasmda ko'rsatilgan soha ichida yotadi. ■

11-m i s o l. $|z - 25i| \leq 15$ shartni qanoatlantiruvchi kompleks sonlar ichidan argumenti eng kichik bo'lgan sonni toping.

Yechish. $|z - 25i| \leq 15$ shartni qanoatlantiruvchi sonlarga mos nuqtalar markazi (0,25) nuqtada va radiusi 15 ga teng bo'lgan yopiq doirani hosil qiladi.





2-rasm

Rasmdan ko'rinib turibdiki, eng kichik argumentli songa M nuqta mos keladi, bunda *OM* to'g'ri chiziq aylanaga urinadi. *OMC* to'g'ri burchakli uchburchakdan

$$OM = \sqrt{OC^2 - MC^2} = \sqrt{25^2 - 15^2} = 20,$$

$$cos\alpha = \frac{MC}{OC} = \frac{15}{25} = \frac{3}{5}; \quad sin\alpha = \frac{OM}{OC} = \frac{20}{25} = \frac{4}{5}.$$

Shuning uchun $x = OM \cdot cos\alpha = 12$, $y = OM\, sin\alpha = 16$, ya'ni izlanayotgan son $z = 12 + 16i$. ■

12-m i s o l. Quyidagi kompleks sonlarni trigonometrik shaklga keltiring: a) $z = 1 - i$; b) $z = -3 - 4i$.

Yechish. a) $a=1$, $b=-1$, $tg\varphi = \frac{b}{a} = \frac{-1}{1} = -1$, $r = \sqrt{a^2 + b^2} = \sqrt{2}$.

Bu z soni ifodalovchi nuqta to'rtinchi chorakka tegishli. Shuning uchun φ argumentning qiymati sifatida $\frac{7}{4}\pi$ yoki $\left(-\frac{\pi}{4}\right)$ ni olish mumkin.

Natijada,

$$1 - i = \sqrt{2}\left[cos(-\frac{\pi}{4}) + i\, sin(-\frac{\pi}{4})\right].$$

Shuni ta'kidlaymizki, $1-i=\sqrt{2}\left[cos\frac{\pi}{4}-i\,sin\frac{\pi}{4}\right]$ tenglik $1-i$ sonning trigonometrik shakli bo'lmaydi.

b) $a=-3$, $b=-4$, shuning uchun $r=5$, $tg\varphi=\frac{4}{3}$. Nuqta uchinchi chorakda yotganligi uchun φ argumentning qiymati $tg\varphi=\frac{4}{3}$ va $\pi<\varphi<\frac{3}{2}\pi$ shartlardan aniqlanadi, ya'ni $\varphi=\pi+\alpha$, bunda α - burchak $tg\alpha=\frac{3}{4}$ shartni qanoatlantiruvchi o'tkir burchak. Shuning uchun $\alpha=arctg\frac{4}{3}$, ya'ni $-\varphi=\pi+arctg\frac{4}{3}$. U holda $-3-4i=5\left[cos(\pi+arctg\frac{4}{3})+isin(\pi+arctg\frac{4}{3})\right]$. ■

13-m i s o l. $z=1+i\ tg\alpha$ kompleks soni trigonometrik shaklga keltiring, bu yerda α quyidagi shartni qanoatlantiruvchi berilgan burchak:

a) $0<\alpha<\frac{\pi}{2}$, b) $\frac{\pi}{2}<\alpha<\pi$.

Yechish. Berilgan z sonning ko'rinishini o'zgartiramiz:

a) $z=1+i\frac{sin\alpha}{cos\alpha}=\frac{1}{cos\alpha}(cos\alpha+isin\alpha)$.

$0<\alpha<\frac{\pi}{2}$ bo'lganda $\frac{1}{cos\alpha}>0$ bo'lganligi va qavs ichida bita argumentning kosinusi va sinusi turganligi uchun oxirgi ifoda z kompleks sonning trigonometrik shaklidan iborat;

b) $\frac{\pi}{2}<\alpha<\pi$ bo'lganda $\frac{1}{cos\alpha}<0$ va yuqorida olingan ifoda z sonning trigonometrik shakli bo'lmaydi.

z sonning ko'rinishini boshqacharoq o'zgartiramiz:

$$z=-\frac{1}{cos\alpha}\left[(-cos\alpha)+i(-sin\alpha)\right]=-\frac{1}{cos\alpha}\left[cos(\pi+\alpha)+i\,sin(\pi+\alpha)\right]$$

Bu ifoda $\frac{\pi}{2}<\alpha<\pi$ bo'lgan holda z sonning trigonometrik shakli bo'ladi. ■

14-m i s o l. $w=z^2-z$ sonning argumentini toping, bunda

$$z=cos\varphi+i\,sin\varphi,\qquad 0\le\varphi\le 2\pi.$$

Yechish.

$w=(cos\varphi+i\,sin\varphi)^2-(cos\varphi+i\,sin\varphi)=(cos^2\varphi-sin^2\varphi+2i\,sin\varphi cos\varphi)-$

$-(cos\varphi+i\,sin\varphi)=(cos2\varphi-cos\varphi)+i(sin2\varphi-sin\varphi).$

Qavs ichidagi ifodalarni almashtirib, quyidagini hosil qilamiz:

$$w=-2sin\frac{3\varphi}{2}sin\frac{\varphi}{2}+i2sin\frac{\varphi}{2}cos\frac{3\varphi}{2}=2sin\frac{\varphi}{2}(-sin\frac{3\varphi}{2}+i\,cos\frac{3\varphi}{2})=$$
$$=2sin\frac{\varphi}{2}\left[cos(\frac{\pi}{2}+\frac{3\varphi}{2})+i\,sin(\frac{\pi}{2}+\frac{3\varphi}{2})\right].$$

$0<\varphi<2\pi$ bo'lganda $sin\frac{\varphi}{2}>0$ va hosil qilingan ifoda w sonning trigonometrik shaklidan iborat. Shuning uchun $arg\,w=\frac{\pi}{2}+\frac{3\varphi}{2}$.

$\varphi=0$ bo'lganda $sin\frac{\varphi}{2}=0$, bundan $w = 0$. Bu holda w sonning argumenti aniqlanmagan. ■

3.13. Quyidagi kompleks sonlarni ifodalovchi nuqtalarni yasang:
$1; -1; i; -i; -1+i;\ 2-3i;\ -6+3i;\ cos\,30°-i\,sin30°;$
$cos\,150°+i\,sin150°$.

3.14. Kompleks tekislikda berilgan z_1, z_2, z_3 nuqtalar parallelogramning ketma-ket uchlaridan iborat. Bu parallelogramning to'rtinchi uchini toping.

3.15. Kompleks tekislikda $z_1 = 6 + 8i$, $z_2 = 4 - 3i$ nuqtalar berilgan. z_1 va z_2 vektorlar hosil qilgan burchak bissektrisasining nuqtalariga mos keluvchi kompleks sonlarni toping.

3.16. Tenglamani yeching:
a) $|z|-iz=1-2i$; b) $z^2+3|z|=0$; c) $z^2+|z|^2=0$.

3.17. Tenglamalar sistemasini yeching: $|z+1-i|=|3+2i-z|=|z+i|$.

3.18. Tenglamalar sistemasini yeching: $\begin{cases}|z+1|=|z+2|\\|3z+9|=|5z+10i|.\end{cases}$

3.19. Quyidagi nuqtalarga mos kompleks sonlarni toping:
a) markazi koordinatalar boshida, tomonlari koordinata o'qlariga parallel va tomonlarining uzunligi 1 ga teng bo'lgan kvadratning uchlariga;
b) markazi koordinatalar boshida, bir tomoni ordinata o'qiga parallel, bita uchi manfiy haqiqiy yarim o'qda joylashgan va tashqi chizilgan aylana radiusi 1 ga teng bo'lgan muntazam uchburchakning uchlariga;
c) markazi $2+i\sqrt{3}$ nuqtaga joylashgan, tomonlaridan biri abssissa o'qiga parallel va tashqi chizilgan aylana radiusi 2 ga teng bo'lgan muntazam oltiburchakning uchlariga.

3.20. Tekislikda quyidagi shartlarni qanoatlanturvchi z kompleks sonlarga mos keladigan nuqtalar to'plamini tasvirlang:
a) $|z|=1$; b) $arg\,z=\frac{\pi}{3}$; c) $|z|\le 2$; d) $|z-1-i|<1$; e) $|z+3+4i|\le 5$;

f) $3<|z|<5$; g) $1\le|z-2i|<2$; h) $|argz|<\frac{\pi}{6}$; i) $|\mathrm{Re}\,z|\le 1$;

j) $-1<Re\,iz<0$; k) $|\mathrm{Im}\,z|=1$; l) $|Re\,z+Im\,z|<1$; m) $|z-1|+|z+1|=3$; n) $|z+2|-|z-2|=3$; o) $|z-2|=Re\,z+2$; p) $|z+1|<|1-z|$.

3.21. $|z+1-i|\le 1$ shartni qanoatlantiruvchi z kompleks sonlar ichidan eng kichik musbat argumentga ega bo'lgan sonni toping.

3.22. $|z-5i|\le 3$ shartni qanoatlantiruvchi z kompleks sonlar ichidan eng kichik musbat argumentga ega bo'lgan sonni toping.

3.23. *Oxy* tekislikdagi qanday M(x,y) nuqtalar uchun quyidagi tengliklar o'rinli:

a) $\left|\sqrt{2x+y}+i\sqrt{x+2y}\right|=\sqrt{3}$. b) $\left|\sqrt{x^2+4}+i\sqrt{y-4}\right|=\sqrt{10}$?

3.24. Kompleks son moduli va argumentini unga qo'shma bo'lgan son moduli va argumenti orqali ifodalang.

3.25. *A* va *B* nuqtalar *Oxy* tekislikda mos ravishda $a = 6 + 8i$ va $b = 4 - 3i$ sonlarni ifodalaydi. Hech bo'lmaganda bita shunday c soni topingki, unga mos keluvchi *C* nuqta *AOB* burchakning bissektrisasida yotsin.

3.26. Qanday shartlar bajarilganda:

a) $|z_1+z_2|=|z_1|+|z_2|$; b) $|z_1+z_2|=|z_1|-|z_2|$?.

3.27*. (-1) dan farqli va moduli 1 ga teng bo'lgan har qanday z kompleks sonni $z=\frac{1+ti}{1-ti}$, bunda $t\in\boldsymbol{R}$, shaklda tasvirlash mumkinligi ni isbotlang.

3.28. Kompleks sonlarni trigonometrik shaklga keltiring:

a) 7; b) i; c) -3; d) $-5i$; e) $1+i\sqrt{3}$; f) $-1+i\sqrt{3}$ g) $1-i\sqrt{3}$; h) $\sqrt{3}+i$;

i) $-\sqrt{3}+i$; j) $-\sqrt{3}-i$; k) $\sqrt{3}-i$; l) $1+i\frac{\sqrt{3}}{3}$; m) $2+\sqrt{3}+i$;

n) $1-(2+\sqrt{3}\,i)$; o) $cos\alpha - i\,sin\alpha$; p) $sin\alpha + i\,cos\alpha$; q) $\frac{1+i\,tg\alpha}{1-i\,tg\alpha}$;

r) $1+cos\alpha + isin\alpha$; s) $-sin\alpha - i(1+cos\alpha)$.

3.29. Kompleks sonlarni algebraik va trigonometrik shaklga keltiring:

a) $\frac{i(cos\frac{5}{3}\pi+isin\frac{5}{3}\pi)}{cos\frac{\pi}{6}+isin\frac{\pi}{6}}$; b) $\frac{1}{cos\frac{4}{3}\pi-isin\frac{4}{3}\pi}$; c) $\frac{i}{(1+i)^2}$;

d) $\frac{-cos\frac{5}{12}\pi+isin\frac{5}{12}\pi}{cos\frac{13\pi}{12}-isin\frac{13\pi}{12}}$; e) $\frac{(cos\frac{\pi}{3}-isin\frac{\pi}{3})(\frac{1}{2}+i\frac{\sqrt{3}}{3})}{i}$.

3.30. Kompleks sonlarni trigonometrik shaklga keltiring:

a) $\dfrac{5(cos100^0 + isin100^0)i}{3(cos40^0 - isin40^0)}$; b) $\dfrac{sin\frac{2}{5}\pi + i(1 - cos\frac{2}{5}\pi)}{i-1}$.

3.31. Ayniyatni isbotlang: $|x+y|^2 + |x-y|^2 = 2(|x|^2 + |y|^2)$. Bu ayniyat qanday geometrik ma'noga ega?

3-§. Darajaga ko'tarish va ildiz chiqarish

$z = r(cos\varphi + isin\varphi) \neq 0$ bo'lsin. U holda har qanday n butun son uchun

$$z^n = r^n(cos\, n\varphi + i\, sin\, n\varphi)$$

Muavr formulasi o'rinli.

Muavr formulasi kompleks sondan italgan darajali ildiz chiqarish masalasini hal qilishga imkon beradi. w kompleks son z kompleks sonning *n-darajali* $(n \in \boldsymbol{N})$ *ildizi* deyiladi, agar $w^n = z$ bo'lsa. z ning n-darajali ildizini $\sqrt[n]{z}$ orqali belgilaymiz.

Berilgan z kompleks sonning n-darajali ildizi bir nechta qiymatlarga ega, shuning uchun $\sqrt[n]{z} = w$ yozuv w son shu qiymatlardan biri ekanligini bildiradi. Bitta mulohaza davomida $\sqrt[n]{z}$ ifoda z kompleks son n-darajali ildizining faqat bitta qiymatini bildiradi.

Agar tekstdan ildizning aynnan shu qiymati haqida gapirilayotganligi ma'lum bo'lsa, bu haqida alohida eslatilmaydi, masalan, kompleks son modulini hisoblashda.

Agar $z = r(cos\varphi + isin\varphi) \neq 0$ bo'lsa, z ning n-darajali ildizi uchun

$$\sqrt[n]{z} = w_k = \sqrt[n]{r}\,(cos\frac{\varphi + 2\kappa\pi}{n} + isin\frac{\varphi + 2\kappa\pi}{n}),$$

bu yerda $k = 0,1...,n-1$, formula o'rinli. Bu formula z kompleks son n-darajali ildizining n ta har xil qiymatilarini beradi.

1-m i s o l. Hisoblang: $\left(\dfrac{1+\sqrt{3}i}{1-i}\right)^{20}$.

Yechish.

$$1+\sqrt{3}i = 2(cos\frac{\pi}{3} + isin\frac{\pi}{3}),\ 1-i = \sqrt{2}(cos\frac{7}{4}\pi + isin\frac{7}{4}\pi),$$

bo'lganligi sababli

$$\left(\frac{1+\sqrt{3}i}{1-i}\right)^{20}=\left(\frac{2(cos\frac{\pi}{3}+isin\frac{\pi}{3})}{\sqrt{2}(cos(-\frac{\pi}{4})+isin(-\frac{\pi}{4}))}\right)^{20}=\sqrt{2}\left(cos\left(\frac{\pi}{3}+\frac{\pi}{4}\right)+isin\left(\frac{\pi}{3}+\frac{\pi}{4}\right)\right)^{20}=$$

$$=\left(\sqrt{2}\left(cos\frac{7}{12}\pi+isin\frac{7}{12}\pi\right)\right)^{20}=2^{10}\left(cos\frac{140}{12}\pi+isin\frac{140}{12}\pi\right)=$$

$$=2^{10}\left(cos\frac{7}{4}\pi+isin\frac{7}{4}\pi\right)=2^{10}\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)=2^{9}\sqrt{2}(1-i).\blacksquare$$

2-m i s o l. Hisoblang: $(1+cos\alpha+isin\alpha)^{n}$.

Yechish.

$$1+cos\alpha+isin\alpha=\begin{cases}2cos\frac{\alpha}{2}\left(cos\frac{\alpha}{2}+isin\frac{\alpha}{2}\right), & агар\ \ 0\le\alpha\le\pi\ \ булса\\ -2cos\frac{\alpha}{2}\left(cos\frac{\alpha+2\pi}{2}+isin\frac{\alpha+2\pi}{2}\right), & агар\ \ \pi\le\alpha\le 2\pi\ \ булса\end{cases}$$

,

(28 (*r*) mashqqa qarang) bo'lganligi uchun $0\le\alpha<\pi$ bo'lganda

$$(1+cos\alpha+isin\alpha)^{n}=2^{n}cos^{n}\frac{\alpha}{2}\left(cos\frac{\alpha n}{2}+isin\frac{\alpha n}{2}\right),$$

$\pi\le\alpha\le 2\pi$ bo'lganda esa

$$(1+cos\alpha+isin\alpha)^{n}=(-2)^{n}cos^{n}\frac{\alpha}{2}\left(cos\left(\frac{\alpha n}{2}+n\pi\right)+isin\left(\frac{\alpha n}{2}+n\pi\right)\right).\ \blacksquare$$

3-m i s o l. $\sqrt[4]{-16}$ ildizning barcha qiymatlarini toping.

Yechish. $z=-16$ ni trigonometrik shaklga keltiramiz:

$$z=-16=16(cos\pi+isin\pi).$$

U holda ildiz chiqarish formulasiga ko'ra

$$w_{\kappa}=2\left(cos\frac{\pi+2\pi\kappa}{4}+isin\frac{\pi+2\pi\kappa}{4}\right),\ k=0,1,2,3.$$

Natijada,

$$w_0=2\left(cos\frac{\pi}{4}+isin\frac{\pi}{4}\right)=\sqrt{2}+i\sqrt{2},\quad w_1=2\left(cos\frac{3}{4}\pi+isin\frac{3}{4}\pi\right)=-\sqrt{2}+i\sqrt{2},$$

$$w_2=2\left(cos\frac{5}{4}\pi+isin\frac{5}{4}\pi\right)=-\sqrt{2}-i\sqrt{2},\quad w_3=2\left(cos\frac{7}{4}\pi+isin\frac{7}{4}\pi\right)=\sqrt{2}-i\sqrt{2}.$$

■

4-m i s o l. $\sqrt[4]{\frac{-1+i}{1-i\sqrt{3}}}$ to'plam elementlarining trigonometrik shaklini yozing.

Yechish. $-1+i=\sqrt{2}\left(cos\frac{3}{4}\pi+isin\frac{3}{4}\pi\right)$ va

$1-i\sqrt{3}=2\left(cos\left(-\frac{\pi}{3}\right)+isin\left(-\frac{\pi}{3}\right)\right)$ bo'lganligi uchun

$$\frac{-1+i}{1-i\sqrt{3}}=\frac{\sqrt{2}}{2}\left(cos\left(\frac{3}{4}\pi+\frac{\pi}{3}\right)+isin\left(\frac{3}{4}\pi+\frac{\pi}{3}\right)\right)=\frac{\sqrt{2}}{2}\left(cos\frac{13}{12}\pi+isin\frac{13}{12}\pi\right).$$

Natijada,

$$\sqrt[4]{\frac{-1+i}{1-i\sqrt{3}}}=\frac{1}{\sqrt[8]{2}}\left(cos\frac{13\pi+24\kappa\pi}{48}+isin\frac{13\pi+24\kappa\pi}{48}\right),\quad \kappa=0,1,2,3.\ \blacksquare$$

Muavr formulasi ba'zi trigonometrik ifodalarni almashtirishda qulayliklar yaratadi.

5-m i s o l. $tg5\varphi$ ni $tg\varphi$ orqali ifodalang.

Yechish. Darajaga ko'tarish formulasiga ko'ra

$$\cos 5\varphi+i\sin 5\varphi=(\cos\varphi+i\sin\varphi)^5.$$

Nyuton binom formulasini qo'llab, quyidagini hosil qilamiz:

$$cos5\varphi+isin5\varphi=cos^5\varphi+5icos^4\varphi sin\varphi-10cos^3\varphi sin^2\varphi-$$
$$-10icos^2\varphi sin^3\varphi+5cos\varphi sin^4\varphi+isin^5\varphi$$

chunki $i^2=-1$, $i^3=-i$, $i^4=1$, $i^5=i$. Mos ravishda haqiqiy va mavhum qismlarini tenglashtirib,

$$cos5\varphi=cos^5\varphi-10cos^3\varphi sin^2\varphi+5cos\varphi sin^4\varphi,\ sin5\varphi=$$
$$=5cos^4\varphi sin\varphi-10cos^2\varphi sin^2\varphi+sin^5\varphi$$

munosabatlarni hosil qilamiz. Bulardan

$$tg5\varphi=\frac{5cos^4\varphi sin\varphi-10cos^2\varphi sin^2\varphi+sin^5\varphi}{cos^5\varphi-10cos^3\varphi sin^2\varphi+5cos\varphi sin^4\varphi}=\frac{5tg\varphi-10tg^3\varphi+tg^5\varphi}{1-10tg^2\varphi+5tg^4\varphi}.$$

Bu yerda biz kasrning surat va maxrajini $cos^5\varphi$ ga bo'ldik. ■

6-m i s o l. $sin^5\varphi$ ni kaðrali argumentlarning trigonometrik funksiyalari orqali chiziqli ifodalang.

Yechish. $z=cos\varphi+isin\varphi$ bo'lsin, u holda $z^{-1}=cos\varphi-isin\varphi$,

$z^k=cos\,k\varphi+isin\,k\varphi$, $z^{-k}=cos\,k\varphi-isin\,k\varphi$,

$$cos\varphi=\frac{z+z^{-1}}{2},\ sin\varphi=\frac{z-z^{-1}}{2i},\ cos\,k\varphi=\frac{z^k+z^{-k}}{2},\ sin\,k\varphi=\frac{z^k-z^{-k}}{2i}.$$

Bularga ko'ra

$$sin^5\varphi=\left(\frac{z-z^{-1}}{2i}\right)^5=\frac{z^5-5z^3+10z-10z^{-1}+5z^{-3}-z^{-5}}{32i}=$$
$$=\frac{(z^5-z^{-5})-5(z^3-z^{-3})+10(z-z^{-1})}{32i}=\frac{2isin5\varphi-10isin3\varphi+20isin\varphi}{32i}=$$
$$=\frac{sin5\varphi-5sin3\varphi+10sin\varphi}{16}.\ \blacksquare$$

Xuddi shunga o'xshash yo'l Bilan istalgan $cos^{\kappa}\varphi\, sin^{m}\varphi$ ifodani karrali argumentning trigonometrik funksiyalari orqali chiziqli ifodalash mumkin.

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3.32. Hisoblang: a) $\left(\frac{1+i\sqrt{3}}{1-i}\right)^{20}$; b) $\left(1-\frac{\sqrt{3}-i}{2}\right)^{24}$;

c) $\frac{\left(-1+i\sqrt{3}\right)^{15}}{(1-i)^{20}}+\frac{\left(-1-i\sqrt{3}\right)^{15}}{(1+i)^{20}}$; d) $\frac{(1+i)^{2n+1}}{(1-i)^{2n-1}}$, $n\in \boldsymbol{N}$; e) $z=(tg1-i)^4$;

f) $(tg2-i)^4$; g) $\left(sin\frac{6\pi}{5}+i\left(1+cos\frac{6\pi}{5}\right)\right)^5$.

3.33. Isbotlang: $\left(\frac{1+itg\,\alpha}{1-itg\,\alpha}\right)^n=\frac{1+itg\,\alpha n}{1-itg\,\alpha n}$.

3.34. Agar $z+\frac{1}{z}=2cos\alpha$ bo'lsa, $z^m+\frac{1}{z^m}=2cos\,m\alpha$ bo'lishini isbotlang.

3.35. $(1+\omega)^n$ ifodani soddalashtiring, bu yerda $\omega=cos\frac{2}{3}\pi+isin\frac{2}{3}\pi$.

3.36. Ildizning qiymatlarini trigonometrik shaklda yozing:

a) $\sqrt[6]{i}$; b) $\sqrt[10]{512\left(1-i\sqrt{3}\right)}$; c) $\sqrt[8]{8\sqrt{2}(1-i)}$.

3.37. Ildizning qiymatlarini algebraik shaklda yozing:

a) $\sqrt[3]{1}$; b) $\sqrt[4]{1}$; c) $\sqrt[6]{1}$; d) $\sqrt[3]{i}$; e) $\sqrt[4]{-4}$; f) $\sqrt[6]{64}$; g) $\sqrt[8]{16}$;

h) $\sqrt[6]{-27}$; i) $\sqrt[4]{8\sqrt{3}i-8}$; j) $\sqrt[4]{-72\left(1-i\sqrt{3}\right)}$; k) $\sqrt[3]{1+i}$; l) $\sqrt[3]{2-2i}$;

m) $\sqrt[3]{\frac{8+24i}{3-i}}$; n) $\sqrt[3]{\frac{27-54i}{2+i}}$; o) $\sqrt[4]{-\frac{18}{1+i\sqrt{3}}}$; p) $\sqrt[4]{\frac{-32}{9\left(1-i\sqrt{3}\right)}}$.

3.38. Tenglamani yeching: a) $z^5-1-i\sqrt{3}=0$; b) $z^6+64=0$.

3.39. $\sqrt{5+12i}$ va $\sqrt{5-12i}$ sonlarning haqiqiy qismlari manfiy bo'lgan holda $z=\frac{\sqrt{5+12i}+\sqrt{5-12i}}{\sqrt{5+12i}-\sqrt{5-12i}}$ sonning algebraik shaklini yozing.

3.40. $z^{10}-z^5-992=0$ tenglamaning haqiqiy qismlari manfiy bo'lgan yechimlarini toping.

3.41. *cos x* va *sin x* lar orqali ifodalang:

a) *sin* 6 *x* + *cos* 6 *x*; b) *cos* 8 *x*; c) *sin* 8 *x*.

3.42. *tg* 7*x* ni *tg x* orqali ifodalang.

3.43. *tg nx* ni *tg x* orqali ifodalang, bunda n – butun musbat son.

3.44*. $sin^n nx$ va $cos^n nx$ larni (*n* – butun musbat son) *x* ga karali burchaklarning sinusi va kosinuslarining birinchi darajali ko'phadi ko'rinishida ifodalash mumkinligini isbotlang.

3.45. x ga karrali burchaklar trigonometrik funksiyalarining birinchi darajali ko'phadlari ko'rinishida tasvirlang:

a) $sin^3 x$; b) $sin^4 x$; c) $cos^5 x$;
d) $cos^6 x$; e) $sin^3 x\ cos^5 x$; f) $cos^7 x + sin^7 x$.

4-§. Yig'indi va ko'paytmalarni kompleks sonlar yordamida hisoblash

1-m i s o l. Ayniyatni isbotlang:

$$1 + C_n^2 + C_n^4 + ... = C_n^1 + C_n^3 + .. = 2^{n-1}.$$

Yechish. Nyuton binomini qo'lab quyidagi tengliklarni hosil qilamiz:

$$1 + C_n^1 + C_n^2 + C_n^3 + ... + C_n^{n-1} + C_n^n = (1+1)^n = 2^n,$$

$$1 - C_n^1 + C_n^2 - C_n^3 + ...(-1)^n C_n^n = (1-1)^n = 0$$

$$1 - C_n^1 + C_n^2 - C_n^3 + ...(-1)^n C_n^n = (1-1)^n = 0.$$

Bu tengliklarni hadlab qo'shib, undan keyin ayirib, kerakli ayniyatni hosil qilamiz. ■

2-m i s o l. Ayniyatni isbotlang:

$$C_n^0 + C_n^3 + C_n^6 + ... = \frac{1}{3}\left(2^n + 2cos\frac{\pi n}{3}\right).$$

Yechish. Quyidagi tenglikni qaraymiz:

$$(1+x)^n = C_n^0 + C_n^1 x + C_n^2 x^2 + C_n^3 x^3 + ... + C_n^{n-1} x^{n-1} + C_n^n x^n.$$

Bu tenglikga ketma-ket $x = 1,\ \varepsilon,\quad \varepsilon^2$ larni qo'yamiz, bu yerda $\varepsilon^2 + \varepsilon + 1 = 0$. Natijada quyidagi tengliklar hosil bo'ladi:

$$2^n = C_n^0 + C_n^1 + C_n^2 + C_n^3 + ...,$$

$$(1+\varepsilon)^n = C_n^0 + C_n^1\varepsilon + C_n^2\varepsilon^2 + C_n^3\varepsilon^3 + ...,$$

$$(1+\varepsilon^2)^n = C_n^0 + C_n^1\varepsilon^2 + C_n^3\varepsilon^4 + C_n^3\varepsilon^6 + ...$$

Lekin k son 3 ga bo'linmaganda $1 + \varepsilon^\kappa + \varepsilon^{2\kappa} = 0$, k son 3 ga bo'linganda esa $1 + \varepsilon^\kappa + \varepsilon^{2\kappa} = 3$ bo'ladi. Shuning uchun yuqoridagi tengliklarni hadlab qo'shib,

$$2^n + (1+\varepsilon)^n + (1+\varepsilon^2)^n = 3\{C_n^0 + C_n^3 + C_n^6 + ...\}$$

tenglikni hosil qilamiz.

$\varepsilon = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ deb olish mumkin bo'lganligi uchun

$$1 + \varepsilon = -\varepsilon^2 = -cos\frac{4\pi}{3} - isin\frac{4\pi}{3} = cos\frac{\pi}{3} + isin\frac{\pi}{3},$$

$$1 + \varepsilon^2 = -\varepsilon = -cos\frac{2\pi}{3} - isin\frac{2\pi}{3} = cos\frac{\pi}{3} - isin\frac{\pi}{3}.$$

Shuning uchun $2^n + (1+\varepsilon)^n + (1+\varepsilon^2)^n = 2^n + 2\cos\frac{n\pi}{3}$. Bu yerdan

$$C_n^0 + C_n^3 + C_n^6 + ... = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right). \ ■$$

3-m i s o l. Tenglikni isbotlang:

$$1^2 + 2^2 + ... + n^2 = C_{n+1}^2 + 2\left(C_n^2 + C_{n-1}^2 + ... C_2^2\right).$$

Yechish. $C_\kappa^2 = \frac{\kappa(\kappa-1)}{1\cdot 2} = \frac{\kappa^2}{2} - \frac{\kappa}{2}$ bo'lganligi uchun $2C_\kappa^2 = \kappa^2 - \kappa$.

Shuning uchun $2\sum_{\kappa=2}^{n} C_\kappa^2 = \sum_{\kappa=2}^{n} k^2 - \sum_{\kappa=2}^{n} k$.

Bundan yuqoridagi ayniyat kelib chiqadi. ■

4-m i s o l. Yig'indini hisoblang: $\delta = 1 - C_n^2 + C_n^4 - C_n^6 + ...$

Yechish. $(1+i)^n = 1 + C_n^1 i + C_n^2 i^2 + C_n^3 i^3 + ...$ ifodani qaraymiz. Bundan

$$(1+i)^n = \left(1 - C_n^2 + C_n^4 - ...\right) + i\left(C_n^1 - C_n^3 + C_n^5 - ....\right).$$

Lekin $(1+i) = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. Shuning uchun

$$\delta = 1 - C_n^2 + C_n^4 - C_n^6 + ... = 2^{\frac{n}{2}}\cos\frac{n\pi}{4} \quad .$$

Bu yerdan $n = 4m$ bo'lganda $\delta = (-1)^m 2^{2m}$, $n = 4m+1$ bo'lganda $\delta = (-1)^m 2^{2m}$, $n = 4m+3$ bo'lganda $\delta = (-1)^{m+1} 2^{2m+1}$, $n = 4m+2$ bo'lganda $\delta = 0$ bo'lishi kelib chiqadi. ■

5-m i s o l. Ayniyatni isbotlang:

$$1 + \frac{1}{2}C_n^1 + \frac{1}{3}C_n^2 + ... + \frac{1}{n+1}C_n^n = \frac{2^{n+1}-1}{n+1}.$$

Yechish. Quyidagi ko'phadni qaraymiz:

$$(1+x)^{n+1} = 1 + C_{n+1}^1 x + C_{n+1}^2 x^2 + ... C_{n+1}^{n+1} x^{n+1}.$$

Bundan

$$\frac{(1+x)^{n+1} - 1}{n+1} = C_n^0 x + \frac{C_n^1}{2}x^2 + \frac{C_n^2}{2}x^3 + ... + \frac{C_n^n}{n+1}x^{n+1} \qquad (*).$$

Bu tenglikka $x = 1$ ni qo'yib, izlanayotgan ayniyatni hosil qilamiz. ■

6-m i s o l. Ayniyatni isbotlang:

$$C_n^n + C_{n+1}^n + C_{n+2}^n + ... + C_{n+\kappa}^n = C_{n+\kappa+1}^{n+1}.$$

Yechish. Tenglikning chap tomonidagi ifoda

$$S = (1+x)^n + (1+x)^{n+1} + (1+x)^{n+2} + ... + (1+x)^{n+\kappa}$$

ko'phaddagi x^n oldidagi koeffisiyentdan iborat. Bu ko'phadni quyidagicha almashtiramiz:

$$S = (1+x)^n\left[1 + (1+x) + (1+x)^2 + ... + (1+x)^\kappa\right] = (1+x)^n\frac{(1+x)^{\kappa+1} - 1}{x} =$$

$$= \frac{1}{x}\left[(1+x)^{n+\kappa+1} - (1+x)^n\right].$$

kvadrat qavs ichidagi ko'phadda x^{n+1} oldidagi koeffisiyent $C_{n+\kappa+1}^{n+1}$ ga teng bo'ladi. ■

7-m i s o l. $C_n^0 C_m^p + C_n^1 C_m^{p-1} + \ldots + C_n^p C_m^0 = C_{m+n}^p$ tenglik o'rinli bo'lishini ko'rsating.

Yechish. Quyidagi ko'phadlarni qaraymiz:

$$(1+x)^n = \sum_{s=0}^{n} C_n^s x^s \ , \ (1+x)^m = \sum_{t=0}^{m} C_m^t x^t \ .$$

U holda: $(1+x)^n (1+x)^m = \sum_{s=0}^{n} C_n^s x^s \sum_{t=0}^{m} C_m^t x^t = (1+x)^{n+m} = \sum C_{m+n}^p x^p$

tenglikdan talab qilinayotgan tenglik kelib chiqadi. ■

8-m i s o l. Ayniyatni isbotlang:

$$(C_n^0)^2 + (C_n^1)^2 + (C_n^2)^2 + \ldots + (C_n^n)^2 = C_{2n}^n .$$

Yechish. $(1+x)^n (1+x)^n = (1+x)^{2n}$ ko'paytmani qaraymiz. Bu ko'paytmani quyidagicha yozish mumkin: $\sum_{s=0}^{n} C_n^s x^s \sum_{t=0}^{n} C_n^t x^t = \sum_{k=0}^{2n} C_{2n}^{\kappa} x^{\kappa}$.

Bu yerdan $C_{2n}^{\kappa} = \sum_{s+t=\kappa} C_n^s C_n^t$.

Natijada, $C_{2n}^n = \sum_{s+t=n} C_n^s C_n^t = \sum_{s=0}^{n} C_n^s C_n^{n-s} = \sum_{s=0}^{n} (C_n^s)^2$. ■

9-m i s o l. Tenglikni isbotlang:

$$2^{2m} cos^{2m}\varphi = \sum_{\kappa=0}^{m-1} 2C_{2m}^{\kappa} cos2(m-\kappa)\varphi + C_{2m}^m .$$

Yechish. $\cos\varphi + \frac{(\cos\varphi + i\sin\varphi) + (\cos\varphi - i\sin\varphi)}{2}$ tenglikda

$cos\varphi + isin\varphi = z$ deb olamiz. U holda $cos\varphi - isin\varphi = z^{-1}$,

$$cos^{2m}\varphi = \left(\frac{z+z^{-1}}{2}\right)^{2m} = \frac{1}{2^{2m}} \sum_{\kappa=0}^{2m} C_{2m}^{\kappa} z^{-\kappa} z^{2m-\kappa} .$$

Bundan $2^{2m} cos^{2m}\varphi = \sum_{\kappa=0}^{m-1} C_{2m}^{\kappa} z^{2(m-\kappa)} + C_{2m}^m + \sum_{\kappa=m+1}^{2m} C_{2m}^{\kappa} z^{2(m-\kappa)}$. Ikkinchi yig'indida $m-k = -(m-k^1)$ deb olamiz. U holda bu yig'indi quyidagi ko'rinishga keladi:

$$\sum_{\kappa'=m-1`}^{0} C_{2m}^{2m-\kappa'} z^{-2(m-\kappa')} = \sum_{\kappa=0`}^{m-1} C_{2m}^{\kappa} z^{-2(m-\kappa)} .$$

Shunday qilib, $2^{2m} cos^{2m}\varphi = \sum_{\kappa=0}^{m-1} C_{2m}^{\kappa} \left(z^{2(m-\kappa)} + z^{-2(m-\kappa)}\right) + C_{2m}^m$.

Lekin $Z^{2(m-\kappa)} + Z^{-2(m-\kappa)} = 2cos2(m-k)$. Shuning uchun

$$2^{2m} \cos^{2m} \varphi = \sum_{\kappa=0}^{m-1} 2C_{2m}^{\kappa} \cos 2(m-\kappa)\varphi + C_{2m}^m . \ ■$$

10-m i s o l. Tenglikni isbotlang:

$$\sin\varphi+\sin 2\varphi+\ldots+\sin n\varphi=\frac{\sin\frac{n\varphi}{2}\sin\frac{(n+1)\varphi}{2}}{\sin\frac{\varphi}{2}}\left(\varphi\neq 2k\pi,\ k\in \boldsymbol{Z}\right).$$

Yechish. $A=\cos\varphi+\cos 2\varphi+\ldots+\cos n\varphi$ yig'indini kiritamiz. U holda isbot qilinayotgan tenglikning chap tomonini B orqali belgilab

$$A+Bi=(\cos\varphi+i\sin\varphi)+(\cos 2\varphi+i\sin 2\varphi)+\ldots+(\cos n\varphi+i\sin n\varphi)$$

ni hosil qilamiz.

Bu geometrik progressiya hadlarining yig'indisidan iborat. Quyidagi belgilashni kiritamiz: $\alpha=\cos\frac{\varphi}{2}+i\sin\frac{\varphi}{2}$. U holda

$$A+Bi=\alpha^2\alpha^4+\ldots+\alpha^{2n}=\frac{\alpha^{2n+2}-\alpha^2}{\alpha^2-1}.$$

Oxirgi kasrning surat va maxrajida α ning shunday darajalarini qavsdan tashqariga chiqaramizki, qavs ichida α ning qarama-qarshi ko'rsatkichli darajalarining ayirmasi qolsin (buning mumkin bo'lishi uchun biz $\cos\varphi+i\sin\varphi$ ni emas $\cos\frac{\varphi}{2}+i\sin\frac{\varphi}{2}$ ni belgilardik):

$$A+Bi=\frac{\alpha^{n+2}\left(\alpha^n+\alpha^{-n}\right)}{\alpha\left(\alpha-\alpha^{-1}\right)}=$$

$$=\frac{\alpha^{n+1}\left(\alpha^n-\alpha^{-n}\right)}{\alpha-\alpha^{-1}}=\frac{\left(\cos\frac{n+1}{2}\varphi+i\sin\frac{n+1}{2}\varphi\right)2i\sin\frac{n\varphi}{2}}{2i\sin\frac{\varphi}{2}}=\frac{\sin\frac{n\varphi}{2}}{\sin\frac{\varphi}{2}}\left(\cos\frac{n+1}{2}\varphi+i\sin\frac{n+1}{2}\varphi\right).$$

Bu yerdan $B=\dfrac{\sin\frac{n\varphi}{2}\sin\frac{(n+1)\varphi}{2}}{\sin\frac{\varphi}{2}}$ ni va bir vaqtda

$A=\dfrac{\sin\frac{n\varphi}{2}\cos\frac{(n+1)\varphi}{2}}{\sin\frac{\varphi}{2}}$ ni hosil qilamiz. ■

Xuddi shunday

$$a_1\cos b_1+a_2\cos b_2+\ldots+a_n\cos b_n \text{ va } a_1\sin b_1+a_2\sin b_2+\ldots+a_n\sin b_n$$

yig'indilarni ham hisoblash mumkin, agar $b_1,b_2,\ldots,b_n$ argumentlar arifmetik progressiyani, $a_1,a_2,\ldots,a_n$ koeffisiyentlar esa gometrik progressiyani tashkil etsa.

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3.46*. Tengliklarni isbotlang:

a) $C_{2n}^1 + C_{2n}^3 + ... + C_{2n}^{n-1} = 2^{2n-2}$, agar n – juft bo'lsa;

b) $1 + C_{2n}^2 + ... + C_{2n}^{2n-1} = 2^{2n-2}$, agar n – toq bo'lsa.

3.47*. Tengliklarni isbotlang:

a) $C_n^1 + C_n^4 + C_n^7 + \cdots = \frac{1}{3}\left(2^n + 2cos\frac{(n-2)\pi}{3}\right)$;

b) $C_n^2 + C_n^5 + C_n^8 + \cdots = \frac{1}{3}\left(2^n + 2cos\frac{(n-4)\pi}{3}\right)$.

3.48*. Quyidagi yig'indini hisoblang: $C_n^1 - C_n^3 + C_n^5 - C_n^7 + \ldots$.

3.49*. Ayniyatni isbotlang:

$$\kappa C_n^0 + \frac{\kappa^2 C_n^1}{2} + \frac{\kappa^3 C_n^2}{3} + \frac{\kappa^4 C_n^3}{4} + ... + \frac{\kappa^{n+1} C_n^n}{n+1} = \frac{(\kappa+1)^{n+1} - 1}{n+1}.$$

3.50*. Ayniyatni isbotlang: $C_n^0 - C_n^1 + C_n^2 + ...(-1)^\kappa C_n^\kappa = (-1)^\kappa C_{n-1}^\kappa$.

3.51*. $C_n^0 C_n^\kappa + C_n^1 C_n^{\kappa+1} + ... + C_n^{n-\kappa} C_n^n = \frac{2n!}{(n-\kappa)!(n+\kappa)!}$ tenglik o'rinli bo'lishini ko'rsating.

3.52*. Ayniyatni isbotlang:

a) $\left(C_{2n}^0\right)^2 - \left(C_{2n}^1\right)^2 + \left(C_{2n}^2\right)^2 + ... + \left(C_{2n}^{2n}\right)^2 = (-1)^n C_{2n}^n$;

b) $\left(C_{2n+1}^0\right)^2 - \left(C_{2n+1}^1\right)^2 + \left(C_{2n+1}^2\right)^2 - ... - \left(C_{2n+1}^{2n+1}\right)^2 = 0$.

3.53*. Quyidagi tengliklarni isbotlang:

a) $2^{2m} sin^{2m}\varphi = \sum_{\kappa=0}^{m-1}(-1)^{m+\kappa} 2C_{2m}^\kappa cos2(m-\kappa)\varphi + C_{2m}^m$;

b) $2^{2m} cos^{2m+1}\varphi = \sum_{\kappa=0}^{m} C_{2m+1}^\kappa cos(2m-2\kappa+1)\varphi$;

c) $2^{2m} \sin^{2m+1}\varphi = \sum_{\kappa=0}^{m}(-1)^{m+\kappa} C_{2m+1}^\kappa \sin(2m-2\kappa+1)\varphi$.

3.54*. Tengliklarni isbotlang:

a) $\cos\frac{\pi}{n} + \cos\frac{3\pi}{n} + \cos\frac{5\pi}{n} + ... + \cos\frac{(2n-1)\pi}{n} = 0$;

b) $\sin\frac{\pi}{n} + \sin\frac{3\pi}{n} + \sin\frac{5\pi}{n} + ... + \sin\frac{(2n-1)\pi}{n} = 0$.

3.55*. Yig'indilarni toping:

a) $\cos x + C_n^1 \cos 2x + ... + C_n^n \cos(n+1)x$;

b) $\sin x + C_n^1 \sin 2x + ... + C_n^n \sin(n+1)x$.

3.56*. Yig'indilarni toping: $\sin^2 x + \sin^2 3x + ... + \sin^2 (2n-1)x$.

3.57. Isbotlang:

a) $cos^2 x+cos^2 2x+...+cos^2 nx=\frac{n}{2}+\frac{cos(n+1)x\,sin\,nx}{2sin\,x}$;

b) $sin^2 x+sin^2 2x+...+sin^2 nx=\frac{n}{2}-\frac{cos(n+1)x\,sin\,nx}{2sinx}$.

3.58*. Yig'indilarni toping:

a) $\cos x+2\cos 2x+3\cos 3x+...+n\cos nx$;

b) $\sin x+2\sin 2x+3\sin 3x+...+n\sin nx$.

5-§. Birning ildizlari

Har qanday noldan farqli kompleks son kabi 1 sonning ham *n*–darajali ildizi *n* ta qiymatga ega. $1=\cos 0+i\sin 0$ bo'lganligi uchun 1 ning *n–darajali ildizlari* uchun $\varepsilon_\kappa=cos\frac{2\pi\kappa}{n}+isin\frac{2\pi\kappa}{n},\ \ k=0,1,....,n-1$ formula o'rinli.

1 ning *n*–darajali ildizi *boshlang'ich ildiz* deyiladi, agar u 1 ning *n* dan kichik darajali ildizi bo'lmasa. Boshqacha aytganda, ε son 1 ning *n*–darajali boshlang'ich ildizi bo'ladi, agar $\varepsilon^n=1$ bo'lib, istagan $m<n$ uchun $\varepsilon^{\rm m}\neq 1$ bo'lsa. $\varepsilon_1=cos\frac{2\pi}{n}+isin\frac{2\pi}{n}$ sonning 1 ning *n*–darajali boshdang'ich ildizi bo'lishi ravshan, lekin $n>2$ bo'lgandan undan boshqa boshlang'ich ildizlar ham mavjud.

1-m i s o l. $\varepsilon=cos\frac{2\pi\kappa}{n}+isin\frac{2\pi\kappa}{n}$ $(k\geq 1,\ \ n\geq 2$-butun sonlar) son 1 ning $\frac{n}{d}$ darajali boshlang'ich ildizi bo'lishini ko'rsating, bu yerda d son k va n larning eng katta umumiy bo'luvchisidan iborat. Bu yerdan kelib chiqadiki, ε son 1 ning *n*–darajali boshlang'ich ildizi bo'lishi uchun k va n larning o'zaro tub bo'lishi zarur va yetarlidir.

Yechish. $\kappa=\kappa_1 d, n=n_1 d$ bo'lsin, bu yera n_1 va κ_1 o'zaro tub sonlar. U holda $\varepsilon=\cos\frac{2\pi\kappa_1}{n_1}+i\sin\frac{2\pi\kappa_1}{n_1}$. ε ni $m<n_1,\ m>0$ darajaga ko'tarib, $\varepsilon^m=cos\frac{2\pi\,m\kappa_1}{n_1}+isin\frac{2\pi\,m\kappa_1}{n_1}$ ni hosil qilamiz.

Agar $\varepsilon^m=1$ bo'lsa, u holda $\frac{2\pi m\kappa_1}{n_1}=2\pi s$, bunda $s\in\mathbf{Z}$ yoki $\frac{\kappa_1 m}{n_1}=s$, ya'ni $\kappa_1 m$ son n_1 ga bo'linadi. Lekin κ_1 va n_1 o'zaro tub. Shuning uchun *m* son n_1 ga bo'linadi, bu esa $0<m<n_1$ shartga zid.

Shunday qilib, $0 < m <$ n_1 bo'lganda $\varepsilon^m \neq 1$. $m = n_1$ bo'lganda esa $\varepsilon^{n_1} = \cos 2\pi\kappa_1 + i\sin 2\pi\kappa_1 = 1$. Bu yerdan ε - son 1 ning $n_1 = \frac{n}{d}$ darajali boshlang'ich ildizi ekanligi kelib chiqadi. ■

Bu misoldan ko'rinadiki, 1 ning n-darajali boshlang'ich ildizlari soni n dan kichik va n bilan o'zaro tub bo'lgan sonlar soniga, ya'ni Eyler funksiyasining $\varphi(n)$ qiymatiga tengdir.

$X_n(x) = \prod_{k=1}^{\varphi(n)} (x - \varepsilon_\kappa)$ ko'phad, bu yerda ε_κ ($\kappa = 0, 1, ..., \varphi(n)$) - 1 ning boshlang'iya ildizi, *doiraviy ko'phad* deyiladi.

2-m i s o l. Birning 6-darajali ildizlarini toping.

Yechilishi: 1 ning n-darajali ildizlari formulasidan:

$$\varepsilon^\kappa = cos\frac{2\pi\kappa}{6} + isin\frac{2\pi\kappa}{6}, \quad \kappa = 0,1,2,3,4,5.$$

ni hosil qilamiz. Natijada, izlanayotgan ildizlar quyidagilardan iborat bo'ladi:

$$\varepsilon_0 = cos0 + isin0 = 1, \ \varepsilon_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

$$\varepsilon_2 = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \ \varepsilon_3 = \cos\pi + i\sin\pi = -1,$$

$$\varepsilon_4 = \cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \ \varepsilon_5 = \cos\frac{5}{3}\pi + i\sin\frac{5}{3}\pi = \frac{1}{2} - \frac{\sqrt{3}}{2}i. \ ■$$

Agar 1 ning n-darajali ildizi 1 ning δ darajali boshlang'ich ildizi bo'lsa, δ son bu *ildiz tegishli bo'lgan ko'rsatkich* deyiladi.

3-m i s o l. Birning 6-darajali boshlang'ich ildizlarini yozing.

Yechish. 1-misolga ko'ra birning 6-darajali boshlang'ich ildizlari $\varepsilon_{1,}\varepsilon_5$ lardan, ya'ni $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$ lardan iborat. ■

4-m i s o l. $x^5 - 1 = 0$ tenglamani algebraik yo'l bilan yechib, 1 ning 5-darajali ildizlarini toping.

Yechish. Tenglamaning chap tomonini ko'paytuvchilarga ajratamiz:

$$(x-1)(x^4 + x^3 + x^2 + x + 1) = 0.$$

Boshlang'ich ildizlar quyidagi tenglamaning ildizlari bo'ladi:

$$x^4 + x^3 + x^2 + x + 1 = 0.$$

Bu tenglama $x^2 + x^{-2} + x + x^{-1} + 1 = 0$ tenglamaga teng kuchli. $z = x + x^{-1}$ belgilash kiritamiz. $x^2 + 2 + x^{-2} = z^2$ bo'lishini e'tiborga olsak, $z^2 + z - 1 = 0$ tenglama hosil bo'ladi, bundan $z_1 = -\frac{1}{2} + \frac{\sqrt{5}}{2}$, $z_2 = -\frac{1}{2} - \frac{\sqrt{5}}{2}$.

1 ning ildizlari $x^2 - z_1 x + 1 = 0$ va $x^2 - z_2 x + 1 = 0$ tenglamalardan topiladi. Bulardan $x = \frac{z_1 \pm i\sqrt{4 - z_1^2}}{2}$ va $x = \frac{z_2 \pm i\sqrt{4 - z_2^2}}{2}$. z_1 va z_2 larning qiymatlarini qo'yib,

$$x_1 = \frac{\sqrt{5}-1}{4} + i\frac{\sqrt{10+2\sqrt{5}}}{4}, \qquad x_2 = \frac{\sqrt{5}-1}{4} - i\frac{\sqrt{10+2\sqrt{5}}}{4},$$

$$x_3 = -\frac{\sqrt{5}-1}{4} + i\frac{\sqrt{10-2\sqrt{5}}}{4}, \qquad x_4 = -\frac{\sqrt{5}-1}{4} - i\frac{\sqrt{10-2\sqrt{5}}}{4}$$

larni hosil qilamiz.

Bularni $x_\kappa = cos\frac{2\pi\kappa}{5} + isin\frac{2\pi\kappa}{5} = cos\kappa \cdot 72^0 + isin\kappa \cdot 72^0$ formula bilan taqqoslab $\cos 72^0 = \frac{\sqrt{5}-1}{4}$ ni hosil qilamiz.

Bundan r radiusli doiraga ichki chizilgan muntazam o'nburchakning tomoni a_{10} uchun formula keltirib chiqariladi:

$$a_{10} = 2rsin\frac{2\pi}{10\cdot 2} = 2rsin\frac{\pi}{10} = 2rcos\left(\frac{\pi}{2} - \frac{\pi}{10}\right) = 2r\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{2}\cdot r . ■$$

5-m i s o l. Birning 7 ko'rsatkichga tegishli bo'lgan 28-darajali ildizlarini yozing.

Yechish. Ma'lumki, 1 ning 28-darajali ildizlari

$$\varepsilon_k = cos\frac{2\pi\kappa}{28} + isin\frac{2\pi\kappa}{28}, (\ \kappa = 0,1,2,....,27),$$

lardan iborat. Bulardan 7 ko'rsatkichga tegishli bo'lganlari

$$\varepsilon_4, \varepsilon_8, \varepsilon_{12}, \varepsilon_{16}, \varepsilon_{20}, \varepsilon_{24},$$

lardir yoki ularni quyidagicha yozish mumkin:

$$cos\frac{2\pi\kappa}{7} + isin\frac{2\pi\kappa}{7}, \text{ бунда } \quad \kappa = 0,1,2,3,4,5,6 . ■$$

6-m i s o l. $x^n - 1$ ni haqiqiy koeffisiyentli ko'paytuvchilarga ajrating.

Yechish. $n = 2m$ bo'lsin, u holda $x^n - 1 = 0$ tenglama ikkita 1, -1 haqiqiy ildizlarga va $2m - 2$ ta kompleks ildizlarga ega.

Bunda $\varepsilon_\kappa = cos\frac{2\pi\kappa}{2m} + isin\frac{2\pi\kappa}{2m}$ ildiz

$$\varepsilon_{2m-\kappa} = cos\frac{2(2m-\kappa)\pi}{2m} + isin\frac{2(2m-\kappa)\pi}{2m}$$

ildizga qo'shma.

Shunday qilib,

$$x^{2m} - 1 = (x^2 - 1)\left(x - \varepsilon_1\right)(x - \overline{\varepsilon_1})(x - \varepsilon_2)(x - \overline{\varepsilon_2})..(x - \varepsilon_{m-1})(x - \overline{\varepsilon_{m-1}});$$

$$x^{2m} - 1 = (x^2 - 1)[x^2 - (\varepsilon_1 + \overline{\varepsilon_1})x + 1]..[x^2 - (\varepsilon_{m-1} + \overline{\varepsilon_{m-1}})x + 1];$$

$$x^{2m}-1=\left(x^2-1\right)\prod_{\kappa=1}^{m-1}\left(x^2-2xcos\frac{\kappa\pi}{m}+1\right).$$

ni hosil qilamiz.

Agar $n = 2m+1$ bo'lsa, shunga o'xshash yo'l bilan

$$x^{2m+1}-1=\left(x-1\right)\prod_{\kappa=1}^{m}\left(x^2-2xcos\frac{2\kappa\pi}{2m+1}+1\right)$$

ni hosil qilamiz. ■

7-m i s o l. Tenglikni isbotlang:

$$sin\frac{\pi}{2m}\cdot sin\frac{2\pi}{2m}...sin\frac{(m-1)\pi}{2m}=\frac{\sqrt{m}}{2^{m-1}}.$$

Yechish. 6-misolning natijasiga ko'ra:

$$\frac{x^{2m}-1}{x^2-1}=\prod_{\kappa=1}^{m-1}\left(x^2-2x\cos\frac{\kappa\pi}{m}+1\right)$$

ni hosil qilamiz.

$x = 1$ ni qo'yib, $m=2^{m-1}\prod_{\kappa=1}^{m-1}\left(1-cos\frac{\kappa\pi}{m}\right)$ yoki $m=2^{2(m-1)}\prod_{\kappa=1}^{m-1}sin^2\frac{\kappa\pi}{2m}$ va nihoyat $\frac{\sqrt{m}}{2^{m-1}}=\prod_{\kappa=1}^{m-1}sin\frac{\kappa\pi}{2m}$ ni hosil qilamiz. ■

8-m i s o l. Tenglamani yeching: $(x+1)^m-(x-1)^m=0. \quad (m>1).$

Yechish. Tenglamani $\left(\frac{x+1}{x-1}\right)^m=1$ shakldа yozish mumkin.

Bundan $\frac{x+1}{x-1}=\varepsilon_\kappa$, bu yerda $\varepsilon_\kappa=cos\frac{2\pi\kappa}{m}+isin\frac{2\pi\kappa}{m}, \quad \kappa=1,2,...,m-1$. U holda $x=\frac{\varepsilon_\kappa+1}{\varepsilon_\kappa-1}$ bo'ladi. Bu ifodani soddalashtiramiz

$$x=\frac{1+cos\frac{2\pi\kappa}{m}+isin\frac{2\pi\kappa}{m}}{cos\frac{2\pi\kappa}{m}+isin\frac{2\pi\kappa}{m}-1}=-\frac{2cos^2\frac{\pi\kappa}{m}+2isin\frac{\pi\kappa}{m}cos\frac{\pi\kappa}{m}}{2sin^2\frac{\pi\kappa}{m}-2isin\frac{\pi\kappa}{m}cos\frac{\pi\kappa}{m}}=$$

$$=-\frac{2cos\frac{\pi\kappa}{m}\left(cos\frac{\pi\kappa}{m}+isin\frac{\pi\kappa}{m}\right)}{2sin\frac{\pi\kappa}{m}+\left(sin\frac{\pi\kappa}{m}-icos\frac{\pi\kappa}{m}\right)}== \text{-}ctg\frac{\pi\kappa}{m}(-i)=ictg\frac{\pi\kappa}{m}.$$

Shunday qilib, $x=-i\,ctg\frac{\pi\kappa}{m}, \quad \kappa=1,...,m-1.$■

9-m i s o l. Agar *a* va *b* o'zaro tub sonlar bo'lsa, 1 ning *ab*- darajali ildizlari 1 ning *a*-darajali va *b*-darajali ildizlarining ko'paytmasidan iborat bo'lishini isbotlang.

Yechish. α_κ va β_s mos ravishda 1 ning *a*-darajali va *b*-darajali ildizlari bo'lsin, bunda $\kappa = 0,1...,a-1$; $s = 0,1,2,...,b-1$.

Avvalo 1 ning *a*-darajali ildizining *b*-darajali ildiziga ko'paytmasi 1 ning *ab* darajali ildizi bo'lishini ko'rsatamiz. Haqiqatan, $\alpha^a = 1$, $\beta^b = 1$ bo'lsin. U holda $(\alpha\beta)^{ab} = (\alpha^a)^b(\beta^b)^a = 1$.

Endi $\alpha_\kappa\beta_s$ larning har xil bo'lishini ko'rsatish yetarli. Faraz qilaylik, $\alpha_{\kappa 1}\beta_{s1} = \alpha_{\kappa 2}\beta_{s2}$. U holda $\dfrac{\alpha_{\kappa 1}}{\alpha_{\kappa 2}} = \dfrac{\beta_{s2}}{\beta_{s1}}$, ya'ni $\alpha_i = \beta_j$. 13-masalaga ko'ra, $\alpha_i = \beta_j$ =1, ya'ni $\kappa_1 = \kappa_2$, $s_1 = s_2$. ■

10-m i s o l. Tenglamani yeching

$$cos\varphi + C_n^1 cos(\varphi+\alpha)x + C_n^2 cos(\varphi+2\alpha)x^2 + ... + C_n^n cos(\varphi+n\alpha)x^n = 0.$$

Yechish.

$$S = cos\varphi + C_n^1 cos(\varphi+\alpha)x + ... + cos(\varphi+n\alpha)x^n,$$

$$T = sin\varphi + C_n^1 sin(\varphi+\alpha)x + ... + sin(\varphi+n\alpha)x^n$$

bo'lsin.

U holda $S + Ti = \mu(1+\lambda x)^n$, $S - Ti = \overline{\mu}(1+\overline{\lambda}x)^n$, bu yerda $\lambda = cos\alpha + isin\alpha$, $\mu = cos\varphi + isin\varphi$. Bulardan $2S = \mu(1+\lambda x)^n + \overline{\mu}(1+\overline{\lambda}x)^n$. Tenglama $\mu(1+\lambda x)^n + \overline{\mu}(1+\overline{\lambda}x)^n = 0$ ko'rinishga keladi. Bu tenglamani yechib,

$$x_k = -\frac{sin\dfrac{(k+1)\pi - 2\varphi}{2n}}{sin\dfrac{(k+1)\pi - 2\varphi - 2n\alpha}{2n}}; \quad k = 0,1,2,...,n-1$$

ni hosil qilamiz. ■

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3.59. Birning quyidagi darajali ildizlarini toping:

a) 2; b) 3; c) 4; d) 8; e) 12; f) 24.

3.60. Birning quyidagi darajali boshlang'ich ildizlarini toping:

a) 2; b) 3; c) 4; d) 8; e) 12; f) 24 .

3.61. Birning a) 16; b) 20; c) 24 darajali har bir ildizi qaysi ko'rsatkichga tegishli bo'lishini aniqlang.

3.62. ε - 1 ning 2*n*-darajali boshlang'ich ildizi bo'lsa,

$$1 + \varepsilon + \varepsilon^2 + ... + \varepsilon^{n-1}$$

yig'indini hisoblang.

3.63. 1 ning barcha *n*- darajali ildizlari yig'indisini toping.

3.64. ε - 1 ning *n*-darajali ildizi bo'lsa, $1 + 2\varepsilon + 3\varepsilon^2 + ... + n\varepsilon^{n-1}$ yig'indini hisoblang.

3.65. ε - 1 ning *n*-darajali ildizi bo'lsin.

$$1+4\varepsilon+9\varepsilon^2+....+n^2\varepsilon^{n-1}$$

yig'indini hisoblang.

3.66. Yig'indilarni hisoblang:

$$a)\ cos\frac{2\pi}{n}+2cos\frac{4\pi}{n}+...+(n-1)cos\frac{2(n-1)\pi}{n};$$

$$b)\ sin\frac{2\pi}{n}+sin\frac{4\pi}{n}+...+(n-1)sin\frac{2(n-1)\pi}{n}.$$

3.67. 1 ning:

a) 15-chi; b) 24-chi; c) 30-chi darajali boshlang'ich ildizlari yig'indisini topling.

3.68*. λ,μ,a,b kompleks sonlar, n natural son bo'lsin. $\lambda(z-a)^n+\mu(z-b)^n=0.$ tenglamaning ildizlari bitta aylanada yoki to'g'ri chiziqda yotishini isbotlang.

3.69. Tenglamalarni yeching:

a) $(x+2)^n-(x-2)^n=0$;

b) $(x+5i)^n-(x-5i)^n=0$;

c*) $(x+3i)^n+i(x-3i)^n=0$;

d) $(x+ai)^n-(cos\varphi+isin\varphi)(x-ai)^n=0,\ \ \varphi\neq 2\kappa\pi,\ \ a\in\boldsymbol{R}$.

3.70*. Agar A moduli 1 ga teng bo'lgan kompleks son bo'lsa, $\left(\frac{1+ix}{1-ix}\right)^m=A$ tenglamaning barcha ildizlari haqiqiy va har xil bo'lishini isbotlang.

3.71*. Agar a va b o'zaro tub sonlar bo'lsa, x^a-1 va x^b-1 ko'phadlar yagona umumiy ildizga ega bo'lishini ko'rsating.

3.72*. Agar a va b o'zaro tub sonlar bo'lsa, 1 ning a-darajali va b- darajali bolang'ich ildizlarining ko'paytmasi 1 ning ab darajali boshlang'ich ildizi bo'ladi va aksincha. Shu tasdiqni isbotlang.

3.73. a va b o'zaro tub sonlar bo'lsa, $\varphi(ab)=\varphi(a)\,\varphi(b)$ bo'lishini isbotlang, bu yerda $\varphi(n)$ 1 ning n-darajali boshlang'ich ildizlari soni.

3.74*. Agar $n=p_1^{\alpha_1}\,p_2^{\alpha_2}...p_k^{\alpha_k}$, $p_1,p_2,...,p_k$-har xil tub sonlar bo'lsa, $\varphi(n)=n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)...\left(1-\frac{1}{p_k}\right)$. tenglik o'rinli bo'lishini isbotlang.

3.75*. $n>2$ bo'lganda 1 ning n-darajali boshlang'ich ildizlari soni juft son bo'lishini isbotlang.

3.76. n ning quyidagi qiymtalari uchun $X_n(x)$ doiraviy ko'phadni yozing:

a) 1; b) 2; c) 3; d) 4; e) 5; f) 6; g) 7; h) 8;

b) i) 9; j) 10; k) 11; l) 12; m) 15; n) 105.

3.77. p tub son uchun $X_p(x)$ ko'phadni yozing.

3.78*. $X_{p^m}(x)$ ko'phadni yozing, p tub son.

3.79*. $n>1$ toq son uchun $X_{2n}(x)=X_n(-x)$ tenglikni isbotlang.

3.80*. Agar *d* son *n* sonning tub bo'luvchilaridan tashkil topgan bo'lsa, 1 ning *nd*-darajali boshlang'ich ildizi 1 ning *n*-darajali ildizining *d*-darajali ildizi bo'ladi va aksincha. Shuni isbotlang.

3.81*. Agar $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_\kappa^{\alpha_\kappa}$, $p_1, p_2, ..., p_\kappa$ - har xil tub sonlar bo'lsa, $X_n(x) = X_{n'}\left(x^{n''}\right)$, $n' = p_1 p_2 ... p_\kappa$; $n'' = \frac{n}{n'}$ bo'lishini isbotlang.

3.82*. $\mu(n)$ orqali 1 ning *n*-darajali boshlang'ich ildizlari yig'inidsini belgilaymiz. Agar *n* biror tub sonning kvadratiga bo'linsa, $\mu(n) = 0$, agar *n* juft sondagi har xil tub sonlarning ko'paytmasi bo'lsa, $\mu(n) = 1$; agar *n* toq sondagi tub sonlarning ko'paytmasi bo'lsa $\mu(n) = -1$ bo'lishini isbotlang.

3.83*. Agar *d* *n* sonning barcha bo'luvchilari to'plamida o'zgarsa, $n \neq 1$ bo'lganda $\sum_{n \neq 1} \mu(d) = 0$ tenglik o'rinli bo'lishini ko'rsating.

3.84*. $X_n(x) = \Pi\left(x^d - 1\right)^{\mu\left(\frac{n}{d}\right)}$ bo'lishini isbotlang, bu yerda *d* *n* sonning barcha bo'luvchilari to'plamida o'zgaradi.

3.85*. $X_n(1)$ ni toping.

3.86*. $X_n(-1)$ ni toping.

3.87*. Birning ikkitadan olingan *n*-darajali boshlang'ich ildizlari yig'indisini toping.

3.88*. $S = 1 + \varepsilon + \varepsilon^4 + \varepsilon^9 + ... + \varepsilon^{(n-1)^2}$, bunda ε – birning *n*–darajali boshlang'ich ildizi. $|S|$ ni toping.

6-§. Kompleks o'zgaruvchining ko'rsatkichli va logarifmik funksiyalari

z kompleks o'zgaruvchining ko'rsatkili funksiyasi quyidagi Eyler formulasi yordamida aniqlanadi: $e^{a+bi} = e^a(cosb + isinb)$.

Bu formulaga $a = 0$ ni qo'yib, $cosb + isinb = e^{bi}$ ni hosil qilamiz.

b ni –*b* ga almashtirib, $cosb - isinb = e^{-bi}$ ni hosil qilamiz.

Bu tenglamalarni hadlab qo'shib va ayirib, quyidagi formulalarni hosil qilamiz:

$$cosb = \frac{e^{bi} + e^{-bi}}{2}, \quad sinb = \frac{e^{bi} - e^{-bi}}{2i},$$

bular *Eyler formulalari* deb ataladi. Ular trigonometrik va mavhum ko'rsatkichli funksiyalar o'rtasidagi bog'lanishni ifodalaydi.

Kompleks sonning $\alpha = r(cos\varphi + isin\varphi)$ trigonometrik shaklini $re^{\varphi i}$ ko'rinishda yozish mumkin. Kompleks sonning bunday ko'rinishdagi yozuvi uning *ko'rsatkichli shakli* deyiladi. Kompleks sonning ko'rsatkichli shaklini quyidagicha yozish mumkin:

$$\alpha = re^{\varphi i} = e^{\ell nr}e^{\varphi i} = e^{\ell nr+\varphi i}$$

bu esa kompleks sonning natural logarifmini $\ell n\alpha = \ell nr + i\varphi$ formula yordamida aniqlash tabiiy bo'lishini ko'rsatadi, ya'ni kompleks son logarifmining haqiqiy qismi esa uning argumentidan iborat. Bunday kiritilgan logarifmik funksiya noldan farqli barcha kompleks sonlar to'plamida aniqlangan. Shuni ta'kidlash kerakki, kompleks sonning argumenti ko'p qiymatli bo'lganligi sababli logarifmik funksiya ham ko'p qiymatlidir. Xususiy holda, logarifmning – ko'paytmaning logarifmi logarifmlar ko'paytmasiga teng – degan xossasi faqat ko'pqiymatlilikni hisobga olgan holda to'g'ri.

1-m i s o l. $ln 1$ ning qiymatlaridan biri 0 ga teng, ln(-1) ning qiymatlaridan biri esa πi dan iborat, chunki $-1 = cos\pi + isin\pi = e^{\pi i}$. Lekin $ln\left[(-1)(-1)\right] = \pi i + \pi i = 2\pi i$. Bu $ln 1$ ning 0 dan farqli qiymatlaridan biridir (chunki $1 = cos2\pi\kappa + isin2\pi\kappa$). ■

α – noldan farqli kompleks son bo'lsin. U holda $ln\alpha$ ning istalgan qiymati uchun $\alpha = e^{ln\alpha}$ bo'ladi. Shuning uchun ta'rif bo'yicha $\alpha^{\beta} = e^{\beta ln\alpha}$ deb hisoblash tabiiydir. Bu ham $ln\alpha$ ning ko'pqiymatliligi sababli α va β ning ko'p qiymatli funksiyasi bo'ladi va u $2\kappa\pi i$ qo'shiluvchi aniqligida aniqlangan.

2-m i s o l. i^i nimaga teng?

Yechish. $ln\, i = i\left(\frac{\pi}{2} + 2\kappa\pi\right)$ bo'lganligi uchun $i^i = e^{-\left(\frac{\pi}{2}+2\kappa\pi\right)}$. Shunday qilib, biz qandaydir ma'noda paradoksal natijaga keldik, ya'ni «juda mavhum» bo'lgan i^i ifodaning barcha qiymatlari haqiqiydir. ■

M A S H Q L A R

3.89*. Limitni toping: $\lim\limits_{n\to\infty}\left(1+\frac{a+bi}{n}\right)^n$.

3.90*. Eyler formulasiga ko'ra kompleks sonlarni ko'paytirishdagi argumentlarning qo'shilishi qoidasi nimaga o'tadi? Muavr formulasi uchun ham shunday savolga javob bering.

3.91. Hisoblang:

a) $ln(-e)$; b) $ln(-2)$; c) $ln\, i$; d) $ln\,(1+i)$; e) $e^{\pi i}$; f) 2^i; q) $\left(\frac{1+i}{\sqrt{2}}\right)^{-i}$.

3.92. Limitni toping: $ln\left(x + i\sqrt{1-x^2}\right)$, $-1 \le x \le 1$.

3.93*. *arctgx* ni logarifmik funksiya orqali ifodalang.

IV - BOB

Ko'phadlar algebrasi

Kalit so'zlar va ifodalar: *bir o'zgaruvchili ko'phad; ko'phadning koeffisiyentlari; ko'phadning darajasi; ko'phadning bosh koeffisiyenti; ko'phadning bosh hadi; ko'phadning ozod hadi; nol ko'phad; ko'phadlar yig'indisi va ko'paytmasi; bir o'zgaruvchili ko'phadlar xalqasi;qoldiqli bo'lish; bo'linma; qoldiq; ko'phadning bo'luvchisi; ko'phadning karralisi; ko'phadlarning umumiy bo'luvchi; ko'phadlarning eng kata umumiy bo'luvchisi (EKUB); Yevklid algoritmi; ko'phadlar EKUBini chiziqli tasvirlash; ko'phadlarning umumiy karralisi; ko'phadlarning eng kichik umumiy karralisi (EKUK); Yevklid ketma-ketligi; ko'phadning qiymati; Bezu teoremasi; ko'phadning ildizi;karali ildiz; tub ildiz; Viyet formulalari; Lagranjning interpolyasion formulasi; Nyutonning interpolyasion formulasi; Teylor formulasi; keltirimaydigan ko'phad; unitar ko'phad; berilgan maydon ustida ko'phadning kanonik yoyilmasi; ko'phadning keltirilmaydigan* $k-$*karrali ko'paytuvchisi; algebraning asosiy teoremasi; Gauss teoremasi; algebraik yopiq maydon; Eyzenshteyn alomati; ko'phadning rasional ildizlari to'g'risidagi birinchi teorema; ko'phadning rasional ildizlari to'g'risidagi ikkinchi teorema; rasional kasr yoki kasr-rasional funksiya; berilgan maydon ustida rasional kasrlar maydoni yoki kasr-rasional funksiyalar maydoni; qisqarmaydigan, to'g'ri va sodda rasional kasrlar; Lagranj formulasi; hadning darajasi; ko'phadning barcha o'zgaruvchilari bo'yicha darajasi; bir jinsli ko'phad yoki forma; ko'phadning bir o'zgaruvchisi bo'yicha darajasi; leksikografisk yoki lug'atiy tartiblash; ko'phadning yuqori hadi; simmetrik ko'phad; elementar (yoki asosiy) simmetrik ko'phad; monogen ko'phad; darajali yig'indilar; Nyuton formulalari.*

§ 1. Ko'phadlar xalqasi. Ko'phadlarning EKUB va EKUKi

P maydon ustidagi b*ir* x *o'zgaruvchili ko'phad deb* quyidagi ko'rinishdagi ifodaga aytiladi

$$a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n, \qquad (1)$$

bu yerda $a_0, a_1, \cdots, a_{n-1}, a_n - P$ maydonning elementlaridan iborat bo'lib, ular (1) *ko'phadning koeffisiyentlari deyiladi*. Agar $a_0 \neq 0$, bo'lsa, n ko'phadning *darajasi* deb yuritiladi, $a_0 -$ *bosh koeffisiyent deb,* $a_n -$*ozod had deb,* $a_0x^n -$ (1) ko'phadning bosh *hadi* deb yuritiladi.

Barcha koeffisiyentlari nolga teng bo'lgan ko'phad nol ko'phad deyiladi va 0 bilan belgilanadi. Nol ko'phadning darajasi aniqlanmagan.

Ikkita ko'phadning o'zgaruvchining bir xil darajalari oldidagi koeffisiyentlari teng bo'lsa, ular teng ko'phadlar deyiladi.

$$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n=\sum_{i=0}^{n}a_ix^{n-i},$$

$$g(x)=b_0x^m+b_1x^{m-1}+\cdots+a_{m-1}x+b_m=\sum_{k=0}^{m}b_kx^{m-k}.$$

ko'phadlar berilgan bo'lsin.

$f(x)$ va $g(x)$ ko'phadlarning ko'paytmasi deb

$f(x)g(x)=c_0x^{n+m}+c_1x^{n+m-1}+\cdots+c_{n+m-1}x+c_{n+m}=\sum_{j=0}^{n+m}c_jx^{n+m-j}$, ko'phadga aytiladi, bu yerda

$$c_j=\sum_{i+k=j}a_ib_k= =a_0b_j+a_1b_{j-1}+a_2b_{j-2}+\cdots+a_{j-1}b_1+a_jb_o,\quad j=0,1,\ldots,n+m.$$

Agar $n\geq m$ bo'lsa, $f(x)$ va $g(x)$ *ko'phadlarning yig'indisi deb*

$$f(x)+g(x)=(a_n+b_m)+(a_{n-1}+b_{m-1})x+\cdots+(a_{n-m}+b_m)x^m+a_{n-m-1}x^{m+1}+\cdots+a_0x^n.$$

ko'phadga aytiladi. P maydon ustida x o'zgaruvchili barcha ko'phadlar to'plami yuqorida keltirilgan ko'phadlarni qo'shish va ko'paytirish amallariga nisbatan birlik elementli kommutativ xalqa tashkil qiladi. Bu xalqa $P[x]$ bilan belgilanadi va P maydon ustida bir o'zgaruvchili *ko'phadlar xalqasi deb yuritiladi*. Bu xalqaning nol elementi vazifasini Ox^0, nol ko'phad bajaradi, birlik element esa ex^0 ko'phaddan iborat bo'lib, bu yerda e element P maydonning birlik elementidan iborat. $f(x)=a_0x^0+a_1x^1+\cdots+a_nx^n$ – ko'phadga qarama-qarshi ko'phad deb $(-f(x))=(-a_0)\,x^0+(-a_1)\,x^1+\cdots+(-a_n)\,x^n$. ko'phadga aytiladi.

Agari $f(x)\in P[x],\quad 0\neq g(x)\in P[x],$ bo'lsa, $f(x)$ ni $g(x)$ ga qoldiqli bo'lish deb quyidagi munosabatga aytiladi:

$$f(x)=g(x)q(x)+r(x),$$

bu yerda $q(x)$ va $r(x)-$ P, maydon ustidagi ko'phadlar bo'lib, $r(x)$ ning darajasi $g(x)$ ning darajasidan kichik bo'ladi yoki $r(x)=0$ bo'ladi. Bu tasvirlash yagonadir. $q(x)$ ko'phad bo'linma deb, $r(x)-$ esa $f(x)$ ni $g(x)$ bo'lgandagi qoldiq deb yuritiladi.

$r(x)=0$ bo'lsa, $f(x)$ ko'phad $g(x)$ ko'phadga bo'linadi deyiladi va $g(x)|\,f(x)$ (yoki $f(x)\vdots\, g(x)$) ko'rinishda yoziladi, bu holda $g(x)$ ko'phad $f(x)$ *ko'phadning* bo'luvchisi, $f(x)$ *esa* $g(x)$ *ko'phadning karralisi* deb yuritiladi.

Agar $P[x],\quad k>1,$ dan olingan $f_1(x),f_2(x),\ldots,f_k(x)$ ko'phadlardan har biri $\varphi(x),$ ko'phadga bo'linsa, u holda $\varphi(x)$ ko'phad $f_1(x),f_2(x),\ldots,f_k(x).$ *ko'phadlarning umumiy bo'luvchisi deyiladi.*

$P[x],\quad k>1,$ dan olingan $f_1(x),f_2(x),\ldots,f_k(x)$ ko'phadlarning *eng katta umumiy bo'luvchisi* (EKUB) berilgan ko'phadlarning barcha umumiy bo'luvchilariga qoldiqsiz bo'linadigan umumiy bo'luvchiga aytiladi. Bir vaqtda nolga teng bo'lmagan ixtiyoriy ko'phadlar uchun EKUB mavjud bo'lib, u noldan farqli o'zgarmas son ko'paytmasi aniqligida yagona ravishda aniqlangan. Barcha eng katta umumiy bo'luvchilar orasidan bosh koeffisiyenti 1 ga teng bo'lgan ko'phad tanlab olinadi. $f_1(x), f_2(x),\dots, f_k(x)$ ko'phadlarning EKUBi $(f_1(x), f_2(x),\dots, f_k(x))$. bilan belgilanadi

Agar $g(x)$ ko'phad $f(x)$ ning bo'luvchisi bo'lsa, u holda $(f(x), g(x)) = b_0^{-1}\, g(x),$ bo'ladi, bu yerda $b_0 -$ $g(x)$ ko'phadning bosh koeffisiyenti. Agar $f(x)$ ko'phad $g(x)$ ko'phadga bo'linmasa, u holda $f(x)$ va $g(x)$ ko'phadlarnig EKUBi $f(x)$ va $g(x)$ ko'phadlar uchun *Yevklid algoritmi*dagi oxirgi noldan farqli ko'phadga teng bo'lib, uni bosh koeffisiyentiga bo'lib olinadi. Yevklid algoritmi $f(x)$ va $g(x)$ ko'phadlar uchun quyidagicha ketma-ket bo'lish jarayonidan iborat: dastlab $f(x)$ ko'phad $g(x)$ga qoldiqli bo'linadi va $r_1(x)$ qoldiq hosil qilinadi; so'ngra $g(x)$ ko'phad $r_1(x)$ga qoldiqli bo'linadi va $r_2(x)$ qoldiq hosil qilinadi; agar $r_2(x) \neq 0,$ bo'lsa, $r_1(x)$ ko'phad $r_2(x)$ ga bo'linadi va hokazo bu jarayon qoldiqda nol hosil bo'lguncha davom ettiriladi. Oxirgi noldan farqli $r_k(x)$ qoldiq $f(x)$ va $g(x)$ ko'phadlarning EKUBidan iborat bo'ladi .

Uchta va uchtadan ko'p ko'phadlarning EKUBini topish quyidagi tenglikka asosan ikkita ko'phadning EKUBini topishga keltiriladi:

$$(f_1(x), f_2(x),\dots, f_k(x)) = ((f_1(x), f_2(x),\dots, f_{k-1}(x)), \quad f_k(x)), \quad k \geq 3.$$

Agar $(f_1(x), f_2(x),\dots, f_k(x)) = d(x),$ bo'lsa, u holda $P[x]$ xalqada shunday $g_i(x), \quad i = \overline{1,k},$ ko'phadlar mavjudki, ular uchun

$$d(x) = \sum_{i=1}^{k} f_i(x) g_i(x) = f_1(x) g_1(x) + \dots + f_k(x) g_k(x). \qquad (2)$$

tenglik o'rinli bo'ladi. (2) tenglik *ko'phadlar EKUBining chiziqli tasviri deyiladi.*

Agar bir necha ko'phadlarning EKUBi birga teng bo'lsa, ular o'zaro tub deyiladi. $P[x]$ xalqadan olingan $f_1(x), f_2(x),\dots, f_k(x)$ ko'phadlar uchun $P[x]$ xalqada shunday $g_i(x), \quad i = \overline{1,k},$ ko'phadlar mavjud bo'lib,

$$f_1(x) g_1(x) + f_2(x) g_2(x) + \dots + f_k(x) g_k(x) = 1.$$

tenglik o'rinli bo'lgandagina o'zaro tub bo'ladi.

Agar $h(x) \in P[x]$ ko'phad $P[x]$ dan olingan nol bo'lmagan $f_1(x), f_2(x),\dots, f_k(x)$ ko'phadlarning har biriga qoldiqsiz bo'linsa, $h(x)$ ko'phad $f_1(x), f_2(x),\dots, f_k(x)$ *ko'phadlarning umumiy karralisi deyiladi.* $P[x]$ dan olingan nol bo'lmagan $f_1(x), f_2(x),\dots, f_k(x), \quad k > 1,$ ko'phadlarning *eng kichik karralisi (EKUK)* deb, ularning shunday ummiy karralisiga aytiladiki, u boshqa ixtiyoriy umumiy karrali ko'phadning bo'luvchisi bo'ladi. Odatda barcha EKUKlar orasidan bosh koeffisiyenti 1ga teng bo'lgan ko'phad EKUK sifatida olinadi. $f_1(x), f_2(x),\dots, f_k(x)$ ko'phadlarning EKUKi $[f_1(x), f_2(x),\dots, f_k(x)]$. bilan belgilanadi. Ikkita ko'phadning EKUKi quyidagi formula bilan topiladi:

$$[f(x), g(x)] = \frac{f(x)\, g(x)}{a_0\, b_0\ (f(x), g(x))},$$

bu yerda $a_0, b_0 -$ mos ravishda $f(x)$ va $g(x)$ ko'phadlarning bosh koeffisiyentlari.

Uchta va undan ortiq ko'phadlarning EKUKini topish quyidagi tenglikka asosan ikkita ko'phadning EKUKini topishga keltiriladi:

$$[f_1(x), f_2(x), \ldots, f_k(x)] = [[f_1(x), f_2(x), \ldots, f_{k-1}(x)], \quad f_k(x)], \quad k \geq 3.$$

1-m i s o l. $\boldsymbol{Q}(x)$ xalqada $f(x) = 2x^4 + x^3 + x^2 - x - 3$ ko'phadni $g(x) = x^3 + 2x^2 - 1.$ ko'phadga bo'lgandagi $q(x)$ bo'linma va $r(x)$ qoldiqni toping.

Yechish. Qoldiqli bo'lish algoritmiga asosan:

$$\begin{array}{l|l} 2x^4 + x^3 + x^2 - x - 3 & x^3 + 2x^2 - 1 \\ \underline{2x^4 + 4x^3 - 2x} & 2x - 3 \\ \quad -3x^3 + x^2 + x - 3 & \\ \quad \underline{-3x^3 - 6x^2 + 3} & \\ \qquad 7x^2 + x - 6 & \end{array}$$

bu yerdan $f(x) = g(x)\,q(x) + r(x)$ ga asosan $q(x) = 2x - 3; \quad r(x) = 7x^2 + x - 6.$ ni hosil qilamiz. ■

2-m i s o l. $\boldsymbol{Z}_7\,[x]$ xalqada $f(x) = 2x^4 + x^3 + x^2 + 6x + 4$ ko'phadni $g(x) = x^3 + 2x^2 + 6.$ ko'phadga bo'lgandagi bo'linma $q(x)$ va qoldiq $r(x)$ ni toping.

Yechish. Qoldiqli bo'lish sxemasi bo'yicha hosil qilamiz :

$$\begin{array}{l|l} 2x^4 + x^3 + x^2 + 6x + 4 & x^3 + 2x^2 + 6 \\ \underline{2x^4 + 4x^3 + 5x} & 2x + 4 \\ \quad 4x^3 + x^2 + x + 4 & \\ \quad \underline{4x^3 + x^2 + 3} & \\ \qquad\qquad x + 1 & \end{array}$$

bu yerdan $f(x) = g(x)\,q(x) + r(x)$ ga asosan $q(x) = 2x + 4; \quad r(x) = x + 1.$ •

3-m i s o l. $\boldsymbol{Q}\,[x]$ xalqada $f(x) = 2x^4 + x^3 + x^2 - x - 3$ va $g(x) = x^3 + 2x^2 - 1.$ ko'phadlarning EKUBini toping

Yechish. Bu ko'phadlar uchun Yevklid algoritmi quyidagicha bo'ladi:

$$
\begin{array}{l|l}
2x^4+x^3+x^2-x-3 & x^3+2x^2-1 \\ \cline{2-2}
2x^4+4x^3-2x & 2x-3 \\ \cline{1-1}
\quad -3x^3+x^2+x-3 & \\
\quad -3x^3-6x^2+3 & \\ \cline{1-1}
\end{array}
$$

$$
\begin{array}{l|l}
x^3+2x^2-1 & 7x^2+x-6 \\ \cline{2-2}
x^3+\frac{1}{7}x^2-\frac{6}{7}x & \frac{1}{7}x+\frac{13}{49} \\ \cline{1-1}
\quad \frac{13}{7}x^2+\frac{6}{7}x-1 & \\
\quad \frac{13}{7}x^2+\frac{13}{49}x-\frac{78}{49} & \\ \cline{1-1}
\end{array}
$$

$$
\begin{array}{l|l}
7x^2+x-6 & \frac{29}{49}x+\frac{29}{49} \\ \cline{2-2}
7x^2+7x & \frac{343}{29}x-\frac{294}{29} \\ \cline{1-1}
\quad -6x-6 & \\
\quad -6x-6 & \\ \cline{1-1}
\qquad 0 & \\
\end{array}
$$

Demak, $f(x)$ va $g(x)$ ko'phadlar uchun *Yevklid algoritmi* ketma-ketligi quyidagi ko'rinishda bo'ladi:

$$f(x)=g(x)(2x-3)+r_1(x) \qquad (r_1(x)=7x^2+x-6),$$

$$g(x)=r_1(x)\left(\frac{1}{7}x+\frac{13}{49}\right)+r_2(x) \qquad \left(r_2(x)=\frac{29}{49}x+\frac{29}{49}\right),$$

$$r_1(x)=r_2(x)\left(\frac{343}{29}x-\frac{294}{29}\right).$$

Bu yerdan $f(x)$ va $g(x)$ ko'phadlarning EKUBi $r_2(x)$ ga tengligi kelib chiqadi. Endi $r_2(x)$ ni uning bosh koeffisiyenti $\frac{29}{49}$ ga bo'lib, $\left(f(x), g(x)\right)=x+1.$ ni hosil qilamiz. ■

4-m i s o l. Yevklid algoritmidan foydalanib $\boldsymbol{Q}(x)$ xalqada $f(x)=x^4+2x^3-x^2-4x-2,$ va $g(x)=x^4+x^3-x^2-2x-2.$ ko'phadlar uchun $\left(f(x),g(x)\right)=f(x)\varphi(x)+g(x)\psi(x)$: tenglikni qanoatlantiradigan $\varphi(x)$ va $\psi(x)$, ko'phadalrni toping.

Yechish. $f(x)$ va $g(x)$ ko’phadlar uchun Yevklid algoritmi quyidagi ko’rinishda bo’ladi:

$$f(x)=g(x)\cdot 1+(x^3-2x)$$
$$g(x)=(x^3-2x)(x+1)+(x^2-2)$$
$$x^3-2x=(x^2-x)\cdot x.$$

Demak, $(f(x), g(x))=x^2-2$.

Yevklid algoritmining ikkinchi tengligidan quyidagini hosil qilamiz:

$$x^2-2=g(x)-(x^3-2x)(x+1).$$

Bu tenglikda (x^3-2x) ning o’rniga uning birinchi tenglikdan topilgan qiymatini qo’yib:

$$x^2-2=g(x)-(f(x)-g(x))(x+1)=g(x)-(x+1)f(x)+(x+1)g(x)$$

ni hosil qilamiz. Shunday qilib,

$$(f(x),g(x))=x^2-2=-(x+1)f(x)+(x+2)g(x),\quad \text{ya'ni}$$
$$\varphi(x)=-(x+1),\quad \psi(x)=x+2. \ ■$$

Agar $f(x)$ va $g(x)$ ko’phadlarning darajalari noldan katta bo’lsa, u holda quyidagi tenglikda

$$f(x)\varphi(x)+g(x)\psi(x)=(f(x),g(x)) \qquad (3)$$

$\varphi(x)$ va $\psi(x)$ ko’phadlarni shunday tanlab olish mumkinki, $\varphi(x)$ ning darajasi $g(x)$ ning darajasidan, $\psi(x)$ niki esa – $f(x)$ nikidan kichik bo’ladi.

Amaliyotda $\varphi(x)$ va $\psi(x)$ ko’phadlarni topish uchun $f(x)$ va $g(x)$ ko’phadlarning o’rniga $f_1(x)=\dfrac{f(x)}{(f(x),g(x))}$, $g_1(x)=\dfrac{f(x)}{(f(x),g(x))}$ ko’phadlarni olish qulaydir. Bu holda dastlab $\varphi(x)$ va $\psi(x)$ ko’phadlar shunday tanlanadiki,

$$1=f_1(x)\varphi(x)+g_1(x)\psi(x). \qquad (4)$$

tenglik o’rinli bo’lsin. Bu quyidagicha amalga oshiriladi. Agar $f_1(x)$ yoki $g_1(x)-$ ko’phad nolinchi darajali bo’lsa, u holda $\varphi(x)$ va $\psi(x)$ ko’phadlar osongina tanlanadi: masalan, $g_1(x)=a\neq 0$ bo’lganda $\varphi(x)=0, \psi(x)=\dfrac{1}{a}$ deb olish mumkin. Agar $f_1(x)$ va $g_1(x)$ larning darajalari musbat bo’lsa, u holda $\varphi(x)$ va $\psi(x)$ ko’phadlarni aniqmas koeffisiyentlar orqali ifodalanadi (bu holda $\varphi(x)$ ning darajasi $g_1(x)$ darajasidan, $\psi(x)$ ning darajasi esa $f_1(x)$ ning darajasidan kichik hisoblanadi) va (4) tenglikning chap va o’ng tarafidagi x ning bir xil darajalari oldidagi koeffisiyentlar tenglashtiriladi (yoki x o’zgaruvchiga har xil qiymatlar berilib, $f_1(x)\varphi(x)+g_1(x)\psi(x)$); ko’phadning qiymati 1 ga tenglashtiriladi, natijada $\varphi(x)$ va $\psi(x)$ ko’phadlarning koeffisiyentlariga nisbatan chiziqli tenglamalar sistemasi hosil bo’ladi). Agar (4) tenglikning ikkala tomonini $(f(x), g(x))$ ga ko’paytirsak, u holda (3) tenglikni hosil qilamiz ((4) tenglikdagi $\varphi(x)$ va $\psi(x)$ ko’phadliar bilan). $\varphi(x)$ va $\psi(x)$ larni bunday topishda $\varphi(x)$ ning darajasi $g(x)$

va $(f(x),\ g(x))$ lar darajalari ayirmasidan kichik, $\psi(x)$ ning darajasi esa $f(x)$ va $(f(x),\ g(x))$ lar darajalari ayirmasidan kichik bo'ladi. (3) tenglikka kirgan $\varphi(x)$ va $\psi(x)$ ko'phadlarning darajalariga qo'yilgan bunday shartlarda ular bir qiymatli ravishda aniqlanadi.

5 - m i s o l . Aniqmas koeffisiyentlar usuli bilan
$f(x) = 3x^5 - 4x^4 - 3x^3 - 3x^2 + 4x - 1,\ g(x) = 3x^5 + 5x^4 + x^3 - x^2 - 3x + 1.$

ko'phadlar uchun $\mathbf{Q}\,[x]$ xalqada

$f(x)\varphi(x) + g(x)\psi(x) = (f(x), g(x))$

tenglikni qanoatlantiradigan $\varphi(x)$ va $\psi(x)$ ko'phadlar topilsin.

Yechish. Yevklid algoritmidan foydalanib,

$$(f(x), g(x)) = 3x^3 + 2x^2 + 2x - 1.$$

ekanligini topamiz.

U holda $f_1(x) = x^2 - 2x + 1, \quad g_1(x) = x^2 + x - 1.$

$f_1(x)$ va $g_1(x)$ ko'phadlar uchun (4) tenglikni aniqmas koeffisiyentli $\varphi(x)$ va $\psi(x)$ ko'phadlar orqali yozamiz:

$$1 = (x^2 - 2x + 1)(ax + b) + (x^2 + x - 1)(a_1 x + b_1).$$

x ning bir xil darajalari oldidagi koeffisiyentlarni tenglashtirib quyidagi tenglamalar sistemani hosil qilamiz:

$$\begin{cases} a + a_1 = 0 \\ b - 2a + b_1 + a_1 = 0 \\ -2b + a + b_1 - a_1 = 0 \\ b - b_1 = 0. \end{cases}$$

Bu sistemani yechib: $a = 3, \quad a_1 = -3, \quad b = 5, \quad b_1 = 4,$ larni topamiz, demak, $(f(x), g(x)) = f(x)(3x + 5) + g(x)(-3x + 4).$ ■

6 - M i s o l. Aniqmas koeffisiyentlar usuli bilan $\mathbf{Q}\,[x]$ xalqada shunday $\varphi(x)$ va $\psi(x)$ ko'phadlar topilsinki,

$$f(x)\varphi(x) + g(x)\psi(x) = (f(x), g(x))$$

tenglik o'rinli bo'lsin, bu yerda
$f(x) = 2x^4 - 4x^3 - 3x^2 + 7x - 2,\ g(x) = 6x^3 + 4x^2 - 5x + 1.$

Yechish. Yevklid algoritmidan foydalanib, $(f(x), g(x)) = 2x^2 + 2x - 1.$ ekanligini topamiz. U holda $f_1(x) = x^2 - 3x + 2, \quad g_1(x) = 3x - 1.$

(4) tenglikni quyidagicha tuzib olamiz:

$$1 = (x^2 - 3x + 2)a + (3x - 1)(bx + c).$$

x ga ketma-ket 0, 1, 2 qiymatlarni berib, Ushbu tenglamalar sistemasini hosil qilamiz:

$$\begin{cases} 2a - c = 1 \\ 2b + 2c = 1 \\ 10b + 5c = 1. \end{cases}$$

Bu sistemani yechib: $a = 0{,}9$; $a_1 = -0{,}3$; $c = 0{,}8$ larni topamiz, demak $(f(x), g(x)) = f(x) \cdot 0{,}9 + g(x)(-0{,}3x + 0{,}8)$. ■

$(f(x), g(x))$ dan farqli bo'lgan $h(x)$, ko'phadni

$$h(x) = f(x)\varphi_1(x) + g(x)\psi_1(x), \qquad (5)$$

ko'rinishda tasvirlash uchun $h(x)$ ning $(f(x), g(x))$ ga qoldiqsiz bo'linishi zarur va yetarlidir, bu yerda $\varphi_1(x), \psi_1(x) -$ lar qandaydir ko'phadlar. Agar bu holda $h(x)$ ning darajasi $f(x)$ va $g(x)$ larning darajalari yig'indisidan kichik bo'lsa, unda $\varphi_1(x)$ и $\psi_1(x)$ ko'phadlarni shunday tanlash mumkinki, $\varphi_1(x)$ ning darajasi $g(x)$, ning darajasidan kichik, $\psi_1(x) -$ niki esa $f(x)$ ning darajasidan kichik bo'ladi. Bu holda $\varphi_1(x)$ и $\psi_1(x)$ larni topish uchun, ularni aniqmas koeffisiyentlar orqali ifodalab, (5) tenglikning chap va o'ng tarafidagi x ning bir xil darajalari oldidagi koeffisiyentlar tenglashtiriladi ($\varphi_1(x)$ и $\psi_1(x)$ lar yagona ravishda topilmasligi ham mumkin). Agar $h(x)$ ning darajasi $f(x)$ va $g(x)$ lar darajalari yig'indisidan katta yoki teng bo'lsa va $h(x)$ =$(f(x), g(x))$ $m(x)$, bo'lsa, u holda dastlab $(f(x), g(x))$ ni (3) ko'rinishda tasvirlab, so'ngra $\varphi_1(x) = \varphi(x)m(x)$, $\psi_1(x) = \psi(x)m(x)$. deb olish kerak.

7 - M i s o l. $\boldsymbol{Q}[x]$ xalqada eng kichik darajali shunday $\varphi(x)$ va $\psi(x)$ ko'phadlar topilsinki,

$$(x^3 - x^2 - x + 1)\varphi(x) + (x^2 + x - 2)\psi(x) = x^4 - 1.$$

tenglik o'rinli bo'lsin.

Yechish. Yevklid algoritmidan foydalanib, $(f(x), g(x)) = x - 1$. ni topamiz. U holda $x^4 - 1 = (f(x), g(x)) \cdot m(x)$, bo'ladi, bu yerda $m(x) = x^3 + x^2 + x + 1$. $h(x) = x^4 - 1$ ning darajasi $f(x)$ va $g(x)$ lar darajalari yig'indisidan kichik bo'lganligi uchun quyidagi tenglikni yozib olamiz

$$x^4 - 1 = (x^3 - x^2 - x + 1)(ax + b) + (x^2 + x - 2)(a_1x^2 + b_1x + c_1).$$

x ning bir xil darajalari oldidagi koeffisiyentlarni tenglashtirib,

$$\begin{cases} a + a_1 = 1 \\ -a + b + a_1 + b_1 = 0 \\ -a - b - 2a_1 + b_1 + c_1 = 0 \\ a - b - 2b_1 + c_1 = 0 \\ b - 2c_1 = -1. \end{cases}$$

sistemani hosil qilamiz. Bu sistema cheksiz ko'p yechimga ega. Ulardan ixtiyoriylari (5) tenglikni qanoatlantiruvchi $\varphi_1(x)$ и $\psi_1(x)$ ko'phadlarning koeffisiyentlarini beradi. Masalan,

$$\varphi_1(x) = x + \frac{1}{3}, \quad \psi_1(x) = \frac{2}{3}(x+1).$$ deb olish mumkin■

$f(x)$ va $g(x)$ ko'phadlar EKUBini (3) ko'rinishda tasvirlash quyidagi masalani yechishda qo'llaniladi.

$\frac{m(\alpha)}{g(\alpha)}$, ko'rinishdagi ifoda berligan bo'lsin, bu yerda $m(x)$ va $g(x)-$ lar rasional koeffisiyentli ko'phadlar, $\alpha-$ esa $g(x)$ bilan o'zaro tub bo'lgan rasional koeffisiyentli $f(x)$ ko'phadning ildizi. $\frac{m(\alpha)}{g(\alpha)} = N(\alpha).$ tenglikni qanoatlantiradigan rasional koeffisiyentli $N(x)$ ko'phadni topish talab qilinadi. Bu masala kasr maxrajidagi irrasionallikni yo'qotish masalasi ham deyiladi.

Bu masalani yechish uchun $(f(x), g(x)) = 1$ tenglikni (3) ko'rinishda tasvirlab olamiz va $N(x)$ sifatida $m(x)v(x)$ ko'phadni olamiz (yoki bu ko'phadni $f(x)$ ga bo'lganda hosil bo'ladigan qoldiqni olamiz).

8 - M i s o l. Quyidagi ifodaning maxrajidagi irrasionallik yo'qotilsin

$$\frac{10\sqrt[3]{2} - 10}{\sqrt[3]{4} - 2\sqrt[3]{2} + 2}.$$

Yechish. Bu yerda

$$\alpha = \sqrt[3]{2}, \quad m(x) = 10x - 10, \quad g(x) = x^2 - 2x + 2, \quad f(x) = x^3 - 2.$$

Yevklid algoritmidan foydalanib $(f(x), g(x)) = 1.$ ni topamiz. So'ngra EKUB ning chiziqli ifodasini topamiz:

$1 = f(x)(-0{,}1x - 0{,}1) + g(x)(0{,}1x^2 + 0{,}3x + 0{,}4),$ ya'ni, $\psi(x) = 0{,}1x^2 + 0{,}3x + 0{,}4.$

Bu yerdan $m(x)\psi(x) = x^3 + 2x^2 + x - 4 = (x^3 - 2)\cdot 1 + 2x^2 + x - 2.$

$N(x) = 2x^2 + x - 2$ deb olamiz va $\frac{10\sqrt[3]{2} - 10}{\sqrt[3]{4} - 2\sqrt[3]{2} + 2} = 2\sqrt[3]{4} + \sqrt[3]{2} - 2.$ ni hosil qilamiz.■

M A SH Q L A R

4.1. $\boldsymbol{R}[x]$ xalqada

a) $x^4 - 4x^3 + 5x^2 + x - 1$ ko'phadni $x^2 - 2x - 3$ ko'phadga;

b) $5x^4 - x^2 + 6$ ko'phadni $x^2 + 3x + 2;$ ko'phadga;

c) $2x^2 - 3x + 1$ *ni* ko'phadni $x^3 + 4;$ ko'phadga

d) $2x^4 - 3x^3 + 4x^2 - 5x + 6$ на ko'phadni $x^2 - 3x + 1.$ ko'phadga bo'lishdan hosil bo'lgan $q(x)$ bo'linmani va $r(x)$ qoldiqni toping:

4.2. $a,\ p$ va q larning qanday qiymatlarida $\boldsymbol{Q}[x]$ xalqada quyidagi ko'phadlar x^2+ax+1 ko'phadga qolddiqsiz bo'linadi:

a) x^4+q; b) $x^4-21x+q$; c) x^4+px+q; d) x^4-7x^2+q.

4.3. $\boldsymbol{Z_3}[x], \boldsymbol{Z_5}[x]$ va $\boldsymbol{Q}[x]$ xalqalarda

a) x^5+x^2-x-1 *ni* ko'phadni x^3-2x+1; ko'phadga;

b) $2x^4+x^2+2x$ *ni* ko'phadni x^2-2. ko'phadga bo'lishdan hosil bo'lgan $q(x)$ bo'linmani va $r(x)$ qoldiqni toping

4.4. $(f(x),g(x))=(\alpha f(x),\beta g(x)),\ 0\neq\alpha,\beta\in P$. tenglikni isbotlang.

4.5. $\boldsymbol{R}[x]$ xalqada $f(x)$ ва $g(x)$ ko'phadlarning EKUBi va EKUK ini toping:

a) $f(x)=x^4+x^3-3x^2-4x-1,\quad g(x)=x^3+x^2-x-1$;

b) $f(x)=x^5+x^4-x^3-2x-1,\quad g(x)=3x^4+2x^3+x^2+2x-2$;

c) $f(x)=x^6-7x^4+8x^3-7x+7,\quad g(x)=3x^5-7x^3+3x^2-7$;

d) $f(x)=x^4-4x^3+1,\quad g(x)=x^3-3x^2+1$;

e) $f(x)=x^4+2x^3+2x+2,\quad g(x)=x^3+3x+2$.

4.6. Yevklid algoritmidan foydalanib, $\boldsymbol{Q}[x]$ xalqada $f(x)$ va $g(x)$ ko'phadlar uchun shunday $\varphi(x)$ va $\psi(x)$, ko'phadlarni topingki,

$(f(x),g(x))=f(x)\varphi(x)+g(x)\psi(x)$: tenglik o'rinli bo'lsin.

a) $f(x)=x^4+2x^3-x^2-4x-2,\quad g(x)=x^4+x^3-x^2-2x-2$;

b) $f(x)=x^5+3x^4+x^3+x^2+3x+1,\quad g(x)=x^4+2x^3+x+2$;

c) $f(x)=4x^4-2x^3-16x^2+5x+9,\quad g(x)=2x^3-x^2-5x+4$;

d) $f(x)=3x^3-2x^2+x+2,\quad g(x)=x^2-x+1$;

e) $f(x)=x^4-x^3-4x^2+4x+1,\quad g(x)=x^2-x-1$;

f) $f(x)=x^5-5x^4-2x^3+12x^2-2x+12,\quad g(x)=x^3-5x^2-3x+17$.

4.7. Agar $f(x)$ va $g(x)-$ lar mos ravishda n ва m, darajali o'zaro tub ko'phadlar bo'lsa, u holda $f(x)\varphi(x)+g(x)\psi(x)=1$, tenglikni qanoatlantiradigan shunday darajasi $m-1$ dan katta bo'lmagan $\varphi(x)$ va darajasi $n-1$, dan katta bo'lmagan $\psi(x)$ faqat bir juft ko'phadlarni topish mumkinligini isbotlang.

4.8. n - darajali $f(x)$ va m - darajali $g(x)$ ko'phadlarning EKUBi k - darajali $d(x)-$ ko'phad bo'lsin. Darajasi $m-k-1$ dan oshmaydgan $\varphi(x)$ va darajasi $n-k-1$, dan oshmaydigan $\psi(x)$ ko'phadlarni shunday tanlab olish mumkinki (yagona ravishda),

$$f(x)\varphi(x)+g(x)\psi(x)=d(x).$$

tenglik o'rinli bo'lishini isbotlang.

4.9. Aniqmas koeffisiyentlar usuli bilan $\boldsymbol{Q}[x]$ xalqada shunday $\varphi(x)$ va $\psi(x)$ ko'phadlarni tanlab olingki, $f(x)\varphi(x)+g(x)\psi(x)=1$: tenglik o'rinli bo'lsin:

a) $f(x)=x^3,\qquad g(x)=(1-x)^2$;

b) $f(x)=x^4,\qquad g(x)=x^3-3x^2+4$;

c) $f(x)=x^3+3x+3, \quad g(x)=x^2-x-2;$

d) $f(x)=x^4-4x^3+1, \quad g(x)=x^3-3x^2+1.$

4.10. $\boldsymbol{Q}[x]$ xalqada

a) $(x-1)^2$ ga bo'lganda 1 qoldiq qoladigan va na $(x+1)^2;$ ga bo'lganda 5 qoldiq qoladigan;

b) $(x-1)^2$ ga bo'lganda $2x$ va $(x-2)^3;$ ga bo'lganda esa $3x$ qoldiq qoladigan;

c) $(x-1)^2$ ga bo'lganda $1-2x$ va $(x+1)^2.$ ga bo'lganda $1+2x$ qoldiq qoladigan eng kichik darajali ko'phadni toping.

4.11. Quyidagi kasrlarning maxrajidagi irrasionallikni yo'qoting:

a) $\dfrac{1}{1+\sqrt[3]{2}+2\sqrt[3]{4}}$; b) $\dfrac{7}{1-\sqrt[4]{2}+\sqrt{2}}$; c) $\dfrac{1}{\sqrt[3]{2}+\sqrt{2}-1}$.

4.12. Eng kichik darajali $\varphi(x)$ ва $\psi(x)$ ko'phadlarni shunday tanlab olingki, quyidagi tengliklar o'rinli bo'lsin:

a) $(x^4-2x^3-4x^2+6x+1)\varphi(x)+(x^3-5x-3)\psi(x)=x^4;$

b) $(x^4+2x^3+x+1)\varphi(x)+(x^4+x^3-2x^2+2x-1)\psi(x)=x^3-2x.$

4.13 *. $x^m\varphi(x)+(1-x)^n\psi(x)=1.$ tenglikni qanoatlantiradigan $\varphi(x)$ ва $\psi(x)$ ko'phadlarni toping.

4.14. $\boldsymbol{Z_3}[x], \boldsymbol{Z_5}[x]$ va $\boldsymbol{Q}[x]$ xalqalarda $f(x)$ va $g(x)$ ko'phadlarning EKUBi va EKUKlarini toping:

a) $f(x)=x^3+x^2+2x+2, \quad g(x)=x^2+x+1;$

b) $f(x)=x^4+x^3-x^2+1, \quad g(x)=x^3-2x-1;$

c) $f(x)=x^4-2x^2+2x-2, \quad g(x)=x^3-x^2+1.$

4.15. $\boldsymbol{Z_2}[x]$ xalqada $f(x)$ va $g(x)$ ko'phadlarning EKUBini va $(f(x)g(x))=f(x)\varphi(x)+g(x)\psi(x)$ tenglikni qanoatlantiradigan $\varphi(x)$ ва $\psi(x)$ ko'phadlarni toping:

a) $f(x)=x^5+x^4+1, \quad g(x)=x^4+x^2+1;$

b) $f(x)=x^5+x^3+x+1, \quad g(x)=x^4+1;$

s) $f(x)=x^5+x+1, \quad g(x)=x^4+x^3+1;$

d) $f(x)=x^5+x^3+x, \quad g(x)=x^4+x+1.$

§ 2. Ko'phadlarning ildizlari

$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n\in P[x], \alpha\in P.$ bo'lsin. $f(x)$ *ko'phadning* $x=\alpha$ dagi qiymati deb, P maydonning $a_0\alpha^n+a_1\alpha^{n-1}+\cdots+a_{n-1}\alpha+a_n$ elementiga aytiladi va $f(\alpha).$ bilan belgilanadi.

$f(x)$ ko'phadni $x-\alpha$ ga bo'lgandagi qoldiqn $f(\alpha)$ ga teng (*Bezu teorema*).

Agar

$$f(x)=(x-\alpha)g(x)+f(\alpha),\quad g(x)=b_0x^{n-1}+b_1x^{n-2}+\cdots+b_{n-2}x+b_{n-1},$$

bo'lsa, $g(x)$ ko'phadning koeffisiyentlarini va $f(\alpha)$ ning qiymatini Gorner sxemasi yordamida topish qulay bo'ladi:

	a_0	a_1	$\dots$	a_{n-1}	a_n
α	b_0	b_1	$\dots$	b_{n-1}	$f(\alpha)$

Bu yerda

$$b_0=a_0,\quad b_1=\alpha b_1+a_1,\dots,b_k=\alpha\, b_{k-1}+a_k,\dots,b_{n-1}=\alpha\, b_{n-2}+a_{n-1},$$
$$f(\alpha)=\alpha\, b_{n-1}+a_n.$$

Agar $f(\alpha)=0$ bo'lsa, $\alpha\in P$ element $f(x)\in P[x]$, *ko'phadning ildizi* deyiladi. Bezu teoremasidan $x-\alpha$ chiziqli ko'phad $f(x)$ ko'phadning bo'luvchisi bo'lganda va faqat shu holdagina α soni $f(x)$ ko'phadning ildizi ekanligi kelib chiqadi.

Agar $(x-\alpha)^k$, $k\in \mathbf{N}$, ko'phad $f(x)$, ning bo'luvchisi bo'lsa, lekin $(x-\alpha)^{k+1}$ ko'phad esa $f(x)$ ning bo'luvchisi bo'lmasa, u holda α soni $f(x)$ ko'phadning *k-karrali ildizi deyiladi*. $k=1$ bo'lgan holda ildiz *sodda ildiz* deyiladi.

$f(x)=a,\ a\in P$ ko'phad o'zgarmas ko'phad deyiladi. Har qanday o'zgarmas bo'lmagan $f(x)\in P[x]$ ko'phad P, maydonning shunday kengaytmasi mavjudki, bu kengaytmada $f(x)$ ko'phadning barcha ildizlari yotadi, boshqacha qilib aytganda bu kengaytmada $f(x)$ ko'phad chiziqli ko'paytuvchilarga ajraladi, ya'ni $f(x)=a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$, bu yerda a_0- $f(x)$ ko'phadning bosh koeffisiyenti.

$f(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n,\quad a_0\neq 0,$ ko'phadning ildizlari $\alpha_1,\alpha_2,\dots,\alpha_n$ uning koeffisiyentlari bilan quyidagi *Viyet formulalari orqali bog'langan:*

$$-\frac{a_1}{a_0}=\alpha_1+\alpha_2+\cdots+\alpha_n=\sum_{i=1}^{n}\alpha_i,$$

$$\frac{a_2}{a_0}=\alpha_1\alpha_2+\alpha_1\alpha_3+\cdots+\alpha_1\alpha_n+\alpha_2\alpha_3+\cdots+\alpha_{n-1}\alpha_n=\sum_{i_1<i_2}^{n}\alpha_{i_1}\,\alpha_{i_2},$$

....

$$\frac{a_k}{a_0}=(-1)^k\sum_{i_1<i_2<\cdots<i_k}\alpha_{i_1}\,\alpha_{i_2}\cdots\alpha_{i_k},$$

$$-\frac{a_n}{a_0}=(-1)^n\,\alpha_1\,\alpha_2\cdots\alpha_n.$$

$f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n\in P[x]$, $a_0\neq 0$, $n\geq 1$ - ko'phadning hosilasi deb,

$f'(x)=na_0x^{n-1}+(n-1)a_1x^{n-2}+\cdots+2a_{n-2}x+a_{n-1}$ ko'phadga aytiladi.

Birinchi hosiladan yana bir marta olingan hosila $f(x)$ ko'phadning ikkinchi tartibli hosilasi deyiladi va $f''(x)$ orqali belgilanadi. O'zgarmas ko'phadning hosilasi ta'rifga binoan nol ko'phadga teng deb hisoblanadi. *Agar* $f(x)\in P[x]$ *ko'phad o'zining birinchi tartibli hosilasi bilan o'zaro tub bo'lsa, u* P, *maydonning o'zida ham va uning har qanday kengaytmasida ham ildizga ega bo'lmaydi.*

Nol xarakteristikali maydon ustidagi ko'phadlar uchun quyidagi tasdiq o'rinli: ko'phadning k–karrali ildizi uning hosilasining $(k-1)-$karrali ildizi bo'ladi.

$f(x)=a_0(x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}\cdots(x-\alpha_s)^{k_s}\in P[x],$ va $i\neq j,$ $k_i>0.$ bo'lganda $char\ P=0,\ \alpha_i\neq\alpha_j$ bo'lsin. U holda

$$(f(x),f'(x))=(x-\alpha_1)^{k_1-1}(x-\alpha_2)^{k_2-1}\cdots(x-\alpha_s)^{k_s-1}$$

va $f_1(x)=\dfrac{f(x)}{(f(x),f'(x))}=a_0(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_s)$ ko'phadlar $f(x)$ ko'phad bilan bir xil ildizlarga ega bo'ladi va bu ildizlarning barchasi sodda ildizlardan iborat bo'ladi.

Agar $\alpha_1,\alpha_2,\cdots,\alpha_n$ lar P, maydonning har xil elementlari bo'lsa, $\beta_1,\beta_2,\cdots,\beta_n-$ lar esa P, maydonning ixtiyoriy elementlari bo'lsa, u holda $P[x]$ xalqada $f(\alpha_i)=\beta_i,$ $i=\overline{0,n}.$ tenglikni qanoatlantiradigan darajasi n dan oshmaydigan faqat va faqat bitta $f(x)$ ko'phad mavjud bo'ladi. Bu ko'phad quyidagi formula orqali beriladi:

$$f(x)=\sum_{i=0}^{n}\beta_i\frac{(x-\alpha_0)\cdots(x-\alpha_{i-1})(x-\alpha_{i+1})\cdots(x-\alpha_n)}{(\alpha_i-\alpha_0)\cdots(\alpha_i-\alpha_{i-1})(\alpha_i-\alpha_{i+1})\cdots(\alpha_1-\alpha_n)}. \qquad (*)$$

(*) formula Lagranjning interpolyasion formulasi deb yuritiladi.

$f(x)$ ko'phadni (talab qilingan xossalari bilan) *Nyutonnig interpolyasion formulasi* orqali ham hosil qilish mumkin:

$$f(x)=\lambda_0+\lambda_1(x-\alpha_0)+\lambda_2(x-\alpha_0)(x-\alpha_1)+\cdots$$

$$\cdots+\lambda_n(x-\alpha_0)(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-1}),$$

bu yerda $\lambda_0,\lambda_1,\ldots,\lambda_n$ koeffisiyentlar x ning o'rniga ketma-ket $\alpha_1,\alpha_2,\cdots,\alpha_n$ qiymatlarni quyish bilan aniqlanadi.

1 - m i s o l. $C[x]$ xalqada $f(x)=x^4+(2+i)x^2+(1+2i)x-i$ ko'phadni $x+i.$ ga bo'lganda hosil bo'ladigan $q(x)$ bo'linma va $r(x)$ qoldiqni toping.

Yechish. Gorner sxemasini tuzamiz:

	1	0	$2+i$	$1+2i$	$-i$
$-i$	1	$-i$	$1+i$	$2+i$	$1-3i$

Demak, $q(x)=x^3-ix^2+(1+i)x+2+i,$ $r(x)=1-3i$. ■

2 - M i s o l. Gorner sxemasidan foydalanib, $f(x)=x^4-2x^3+x^2+x+1.$ ko'phad uchun $f(-3),$ ni hisoblang.

Yechish. Gorner sxemasini tuzamiz:

	1	-2	1	1	1
-3	1	-5	16	-47	42

Demak, $f(-3) = 42.$ ■

Har qanday $f(x) = a_0x^4 + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ ko'phad va ixtiyoriy α son uchun $f(x)$ ning $x - \alpha$: ayirmaning darajalari orqali yoyilmasini yozish mumkin:

$$f(x) = b_0(x-\alpha)^n + b_1(x-\alpha)^{n-1} + \cdots + b_{n-1}(x-\alpha) + b_n.$$

Bu yoyilmaning koeffisiyentalirini topish uchun dastlab $f(x)$ ni $x - \alpha.$ga qoldiqli bo'lish kerak. Natijada qoldiqda b_n, hosil bo'ladi, bo'linma esa qandaydir $q(x).$ ko'phad bo'ladi. So'ngra $q(x)$ bo'linma $x-\alpha;$ ga bo'linadi, qoldiqda b_{n-1}, bo'linma esa- $q_1(x).$ bo'ladi. Keyin $q_1(x)$ ni $x-\alpha,$ ga bo'lamiz va b_{n-2} qoldiqni hosil qilamiz va hakoza shu kabi davom etib yoyilmaning barcha koeffisiyentlarini hosil qilamiz.

Bu yoyilmaning koefffisiyentlarini Gorner sxemasi yordamida hisoblash juda qulay bo'lib, u barcha hisoblashlarni bita jadvalga birlashtiradi..

3-m i s o l. $f(x) = x^4 + 2x^3 - x - 1$ ko'phad $x + 2.$ ning darajalari bo'yicha yoying.

Yechish. Gorner sxemasini tuzamiz, uning birinchi satriga $f(x)$ ko'phadning koeffisiyentlarini yozib chiqamiz. Ikkinchi satrida $f(x)$ ni $x+2,$ ga bo'linganda hosil bo'ladigan $q(x)$ bo'linmaninng koeffisiyentlarini va b_4 qoldiqni yozamiz, uchinchi satriga $q(x)$ ni $x+2$ ga bo'lingandagi $q_1(x)$ bo'linmaning koeffisiyentlarini va b_3 qoldiqni yozamiz va hakoza shu yo'sinda davom etamiz :

	1	2	0	-1	-1
-2	1	0	0	-1	1
-2	1	-2	4	-9	
-2	1	-4	12		
-2	1	-6			
-2	1	.			

Demak, $f(x) = (x+2)^4 - 6(x+2)^3 + 12(x+2)^2 - 9(x+2) + 1.$ ■

4-m i s o l. Agar $f(x) = x^4 - 5x^3 - 3x^2 + 9.$ bo'lsa, Gorner sxemasi yordamida $f(x+3),$ ko'phadni x ning darajalari bo'yicha yoying.

Yechish. $f(x+3)$ ko'phadni x ning darajalari bo'yicha yoyish uchun dastlab $f(x)$ ni $x-3,$ ning darajalari bo'yicha yoyamiz, so'ngra bu yoyilmada x ni $x+3.$ ga almashtiramiz.

Gorner sxemasini tuzamiz:

	1	-5	-3	0	9
3	1	-2	-9	-27	-72
3	1	1	-6	-45	
3	1	4	6		

3	1	7
3	1	

Demak, $f(x) = (x-3)^4 + 7(x-3)^3 + 6(x-3)^2 - 45(x-3) - 72.$

Bu yerdan esa $f(x-3) = x^4 + 7x^3 + 6x^2 - 45x - 72.$ ■

5-m i s o l. Gorner sxemasi yordamida $f(x) = x^4 - 7x^3 + 9x^2 + 8x + 16.$ ko'phadning 4 ga teng bo'lgan ildizining necha karrali ekanligini aniqlang.

Yechish. α soni $f(x)$ ko'phadning necha karrali ildizi bo'lishini tekshirish uchun Gorner sxemasidan quyidagicha foydalanish mumkin. Avval $f(x)$ ko'phad $x-\alpha$, ga bo'linadi, agar qoldiq nolga teng bo'lsa, hosil qilingan bo'linma yana $x-\alpha$ ga bo'linadi va hakoza bu jarayon qoldiq noldan farqli bo'lguncha davom ettiriladi.

Shunga asosan Gorner sxemasini tuzamiz:

	1	-7	9	8	16
4	1	-3	-3	-4	0
4	1	1	1	0	
4	1	5	21	.	

Demak, $\alpha = 4-$ soni berilgan ko'phadning ikki karrali ildizi ekan. ■

Ko'phadni $x-\alpha$ ning darajalari bo'yicha yoyilmasi maxraji chiziqli ikki hadning darajasidan iborat bo'lgan rasional kasrni sodda kasrlarga yoyishda ham ishlatilishi mumkin.

6-m i s o l. Gorner sxemasidan foydalanib $\frac{x^3 + x - 1}{(x+2)^5}.$ kasrni sodda kasrlarga yoying.

Yechish. $f(x) = x^3 + x - 1$ ko'phadni $x - (-2) = x + 2$: ayirmaning darajalari bo'yicha yoyamiz:

	1	0	1	-1
-2	1	-2	5	-11
-2	1	-4	13	
-2	1	-6		
-2	1	.		

Demak, $f(x) = (x+2)^3 - 6(x+2) + 13(x+2) - 11.$

Bu yerdan izlangan yoyilmani hosil qilamiz

$$\frac{x^3 + x - 1}{(x+2)^2} = \frac{1}{(x+2)^2} - \frac{6}{(x+2)^3} + \frac{13}{(x+2)^4} - \frac{11}{(x+2)^5}.$$ ■

$f(x) = b_0(x-\alpha)^n + b_1(x-\alpha)^{n-1} + \cdots + b_{n-1}(x-\alpha) + b_n$ ko'phadning

$x-\alpha$ ning darajalari bo'yicha yoyilmasi orqali bu ko'phadning ixtiyoriy tartibli hosilalarining α dagi qiymatlarini topish mumkin, ya'ni:

$f^{(k)}(\alpha) = k! \quad b_{n-k}.$ (* *)

7-m i s o l. $f(x)=x^4+2x^3-x-1$ ko'phad hosilalarining $x=-2.$dagi qiymatlarini hisoblang.

Yechish. 3-misoldagi Gorner sxemasi $f(x)=x^4+2x^3-x-1$ ko'phad hosilalarining $x=-2$: qiymatlarini hisoblashga imkon beradi:

$$f'(-2)=-9, \quad f''(-2)=12\cdot 2!=24,$$

$$f'''(-2)=-6\cdot 3!=-36, \quad f^{('''')}(-2)=1\cdot 4!=24. \ ■$$

(* *) munosabatdan $f(x)$ ko'phadning $x-\alpha$: ning darajalari bo'yicha yoyish uchun *Teylor formulasi* kelib chiqadi:

$$f(x)=f(\alpha)+\frac{f'(\alpha)}{1!}(x-\alpha)+\frac{f''(\alpha)}{2!}(x-\alpha)^2+\cdots+\frac{f^{(n)}(\alpha)}{n!}(x-\alpha)^n.$$

8-m i s o l. $f(x)$ ko'phad $x-2$ va $x-3$ larga bo'linganda mos ravishda 5 va 7 qoldiqlar hosil bo'ladi. $f(x)$ ni $(x-2)(x-3)$ ga bo'lganda hosil bo'ladigan qoldiqni toping.

Yechish. $f(x)$ ni $(x-2)(x-3)$ larga bo'lganda $r(x)$ qoldiq $ax+b,$ ko'rinishda bo'ladi, ya'ni $f(x)=(x-2)(x-3)q(x)+(ax+b).$

$x=2$ va $x=3$ (2 va 3 $(x-2)$ va $(x-3)$ larning ildizlari) larniyuqoridagi tenglikka qo'yib a va b: larni topish uchun quyidagi chiziqli tenglamalar sistemasini hosil qilamiz:

$$\left.\begin{array}{l} f(2)=2a+b=5 \\ g(3)=3a+b=7 \end{array}\right\}$$

Bu sistemadan $a=2,\ \ b=1.$ Demak, $f(x)$ ni $(x-2)(x-3)$ ga bo'lganda hosil bo'ladigan qoldiq $2x+1.$ ga teng. ■

9-m i s o l. Qoldiqli bo'lish algoritmini qo'llamasdan $f(x)=x^{128}+x^{64}+x^{32}+x^8+x^4+x^2+x+1$ ko'phadni $x^2-1.$ga bo'lganda hosil bo'ladigan qoldiqni toping.

Yechish. $f(x)$ ni x^2-1 ga bo'lganda hosil bo'ladigan $r(x)$ qoldiq $ax+b.$ ko'rinishda bo'ladi. $x=1$ va $x=-1$ (1 va –1 x^2-1 ning ildizlari) larda $f(x)$ ko'phadning qiymatlarini hisoblab, a va b:larni topish uchun quyidagi tenglamalar sistemasini hosil qilamiz:

$$\left.\begin{array}{l} f(1)=8=a+b \\ f(-1)=6=-a+b \end{array}\right\}.$$

Bu sistemaning yechimlari $a=1,\ \ b=7.$ Demak, $f(x)$ ni x^2-1ga bo'lgand-agi qoldiq $x+7$ ga teng. ■

10-m i s o l. Qanday p,q,r larda $f(x)=x^3+px^2+qx+r$ ko'phad $(x-1)^3$ ga $\boldsymbol{R}[x]$ xalqada qoldiqsiz bo'linadi.

Yechish. $f(x)$ ning $(x-1)^3$ ga qoldiqsiz bo'linishi $x=1$ning $f(x)$ ko'phadning 3-karrali ildizi ekanligiga teng kuchli. Demak, $x=1$ soni $f(x)$, $f'(x)$ va $f''(x).$ larning ildizlari bo'ladi. $f'(x)$ va

$f''(x)$: $\quad f'(x)=3x^2+2px+q, \quad f''(x)=6x+2p. \quad x=1$ ni $\quad f(x), \quad f'(x)$ ва $f''(x)$ larga qo'yib, p,q va r larni topish quyidagi uchta qiziqli tenglamalar sistemasini hosil qilamiz:

$$\left.\begin{aligned} f(1)&=1+p+q+2=0\\ f'(1)&=3+2p+q=0\\ f''(1)&=6+2p=0 \end{aligned}\right\}.$$

Bu sistemadan: $\quad p=-3, \quad q=3, \quad r=-1$. ■

11-m i s o l. 1 soni $f(x)=x^{2n}-nx^{n+1}+nx^{n-1}-1$. ko'phadning uch karrali ildizi ekanligini isbotlang.

Yechish. $x=1$ ning $f(x)$ ko'phadning uch karrali ildizi ekanligi $x=1$ ning $f(x), \quad f'(x)$ ва $f''(x)$ ko'phadlarlarning ildizi ekanligiga teng kuchli.

$f'(x)$ va $f''(x)$: larni topamiz:

$$f'(x)=2nx^{2n-1}-n(n+1)x^n+n(n-1)x^{n-2},$$

$$f''(x)=2n(2n-1)x^{2n-1}-n^2(n+1)x^{n-1}+n(n-1)(n-2)x^{n-3}.$$

U holda

$$f(1)=1-n+n-1=0,$$

$$f'(1)=2n-n(n+1)+n(n-1)=2n-n^2-n+n^2-n=0,$$

$$f''(1)=2n(2n-1)-n^2(n+1)+n(n-1)(n-2)=$$

$$=4n^2-2n--n^3-n^2+n^3-n^2-2n^2+2n=0.$$

Shunday qilib, 1 soni haqiqatdan ham $x^{2n}-nx^{n-1}+nx^{n-1}-1$ kKo'phadning uch karrali ildizi ekan. ■

12-m i s o l. Nayti kotorыy imeyet te je korni, chto i $f(x)=x^6-6x^4-4x^3+9x^2+12x+4$, ko'phadning ildizlariga ega bo'ladigan, lekin karali ildizlarga ega bo'lmaydigan $\varphi(x)$, ko'phadni toping ($f(x)$), ko'phadning karali ildizlarini ajratingva $f(x)$ ko'phadni S maydon ustida chiziqli ko'paytuvchilarga ajrating).

Yechish. Berilgan ko'phadning hosilasini topamiz $f'(x)$:

$$f'(x)=6x^5-24x^3-12x^2+18x+12.$$

Yevklid algoritmidan foydalanib $f(x)$ va $f'(x)$ ko'phadlarning EKUBini topamiz:

$(f(x), f'(x))=x^4+x^3-3x^2-5x-2.$ Bu holda

$$\varphi(x)=\frac{f(x)}{(f(x), f'(x))}=x^2-x-2=(x+1)(x-2)$$

ko'phad $f(x)$ ko'phadning ildizlariga ega bo'ladi va $x_1=-1, \quad x_2=2$.

Gorner sxemasidan foydalanib $x_1=-1$ ildiz $f(x)$ ko'phadning 4 karrali ildizi ekanligini topamiz, $x_2=2$ ildiz esa $f(x)$ ning 2 karrali ildizi bo'ladi. Demak, $f(x)$ quyidagicha chiziqli ko'paytuvchilarga ajraladi: $f(x)=(x+1)^4(x-2)$. ■

13-m i s o l. a ning qiymatini shunday aniqlangki, $x^3-21x+a$ ko'phadning bir ildizi ikkinchisining ikkkilanganligiga teng bo'lsin.

Yechish. $\alpha_1,\alpha_2,\alpha_3-$ lar $x^3-21x+a.$ ko'phadning ildizlari bo'lsin.

$\alpha_1=2\alpha_2.$ bo'lsin. U holda Viyet formulalariga asosan:

$$\left.\begin{array}{l}\alpha_1+\alpha_2+\alpha_3=0\\ \alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3=-21\\ \alpha_1\alpha_2\alpha_3=-a\end{array}\right\},\qquad \left.\begin{array}{l}2\alpha_2+\alpha_2+\alpha_3=0\\ 2\alpha_2^2+3\alpha_2\alpha_3=-21\\ 2\alpha_2^2\alpha_3=-a\end{array}\right\},$$

$$\left.\begin{array}{l}3\alpha_2+\alpha_3=0\\ 2\alpha_2+3\alpha_2\alpha_3=-21\\ 2\alpha_2^2\alpha_3=-a\end{array}\right\},\qquad \left.\begin{array}{l}\alpha_3=-3\alpha_2\\ 2\alpha_2^2-9\alpha_2^2=-21\\ -6\alpha_2^3=-a\end{array}\right\},\quad \alpha_2^2=3,\quad \alpha_2=\pm\sqrt{3}.$$

Bu yerdan $a=6\cdot(\pm3\sqrt{3})=\pm18\sqrt{3}.$

Shunday qilib, $a=\pm18\sqrt{3}$ bo'lganda $x^3-21x+a$ ko'phadning bir ildizi ikkinchisining ikkkilanganligiga teng ekan. ■

14-m i s o l. 1, -1, 3 ildizlarga ega bo'lgan bosh koeffisiyenti birga teng bo'lgan uchinchi darajali ko'phad quring.

Yechish. Izlanayotgan ko'phad quyidagicha bo'lsin:
$f(x)=x^3+a_1x^2+a_2x+a_3.$

U holda Viyet formulalariga asosan:

$$a_1=-(\alpha_1+\alpha_2+\alpha_3)=-(1-1+3)=-3$$

$$a_2=\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3=1\cdot(-1)+1\cdot3+(-1)\cdot3=-1$$

$$a_3=-\alpha_1\alpha_2\alpha_3)=-1\cdot(-1)\cdot3=3.$$

Demak, izlanayotgan ko'phad $f(x)=x^3-3x^2-x+3$ dan iborat. ■

15-m i s o l. $f(x)=5x^4-3x^3+2x-1.$ ko'phadning kompleks ildizla-riga teskari bo'lgan sonlarning yig'indisini toping.

Yechish. $\alpha_1,\alpha_2,\alpha_3,\alpha_4-$ lar $f(x).$ ning ildizilari bo'lsin.

$\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}+\frac{1}{\alpha_4}$ yig'inidini topish kerak. Bu holda

$$\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}+\frac{1}{\alpha_4}==\frac{\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_3\alpha_4+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2\alpha_3\alpha_4}.$$

Viyet formulalariga asosan oxirgi kasrning maxraji $\left(-\frac{a_3}{a_0}\right)=-\frac{2}{5}$ ga teng, su-rati esa $\frac{a_4}{a_0}=-\frac{1}{5}.$ ga teng. Demak,

$$\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}+\frac{1}{\alpha_4}=\left(-\frac{2}{5}\right):\left(-\frac{1}{5}\right)=2\,.\ ■$$

16-m i s o l. Lagranj interpolyasion formulsidan foydalanib qiymatlari quyidagi jadval bo'yicha berilgan ko'phadni tuzing:

x	1	3	4
$f(x)$	2	-2	-1

Yechish. Izlanayotgan ko'phadni ko'rinishi quyidagicha bo'ladi: $f(x)=2\cdot\frac{(x-3)(x-4)}{(1-3)(1-4)}-2\frac{(x-1)(x-4)}{(3-1)(3-4)}-\frac{(x-1)(x-3)}{(4-1)(4-3)}=x^2-6x+7.$ ■

17-m i s o l. Nyuton interpolyasion formulasidan foydalanib qiymatlari quyidagi jadval bo'yicha berilgan eng kichik darajali ko'phadni tuzing:

x	1	-2	3
$f(x)$	4	7	12

Yechish. Nyuton formulasidan foydalanib:

$f(x)=4+\lambda_1(x-1)+\lambda_2(x-1)(x+2)$.

$x=-2,$ deb olib $7=4+\lambda_1(-3),\quad \lambda_1=-1$ larga ega bo'lamiz. $x=3,$ da esa $12=4-(3-1)+\lambda_2(3-1)(3+2),\quad \lambda_2=1$. Izlanayotgan ko'phad $f(x)=4-(x-1)+(x-1)(x+2)=x^2+3$ bo'ladi. ■

MAShQLAR

4..16. $f(x)\in \mathbf{Q}[x]$ ko'phadning koeffisiyentlari yig'indisini toping:

a) $f(x)=(2-5x+x^3)^{211}(3-7x+9x^2-5x^3)^{135}$;

b) $f(x)=(7-3x-3x^5)^{100}(5-x^2-5x^7)^{1000}$.

4.17. $\mathbf{C}[x]$ xalqada $q(x)$ bo'linma va $r(x)$ qoldiqni toping:

a) $x^4-2x^3+4x^2-6x+8$ на $x-1$;

b) $2x^5-5x^3-8x$ на $x+3$;

s) $4x^3+x^2$ на $x+1+i$;

d) x^3-x^2-x на $x-1+2i$.

4.18. Gorner sxemasidan foydalanib $f(\alpha)$, ni hisoblang:

a) $f(x)=x^4-3x^3+6x^2-10x+16,\ \alpha=4$;

b) $f(x)=5x^4-7x^3+8x^2-3x+7,\ \alpha=3$;

s) $f(x)=2x^5+2x^4-3x^3+4x^2-6x+5,\ \alpha=-\frac{1}{2}$;

d) $f(x)=x^5+(1+2i)x^4-(1+3i)x^2+7,\ \alpha=-2-i$;

ye) $f(x)=x^5+(1-2i)x^4-(3+i)x^2+7,\ \alpha=-1+2i$.

4.19. Gorner sxemasidan foydalanib $f(x)$ ko'phadni $x-\alpha$ ning darajalari bo'yicha yoying:

a) $f(x)=x^4+2x^3-3x^2-4x+1,\ \alpha=-1$;

b) $f(x)=x^5,\ \alpha=1$;

s) $f(x)=x^4-8x^3+24x^2-50x+90,\ \alpha=2$;

d) $f(x)=x^4+2ix^3-(1+i)x^2-3x+7+i,\quad \alpha=-i$;

e) $f(x)=x^4+(3-8i)x^3-(21+18i)x^2-(33-20i)x+7+18i,\quad \alpha=-1+2i$.

4.20. Gorner sxemasidan foydalanib x ning darajalari bo'yicha yoying:

a) $f(x+3)$, $f(x)=x^4-x^3+1$;

b) $f(x+2)$, $f(x)=2x^4-3x^3+5x^2+6x-1$;

s) $f(x)=(x-2)^4+4(x-2)^3+6(x-2)^2+10(x-2)+20$;

d) $f(x)=(x+3)^5-2(x+3)^3+3(x+3)^2+7(x+3)-8$.

4.21. Gorner sxemasidan foydalanib ildizning necha karrali ekanligini aniqlang:

a) $x^5-5x^4+7x^3-2x^2+4x-8$; ko'phad uchun 2 soni;

b) $x^5+7x^4+16x^3+8x^2-16x-16$; ko'phad uchun -2 soni;

c) $x^4-6x^3+10x^2-6x+9$. ko'phad uchun 3 soni.

4.22. $f(x)$ ko'phad $x-1$ va $x-2$ larga bo'linganda qoldiqlar mos ravishda bir va ikkiga teng. $f(x)$ ni $(x-1)(x-2)$ ga bo'lgandagi qoldiqni toping.

4.23. $f(x)$ ko'phad $x+1$, $x-1$ va $x+3$ Larga bo'linganida qoldiqlar mos ravishda 5, - 4 va 6 ga teng. $f(x)$ ni $(x^2-1)(x+3)$. ga bo'lgandagi qoldiqni toping.

4.24. $f(x)$ ko'phad $x-1$, $x-2$, $x-3$ va $x-4$ Larga bo'linganida qoldiqlar mos ravishda 1, 3, 5, va 6 ge teng. $f(x)$ ni $(x-1)(x-2)(x-3)(x-4)$. ga bo'lgandagi qoldiqni toping.

4.25. Qoldiqli bo'lish algoritmini qo'llamasdan $f(x)=x^{243}+x^{81}+x^{27}+x^9+x^3+x+1$ ko'phadni: a) x^2-1; b) x^2+1; c) x^4-1. larga bo'lgandagi qoldiqni toping.

4.26. $f(x)\in P[x]$, $char\,P=0$, ko'phadning α ildizi faqat va faqat $f'(\alpha)=...=f''(\alpha)=...=f^{(k-1)}(\alpha)=0$, bo'lib, lekin $f^{(k)}(\alpha)\neq 0$. bo'lgadagina k karrali bo'lishini isbotlang.

4.27. 1 soni quyidagi ko'phadlarningn uch karali ildizi ekanligini ko'rsating:

a) $x^{2n+1}-(2n+1)x^{n+1}+(2n+1)x^n-1$;

b) $(n-2m)x^n-nx^{n-m}+nx^m-(n-2m)$.

4.28. $x^5+10ax^3+5bx+c$ ko'phadni noldan farqli uch karrali ildizga ega bo'lishi shartlarini toping.

4.29. a koeffisiyentni shunday aniqlangki, x^5-ax^2-ax+1 ko'phad uchun (-1) soni ikkidan kam bo'lmagan karralikga ega bo'lsin.

4.30. a va b larni shunday aniqlangki, ax^4+bx^3+1 ko'phad $\boldsymbol{R}[x]$. xalqada $(x-1)^2$ ga qoldiqsiz bo'linsin.

4.31. Qanday p,q,r larda quyidagi ko'phadlarning har biri $\boldsymbol{R}[x]$: xalqada $(x-1)^3$ ga qoldiqsiz bo'linadi:

a) $f(x)=x^4+px^3+qx^2+r$;

b) $f(x)=px^4+qx^2+rx+1$?

4.32. a ning qanday qiymatlarida $f(x)$ ko'phad karali bo'lgan 1ildizga ega bo'ladi va uning karraligi nechaga teng bo'ladi:

a) $f(x)=x^3+ax^2+3x-1$;

b) $f(x)=2x^3-x^2+ax+3;$

s) $f(x)=3x^4-6x^3+ax^2-2x+1?$

4.33.

$$f(x)=x^{2n+1}-\frac{n(n+1)(2n+1)}{6}x^{n+2}+\frac{(n-1)(n+2)(2n+1)}{2}x^{n+1}-$$
$$-\frac{(n-1)(n+2)(2n+1)}{2}x^{n}+\frac{n(n+1)(2n+1)}{6}x^{n-1}-1$$

ko'phadni $(x-1)^5$ ga bo'linishini, $(x-1)^6$ ga esa bo'linmasligini ko'rsating.

4.34*. $f(x)=a_0x^n+a_1x^{n-1}+...+a_n$ ko'phad $(x-1)^{k+1},$ ga bo'linishi uchun

$$a_0+a_1+a_2+\cdots+a_n=0,$$
$$a_1+2a_2+\cdots+na_n=0,$$
$$a_1+4a_2+\cdots+n^2a_n=0,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_1+2^ka_2+\cdots+n^ka_n=0.$$

shartning bajarilishi zarur va yetarli ekanligini ko'rsating.

4.35*. $x^n+ax^{n-m}+b$ ko'phad noldan farqli karraliligi ikkidan yuqori bo'lmagan ildizlarga ega bo'lmaslagini isbotlang.

4.36*. Qanday shartlarda $x^n+ax^{n-m}+b$ ko'phad noldan farqli ikki karali ildizga ega bo'ladi.

4.37*. $a_1x^{p_1}+a_2x^{p_2}+\cdots+a_kx^{p_k}$ $k-$hadli ko'phadni $(k-1)-$dan yuqori karali ildizlarga ega emasligini isbotlang.

4.38*. $a_1x^{m_1}+a_2x^{m_2}+\cdots+a_kx^{m_k}$ ko'phadning har bir noldan farqli $(k-1)-$ karali ildizi

$$a_1x^{m_1}\varphi'(m_1)=a_2x^{m_2}\varphi'(m_2)=\cdots=a_kx^{m_k}\varphi'(m_k),$$

tenglamalarni qanoatlantirishini isbotlang, bu yerda $\varphi(t)=(t-m_1)(t-m_2)\cdots(t-m_k).$

4.39*. Ko'phad faqat va faqat u $a_0(x-x_0)^n.$ ko'phaddan iborat bo'lgandagina o'zining hosilasiga bo'linishini isbotlang.

4.40*.

$$1+\frac{x}{1}+\frac{x^2}{1\cdot 2}+\cdots+\frac{x^n}{n!}$$

ko'phadni karali ildizlarga ega emasligini isbotlang.

4.41*. $f(x)=\dfrac{\varphi(x)}{\omega(x)},$ kasr-rasional funksiyaning maxraji $\omega(x)$ nolga aylanmaydigan x_0 soni uning suratining k karrali ildizi bo'lishi uchun $f(x_0)=f'(x_0)=\cdots=f^{(k-1)}(x_0)=0,\quad f^{k}(x_0)\neq 0.$ tengshliklarning bajarilishi zarur va yetarli ekanligini isbotlang.

4.42*. $f(x)=\dfrac{\varphi(x)}{\omega(x)}$ kasr rasional funksiya

$$f(x)=f(x_0)+\frac{f'(x_0)}{1}(x-x_0)+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{F(x)}{\omega(x)}(x-x_0)^{n+1},$$ bu yerda $F(x)$ - ko'phad, ko'rinishda tasvirlanmasligini isbotlang. $\omega(x_0)\neq 0$ deb olinadi (kasr rasional funksiya uchun uchun *Teylor formulsi).*

4.43*. Agar x_0 $f_1(x)f_2^{'}(x)-f_2(x)\,f_1^{'}(x),$ ko'phadning k karrali ildizi bo'lsa, u holda x_0 aynan nolga teng bo'lmagan $f_1(x)f_2\,(x_0)-f_2(x)\,f_1\,(x_0),$ ko'phadning $k+1$ karrali ildizi bo'ladi va aksincha. Shu tasdiqni isbotlang.

4.44*. Agar $f(x)$ karrali ildizlarga ega bo'lmasa, u holda $[f'(x)]^2-f(x)f''(x)$ ko'phad $n-1,$ dan yuqori karrali ildizlarga ega bo'lmasligini isbotlang, bu yerda $n-$ $f(x)$ ko'phadning darajasi.

4.45*. $n,$ darajali shunday $f(x)$ ko'phadni quringki, uning uchun $[f'(x)]^2-f(x)f''(x)$ ko'phad $n-1,$ karrali x_0 ildizga ega bo'lsin, lekin $f(x)$ ning ildizi bo'lmasin.

4.46*. $f(z)-$ kompleks koeffisiyentli ko'phad bo'lsin va $f(x+yi)=u(x,y)+iv(x,y),$ bu yerda $u(x,y)$ va $v(x,y)-$ko'phadlar haqiqiy koeffisiyentli. $u(x,y)=0,$ $v(x,y)=0$ tenglamalar sistemasining barcha yechimlarini (haqiqiy va kompleks) $f(z)$ ko'phadning ildizlari orqali ifodalang.

4.47. $\boldsymbol{C}[x]$ xalqada EKUB $(f(x)f'(x)),$ ni toping, agar $f(x)$ quyidagicha bo'lsa:

a) $(x-1)(x^2-1)(x^3-1)(x^4-1);$ b) $(x^2-4)^3(x^2+4)^2(x^4-16);$

s) $(x^2+1)^2(x^4+1)^4(x^6+1)^6(x^8+1)^8;$ d) $x^{k+1}-x^k-x^l+1.$

4.48. $f(x)=3x^4+x^3+2x^2-x+1,$ $g(x)=3x^4-5x^3+6x^2-3x+1$ ko'phad-larning kompleks ildizlarini ularning EKUBini topish va kvadrat tenglamani yechish usuli orqali toping.

4.49. Agar $f(x)$ quyidagi ko'phadlardan iborat bo'lsa, $f(x)$ ko'phadning ildizlariga ega bo'lgan, lekin karrali ildizlarga ega bo'lmagan $\varphi(x)$ ko'phadni toping (ya'ni, $f(x)$ ko'phadning karrali ildizlarini ajrating) va $f(x)$ ni $\boldsymbol{C},$ maydon ustida chiziqli ko'paytuvchilarga ajrating:

a) $x^6-2x^5-9x^4+4x^3+31x^2+30x+9;$

b) $x^6-2x^5-x^4+4x^3-x^2-2x+1;$

c) $x^6-4x^5-7x^4-8x^3+7x^2-4x+1;$

d) $x^5-8x^4+25x^3-38x^2+28x-8.$

4.50. Bosh koeffisiyenti birga teng bo'lgan va quyidagi ildizlarga ega bo'lgan to'rtinchi darajali ko'phadni tuzing:

a) ildizlari 1, 2, -3, -4 sonlardan iborat;

b) (-1) uch karrali ildiz, i esa tub ildiz;

c) ildizlari $2,$ $-1,$ $1+i$ va $1-i;$

d) 3 soni ikki karrali ildiz, –2 va –4 lar esa tub ildiz.

4.51. Quyidagi ko'phadning barcha kompleks ildizlarining kvadratlari yig'indisini va ularning ko'paytmasini toping:

a) $3x^5 - x^3 + x + 2$; b) $x^n - ax^{n-1} + b \quad (n \geq 3)$.

4.52. Birning n darajali kompleks ildizlarining yig'indisini va ko'paytmasini toping.

4.53. a ni shunday aniqlangki,

a) $x^3 + 12x^2 + a$ ko'phadning ikkita ildizining yig'indisi uning uchinchi ildiziga teng bo'lsin;

b) $x^3 - 20x + a$ ko'phadning ikkita ildizining ko'paytmasi uning uchinchi ildiziga teng bo'lsin.

4.54. $x^4 + px^3 + qx^2 + rx + s$ ko'phadning ikkita ildizining yig'indisi uning boshqa ikkita ildizlarining yig'indisiga teng bo'lsa, berilgan ko'phadning koeffisiyentlari qanday shartni qanoatlantirishini toping.

4.55. $x^4 + px^3 + qx^2 + rx + s$ ko'phadning ikkita ildizining ko'paytmasi uning boshqa ikkita ildizlarining ko'paytmasiga teng bo'lsa, berilgan ko'phadning koeffisiyentlari qanday shartni qanoatlantirishini toping.

4.56. $x^3 + ax^2 + bx + c$ ko'phadning ildizlari faqat $b^3 = a^3c.$ shart bajarilgandagina geometrik progressiya tashkil qilishini ko'rsating.

4.57. Agar $x^4 - 16x^3 + 86x^2 - 176x + 105,$ ko'phadning ildizlari arifmetik progressiyani tashkil etsa, uning barcha ildizlarini toping.

4.58. $f(x) = x^n + ax^{n-1} + \cdots + a_n.$ ko'phad ildizlarining kvadratlari yig'indisini va kublari yig'indisini toping.

4.59. Ildizlari $\alpha_1^2, \alpha_2^2, \alpha_3^2,$ sonlardan iborat bo'lgan uchinchi darajali ko'phad tuzing, bu yerda $\alpha_1, \alpha_2, \alpha_3$ –lar $3x^3 - 4x^2 + 6x + 10.$ko'phadning ildizlaridan iborat.

4.60. Agar $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ ko'phad $\alpha_1, ..., \alpha_n.$ildizlarga ega bo'lsa, quyidagi ko'phadlarning ildizlarini toping:

a) $a_0x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n$;

b) $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$;

c) $f(a) + \dfrac{f'(a)}{1!}x + \dfrac{f''(a)}{2!}x^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}x^n$;

d) $a_0x^n + a_1bx^{n-1} + a_2b^2x^{n-2} + \cdots + a_nb^n$?

4.61. Agar $2x^3 - x^2 - 7x + \lambda = 0$ tenglamaning ikkita ildizining yig'indisi 1 ga teng bo'lsa, λ ni toping.

4.62. Agar $x^3 + px + q = 0,$ tenglamaning ildizlari $x_3 = \dfrac{1}{x_1} + \dfrac{1}{x_3}.$ shartni qanoatlantirsa, uning koeffisiyentlari orasidagi munosabatni toping.

4.63*. Ildizlari $\alpha, \ \dfrac{1}{\alpha}, \ -\alpha, \ -\dfrac{1}{\alpha}.$ sonlardan iborat bo'lgan to'rtinchi darajali tenglama tuzing.

4.64*. Ildizlari $\alpha,\ \frac{1}{\alpha},\ 1-\alpha,\ \frac{1}{1-\alpha},\ 1-\frac{1}{\alpha},\ \frac{1}{1-\frac{1}{\alpha}}.$ sonlardan iborat bo'lgan oltinchi darajali tenglama tuzing.

4.65. Agar $\alpha_1,\ \ \alpha_2-$lar butun koeffisiyentli $g(x)=x^2+ax+b$ ko'phadning ildizlari bo'lsa, $f(\alpha_1)+f(\alpha_2)-$ butun sondan iborat ekanligini isbot qiling.

4.66. Lagranjning interpolyasion formulasidan foydalanib qiymatlari quyidagi jadval bilan berilgan ko'phad tuzing:

a)

x	1	2	3	4
$f(x)$	2	1	4	3

b)

x	1	i	-1	$-i$
$f(x)$	1	2	3	4

c)

x	0	1	2	3
$f(x)$	1	-1	-3	1

d)

x	-2	-1	0	1	2
$f(x)$	13	1	1	1	13

4.67. Nyutonning interpolyasion formulasidan foydalanib, qiymatlari quyidagi jadval bilan berilgan eng kichik darajali ko'phad tuzing:

a)

x	0	1	2	3	4
$f(x)$	1	2	3	4	6

b)

x	-1	0	1	2	3
$f(x)$	6	5	0	3	2

c)

x	1	$\frac{9}{4}$	4	$\frac{25}{4}$
$f(x)$	1	$\frac{3}{2}$	2	$\frac{5}{2}$

d)

x	1	2	3	4	6
$f(x)$	5	6	1	-4	10

4.68*. Qiymatlari quyidagi jadval bilan berilgan $f(x)$ ko'phadni toping:

x	1	ε_1	ε_2	…	ε_{n-1}
$f(x)$	1	2	3	…	n

bu yerda $\varepsilon_k=\cos\frac{2\pi k}{n}+i\sin\frac{2\pi k}{n}.$

4.69. Darajasi n-1 dan oshmaydigan $f(x)$ ko'phad birning n-darajali ildizlariga teng bo'lgan $y_1, y_2, \ldots, y_n$ qiymatlarni qabul qilsa, $f(0).$ ni toping.

4.70*. Agar $\varphi(x)$ ko'phadning ildizlari $x_1, x_2, \ldots, x_n$ hammasi har xil bo'lsa, $0 \le s \le n-2.$ bo'lganda $\sum_{i=1}^{n} \frac{x_i^s}{\varphi'(x_i)} = 0$ tenglikni isbot qiling.

4.71*. $\sum_{i=1}^{n} \frac{x_i^{n-1}}{\varphi'(x_i)}$ yig'indini toping (70 masalaning belgilashlariga qarang).

4.72*. Quyidagi jadvallar bilan berilgan eng kichik darajali ko'phad tuzing:

a)

x	0	1	2	…	n
y	1	2	4	…	2^n

b)

x	0	1	2	…	n
y	1	a	a^2	…	a^n

4.73*. Darajasi 2p teng bo'lgan $x(x-2)\cdots(x-2n)$ ko'phadga bo'linganida 1 qoldiq qoladigan va $(x-1)(x-3)\cdots[x-(2n-1)]$ ko'phadga bo'linganida esa (-1) qoldiq qoladigan ko'phadni toping.

4.74*. Quyidagi jadval bilan berilgan eng kichik darajali ko'phad tuzing:

x	1	2	3	…	n
y	1	$\frac{1}{2}$	$\frac{1}{3}$	…	$\frac{1}{n}$

4.75*. Darajasi $(n-1)-$dan oshmaydigan va $x_1, x_2, ..., x_n,\quad x_i \ne a_i,\quad i = \overline{1,n}.$ nuqtalarda $f(x) = \frac{1}{x-a}$ shartni qanoatlantiradigan ko'phadni toping.

4.76*. Darajasi $k \le n,$ bo'lgan erkli o'zgaruvchining $n+1$ ta ketma-ket butun qiymatlarida butun qiymatlarni qabul qiladigan ko'phad erkli o'zgaruvchining barcha butun qiymatlarida butun qiymatlarni qabul qilishini isbotlang.

4.77*. Darajasi p ga teng bo'lgan va $x = 0, 1, 4, 9, ..., n^2,$ qiymatlarda butun qiymatlarni qabul qiladigan ko'phad barcha natural sonlarning kvadratlarida ham butun qiymatlarni qabul qilishini isbotlang.

§ 3. Keltirilmaydigan ko'paytuvchilarga yoyish. C, R va Q maydonlar ustidagi ko'phadlar

Darajasi $n \geq 1$ bo'lgan va koeffisiyentlari P maydonga tegishli bo'lgan $f(x)$ ko'phad P maydonda keltirilmaydigan deyiladi, agar u darajasi n dan kichik bo'lgan va koeffisiyentlari P maydonga tegishli bo'lgan ko'phadlarning ko'paytmasi shaklida tasvirlanmasa. Aks holda $f(x)$ ko'phad P maydonda keltiriladigan yoki ko'paytuvchilarga ajraladigan ko'phad deyiladi. O'zgarmas ko'phadlar ta'rif bo'yicha keltiriladigan ko'phadlarga ham keltirilmaydigan ko'phadlarga kiritilmaydi. Birinchi darajali ko'phadlar ixtiyoriy maydonda keltirilmaydigan ko'phadlardan iborat. Ko'phadning keltirilmasligi uning koeffisiyentlari tegishli bo'lgan maydonga bog'liq. Biror maydonda keltirilmaydigan ko'phad shu maydonning chekli algebraik kengaytmalarida keltiriladigan ko'phaddan iborat bo'lishi mumkin.

Koeffisiyentlari P maydonga tegishli bo'lgan har qanday o'zgarmasdan farqli ko'phad shu maydonda keltirilmaydigan ko'phadlarning ko'paytmasiga yoyiladi. Agar o'zgarmas ko'paytuvchilarni va yoyilmadagi ko'paytuvchilarning joylashish tartibi e'tiborga olinmasa, bu yoyilma yagona bo'ladi.

$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n \in P[x], \quad n \geq 1$ bo'lsin. $f(x)$ ko'phadning

$$f(x) = a_0 p_1^{k_1}(x) p_2^{k_2}(x) \cdots p_s^{k_s}(x),$$

ko'rinishdagi yoyilmasi uning *P maydon ustidagi kanonik yoyilmasi deyilad*, bu yerda $p_1(x), p_2(x), \cdots, p_s(x) -$ bosh koeffisiyentlari 1 ga teng bo'lgan P maydonda keltirilmaydigan ko'phadlardan (*unitar ko'phadlar*) iborat. $f(x)$ ko'phadning kanonik yoyilmasi ko'paytuvchilarni joylashish tartibi aniqligida bir qiymatlidir.

$p(x) -$ P maydonda keltirilmaydigan ko'phad, $f(x) \in P[x]$ bo'lsin. Agar $p^k(x)$ ($k \in \mathbf{N}$) ko'phad $f(x)$ ningn bo'luvchisidan iborat bo'lib, $f(x)$ ko'phad $p^{k+1}(x)$ ga bo'linmasa, u holda $p(x)$ ko'phad $f(x)$ *ko'phadning $k-$karrali keltirilmaydigan ko'paytuvchisi deyiladi. Agar* $char\, P = 0$ bo'lsa, u holda $f(x)$ ko'phadning $k-$karrali keltirilmaydigan ko'pavytuvchisi uning hosilasi $f'(x)$ ko'phadning $(k-1)-$karrali ko'paytuvchisidan iborat bo'ladi. Xususiy holda, $k = 1$ bo'lsa $f'(x)$ ko'phad $p(x)$ ga bo'linmaydi.

$f(x) = a_0 p_1^{k_1}(x) p_2^{k_2}(x) \cdots p_s^{k_s}(x) -$ yoyilma $f(x)$ ko'phadning P, $char\, P = 0.$ maydondagi kanonik yoyilmasidan iborat bo'lsin.

U holda $(f(x), f'(x)) = p_1^{k_1-1}(x) p_2^{k_2-1}(x) \cdots p_s^{k_s-1}(x)$ va

$$f_1(x) = \frac{f(x)}{(f(x), f'(x))} = a_0 p_1(x) p_2(x) \cdots p_s(x)$$

ko'phadlar ham $f(x)$ yoyilmasidagi keltirilmaydigan ko'phadlarga ega bo'lishadi, lekin k $(k \geq 2)$ - karrali keltirilmaydigan ko'paytuvchilarga ega bo'lishmaydi

Darajasi $n \geq 1$ bo'lgan va koeffisiyentlari kompleks sonlardan iborat bo'lgan $f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$ ko'phad S maydonda kamida bitta ildizga ega bo'ladi (algebraning asosiy teoremasi, *Gauss teoremasi*).

Agar $P[x]$ dan olingan darajasi $n \geq 1$ bo'lgan har bir $f(x)$ ko'phad P maydonda ildizga ega bo'lsa, u holda bu maydon algebraik yopiq maydon deyiladi.

Shunday qilib, algebraning asosiy teoremasi kompleks sonlar maydoni *S* ning algebraik yopiqligini anglatadi. Algebraik yopiq maydonda faqat birinchi darajali ko'phadlar keltirilmaydigan ko'phadlardan iboart bo'ladi.

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n \quad (n \geq 1)$$

ko'phadning algebraik yopiq maydon P (*S maydon ustida ham*) kanonik yoyilmasi quyidagi ko'rinishda bo'ladi:

$$f(x) = a_0(x-\alpha_1)^{k_1}(x-\alpha_2)^{k_2}\cdots(x-\alpha_s)^{k_s},$$

bu yerda $\alpha_1, \alpha_2, ..., \alpha_s -$ $f(x)$ ko'phadning har xil ildizlari, $k_1 + k_2 + \cdots + k_s = n.$

Haqiqiy sonlar maydoni *R* ning ustida faqat birinchi darajali ko'phadlar va diskriminanti manfiy bo'lgan ikkinchi darajali ko'phadlar keltirilmaydigan ko'phadlardan iborat.

Haqiqiy sonlar maydoni *R* ning ustida $n-$darajali $f(x)$ ko'phadning kanonik yoyilmasi quyidagi ko'rinishda bo'ladi:

$$f(x) = a_0(x-\alpha_1)^{k_1}\cdots(x-\alpha_s)^{k_s}(x^2+p_1x+q_1)^{l_1}\cdots(x^2+p_tx+q_t)^{l_t},$$

bu yerda $\alpha_1, ..., \alpha_s -$har xil haqiqiy sonlar; $x^2 + p_1x + q_1, ..., x^2 + p_tx + q_t$ - *R* maydon ustidagi diskrminanti manfiy bo'lgan ikkinchi darajali har xil ko'phadlardan iborat.

$f(x) \in \boldsymbol{Q}[x]$ ko'phad *Q* maydon ustida keltirilmaydigan ko'phad bo'lishi uchun, uning koeffisiyentlari maxrajlarining eng kichik umumiy karralisiga ko'paytirish natijasida hosil bo'lgan butun koeffisiyentli ko'phad keltirilmaydigan ko'phaddan iborat bo'lishi kerak. Butun koeffisiyentli ko'phad rasional sonlar maydoni ustida keltirilmaydigan bo'lishi uchun u butun koeffisiyentli o'zgarmas bo'lmagan ikkita ko'phadning ko'paytmasi shaklida tasvirlanmasligi kerak. *Q* maydon ustida butun koeffisiyentli ko'phadning keltirilmasligini quyidagi *Eyzenshteyn alomati* bilan ham o'rnatish mumkin.

$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n -$butun koeffisiyentli ko'phad bo'lsin. Agar a_0 bo'linmaydigan shunday *p* tub son mavjud bo'lib, $f(x)$ ko'phadning barcha boshqa koeffisiyentlari *p* ga bo'linsa, a_n esa *p* ga bo'linib, p^2 ga bo'linmasa, u holda $f(x)$ko'phad rasional sonlar maydoni *Q* da keltirilmaydigan ko'phaddan iborat bo'ladi.

Misol 1. $f(x) = x^4 + x^3 + x + 2$ ko'phadni $\boldsymbol{Z}_3$maydon ustida keltirilmaydigan ko'phadlar ko'paytmasiga ajrating.

Yechish. $f(x)$ ni quyidagi ko'rinishda yozib olamiz:

$$f(x) = x^4 + x^3 + x - 1 = (x^4 - 1) + x(x^2 + 1) = (x^2+1)(x^2-1) + x(x^2+1) =$$
$$= (x^2+1)(x^2+x-1) = (x^2+1)(x^2+x+2). \blacksquare$$

M i s o l 2. $f(x) = x^6 - 27$ ko'phadni *C*, *R* va *Q* maydonlar ustida keltirilmaydigan ko'phadlar ko'paytmasiga yoying.

Yechish. $f(x)$ ning *C* maydon ustida kanonik yoyilmasini quyidagicha topamiz:

$$f(x)=x^6-27=$$

$$(x^2)^3-3^3=(x^2-3)(x^4+3x^3+9)=(x^2-3)((x^2+3)^2-(\sqrt{3}x)^2)=$$

$$=(x-\sqrt{3})(x+\sqrt{3})(x^2+\sqrt{3}x+3)(x^2-\sqrt{3}x+3)=$$

$$=(x-\sqrt{3})(x+\sqrt{3})\left(x-\frac{-\sqrt{3}+3i}{2}\right)\left(x-\frac{-\sqrt{3}-3i}{2}\right)\left(x-\frac{\sqrt{3}+3i}{2}\right)\left(x-\frac{\sqrt{3}-3i}{2}\right)$$

$f(x)$ ning R maydon ustidagi kanonik yoyilmasi quyidagicha bo'ladi:

$$f(x)=(x-\sqrt{3})(x+\sqrt{3})(x^2+\sqrt{3}x+3)(x^2-\sqrt{3}x+3).$$

$f(x)$ ning Q maydon ustidagi kanonik yoyilmasi esa quyidagicha bo'ladi: :

$$f(x)=(x^2-3)(x^4+3x^2+9).\ ■$$

M i s o l 3. Haqiqiy koeffisiyentlarga ega bo'lgan 1 soni ikki karrali ildizi, 2, 3 va $1+i$ lar esa tub ildizi bo'lgan eng kichik darajali ko'phad tuzing.

Yechish. Agar haqiqiy koeffisiyentli ko'phad kompleks ildizga ega bo'lsa, u holda bu kompleks sonning qo'shmasi ham shu ko'phadning ildizi bo'ladi. Shuninng uchun $1-i$. soni ham ko'phadning ildizi bo'ladi. U holda izlanayotgan ko'phad quyidagi ko'paytmadan iborat bo'ladi:

$$f(x)=(x-1)^2(x-2)(x-3)(x-1-i)(x-1+i)=$$

$$=(x-1)^2(x-2)(x-3)(x^2-2x+2)=$$

$$=x^6-9x^5+33x^4-65x^3+74x^2-46x+12.\ ■$$

Agar $f(x)$ va $g(x)$ ko'phadlar chiziqli ko'paytuvchilarga ajratilgan bo'lib, ularning ko'rinishi quyidagicha bo'lsa,

$$f(x)=a_0(x-\alpha_1)^{k_1}\cdots(x-\alpha_p)^{k_p}(x-\beta_1)^{u_1}\cdots(x-\beta_t)^{u_t},$$

$$g(x)=b_0(x-\alpha_1)^{l_1}\cdots(x-\alpha_p)^{l_p}(x-\gamma_1)^{v_1}\cdots(x-\gamma_s)^{v_s}$$

($\alpha_i,\beta_q,\gamma_r$ sonlar o'zaro har xil), u holda bu ko'phadlarninng eng katta umumiy bo'luvchisi

$$(f(x),g(x))=(x-\alpha_1)^{m_1}\cdots(x-\alpha_p)^{m_p}$$

Formula bilan topiladi, bu yerda har bir $j\ (j=\overline{1,p})$ uchun m_j- k_j,l_j sonlarning eng kichigidan iborat.

Misol 4. $f(x)$ va $g(x)$ ko'phadlarning EKUBini toping.

$$f(x)=(x-1)^3(x+2)^2(x-5),\quad g(x)=(x-1)(x+2)^4(x+7)(x+1)^2.$$

Yechish. $(f(x),g(x))=(x-1)(x+2)^2$. ■

Misol 5. $\boldsymbol{R}[x]$ xalqada $x^{3m}+x^{3n+1}+x^{3p+2}$ ko'phadning x^2+x+1 ko'phadga qoldiqsiz bo'linishini ko'rsating

Yechish. Agar $\alpha-$ son x^2+x+1 ko'phadning ildizi bo'lsa, $\alpha^3=1$ bo'ladi. Demak, $\alpha^{3m}+\alpha^{3n+1}+\alpha^{3p+2}=1+\alpha+\alpha^2=0$. ■

M i s o l 6. Qanday shartlarda $x^{3m}-x^{3n+1}+x^{3p+2}$ ko'phad x^2-x+1ko'phadga qoldiqsiz bo'linadi?

Yechish. Agar α soni x^2-x+1 ko'phadning ildizi bo'lsa, $\alpha^3=-1$ bo'ladi. Demak,

$$\alpha^{3m}-\alpha^{3n+1}+\alpha^{3p+2}=(-1)^m-(-1)^n\alpha+(-1)^p\alpha^2=(-1)^m-(-1)^p+\alpha[(-1)^p-(-1)^n].$$

Oxirgi ifoda $(-1)^m=(-1)^p=(-1)^n$ holdagina nolga teng bo'ladi, ya'ni $m,n,p-$bir vaqtda juft yoki bir vaqtda toq sonlardan iborat bo'lishi kerak. ■

Endi quyidagi masalani qaraymiz: rasional koeffisiyentli $f(x)$ ko'phadning rasional ildizlari topilsin. Ma'lumki, agar $f(x)$- ko'phadning koeffisiyentlari rasional sonlardan iborat bo'lsa, u holda shunday butun λ soni mavjudki, $\lambda\, f(x)$ ko'phadning koeffisiyentlari butun sonlardan iboart bo'ladi. $f(x)$ va $\lambda\, f(x)$ ko'phadlar bir xil ildizlarga ega bo'ladi. Shuning uchun faqat koeffisiyentlari butun sonlardan iborat bo'lgan $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n$ ko'phadni qarash yetarli.

Agar qisqarmaydigan $\frac{k}{l}$ kasr $f(x)$ ko'phadning ildizi bo'lsa, u holda

$$a_0k^n+a_1k^{n-1}l+\cdots+a_{n-1}kl^{n-1}+a_nl^n=0.$$

Bu yerdan *ko'phadning rasional ildizlari to'g'risidagi birinchi teorema* kelib chiqadi*: agar qisqarmaydigan* $\frac{k}{l}$ *kasr koeffisiyentlari butun sonlardan iborat bo'lgan* $f(x)$ *ko'phadning ildizidan iborat bo'lsa, u holda* $k-$*ko'phad ozod hadining bo'luvchisidan,* $l-$ *ko'phad bosh koeffisiyentining bo'luvchisidan iborat bo'ladi.*

Shunday qilib, $f(x)$ ko'phadning rasional ildizlarini topish masalasi a_n ning barcha bo'luvchilari k-larda va a_0 ning barcha bo'luvchilari l ning qiymatlarida $f\left(\frac{k}{l}\right)$ ning chekli qiymatlari to'plamini tekshirishga keltiriladi.

Misol 7.

$$f(x)=2x^3-3x^2+3x-1$$

ko'phadning barcha rasional ildizlari topilsin.

Yechish. k ning mumkin bo'lgan qiymatlari $1,-1.$ l ning mumkin bo'lgan qiymatlari $1,2$ lardan iborat (ishorani suratga yozamiz). Bu yerdan $\frac{k}{l}:1,-1,\frac{1}{2},-\frac{1}{2}$ qiymatlardan iborat ekanligini hosil qilamiz. Bu qiymatlarda ko'phadning qiymatlarini tekshiramiz:

$$f(1)=1,\quad f(-1)=-9,\quad f\left(\frac{1}{2}\right)=0,\quad f\left(-\frac{1}{2}\right)=-\frac{7}{2},$$

demak, $\frac{1}{2}-$ $f(x)$ ko'phadning ildizidan iborat ekan.

Shunday qilib, $\frac{1}{2}-$son $f(x)$ ko'phadning tub ildizidan iborat ekan.■

Ildizlarni hisoblash jarayonini ko'phadning rasional ildizlari haqidagi ikkinchi teoremadan foydalanib, ancha soddalashtirish mumkin*: Agar* *qisqarmaydigan $\frac{k}{l}$ kasr koeffisiyentlari butun sonlardan iborat bo'lgan $f(x)$ ko'phadning ildizidan iborat bo'lsa, u holda $\frac{k}{l}$ ga teng bo'lmagan har qanday butun m da $f(m)$ soni $k-ml$ ga qoldiqsiz bo'linadi.*

Misol 8. $f(x)=12x^4+32x^3+23x^2+15x+18$ ko'phadning rasional ildizlari topilsin.

Yechish. Agar $\frac{k}{l}$ – kasr $f(x)$ ko'phadning ilidizdan iborat bo'lsa, u holda k va l sonlar quyidagi qiymatlarni qabul qilishi mumkin:

$$k:\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18; \quad l:1, 2, 3, 4, 6, 12.$$

Yuqoridagi teoremaga ko'ra $f(1)=100$ $(k-l)$ ga $f(-1)=6$ esa $(k+l)$ ga qoldiqsiz bo'linadi. Bu shartni faqat quyidagi sonlar $\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, 2, -\frac{2}{3}, -3, -\frac{3}{2}$ qanoatlantiradi. $f(x)$ ko'phadning ildizlarini shu sonlar ichidan izlash kerak.

Gorner sxemasidan foydalanib, $\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, 2, -\frac{2}{3}, -3, -\frac{3}{2}$ sonlardan qaysi biri $f(x)$ ko'phadning ildizlarini ekanligini tekshiramiz (ushbu holda ildiz faqat manfiy son bo'lishi mumkinligini hisobga olamiz):

	12	32	23	15	18
$-\frac{1}{3}$	12	28	$\frac{41}{3}$	$\frac{94}{9}$	$\frac{392}{27}$
$-\frac{1}{4}$	12	29	$\frac{63}{4}$	$\frac{177}{16}$	$\frac{211}{6}$
$-\frac{2}{3}$	12	24	7	$\frac{31}{3}$	$\frac{100}{9}$
-3	12	- 4	35	- 90	288
$-\frac{3}{2}$	12	14	2	12	0
$-\frac{3}{2}$	12	- 4	8	0	
$-\frac{3}{2}$	12	-22	41		

Demak, $-\frac{3}{2}$- $f(x)$ ko'phadning ikki karrali ildizi ekan va bu ko'phad boshqa rasional ildizlarga ega emas. ■

Misol 9. Eyzenshteyn alomatidan foydalanib quyidagi ko'phadlarninng Q rasional sonlar maydoni ustida keltirilmaydigan ko'phadlar ekanligini ko'rsating:

a) $f(x)=3x^7-4x^6+2x^5-6x^3-8x-2;$

b) $f(x)=x^4+x^3+x^2+x+1.$

Yechish. a) Eyzenshteyn alomatiga asosan $p=2$ bo'lganda $f(x)$ ning Q maydonda keltirilmasligini hosil qilamiz.

b) Berilgan ko'phadga Eyzenshteyn alomatini bevosita qo'llab bo'lmaydi, shuning uchun quyidagi almashtirishni olamiz $x=y+1,$ natijada

$$h(y)=y^4+5y^3+10y^2+10y+5,$$

ko'phadni hosil qilamiz. Endi Eyzenshteyn alomatiga asosan $p=5$ da $f(x)$ ko'phadning Q maydonda keltirilmasligini hosil qilamiz. ■

Q maydon ustida ko'phadlarning keltirilmasligi yoki ko'paytuvchilarga ajratilishinig yana bir usulini keltiramiz. Ukajem yeщye odin sposob ustanovleniya privodimosti ili neprivodimosti ko'phadning nad polem peremennomu x o'zgaruvchiga t m ta butun $x_1,...,x_m$ qiymatlar beriladi, bu yerda, $m=\frac{n}{2}+1$ deb, n toq bo'lganda esa $m=\frac{n+1}{2}$ deb olinadi. So'ngra $c_1,c_2,...,c_m$, bu yerda $c_i\,(i=\overline{1,m})$ yest $f(x_i)$ ning bo'luvchisi, sonlarning mumkin bo'lgan to'plamlari tuziladi (hammasi bo'lib $S=2^m s_1,...,s_m$ ta to'plam hosil bo'ladi, bu yerda s_i – $f(x_i)$ ning barcha musbat bo'luvchilari soni). Ana shunday tuzilgan har bir to'plam uchun darajasi $m-1$ ga teng yoki kichik bo'lgan $h_j(x)$ $(j=\overline{1;S})$ ko'phad tuziladi. Bu ko'phad quyidagi xossaga ega bo'ladi: $h_j(x_i)=c_i$, bu yerda $i=\overline{1,m},$ c_i sonlar esa $j-$to'plamdan olingan. Shundan so'ng darajasi musbat va koeffisiyentlari buitun sonlardan iborat bo'lgan $h_j(x)$ ko'phadlar uchun bevosita $f(x)$ ni $h_j(x)$ga bo'linishi tekshirib ko'riladi. Agar $f(x)$ ko'phad hosil qilingan $h_j(x)$ ko'phadlardan birortasiga ham bo'linmasa, u holda $f(x)$ Q maydon ustida keltirilmaydigan ko'phaddan iborat bo'ladi. Aks holda esa $f(x)$ ko'phadning darajasi uning darajasidan kichik bo'lgan rasional koeffisiyentli ko'phadlarga yoyilmasini hosil qilamiz.

Yuqoridagi usulni $f(x)$ ko'phadga qo'lanilishida quyidagi soddalashtirishlarga erishish mumkin:

Agar sonlarning ikkita $c_1,c_2,...,c_m$ va $c'_1,c'_2,...,c'_m$ to'plamlari bir-biridan faqat ishoralarga farq qilsa, u holda $h_j(x)$ ko'phad faqat bitta to'plam uchun tuziladi.

O'zgaruvchining yana bir qancha butun qiymatlari $x_{m+1},x_{m+2},...,x_{m+l}$ ham olish mumkin, agar ularning birortasi uchun $f(x_{m+k})$ qiymat $h_j(x_{m+k})$ ga bo'linmasa, u holda $f(x)$ ni $h_j(x)$ bo'linishini tekshirish shart emas.

M i s o l 10. Yuqorida keltirilgan usul bilan quyidagi ko'phadlarni Q maydon ustida keltirmasligini ko'rsating yoki ularning keltirilmaydigan ko'phadlarga yoyilmasini toping:

a) $f(x)=x^5-2x^4-3x^3+6x^2+x-1;$

b) $f(x)=x^4+x^3-6x^2-3x+9.$

Yechish. a) Bu yerda $f(0)=-1,\quad f(1)=2,\quad f(-1)=4,\quad f(2)=1.$ Ular orasidan bo'luvchilari soni kam bo'lgan $f(0),\quad f(1),\quad f(2)$ qiymatlarni tanlab olamiz va quyidagi to'plamlarni tuzamiz:

$c_1=1$	1	1	-1	1	1	1	-1
$c_2=1$	1	-1	1	2	2	-2	2
$c_3=1$	-1	1	1	1	-1	1	1

(keltirilgan to'plamlardan faqat ishorasi bilan farq qiladiganlarini olmaymiz).

Bu to'plamlar uchun Nyutonning interpolyasion formulasiga asosan $h_j(x)\quad (j=\overline{1,8})$ ko'phadlarni tuzamiz:

$$h_1(x)=1,\quad h_2(x)=-x^2+x+1,\quad h_3(x)=2x^2-4x+1,$$
$$h_4(x)=-x^2+3x-1,\quad h_5(x)=-x^2+2x+1,\quad h_6(x)=-2x^2+3x+1,$$
$$h_7(x)=3x^2-6x+1,\quad h_8(x)=-2x^2+5x-1,$$

Bu yerda $h_j(x)-$ $x=0,$ 1, 2 larda qiymat qabul qiladigan va yuqoridagi jadvalning $j-$nchi ustunida joylashgan ko'phad.

$h_3(-1)=7,\quad h_4(-1)=-5,\quad h_7(-1)=10$ va $h_8(-1)=-8$ sonlar $f(-1)=4$ ning bo'luvchilari emas, shuning uchun $h_3(x),\quad h_4(x),\quad h_7(x)$ va $h_8(x)$ ko'phadlarni qaramaymiz. Yana qo'shimcha ravishda $f(-2)=-19$ qiymatni olamiz. Bu qiymat $h_2(-2)=-5,\quad h_5(-2)=-7$ va $h_6(-2)=-13$ sonlarga bo'linmaydi. Shunday qilib, $f(x)$ ko'phad Q maydonda keltirilmaydi.

b) $f(-2)=-1,\quad f(1)=2,\quad f(2)=3.$

Bu qiymatlarning bo'luvchilari to'plamini tuzamiz:

$c_1=1$	-1	1	1	1	-1	1	1	1	-1	1	1	1	1	-1	1	1
$c_2=1$	1	-1	1	2	2	-2	2	1	1	-1	-1	1	2	2	-2	2
$c_3=1$	1	1	-1	1	1	1	-1	3	3	3	3	-3	3	3	3	-3

(faqat ishorasi bilan farq qiladigan to'plamlarni qaramaymiz).

Bu to'plamlarga mos keladigan ko'phadlarni Nyutonning interpolyasion formulasi bilan tuzamiz:

$h_1(x)=1;\quad h_2(x),\quad h_3(x),\quad h_4(x),\quad h_5(x),\quad h_6(x)-$ kasr koeffisiyentli ko'phadlar; $h_7(x)=x^2-3-$esa $f(x)$ ning bo'luvchisi, ya'ni $f(x)=(x^2-3)(x^2+x-3)$, bu yerda x^2-3 va x^2+x-3 ko'phadlar Q maydonda keltirilmaydigan ko'phadlardan iborat (chunki ular Q da ildizga ega emas). ■

MAShQLAR

4.78. Quyidagi ko'phadlarni keltirilmaydigan ko'phadlar ko'paytmasiga yoying:

a) x^3+x+1 ni $\boldsymbol{Z_2}$ maydonda;

b) $x^5+x^3+x^2+1$ ni $\boldsymbol{Z_3}$ maydonda;

c) x^3+2x^2+4x+1 ni $\boldsymbol{Z_5}$ maydonda;

d) $x^4+3x^3+2x^2+x+4$ ni $\boldsymbol{Z_5}$ maydonda;

4.79. $\boldsymbol{Z_3}$ maydonda bosh koeffisiyenti birga teng bo'lgan ikkinchi darajali barcha keltirilmaydigan ko'phadlarni toping.

4.80. $\boldsymbol{Z_3}$ maydonda bosh koeffisiyenti birga teng bo'lgan uchinchi darajali barcha keltirilmaydigan ko'phadlarni toping.

4.81. S va R maydonlar ustida quyidagi ko'phadlarni keltirilmaydigan ko'phadlar ko'paytmasiga yoying.

a) x^3-8; b) x^3+8; c) x^4-16; d) x^4+16; ye) x^6+27;

f) x^8-6x^4+9; g) $x^{2n}-2x^n+2$; h) $x^{2n}+x^n+1$;

i) $x^{2n}-1$; j) $x^{2n+1}-1$.

4.82. Ildizlari birning barcha n-darajali ildizining qiymatlaridan iborat bo'lgan eng kichik darajali ko'phadni tuzing.

4.83. C va R maydonlar ustida berilgan ildizlar bo'yicha eng kichik darajali ko'phad tuzing:

a) 1 - ikki karrali ildiz, i va –1 lar tub ildizlar;

b) 1-2i uch karrali ildiz;

c) i ikki karrali ildiz va -1-i tub ildiz;

d) –1–i va -2+i lar ikki karrali ildizlar;

ye) 1, -1 va i tub ildizlar.

4.84. Quyidagi ko'phadlarning EKUBini toping:

a) $(x-1)^3(x+2)^2(x-3)(x-4)$ и $(x-1)^2(x+2)(x+5)$;

b) $(x-1)(x^2-1)(x^3-1)(x^4-1)$ и $(x+1)(x^2+1)(x^3+1)(x^4+1)$;

c) $(x^3-1)(x^2-2x+1)$ и $(x^2-1)^2$.

4.85*. Quyidagi ko'phadlarning EKUBini toping: x^m-1 va x^n-1.

4.86. Quyidagi ko'phadlarning EKUBini toping: x^m+a^m va x^n+a^n.

4.87. m ning qanday qiymatlarida $\boldsymbol{R}[x]$ xalqada quyidagi ko'phadlar x^2+x+1 ko'phadga qoldiqsiz bo'linadi:

a) $x^{2m}+x^m+1$; b) $(x+1)^m-x^m-1$; c) $(x+1)^m+x^m+1$.

4.88. m ning qanday qiymatlarida $\boldsymbol{R}[x]$ xalqada quyidagi ko'phadlar $(x^2+x+1)^2$ ko'phadga qoldiqsiz bo'linadi:

a) $(x+1)^m-x^m-1$; b) $(x+1)^m+x^m+1$.

4.89. $(x+1)^m + x^m + 1$ va $(x+1)^m - x^m - 1$ ko'phadlar $(x^2+x+1)^3$ ko'phadga qoldiqsiz bo'linishi mumkinmi?

4.90. Agar α soni butun koeffisiyentli $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ ko'phadning ildizi bo'lsa, quyidagilarni isbot qiling: a) $\alpha-$ soni a_n ning bo'luvchisidan iborat; b) $\alpha - m$ - soni ixtiyoriy butun m da $f(m)$ ning bo'luvchisidan iborat. Xususiy holda, $\alpha - 1$ - $f(1)$ ning, $\alpha + 1-$ esa $f(-1)$ ning bo'luvchisidan iborat.

4.91. Quyidagi ko'phadlarning butun ildizlarini toping:

a) $6x^4 + 19x^3 - 7x^2 - 26x + 12$;

b) $3x^4 + 5x^3 + x^2 + x - 2$;

c) $2x^5 + 7x^4 + 3x^3 - 11x^2 - 16x - 12$;

d) $3x^6 - 5x^4 - 10x^3 - 8x^2 + x - 2$;

ye) $6x^4 + x^3 + 2x^2 - 4x + 1$.

4.92*. Agar $\dfrac{k}{l}-$ qisqarmaydigan rasional kasr koeffisiyentlari butun sonlardan iborat bo'lgan $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ ko'phadning ildizi bo'lsa, quyidagilsharni isbot qiling:

a) $l-$ a_0 ning bo'luvchisi;

b) $k-$ a_n ning bo'luvchisi;

s) $k - ml-$ ixtiyoriy butun m da $f(m)$ ning bo'luvchisidan iborat. Xususiy holda, $k - l-$ $f(1)$ ning, $k + l-$ esa $f(-1)$ ning bo'luvchisidan iborat.

4.93. Quyidagi ko'phadlarning rasional ildizlarini toping:

a) $x^3 - 6x^2 + 15x - 14$; b) $x^4 - 2x^3 - 8x^2 + 13x - 24$;

c) $x^5 - 7x^3 - 12x^2 + 6x + 36$; d) $6x^4 + 19x^3 - 7x^2 - 26x + 12$;

ye) $24x^4 - 42x^3 - 77x^2 + 56x + 60$; f) $x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6$;

g) $10x^4 - 13x^3 + 15x^2 - 18x - 24$; h) $3x^4 - 2x^3 + 4x^2 - x + 2$;

l) $4x^4 - 7x^2 - 5x - 1$.

4.94. Q maydon ustida quyidagi ko'phadlarni keltirilmaydigan ko'phadlar ko'paytmasiga yoying:

a) $x^3 + 6x^2 - 8x + 12$; b) $3x^3 + 5x^2 + 5x + 2$; c) $30x^3 + 19x^2 - 1$;

d) $3x^3 + 4x^2 + 4x + 4$.

4.95*. Eyzenshteyn alomatidan foydalanib, quyidagi ko'phadlarni Q maydon ustida keltirilmasligini isbot qiling:

a) $x^4 - 8x^3 + 12x^2 - 6x + 2$; b) $x^5 - 12x^3 + 36x - 12$; c) $x^4 - x^3 + 2x + 2$;

d) $x^n + p$, bu yerda $p-$ tub son; ye) $\dfrac{x^p - 1}{x - 1}$, bu yerda $p-$ tub son.

4.96*. Agar $f(0)$ va $f(1)-$ toq sonlardan iborat bo'lsa, butun koeffisiyentli $f(x)$ ko'phad butun ildizlarga ega emasligini isbot qiling.

4.97*. Agar koeffisiyentlari butun sonlardan iborat bo'lgan $f(x)$ ko'phad erkli o'zgaruvchining ikkita x_1 va x_2 qiymatlarida ± 1 qiymatlarni qabul qilsa, u holda $|x_1 - x_2| > 2$ bo'lganda $f(x)$ ko'phad rasional ildizlarga ega emasligini isbot qiling. Agarda $|x_1 - x_2| \le 2$ bo'lsa, u holda $f(x)$ ning rasional ilidizi faqat $\frac{1}{2}(x_1 + x_2)$ dan iborat ekanligini ko'rsating.

4.98. Q maydon ustida $x^5 + 2x^3 + 3x^2 - 6x - 5$ ko'phadning keltirilmasligini 2 modul bo'yicha reduksiya yordamida isbot qiling.

4.99*. Q maydon ustida $x^5 - 6x^3 + 2x^2 - 4x + 5$ ko'phadning keltirilmasligini 2 va 3 modullar bo'yicha reduksiya yordamida isbot qiling.

4.100*. $f(x)$-- $\boldsymbol{Z}_p$ maydon ustida keltirilmaydigan ko'phad bo'lsin. $f(x),\quad f(x+1),\ldots, f(x+p-1)$ ko'phadlarning yo juft-jufti bilan har xil ekanligini, yoki ularning hammasi ustma-ust tushishini isbot qiling.

4.101*. $f(x) = x^p - x - a$ ko'phadni $\boldsymbol{Z}_p$ maydonda a soni p ga bo'linmagan holda keltirilmasligini isbot qiling.

4.102. Quyidagi ko'phadlarni $\boldsymbol{Q}$ maydon ustida o'zgaruvchining butun qiymatlarini ko'paytuvchilarga ajratish usuli bilan ko'paytuvchilarga ajrating yoki ularning keltirilmasligini isbot qiling:

a) $x^4 - 3x^2 + 1$; b) $x^4 + 5x^3 - 3x^2 - 5x + 1$;

c) $x^4 + 3x^3 - 2x^2 - 2x + 1$; d) $x^4 - x^3 - 3x^2 + 2x + 2$.

4.103*. Butun koeffisiyentli to'rtinchi darajali $x^4 + ax^3 + bx^2 + cx + d$ ko'phadni, agar u $x^2 + \frac{cm - am^2}{d - m^2} \cdot x + m,$ bu yerda $m-$soni d ning bo'luvchisi, ko'rinishdagi ko'phadlardan birortasiga bo'linmasa, $\boldsymbol{Q}$ maydon ustida keltirilmasligini sboott qiling. Kasr koeffisiyentli ko'phadlar e'tiborga olinmasa ham bo'ladi. Bu holda cheklashlar faqat koeffisiyentlari $d = k^2,\ \ c = ak$ shartlarni qanoatlantiradigan ko'phadlar uchun o'rinli.

4.104*. Butun koeffisiyentli beshinchi darajali

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e \text{ s}$$

ko'phad butun ildizlraga ega bo'lmasa va u quyidagi ko'rinishdagi butun koeffisiyentli ko'phadlardan

$$x^2 + \frac{am^3 - cm^2 - dn + be}{m^3 - n^2 + ae - dm} \cdot x + m,$$

bu yerda $m-$soni e ning bo'luvchisi va $n = \frac{e}{m}$, birortasiga bo'linmasa, uni $\boldsymbol{Q}$ maydon ustida keltirilmasligini isbot qiling:

4.105. Quyidagi ko'phadlarni 103, 104 masalalardan foydalanib $\boldsymbol{Q}$ maydon ustida ko'paytuvchilarga ajrating yoki ularning keltirilmasligini isbot qiling:

a) $x^4 - 3x^3 + 2x^2 + 3x - 9$; b) $x^4 - 3x^3 + 2x^2 + 2x - 6$;

c) $x^4 + 4x^3 - 6x^2 - 23x - 12$; d) $x^5 + x^4 - 4x^3 + 9x^2 - 6x + 6$.

4.106. a va b butun sonlar bo'lganda Q maydon ustida x^5+ax^3+bx+1 ko'rinishdagi barcha keltiriladigan ko'phadlarni toping.

4.107. Rasional koeffisiyentli x^4+px^2+q ko'phadning Q maydon ustida keltiriladigan ko'phaddan iborat bo'lishi uchun zaruriy va yetarli shartlarni toping.

4.108. Q maydon ustida $f(x)=(x-a_1)(x-a_2)...(x-a_n)-1;$ $a_1,a_2,...,a_n-$ o'zaro har xil butun sonlar, ko'phadning keltirilmasligini isbot qiling.

4.109*. Agar $n-$darajali butun koeffisiyentli ko'phad o'zgaruvchining $2m$ dan($n=2m$ yoki $2m+1$) ortiq butun qiymatlarida ±1 qiymatni qabul qilsa, uning Q maydonda keltirilmasligini isbot qiling.

4.110*. Agar $a_1,a_2,...,a_n-$ o'zaro har xil butun sonlar bo'lsa, $f(x)=(x-a_1)^2(x-a_2)^2...(x-a_n)^2+1$ ko'phadning Q maydon ustida keltirilmasligini isbot qiling.

4.111*. Agar butun koeffisiyentli ax^2+bx+1 ko'phad Q maydon ustida keltirilmaydigan ko'phad bo'lsa, u holda $a[\varphi(x)]^2+b\varphi(x)+1$ ko'phad ham keltirilmasligini isbot qiling, bu yerda $\varphi(x)=(x-a_1)(x-a_2)...(x-a_n)$, $n\ge 7$ bo'lganda va $a_1,a_2,...,a_n-$ o'zaro har xil butun sonlar.

§ 4. Rasional kasrlar

P maydonda r*asional kasr yoki* kasr-*rasional funksiya* deb $\frac{f(x)}{g(x)}$ ko'rinishdagi ifodaga aytiladi, bu yerda *f(x)* va *g(x)* ($g(x)\neq 0$) koeffisiyentlari P maydondan olingan ko'phadlardan iborat. Agar $P[x]$ xalqada $f(x)\psi(x)=g(x)\psi(x)$ tenglik o'rinli bo'lsa, P maydon ustidagi $\frac{f(x)}{g(x)}$ va $\frac{\varphi(x)}{\psi(x)}$ rasional kasrlar teng deyiladi.

P maydon ustidagi barcha rasional kasrlar to'plamida *algebraik amallarni (qo'shish va ko'paytirish)* quyidagi tengliklar bilan berish mumkin

$$\frac{f(x)}{g(x)}+\frac{\varphi(x)}{\psi(x)}=\frac{f(x)\psi(x)+g(x)\varphi(x)}{g(x)\psi(x)},$$

$$\frac{f(x)}{g(x)}\cdot\frac{\varphi(x)}{\psi(x)}=\frac{f(x)\varphi(x)}{g(x)\psi(x)}.$$

Bu amallarga nisbatan P maydon ustidagi barcha rasional kasrlar to'plami maydon tashkil qiladi, bu maydon rasional kasrlar maydoni yoki P maydon ustidagi kasr-rasional funksiyalar maydoni deyiladi va $P(x)$ bilan belgilanadi.

Agar $f_1(x)$ va $g_1(x)$ ko'phadlar o'zaro tub bo'lsa, $\frac{f_1(x)}{g_1(x)}$ rasional kasr qisqarmaydigan rasional kasr deyiladi. Har qanday rasional kasr uni surati $f(x)$ va maxraji $g(x)$ ning eng katta umumiy bo'luvchisiga qiqartirgandan so'ng qisqarmaydigan rasional kasrga teng bo'ladi.

Agar rasional kasrda suratining darajasi maxrajining darajasidan kichik bo'lsa, bunday rasional kasrga *to'g'ri rasional kasr* deyiladi.

M i s o l 1. $\dfrac{x^2 - x}{x^3 + 2x^2 - x} = \dfrac{x-1}{x^2 + 2x - 1}$. ■

Ko'phaddan farqli har qanday rasional kasr (ya'ni, kasrning qisqarmaydigan shaklida maxrajning darajasi noldan katta), yagona ravishda biror ko'phad va to'g'ri rasional kasrning yig'indisi shaklida tasvirlanadi. $\dfrac{f(x)}{g(x)}$ kasrning ana shunday tasvirini hosil qilish uchun $f(x)$ ko'phadni $g(x)$ ga qoldiqli bo'lish kerak. Agar shundan keyin $q(x)$ bo'linma va $r(x)$ qoldiq hosil bo'lsa,

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

bo'ladi, bu yerda $\dfrac{r(x)}{g(x)}$ − to'g'ri kasrdan iborat.

Misol 2. $\dfrac{f(x)}{g(x)} = \dfrac{3x^4 + 2x^2 - 4x + 6}{x^2 - 2x + 3}$.

$3x^4 + 2x^2 - 4x + 6 = (x^2 - 2x + 3)(3x^2 + 6x + 5) - 12x - 9$, bu yerdan

$$\frac{f(x)}{g(x)} = 3x^2 + 6x + 5 + \frac{-12x - 9}{x^2 - 2x + 3}. \ ■$$

Agar $g(x)$ ko'phad P maydonda keltirilmaydigan qandaydir $p(x)$ ko'phadning darajasidan iborat bo'lsa, $\dfrac{f(x)}{g(x)}$ to'g'ri rasional kasr sodda kasr deyiladi, bu yerda $f(x)$ ko'phadning darajasi $p(x)$ ning darajasidan kichik bo'ladi.

Rasional kasrlar to'g'risidagi asosiy teorema quyidagicha: $P(x)$ maydondan olingan har qanday to'g'ri rasional kasr yagona ravishda sodda rasional kasrlarning yig'indisi ko'rinishida tasvirlanishi mumkin.

$\dfrac{f(x)}{g(x)}$ rasional kasrni P maydon ustida yoyilmasini hosil qilish uchun dastavval $g(x)$ ko'phad P maydonda keltirilmaydigan ko'phadlar ko'paytmasiga yoyiladi va agar yoyilma $g(x) = p_1^{s_1}(x)...p_t^{s_t}(x)$ $(p_1(x),...,p_t(x)$ − keltirilmaydigan ko'phadlar) shaklda bo'lsa, u holda:

$$\frac{f(x)}{g(x)} = \frac{U_1^{(1)}(x)}{p_1^{s_1}(x)} + \frac{U_2^{(1)}(x)}{p_1^{s_1-1}(x)} + \cdots + \frac{U_{s_1}^{(1)}(x)}{p_1(x)} + \cdots + \frac{U_1^{(t)}(x)}{p_t^{s_t}(x)} + \frac{U_2^{(t)}(x)}{p_t^{s_t-1}(x)} + \cdots + \frac{U_{s_t}^{(t)}(x)}{p_t(x)},$$

bo'ladi, bu yerda $U_j^{(i)}(x)$ $(i = \overline{1,t};\ j = \overline{1,s_i}$ tayinlangan $i\,da)$ − aniqmas koeffisiyentli ko'phadning darajasi $p_i(x)$ ko'phadning darajasidan kichik. So'ngar yuqoridagi tenglikning o'ng tomonidagi kasrlar umumiy maxraj $g(x)$ ga keltiriladi va suratda hosil bo'lgan yig'indi $f(x)$ ko'phadga tenglashtiriladi. Hosil bo'lgan tenglikning o’ng va chap qismidagi x ning bir xil darajalari oldidagi koeffisiyentlar tenglashtirilib (yoki x o’zgaruvchiga sonli qiymatlar berib), chiziqli tenglamalar sistemasi hosil qilinadi. Hosil qilingan sistema yechilib $U_j^{(i)}(x)$ ko’phadlarninng koeffisiyentlari topiladi.

Agar $P = \boldsymbol{C}$ – kompleks sonlar maydonidan iborat bo’lsa, u holda sodda kasrlar $\frac{\alpha}{(x-\beta)^k}$ ko’rinishda bo’ladi, bu yerda α va $\beta -$ kompleks sonlardan iborat va $k \geq 1$.

Agar $K = \boldsymbol{R}$ – haqiqiy sonlar maydonidan iborat bo’lsa, u holda sodda kasrlar $\frac{\alpha}{(x-\beta)^m}$ (α va $\beta -$haqiqiy sonlar, $m \geq 1$) va $\frac{\alpha_1 x + \alpha_2}{(x^2 + \beta_1 x + \beta_2)^n}$, bu yerda $x^2 + \beta_1 x + \beta_2 -$ haqiqiy koeffisiyentli ko’phaddan iborat bo’lib, u haqiqiy ildizlarga ega emas, $\alpha_1, \alpha_2 -$ haqiqiy sonlar, $m \geq 1$.

Misol 3. R va C maydonlar ustida $\frac{f(x)}{g(x)} = \frac{2x^4 - 2x^3 - 6x^2 + 2x + 7}{x^5 + 2x^4 + 2x^3 + 4x^2 + x + 2}$ kasrni sodda rasional kasrlar yoying.

Yechish. Dastavval R maydonni qaraymiz.

Bu holda $g(x) = (x+2)(x^2+1)^2$. Bu yerdan

$$\frac{f(x)}{g(x)} = \frac{a}{x+2} + \frac{a_1 x + a_2}{(x^2+1)^2} + \frac{c_1 x + c_2}{x^2+1}.$$

Tenglikning o’ng qismidagi kasrlarni umumiy maxraj $g(x)$ga keltirib, quyidagi tenglikni hosil qilamiz:

$f(x) = a(x^2+1)^2 + (a_1 x + a_2)(x+2) + (c_1 x + c_2)(x+2)(x^2+1)$, yoki

$2x^4 - 2x^3 + 6x^2 + 2x + 7 = (a + c_1)x^4 + (2c_1 + c_2)x^3 + (2a + a_1 + c_1 + 2c_2)x^2 +$
$+ (2a_1 + a_2 + 2c_1 + c_2)x + (a + 2a_2 + 2c_2).$

x ning bir xil darajalari oldidagi koeffisiyentlarni tenglashtirib, quyidagi tengamalar sistemasini hosil qilamiz:

$$\begin{cases} a + c_1 = 2 \\ 2c_1 + c_2 = -2 \\ 2a + a_1 + c_1 + 2c_2 = 6 \\ 2a_1 + a_2 + 2c_1 + c_2 = 2 \\ a + 2a_2 + 2c_2 = 7. \end{cases}$$

Bu sistemadan $a = 3, \quad a_1 = 1, \quad a_2 = 2, \quad c_1 = -1, \quad c_2 = 0$ larni topamiz. Shunday qilib,

$\frac{f(x)}{g(x)} = \frac{3}{x+2} + \frac{x+2}{(x^2+1)^2} - \frac{x}{x^2+1} -$ berligan kasrning R maydon ustida sodda kasrldarga yoyilmasidan iborat.

Kompleks sonlar mydonida esa quyidagilarga ega bo’lamiz:

$g(x) = (x+2)(x+i)^2(x-i)^2$. Bu yerdan

$$\frac{f(x)}{g(x)}=\frac{a}{x+2}+\frac{b}{(x+i)^2}+\frac{c}{x+i}+\frac{d}{(x-i)^2}+\frac{e}{x-i}.$$

Tenglikning o’ng tomonidagi kasrlarni umumiy maxraj $g(x)$ ga keltirib, quyidagi tenglikni hosil qilamiz:

$$2x^4-2x^3+6x^2+2x+7=a(x+i)^2(x-i)^2+b(x+2)(x-i)^2+$$
$$+c(x+2)(x+i)(x-i)^2+d(x+2)(x+i)^2+e(x+2)(x+i)^2(x-i).$$

$x=-2$ da: $75=a\cdot 25$, ya’ni $a=3$. $x=-i$ da: $3-4i=b(2-i)(-4)$, bu yerdan $b=-\frac{1}{2}+\frac{1}{4}i$. $x=i$ da: $3+4i=d(2+i)(-4)$, bu yerdan. $b=-\frac{1}{2}-\frac{1}{4}i$. Endi ketma-ket $x=0$ va $x=-1$ deb olib, yuqorida topilgan a,b,d koeffisiyentlarning qiymatlarini hisobga olib,

$$\begin{cases} 7=3+\left(1-\frac{1}{2}i\right)-2ci+\left(1+\frac{1}{2}i\right)+2ei, \\ 15=12+\left(-\frac{1}{2}-i\right)+c(-2-2i)+\left(-\frac{1}{2}+i\right)+e(-2+2i) \end{cases}$$

yoki

$$\begin{cases} -2ci+2ei=2 \\ c(-2-2i)+e(-2+2i)=4, \end{cases}$$

bu yerdan $c=\frac{-1+i}{2}$, $e=\frac{-1-i}{2}$.

Shunday qilib,

$$\frac{f(x)}{g(x)}=\frac{3}{x+2}-\frac{2-i}{4(x+i)^2}-\frac{1-i}{2(x+i)}-\frac{2+i}{4(x-i)^2}-\frac{1+i}{2(x-i)}-$$

berilgan kasrning C maydonda sodda kasrlarga yoyilmasini hosil qilamiz. ■

Rasional kasrlarning maxrajlari o’zaro tub chiziqli ko’phadlar yoyilmasidan iborat bo’lgan holni qaraymiz.

Quyidagi rasional kasr berilgan bo’lsin:

$$\frac{f(x)}{(x-x_1)(x-x_2)\cdots(x-x_n)}, \quad x_i\neq x_j.$$

Uning sodda rasional kasrlarga yoyilmasi

$$\frac{f(x)}{(x-x_1)(x-x_2)\cdots(x-x_n)}=\frac{a_1}{x-x_1}+\frac{a_2}{x-x_2}+\cdots+\frac{a_n}{x-x_n}$$

ko’rinishda bo’ladi. Bu tenglikdagi koeffisiyentlarni topish uchun uning ikkala tomonini maxrajiga ko’paytirib yuboramiz:

$$f(x)=a_1(x-x_2)\cdots(x-x_n)+a_2(x-x_1)(x-x_3)\cdots(x-x_n)+\cdots+$$
$$+a_n(x-x_1)\cdots(x-x_{n-1}).$$

Endi ketma – ket $x=x_1, x=x_2,..., x=x_n$ qiymatlarni berib:

$$f(x_1) = a_1(x_1 - x_2)(x_1 - x_3)\cdots(x_1 - x_n),$$
$$f(x_2) = a_2(x_2 - x_1)(x_2 - x_3)\cdots(x_2 - x_n),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$f(x_n) = a_n(x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1}).$$

tengliklarni hosil qilamiz . Tengliklarning o'ng tomonlaridagi ko'paytuvchilarningn hammasi noldan farqli va ular $F(x) = (x - x_1)(x - x_2)\cdots(x - x_n)$ ko'phadning hosilalari orqali osongina topiladi. Haqiqatdan ham,

$$F'(x) = (x - x_2)\ldots(x - x_n) + (x - x_1)(x - x_3)\ldots(x - x_n) + \ldots +$$
$$+ (x - x_1)(x - x_2)\ldots(x - x_{n-1}).$$

Bu tenglikkka ketma-ket $x = x_1, x = x_2, ..., x = x_n$ qiymatlarni berib, quyidagilarni hosil qilamiz:

$$F'(x_1) = (x_1 - x_2)(x_1 - x_3)\cdots(x_1 - x_n),$$
$$F'(x_2) = (x_2 - x_1)(x_2 - x_3)\cdots(x_2 - x_n),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$F'(x_n) = (x_n - x_1)(x_n - x_2)\cdots(x_n - x_{n-1}).$$

Bu yerdan $a_1 = \dfrac{f(x_1)}{F'(x_1)}, \quad a_2 = \dfrac{f(x_2)}{F'(x_2)}, ..., a_n = \dfrac{f(x_n)}{F'(x_n)},$ va berilgan kasrni sodda kasrlarga yoyilmasi uchun quyidagi formulani hosil qilamiz

$$\frac{f(x)}{F(x)} = \sum_{k=1}^{n} \frac{f(x_k)}{F'(x_k)(x - x_k)} \quad \textit{(Lagranj formulasi).}$$

M i s o l 4. Quyidagi kasrni sodda kasrlarga yoying

$$\frac{x^3 + 5x + 7}{(x+2)(x+1)x(x-1)(x-2)}.$$

Yechish. Bu yerda

$$F(x) = (x+2)(x+1)x(x-1)(x-2),$$
$$F'(-2) = (-2+1)(-2)(-2-1)(-2-2) = 24,$$
$$F'(-1) = (-1+2)(-1)(-1-1)(-1-2) = -6,$$
$$F'(0) = (0+2)(0+1)(0-1)(0-2) = 4,$$
$$F'(1) = -6, \quad F'(2) = 24.$$

Demak,

$$\frac{x^3 + 5x + 7}{(x+2)(x+1)x(x-1)(x-2)} = -\frac{11}{24(x+2)} - \frac{1}{6(x+1)} + \frac{7}{4x} - \frac{13}{6(x-1)} + \frac{25}{24(x-2)}.$$

Miso l 5. R maydon ustida drob $\dfrac{1}{x^{2n}+1}$ kasrni sodda kasrlarga yoying.

Yechish. Dastavval berilgan kasrning C maydon ustida yoyilmasini topamiz. $F(x) = x^{2n} + 1$ ko'phadning ildizlari birlik aylanada yotadi va juft-jufti bilan o'zaro qo'shma kompleks sonlardan iborat. Ya'ni

$$x_k = \cos\frac{(2k-1)\pi}{2n} + i\sin\frac{(2k-1)\pi}{2n}, \quad k = \overline{1,n},$$

ildizlar $\bar{x}_k = x_{2n+1-k}$ ildizlar bilan o'zaro qo'shma. Ildizlar juft-jufti bilan har xil bo'lganligi uchun Lagranj formulasinin qo'llash mumkin. Endi $F'(x) = 2nx^{2n-1}$, bu yerdan

$F'(x_k) = 2nx_k^{2n-1} = 2nx_k^{-1}x_k^{2n} = -2nx_k^{-1}.$

Lagranj formulasiga asosan

$$\frac{1}{x^{2n}+1} = -\frac{1}{2n}\sum_{k=1}^{n}\frac{x_k}{x-x_k} - \frac{1}{2n}\sum_{k=1}^{n}\frac{\bar{x}_k}{x-\bar{x}_k}.$$

Endi kompleks qo'shma qo'shiluvchilarni birlashtirib, quyidagini hosil qilamiz

$$\frac{1}{x^{2n}+1} = -\frac{1}{2n}\sum_{k=1}^{n}\left(\frac{x_k}{x-x_k} + \frac{\bar{x}_k}{x-\bar{x}_k}\right) =$$

$$= -\frac{1}{2n}\sum_{k=1}^{n}\frac{(x_k+\bar{x}_k)x-2}{x^2-(x_k+\bar{x}_k)x+1} = \frac{1}{n}\sum_{k=1}^{n}\frac{1-x\cos\frac{(2k-1)\pi}{2n}}{x^2-2x\cos\frac{(2k-1)\pi}{2n}+1}. \blacksquare$$

Misol 6. $\boldsymbol{Z}_\mathbf{p}$ - p modul bo'yicha chegirmalar maydoni ustida $\frac{1}{x^p - x}$ kasrni sodda kasrlarga yoying.

Yechish. $\boldsymbol{Z}_\mathbf{p}$ maydoning elementlari $0,1,\ldots,p-1$ lar $F(x) = x^p - x$ ko'phadning ildizxlari bo'lganligi uchun $x^p - x = x(x-1)\cdots(x-p-1)$. U holda $F'(x) = px^{p-1} - 1 = -1$. Demak, $\frac{1}{x^p - x} = -\sum_{k=0}^{p-1}\frac{1}{x-k}$. ■

MAShQLAR

4.112*. Gorner sxemasidan foydalanib, quyidagi kasrlarni sodda kasrlar yig'indisiga yoying:

a) $\frac{x^2+x+1}{(x+2)}$; b) $\frac{x^3-x+1}{(x-2)^5}$; c) $\frac{x^4-2x^2+3}{(x+1)^5}$.

4.113*. $\boldsymbol{C}$ maydon ustida sodda kasrlar yig'indisiga yoying:

a) $\frac{x^2}{(x-1)(x+2)(x+3)}$; b) $\frac{1}{(x-1)(x-2)(x-3)(x-4)}$;

c) $\frac{3+x}{(x-1)(x^2+1)}$; d) $\frac{x^2}{x^4-1}$; e) $\frac{1}{x^3-1}$; f) $\frac{1}{x^4+4}$;

g) $\frac{1}{x^n-1}$; h) $\frac{1}{x^n+1}$; i) $\frac{n!}{x(x-1)(x-2)\cdots(x-n)}$;

j) $\dfrac{(2n)!}{x(x^2-1)(x^2-4)\cdots(x^2-n^2)}$.

4.114*. $\boldsymbol{C}$ maydon ustida sodda kasrlar yig'indisiga yoying:

a) $\dfrac{x}{(x^2-1)^2}$; b) $\dfrac{1}{(x^2-1)^2}$; c) $\dfrac{5x^2+6x-23}{(x-1)^3(x+1)^2(x-2)}$;

d) $\dfrac{1}{(x^n-1)^2}$; e) $\dfrac{1}{x^m(1-x)^n}$; f) $\dfrac{1}{(x^2-a^2)^n}$, $a\neq 0$; g) $\dfrac{1}{(x^2+a^2)^n}$.

4.115*. $\boldsymbol{R}$. maydon ustida sodda kasrlar yig'indisiga yoying:

a) $\dfrac{1}{x^3-1}$; b) $\dfrac{x^2}{x^4-16}$; c) $\dfrac{1}{x^4+4}$; d) $\dfrac{x^2}{x^6+27}$;

e) $\dfrac{x^m}{x^{2n+1}-1}$, $m<2n+1$; f) $\dfrac{x^m}{x^{2n+1}+1}$, $m<2n+1$; g) $\dfrac{1}{x^{2n}-1}$;

h) $\dfrac{x^{2m}}{x^{2n}+1}$, $m<n$; i) $\dfrac{1}{x(x^2+1)(x^2+4)\cdots(x^2+n^2)}$.

4.116. $\boldsymbol{R}$. maydon ustida sodda kasrlar yig'indisiga yoying:

a) $\dfrac{x}{(x+1)(x^2+1)^2}$; b) $\dfrac{2x-1}{x(x+1)^2(x^2+x+1)^2}$;

c) $\dfrac{1}{(x^4-1)^2}$; d) $\dfrac{1}{(x^{2n}-1)^2}$.

4.117. $\varphi(x)=(x-x_1)(x-x_2)\cdots(x-x_n)$. bo'lsin. $\varphi(x)$ ni quyidagi summalar orqali ifodasini toping: a) $\sum\dfrac{1}{x-x_i}$; b) $\sum\dfrac{x_i}{x-x_i}$; c) $\sum\dfrac{1}{(x-x_i)^2}$.

4.118*. x_1,x_2,x_3 −lar $\varphi(x)$ ko'phadning ildizlari ekanligini bilgan holda quyidagi yig'indilarni hisoblang: :

a) $\dfrac{1}{2-x_1}+\dfrac{1}{2-x_2}+\dfrac{1}{2-x_3}$, $\varphi(x)=x^3-3x-1$;

b) $\dfrac{1}{x_1^2-3x_1+2}+\dfrac{1}{x_2^2-3x_2+2}+\dfrac{1}{x_3^2-3x_3+2}$, $\varphi(x)=x^3+x^2-4x+1$;

s) $\dfrac{1}{x_1^2-2x_1+1}+\dfrac{1}{x_2^2-2x_2+1}+\dfrac{1}{x_3^2-2x_3+1}$, $\varphi(x)=x^3+x^2-1$.

§ 5. Bir necha o'zgaruvchili ko'phadlar

P maydon ustida $x_1,x_2,\ldots,x_n$ o'zgaruvchilardan bog'liq bo'lgan $f(x_1,x_2,\ldots,x_n)$ ko'phad deb

$$a\,x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}, \qquad (*)$$

ko'rinishdagi hadlarning chekli sondagi yig'indisiga aytiladi, bu yerda $k_i \geq 0$ $(i=\overline{1,n})$, $a - P$ maydonning elementidan iborat bo'lib, (*) hadning koeffisiyenti deb yuritiladi. $f(x_1, x_2, ..., x_n)$ ko'phadda o'xshash hadlar keltirilgan hisoblanadi va koeffisiyenti nolga teng hadlar yozilmaydi.

Ikkita $f(x_1,...,x_n)$ va $g(x_1,...,x_n)$ ko'phadlar teng deyiladi, agar ularning bir xil hadlari oldidagi koeffisiyentlar teng bo'lsa.

$k_1 + k_2 + ... + k_n$ yig'indi $a\, x_1^{k_1} x_2^{k_2} ... x_n^{k_n}$. hadning darajasi hisoblanadi.

$f(x_1, x_2, ..., x_n)$ *ko'phadning barcha o'zgaruvchilari bo'yicha darajasi deb* uning hadlarining eng bqori darajasiga aytiladi. Nolinchi darajali ko'phadlar – bu eto P sonlar maydoning noldan farqli elementlaridan iborat. Barcha koeffisiyentlari nolbga teng bo'lgan ko'phad nol ko'phad deb yuritiladi. Nol ko'phadning darajasi aniqlanmagan hisoblanadi. Agar $f(x_1,...,x_n)$ ko'phadning hadlari barcha o'zgaruvchilar bo'yicha bir xil m, darajali bo'lsa, bunday ko'phad bir jinsli *ko'phad* yoki $m-$ darjali n o'zgaruvchili *forma* deb yuritiladi.

$f(x_1,...,x_n)$ *ko'phadning bita o'zgaruvchi* x_i $(i=\overline{1,n})$ ga nisbatan darajasi deb, bu ko'phadning hadlariga kirgan x_i ning eng yuqori darajasiga aytiladi (bu daraja nolga teng bo'lishi ham mumkin).

$f(x_1,...,x_n)$ va $g(x_1,...,x_n)$ *ko'phadlarning yig'indisi deb,* koeffisiyentlari f va g; ko'phadlarning mos darajali hadlari koeffisiyentlarining yig'indisidan iborat bo'lgan ko'phadga aytiladi..

$f(x_1,...,x_n)$ va $g(x_1,...,x_n)$ ko'phadalarning *ko'paytmasi deb,* f ni g ga hadma-had kshpaytirib, so'ngra o'xshash hadlari ixchamlangan *ko'phadga* aytiladi. Yuqorida kiritilgan ko'phadlarni qo'shish va ko'paytirish amallariga nisbatan P maydon ustidagi $x_1, x_2, ..., x_n$ o'zgaruvchilardan bog'liq barcha ko'phadlar to'plami kommutativ xalqa tashkil etadi va bu xalqa $P[x_1, x_2, ..., x_n]$. orqali belgilanadi.

$\alpha = a\, x_1^{k_1} x_2^{k_2} ... x_n^{k_n}$ va $\beta = b\, x_1^{l_1} x_2^{l_2} ... x_n^{l_n}$ $-$lar $f(x_1,...,x_n) \in P[x_1, x_2, ..., x_n]$, $a \neq 0$, $b \neq 0$. ko'phadning ikkita har xil hadlari bo'lsin. α had β haddan yuqori (β had esa α haddan quyi) deyiladi, agar shunday i, $1 \leq i \leq n$, mavjud bo'lib, $k_1 = l_1, k_2 = l_2, ..., k_{i-1} = l_{i-1}$, va $k_i > l_i$. bo'lsa.

Agar $f(x_1,...,x_n)$ ko'phadning hamma hadlari shunday tuzilgan bo'lsaki, har bir keyingi had o'zidan oldingi haddan quyi bo'lsa, u holda bu ko'phadning hadlari *leksikografik yoki lug'at bo'yicha yozilgan deyiladi* (yoki $f(x_1,...,x_n)$ ko'phad leksikografik (lug'atiy) ko'phad deyiladi).

Ko'phadning leksikografik yozuvida birinchi o'rinda turgan hadi ko'phadning yuqori hadi deyiladi. Ko'phadlar ko'paytmasining yuqori hadi ular yuqori hadlarining ko'paytmasiga teng.

1 - M i s o l. a) $3x_1^6 x_2^2 x_3^5$ hadning darajasi 13 ga teng;

b) $f = 3x_1 x_2^5 x_3^3 + x_2^3 x_3^8 - 4x_1^6 x_2^7 x_3^8$ ko'phadning darajasi 21 ga teng;

s) $f=2x_1^3x_2^2x_3^4-7x_1x_2^8+11x_2^3x_3^6$ - bir jinsli to'qqizinchi darajali ko'phaddan iborat. ■

$f(x_1,x_2,...,x_n)$ ko'phad $x_1,x_2,...,x_n$ o'zgaruvchilarning o'rinlarini almashtirganda ham o'zgarmasa unga *simmetrik ko'phad* deyiladi. Aniqroq qilib aytadigan bo'lsak, $\alpha -$ S_n; dan olingan o'rniga qo'yish bo'lsin, $f=f(x_1,...,x_n)$ ko'phad uchun $\alpha\ f(x_1,\dots x_n)=f\,(x_{\alpha(1)},\dots x_{\alpha(n)})$ deb olamiz. f ko'phad simmetrik ko'phad deyildai, agar barcha $\alpha\in S_n$. lar uchun $\alpha\ f=f$ tenglik o'rinli bo'lsa.

2-M i s o l. $\boldsymbol{R}[x_1,x_2,x_3,x_4]$ xalqaning quyidagi ko'phadlari simmetrik ko'phadlar bo'ladi:

a) $f=x_1+x_2+x_3+x_4$; b) $g=x_1^3+x_2^3+x_3^3+x_4^3$;

c) $h=x_1^2x_2^2+x_1^2x_3^2+x_1^2x_4^2+x_2^2x_3^2+x_2^2x_4^2+x_3^2x_4^2$. ■

Quyidagi n o'zgaruvchili simmetrichek ko'phadlar

$\sigma_1=x_1+x_2+\dots+x_n$;

$\sigma_2=x_1x_2+x_1x_3+\dots+x_1x_n+x_2x_3+\dots+x_{n-1}x_n$;

$\sigma_3=x_1x_2x_3+x_1x_2x_4+\dots+x_1x_2x_n+\dots+x_{n-2}x_{n-1}x_n$;

....

$\sigma_{n-1}=x_1x_2\cdots x_{n-1}+x_1x_2\cdots x_{n-2}x_n+\dots+x_2x_3\cdots x_n$;

$\sigma_n=x_1x_2\cdots x_n$

elementar (yoki asosiy*) simmetrichek ko'phadlar deyiladi.*

Agar $g(x)=a_0x^n+a_1x^{n-1}+\dots+a_{n-1}x+a_n$ –ko'phad koeffisiyentlari P maydondan olingan bir o'zgaruvchili ko'phad bo'lsa, u holda n o'zgaruvchili elementar simmetrik ko'phadlarning qiymatlari o'zgaruvchilar $g(x)$, ko'phadning ildizlariga teng bo'lgan qiymatlarni qabul qilganda mos ravishda $-\frac{a_1}{a_0},\frac{a_2}{a_0},...,(-1)^n\frac{a_n}{a_0}$. larga teng bo'ladi.

$x_1,x_2,...,x_n$ o'zgaruvchili simmetrik ko'phadning barcha hadlari ulardan bittasining o'zgaruvchilarini o'rnini almashtirish yordamida hosil qilingan bo'lsa, unga *monoge* ko'phad deyiladi. Agar $a\ x_1^{k_1}x_2^{k_2}...x_n^{k_n}$ – monogen ko'phadning yuqori hadi bo'lsa, u holda bu ko'phad $S(a\ x_1^{k_1}...x_n^{k_n})$. bilan belgilanadi.

S i m m e t r i k k o' p h a d l a r t o' g' r i s i d a a s o s i y t e o r e m a

P maydon ustidagi har qanday simmmetrik ko'phadni yagona ravishda koeffisiyentlari P maydon elementlari bo'lgan elementar simmetrik ko'phadlar $\sigma_1,\sigma_2,...,\sigma_n$ ning ko'phadi ko'rinishida tasvirlash mumkin.

Berilgan simmetrik ko'phadning elementar ko'phadlar orqali ifodasini topish uchun dastlab bu ko'phadning barcha o'zgaruvchilari bo'yicha bir xil darajaga ega bo'lgan hadlarini yig'ib bir jinsli qismlarga ajaratish kerak, so'ngra esa hosil bo'lgan har bir bir jinsli qimni alohida elementar simmetrik ko'phadlar orqali ifodalash kerak.. Bir jinsli simmetrik $f(x_1,...,x_n)$ ko'phadni elementarnыye simmetrik ko'phadlar orqali ifodalash uchun uning yuqori hadi $a\,x_1^{k_1}...x_n^{k_n}$, ni olib, bu hadning ko'rsatkichlari $k_1, k_2,...,k_n$ larni yozib chiqish kerak, so'ngra quyidagi xossalarga ega bo'lgan $l_1,l_2,...,l_n$ sonlarning mumkin bo'lgan majmualarini yozib chiqish kerak:

1) Har bir majmuada $l_1+l_2+...+l_n$ yig'indi bir xil bo'lib, u $k_1+k_2+...+k_n$; ga teng bo'lishi kerak.

2) har bir majmuaning sonlari quyidagi tartibda joylashadi $l_1 \geq l_2 \geq ... \geq l_n$

3) $x_1^{l_1}x_2^{l_2}...x_n^{l_n}$ had $x_1^{k_1}x_2^{k_2}...x_n^{k_n}$. haddan yuqori emas.

Shundan so'ng har bir $l_1,l_2,...,l_n$ majmua uchun $\sigma_1^{l_1-l_2}\sigma_2^{l_2-l_3}\cdots\sigma_{n-1}^{l_{n-1}-l_n}\sigma_n^{l_n}$ ko'paytmalarni tuzib chiqiladi va $f(x_1,x_2,...,x_n)$ ko'phad aniqmas koeffisiyentlar orqali tuzilgan ko'paytmalarning yig'indisiga tenglashtiriladi ($\sigma_1^{k_1-k_2}\cdots\sigma_{n-1}^{k_{n-1}-k_n}\sigma_n^{k_n}$ hadning koeffisiyenti a_0 ga teng qilib olinadi). Hosil bo'lgan tenglikning ikkala tomonidagi $x_1, x_2,..., x_n$ o'zgaruvchilarga har xil usullar bilan qiymatlar berilib aniqmas koeffisiyentlar topiladi, natijada $f(x_1,x_2,...,x_n)$ ko'phadning elementar simmetrik ko'phadlar orqali ifodasi topiladi.

3 - M i s o l.

$$f(x_1,x_2,x_3)=x_1^4x_2+x_1^4x_3+x_1x_2^4+x_1x_3^4+x_2^4x_3+x_2x_3^4$$

ko'phadni asosiy simmetrik ko'phadlar orqali ifodasini toping.

Yechish. Bu yerda $f(x_1,x_2,x_3)-$ bir jinsli simmetrik ko'phaddan iborat.

1) f ko'phadning yuqori hadi $x_1^4x_2$; ga teng.

2) ko'rsatkichlarning mumkin bo'lgan barcha majmualari uchun va ularga mos keladigan $\sigma_1^{l_1-l_2}\sigma_2^{l_2-l_3}\cdots\sigma_{n-1}^{l_{n-1}-l_n}\sigma_n^{l_n}$ ko'paytmalar uchun quyidagi jadvalni tuzamiz:

Ko'rsatkichlar majmuasi	$\sigma_1^{l_1-l_2}\cdots\sigma_n^{l_n}$
4 1 0	$\sigma_1^3\sigma_2$
3 2 0	$A\,\sigma_1\sigma_2^2$
3 1 1	$B\,\sigma_1^2\sigma_3$
2 2 1	$C\,\sigma_2\sigma_3$

3) $f=\sigma_1^3\sigma_2+A\,\sigma_1\sigma_2^2+B\,\sigma_1^2\sigma_3+C\,\sigma_2\sigma_3$, bu yerda *A,V,S*—noma'lum koeffisiyentlar

4) $x_1, x_2,...x_n$ o'zgaruvchilarga har xil qiymatlar berib, hosil bo'lgan qiymalrani quyidagi jadvalga kiritamiz:

x_1	x_2	x_3	f	σ_1	σ_2	σ_3
1	1	1	6	3	3	1
1	1	0	2	2	1	0

1	1	-1	2	1	-1	-1

5) 4 va 3 bandlardan quyidagi sistemani hosil qilamiz:

$$\begin{cases} 6 = 81 + 27A + 9B + 3C \\ 2 = 8 + 2A \\ 2 = -1 + A - B + C \end{cases};$$

6) Bu sistemani yechib, $A = -3,\quad B = -1,\quad C = 5;$ larni hosil qilamiz.

7) 3 banddagi ko'phadga koeffisiyentlarning topilgan qiymatlarini qo'yib:

$$f = \sigma_1^3\sigma_2 - 3\sigma_1\sigma_2^2 - \sigma_1^2\sigma_3 + 5\sigma_2\sigma_3.$$

ni hosil qilamiz.■

M i s o l 4.

$$f(x_1, x_2, ..., x_n) = S(x_1^3) + 2x_1^2 + 2x_2^2 + \cdots + 2x_n^2$$

ko'phadni asosiy simmetrik ko'phadlar orqali ifodasini toping.

Yechish. $f(x_1, x_2, ..., x_n)$ ko'phadni bir jinsli qimlarga ajratamizx:

$$f_1(x_1, ..., x_n) = S(x_1^3) \quad \text{va} \quad f_2(x_1, ..., x_n) = S(2x_1^2).$$

$f_1 = x_1^3 + x_2^3 + \cdots + x_n^3$ ni elementar simmetrik ko'phadlar orqali ifodasini topamiz:

1) f_1 ko'phadning yuqori hadi x_1^3;

2) ko'rsatkichlarning mumkin bo'lgan barcha majmualari uchun va ularga mos keladigan ko'paytmalar uchun quyidagi jadvalni tuzamiz:

Ko'rsatkichlar majmuasi	$\sigma_1^{l_1-l_2} \cdots \sigma_n^{l_n}$
3 0 0	σ_1^3
2 1 0	$A\,\sigma_1\sigma_2$
1 1 1	$B\,\sigma_3$

3) $f_1 = \sigma_1^3 + A\,\sigma_1\sigma_2 + B\,\sigma_3$, bu yerda A *va* V – noma'lum koeffisiyentlar;

4) $x_1, x_2, ... x_n$ o'zgaruvchilarga har xil qiymatlar berib quyidagi jadvalni tuzamiz:

x_1	x_2	x_3	…	x_n	f_1	σ_1	σ_2	σ_3
1	1	1	…	1	n	n	C_n^2	C_n^3
1	1	0	…	0	2	2	1	0

5) 4 va 3 bandlardan quyidagi sistemani hosil qilamiz:

$$\begin{cases} n = n^3 + nC_n^2 A + C_n^3 B \\ 2 = 8 + 2A + 0 \cdot B \end{cases};$$

6) 5 banddagi sistemani yechib: $A = -3,\quad B = 3;$ larni topamiz.

7) $f_1 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$

Xuddi shunga o'xshash $f_2(x_1, ..., x_n) = 2\sigma_1^2 - 4\sigma_2.$ ni topamiz.

Shunday qilib, $f(x_1,\dots x_n)=\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3+2\sigma_1^2-4\sigma_2$. ■

Simmetrik ko’phadlar to’g’risidagi asosiy teoremadan foydalanib, $g(x)$ ko’phadning ildizlarini hisoblamasdan turib, ulardan tuzilgan ixtiyoriy simmetrik ko’phadning qiymatini hisoblash mumkin.

M i s o l 5. $g(x)=7x^4-14x^3-7x+2.$ ko’phadning

ildizlari $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ larning kublari yig’inidisini simmetrik ko’phadlar orqali ifodalab, uning qiymatini toping.

Yechish. Quyidagi simmetrik ko’phad

$f(x_1,x_2,x_3,x_4)=x_1^3+x_2^3+x_3^3+x_4^3$ ni elementar simmetrik ko’phadlar orqali ifodasini topamiz:

$$f(x_1,x_2,x_3,x_4)=\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3.$$

O’zgaruvchilarning quyidagi qiymatlari $x_1=\alpha_1,\quad x_2=\alpha_2,\quad x_3=\alpha_3,\quad x_4=\alpha_4$ -ni elnementar simmetrik ko’phadlarga qo’yib $\sigma_1=2,\quad \sigma_2=0,\quad \sigma_3=1.$ ni hosil qilamiz. Bu yerdan $f(\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\alpha_1^3+\alpha_2^3+\alpha_3^3+\alpha_4^3=8-0+3=11$. ■

$S_k=x_1^k+x_2^k+\cdots+x_n^k,\quad k=1,2,....$ simmetrik ko’phadlar darajali yig’indilar deyiladi. Elementar simmetrik ko’phadlar *Nyuton formulalari* bilan quyidagicha bohlangan *:*

$$S_k-S_{k-1}\sigma_1+S_{k-2}\sigma_2-\cdots+(-1)^{k-1}S_1\sigma_{k-1}+(-1)^k k\sigma_k=0,\quad k\le n,$$

$$S_k-S_{k-1}\sigma_1+S_{k-2}\sigma_2-\cdots+(-1)^n S_{k-n}\sigma_n=0,\quad k>n.$$

Bu formulalardan $S_1,S_2,...$ ifodalarni ketmag’ket ravishda $\sigma_1,\sigma_2,...,\sigma_n$ lar orqali topish mumkin va aksincha.

6 - M i s o l . $n\ge 3$ bo’lsin. U holda $S_1=\sigma_1;\quad S_2-S_1\sigma_1+2\sigma_2=0,$ bu yerdan $S_2=\sigma_1^2-2\sigma_2;\quad S_3-S_2\sigma_1+S_1\sigma_2-3\sigma_3=0,$ bu yerdan esa $S_3=\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3$. ■

M A Sh Q L A R

4.119. Quyidagi simmetrik ko’phadlarni elementar simmetrik ko’phadlar orqali ifodalang:

a) $x_1^3+x_2^3+x_3^3+3x_1x_2x_3$;

b) $x_1^2x_2+x_1x_2^2+x_1^2x_3+x_1x_3^2+x_2^2x_3+x_2x_3^2$;

c) $x_1^4+x_2^4+x_3^4-2x_1^2x_2^2-2x_2^2x_3^2-2x_3^2x_1^2$;

d) $x_1^5x_2^2+x_1^2x_2^5+x_1^5x_3^2+x_1^2x_3^5+x_2^5x_3^2+x_2^2x_3^5$;

e) $(x_1+x_2)(x_1+x_3)(x_2+x_3)$;

f) $(x_1^2+x_2^2)(x_1^2+x_3^2)(x_2^2+x_3^2)$;

g) $(2x_1-x_2-x_3)(2x_2-x_1-x_3)(2x_3-x_1-x_2)$;

h) $(x_1-x_2)^2(x_1-x_3)^2(x_2-x_3)^2$.

4.120. Quyidagi ko’phadlarni elementar simmetrik ko’phadlar orqali ifodalang:

a) $(x_1+x_2)(x_1+x_3)(x_1+x_4)(x_2+x_3)(x_2+x_4)(x_3+x_4)$;

b) $(x_1x_2+x_3x_4)(x_1x_3+x_2x_4)(x_1x_4+x_2x_3)$;

c) $(x_1+x_2-x_3-x_4)(x_1-x_2+x_3-x_4)(x_1-x_2-x_3+x_4)$.

4.121. p o'zgaruvchili monogen ko'phadlarni elementar simmetrik ko'phadlar orqali ifodalang:

a) $S(x_1^2)$; b) $S(x_1^2x_2x_3)$; c) $S(x_1^2x_2^2)$; d) $S(x_1^3x_2)$;

e) $S(x_1^4)$; f) $S(x_1^2x_2^2x_3)$; g) $S(x_1^3x_2x_3)$; h) $S(x_1^3x_2^2)$;

i) $S(x_1^4x_2)$; j) $S(x_1^4)$; k) $S(x_1^2x_2^2x_3x_4)$; l) $S(x_1^2x_2^2x_3^2)$;

m) $S(x_1^3x_2x_3x_4)$; n) $S(x_1^3x_2^2x_3)$; o) $S(x_1^3x_2^3)$; p) $S(x_1^4x_2x_3)$;

q) $S(x_1^4x_2^2)$; r) $S(x_1^5x_2)$; s) $S(x_1^6)$.

4.122. $S(x_1^2x_2^2\cdots x_k^2)$. monogen ko'phadni elementar simmetrik ko'phadlar orqali ifodalang.

4.123. Quyidagi kasrlarni elementar simmetrik ko'phadlar orqali ifodalang:

a) $\frac{x_1}{x_2}+\frac{x_2}{x_3}+\frac{x_3}{x_1}+\frac{x_2}{x_1}+\frac{x_3}{x_2}+\frac{x_1}{x_3}$; b) $\frac{(x_1-x_2)^2}{x_1+x_2}+\frac{(x_2-x_3)^2}{x_2+x_3}+\frac{(x_3-x_1)^2}{x_3+x_1}$;

c) $\left(\frac{x_2}{x_1}+\frac{x_3}{x_2}+\frac{x_1}{x_3}\right)\left(\frac{x_1}{x_2}+\frac{x_2}{x_3}+\frac{x_3}{x_1}\right)$.

4.124. Quyidagi yig'indilarni elementar simmetris ko'phadlar orqali ifodaldang: a) $\sum\frac{1}{x_i}$; b) $\sum\frac{1}{x_i^2}$; c) $\sum_{i\neq j}\frac{x_i}{x_j}$.

4.125. Quyidagi tenglama ildizlari kvadratlari yig'indisini hisoblang:

$$x^3+2x-3=0.$$

4.126. $x_1^3x_2+x_1x_2^3+x_2^3x_3+x_2x_3^3+x_3^3x_1+x_3x_1^3$ ifodaning qiymatini hisoblang, bu yerda o'zgaruvchilar $x^3-x^2-4x+1=0$. tenglamaning ildizlaridan iborat.

4.127. $S(x_1^3x_2x_3)$ monogen ko'phadning o'zgaruvchilari $f(x)=x^4+x^2-2x^2-3x+1=0$. ko'phadning ildizlariga teng bo'lganda, uning qiymatini hisoblang.

4.128. Simmetrik $S(x_1,x_2,...,x_n)$ ko'phadning $g(x)$ ko'phad ildizlari orqali qiymatini hisoblang:

a) $f=S(x_1^4x_2)$, $g(x)=3x^3-5x^2+1$;

b) $f=S(x_1^3x_2^3)$, $g(x)=3x^4-2x^2+x-1$;

c) $f=(x_1^2+x_1x_2+x_2^2)(x_2^2+x_2x_3+x_3^2)(x_3^2+x_1x_3+x_1^2)$, $g(x)=5x^3-6x^2+7x-8$.

4.129. Quyidagi simmetrik fnuksiyalarni $a_0x^3+a_1x^2+a_2x+a_3=0$ tenglamaning koeffisiyentlari orqali ifodalang;

a) $a_0^4(x_1-x_2)^2(x_1-x_2)^2(x_2-x_3)^2$;

b) $a_0^4(x_1^2-x_2x_3)(x_2^2-x_1x_3)(x_3^2-x_1x_2)$;

c) $\frac{(x_1-x_2)^2}{x_1x_2}+\frac{(x_1-x_3)^2}{x_1x_3}+\frac{(x_2-x_3)^2}{x_2x_3}$;

d) $a_0^4(x_1^2+x_1x_2+x_2^2)(x_2^2+x_2x_3+x_3^2)(x_3^2+x_3x_1+x_1^2).$

4.130. Nyuton formulalaridan foydalanib S_4, S_5, S_6 larni elementar simmetrik ko'phadlar orqali ifodalang.

4.131. $\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6$ larni Nyuton formulalaridan foydalanib $S_1, S_2,...,$ darajali ifodalar orqali ifodalang.

4.132. Quyidagi ko'phadlar ildizlarining $k-x$ darajalari yig'indisin hisoblang;

a) $x^6-4x^5+3x^3-4x^2+x+1, \quad k=5;$ b) $x^4-x^3-1, \quad k=8;$

c) $x^3-3x+1, \quad k=10;$ d) $x^5-x^4-x-1, \quad k=5;$

e) $x^5-x^4-x^3-1, \quad k=6.$

4.133. $S_1, S_2, ..., S_n$ larni

$$x^n+\frac{x^{n-1}}{1!}+\frac{x^{n-2}}{2!}+\cdots+\frac{1}{n!}=0.$$

tenglamaning ildizlaridan foydalanib toping.

4.134. Agar x_1, x_2, x_3 $-$lar $3x^3-4x^2+6x+10.$ ko'phadning ildizlari bulsa, ildizlari $x_1^2, x_2^2, x_3^2,$ lardan iborat bulgan kupxadni tuzing.

4.135. Agar uchinchi darajali ko'phadning ildizlari $x_1^3+x_1, \quad x_2^3+x_2, \quad x_3^3+x_3,$ ifodalardan iborat bo'lsa, bu ko'phadni tuzing, bu yerda x_1, x_2, x_3 $-$lar $2x^3+6x^2+6x-3.$ ko'phadning ildizlaridan iborat.

4.136. $S_1=S_2=\cdots=S_{n-1}=0.$ shartni qanoatlantiradigan $n-$darajali tenglamani toping.

4.137. $S_2=S_3=\cdots=S_n=0.$ shartni qanoatlantiradigan $n-$darajali tenglamani toping.

4.138*. $S_k=\begin{vmatrix}\sigma_1 & 1 & 0 & \cdots & 0\\ 2\sigma_2 & \sigma_1 & 1 & \cdots & 0\\ 3\sigma_3 & \sigma_2 & \sigma_1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ k\sigma_k & \sigma_{k-1} & \sigma_{k-2} & \cdots & \sigma_1.\end{vmatrix}$ ekanligini isbotlang.

4.139. $\sigma_k=\frac{1}{k!}\begin{vmatrix}S_1 & 1 & 0 & \cdots & 0\\ S_2 & S_1 & 2 & \cdots & 0\\ S_3 & S_2 & S_1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ S_k & S_{k-1} & S_{k-2} & \cdots & S_1.\end{vmatrix}$ tenglikni isbotlang

4.140. $\begin{vmatrix} x^n & x^{n-1} & x^{n-2} & \cdots & 1 \\ S_1 & 1 & 0 & \cdots & 0 \\ S_2 & S_1 & 2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ S_n & S_{n-1} & S_{n-2} & \cdots & n. \end{vmatrix}$ determinantni hisoblang.

JAVOBLAR va KO'RSATMALAR

I - BOB.
Determinantlar nazariyasi

1.1. a) (10,8), (1,9), (2,5), (1,10), (7,10), (7,4), (2,3), (3,6), (4,6);

b) $(2,2n)$, $(4,2n\text{-}2)$,.., hosil bo'lgunga qadar $(2n,\ 2n–2\ ,..,\ 2,1,3,..,2n\ –1)$ transpozisiyalar bajarish lozim. So'ngra $2n–1$ soni $2n$ va $2n – 2$ orasiga joylashguncha qadar o'zidan oldingi sonlar bilan transponirlanadi. Xuddi shunday $2n – 3$ uchun va h. k.

1.2. $i = 8$; $k = 3$.

1.3. 10.

1.4. a) $\frac{1}{2}n(n+1)$ inversiyalar. Faqat $n = 4k$ va $n = 4k + 3$ larda inversiyalar soni juft;

b) $\frac{1}{2}n(n–1)$ inversiyalar, faqat $n = 4k$ va $n = 4k + 1$ larda inversiyalar soni juft;

c) Inversiyalar soni n^2, faqat n juft bo'lganda juft bo'ladi;

d) Inversiyalar soni $\frac{3}{2}n(n–1)$, faqat $n = 4k$ va $n = 4k + 1$ bo'lganda juft bo'ladi;

e) Inversiyalar soni $\frac{1}{2}n(3n+1)$, faqat ixtiyoriy n larda juft bo'ladi ;

f) Inversiyalar osni $3n(n – 1)$, faqat ixtiyoriy n larda juft bo'ladi;

g) Inversiyalar soni $n(5n+1)$, istalgan n larda juft bo'ladi;

1.5. $k–1$.

1.6. $n\text{-}k$.

1.7. C_n^2.

1.8. $n = 4k$, $4k + 1$ larda bir xil, $n = 4k + 2$, $4k + 3$ larda har xildir. Bunda k – ixtiyoriy butun nomanfiy son.

1.9. Inversiyalar soni ±1 ga o'zgaradi.

1.10. $C_n^2 - k$.

1.11. $\frac{1}{2}n!\,C_n^2$. *Ko'rsatma.* Oldingi masaladan foydalaning.

1.12. *Ko'rsatma.* Ketma-ket o'zaro qo'shni transpozisiyalarni bajarib 1 ni birinchi o'ringa, 2 ni ikkinchi o'ringa va h.k. keltiring. Bitta qo'shni transpozisiyani bajarish inversiyalar sonini bittaga o'zgartiradi.

1.13. *Ko'rsatma.* $(1,\ 2\ ,...,\ n)$ o'rin almashtirishdan boshlab quyidagi transpozisiyalardan tuzilgan o'rin almashtirishlar qaralsin: avval 1 ni o'ngda turgan har bir son bilan o'rin almashtirib oxirgi o'ringa keltiriladi, so'ng ikkini xuddi shu turgan joyga keltiramiz va h.k. $n, n – 1$,.., 2, 1 o'rin almashtirishga kelguncha davom ettiriladi.

1.14. *Ko'rsatma*. $(A\ ii_1 .. i_m j\ B)$ o'rin almashtirishda (i, j) transpozisiya bajarilgan bo'lsin. Agar j bilan inversiya tashkil qiluvchi $i_1, i_2,.., i_m$ lar sonini r bilan belgilansa, u holda agar $i<j$ faqat $p+q=m$ bo'lganda inversiyalar bittaga ko'payadi, faqat $p+q=m+1$ bo'lganda 1 taga kamayadi. $i>j$ da $p+q=m-1$ dagina inversiyalar soni 1 taga ko'payadi, $p+q=m$ bo'lgandagina 1 taga kamayadi.

1.15. a) va f) o'rniga qo'yishlar emas.

1.16. a) (1) (278) (345) (6), juft o'rniga qo'yish;

b) (18) (27) (36) (45) (9), juft o'rniga qo'yish;

c) (13) (2) (46) (15)..($3n-2, 3n$)($3n$–1). Dekrement n ga teng. Œrniga qo'yishning juftligi n ning juftligi bilan bir xildir;

d) $(1, k+1, 2k+1,...,nk+1)\ (2, k+2, 2k+2,...., nk-k+2)...(k, 2k, 3k,..., nk)$. Dekrement $nk-k$ ga teng. Juft k larda va toq k va n larda o'rniga qo'yish juft. k toq n esa juft bo'lganda esa o'rniga qo'yish toq.

1.17. a) $\begin{pmatrix}1\,2\,3\,4\,5\,6\\2\,3\,5\,4\,6\,1\end{pmatrix}$; b) $\begin{pmatrix}1\,2\,3\,4\,5\,6\\2\,3\,5\,1\,6\,4\end{pmatrix}$; c) $\begin{pmatrix}1\,2\,3\,4\\3\,4\,1\,2\end{pmatrix}$; d) $\begin{pmatrix}1\,2\,3\,4\,5\\2\,3\,4\,5\,1\end{pmatrix}$

1.18. a) $\begin{pmatrix}1\,2\,3\,4\,5\\5\,3\,4\,2\,1\end{pmatrix}$; b) $\begin{pmatrix}1\,2\,3\,4\,5\\3\,5\,1\,4\,2\end{pmatrix}$; c) $\begin{pmatrix}1\,2\,3\,4\,5\,6\,7\,8\,9\\7\,4\,1\,6\,3\,2\,5\,8\,9\end{pmatrix}$;

d) $\begin{pmatrix}1\,2\,3\,4\,....\,2n-1 & 2n\\2\,1\,4\,3\,....\,2n & 1\end{pmatrix}$; e) $\begin{pmatrix}1\,2\,3\,4\,....\,2n-1 & 2n\\2\,1\,4\,3\,....\,2n & 1\end{pmatrix}$.

1.19. $(\beta_1,...,\beta_n)$. 20. a) i=8, k=5; b) i=5, k=8. 23. A.

1.24. $A^n = \begin{cases} e, agar\ n-juft & bo'lsa \\ A, agar\ n-toq & bo'lsa. \end{cases}$. 25. $x = \begin{pmatrix}1\,2\,3\,4\,5\,6\,7\,8\\4\,6\,7\,2\,8\,1\,5\,3\end{pmatrix}$.

1.28. *Ko'rsatma.* O'rniga qo'yishning birinchi satridagi sonlarni o'sib borish tartibida joylashtirib, so'ng ayniy o'rniga qo'yishdan ikkinchi satrda bir qator transpozisiyalar bajarib berilgan o'rniga qo'yishga o'tish lozim.

1.29. *Ko'rsatma.* O'rniga qo'yishni dekrement soniga teng miqdordagi transpozisiyalar ko'paytmasi ko'rinishida ifodalashda berilgan o'ringa qo'yishni bitta siklga kiruvchi sonlar transpozisiyasiga ko'paytirish va 27 masaladan foydalanish lozim. Transpozisiyalar sonining minimalligini isbotlash uchun bitta transpozisiyaga ko'paytirishda dekrement bittadan ko'pga oshmasligini e'tiborga olish lozim.

1.30. *Yechish.* Agar $X-S$ bilan o'rin almashinuvchi, ya'ni $SX=XS$ shartni qanoatlantiruvchi o'rniga qo'yish bo'lsa, $X^{-1}SX=S$. S ni $S=(12)(34)=$ $X^{-1}(12)XX^{-1}(34)X$ sikllarga yoyamiz. Bevosita hisoblash bilan $X^{-1}(12)X$—uzunligi ikkiga teng bo'lgan (12) siklda 1 va 2 ni X o'rniga ularga keluvchi sonlar bilan almashtirishdan hosil bo'lgan sikl bo'ladi. Bu (34) sikl uchun ham o'rinlidir. Shunday qilib, X o'rniga qo'yish S dagi sikllarni xuddi shu uzunlikdagi sikllarga o'tkazadi, S ni sikllarga yoyilmasining yagonaligiga ko'ra esa, sikllar yoki o'ziga, yoki biri boshqasiga o'tadi. Uzunligi 2 ga teng bo'lgan har bir siklni ikki xil yo'l bilan: (12) = (21), (34) = (43) yozish mumkin bo'lganligi sababli, S bilan o'rin almashinuvchi o'rniga qo'yishlar quyidagilardir:

$$\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix},\ \begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix},\ \begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix},\ \begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix},$$

$$\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix},\ \begin{pmatrix}1&2&3&4\\3&4&2&1\end{pmatrix},\ \begin{pmatrix}1&2&3&4\\4&3&1&2\end{pmatrix},\ \begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}.$$

1.31. $\begin{pmatrix}1&2&3&4&5\\1&2&3&4&5\end{pmatrix},\ \begin{pmatrix}1&2&3&4&5\\1&4&3&2&5\end{pmatrix},\ \begin{pmatrix}1&2&3&4&5\\3&2&5&4&1\end{pmatrix},$

$\begin{pmatrix}1&2&3&4&5\\3&4&5&2&1\end{pmatrix},\ \begin{pmatrix}1&2&3&4&5\\5&2&1&4&3\end{pmatrix},\ \begin{pmatrix}1&2&3&4&5\\5&4&1&2&3\end{pmatrix}.$

1.32. a) -1; b) 0; s) $\sin(\alpha - \beta)$; d) 0; e) 0; f) $a^2 + b^2 + c^2 + d^2$.

1.33. a) -50; b) 16; c) 0; d) $3abc - a^3 - b^3 - c^3$; e) 0;
f) $\sin(\beta - \delta) + \sin(\delta - \alpha) + \sin(\alpha - \beta)$; g) –2; h) 0; i) $3i\sqrt{3}$.

1.34. s) *Ko'rsatma*. Tenglikning chap tomonida turgan determinantning uchinchi ustunini $a + b + c$ ga ko'paytirib, $ab + bc + ca$ ga ko'paytirilgan birinchi ustun ayriladi.

1.35. a) minus ishora bilan kiradi; v) plyus ishora bilan kiradi; s) determinantning hadi emas; d) $(-1)^{n-1}$ ishora bilan kiradi; e) $(-1)^{(n-1)(n-2)/2}$ ishora bilan kiradi; f) $(-1)^{3n} = (-1)^n$ ishora bilan kiradi.

1.36. i=5, k=1. 37. a) $10x^4$, $-2x^2$, $-3x^3$; b) $x^4, x^2, 6x^2, 2x^2$.

1.38. a) $4a - c - d$; b) $2a + b - c - d$; c) $-5a - 5b - 5c - 5d$.

1.39. 0.

1.41. a) $(-1)^n$ga ko'paytiriladi; b)o'zgarmaydi. *Ko'rsatma:* almashtirishni o'rta chiziqlarga nisbatan gorizontal va vertikal ikkita simmetriya bilan, hamda bosh diagonalga nisbatansimmetriya bilan almashtirish mumkin;c) o'zgarmaydi.

1.42. a) o'zgarmaydi. b) o'zgarmaydi; c) nolga aylanadi.

1.45. a) $a_{11}a_{22}...a_{nn}$; b) $(-1)^{\frac{n(n-1)}{2}} a_{1n}a_{2,n-1}...a_{n1}$; c) $ab\,cd$; d) $ab\,cd$; e) 0.

1.46. a) $x = a_1,..., x = a_{n-1}$. $x = a_i$ da ikkita satr ustma-ust tushadi;

b) $-1, -2,..., -n+1$; c) $a_1a_2,...,a_{n-1}$.

1.47. a) –252; b) –3; c) –3; d) –65; e) –1455; f) 8; g) 900; h)-74; i) 54; j) 2: k) 0; l) 394; m) 39; n) 1875; o) –8204; p) $\frac{937}{240}$; q) $\frac{163}{1680}$; r) 1/35. *Ko'rsatma.* Har bir satr elementlarini umumiy maxradga keltirish va uni determinant belgisidan tashqari chiqarish lozim;

s) $\sqrt{14}(\sqrt{14}+\sqrt{15}-5)$; t) $2(\sqrt{5}+\sqrt{15}-\sqrt{21})^2$.

1.48. a) $n(-1)^{\frac{n(n-1)}{2}}$; b) $x_1(x_2 - a_{12})(x_3 - a_{23}) ... (x_n - a_{n-1,n})$;

c) $(-1)^{\frac{n(n-1)}{2}} b_1 b_2 ... b_n$; d) $2n + 1$; e) $(a_0 + a_1 + a_2 + ... + a_n)x^n$;

f) $x_1 x_2 ... x_n(a_1/x_1 + a_2/x_2 + ... + a_n/x_n)$.

Ko'rsatma. Determinantning i-ustunidan x_i ni chiqarish va har bir ustunga barcha keyingi ustunlarni qo'yish lozim.

1.49. 1.

1.50. $n(-1)^{n-1}$.

1.51. $(-1)^{n-1}(n-1)2^{n-2}$. *Ko'rsatma.* Har bir satrdan oldingi satr ayrilib, so'ng oxirgi ustun qolganlarga qo'shiladi.

1.52. a) $(-1)^n(x-1)(x-2)...(x-n)$; b) $a_0(x-a_1)(x-a_2)...(x-a_n)$; c) $(x^2-1)(x^2-4)$;

d) x^2z^2. *Ko'rsatma.* Birinchi ikki satr va birinchi ikki ustunlarni o'rnini almashtirib, determinantda x ni $(-x)$ ga almashtirganda uning o'zgarmasligi isbotlanadi. $x = 0$ da determinant nolga aylanishini tekshirish va uni x^2 ga bo'linishini isbotlash lozim. Xuddi shu mulohazalarni z uchun ham o'tkazish kerak.

1.53. a) $n+1$; b) $2^{n+1}-1$; c) $\frac{5^{n+1}-2^{n+1}}{3}$; d) $9-2^{n+1}$; e) $5\cdot 2^{n-1}-4\cdot 3^{n-1}$;

f) $\alpha \neq \beta$ da $\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}$; $\alpha = \beta$ da $(n+1)\alpha^n$; g) $\prod_{k=1}^{n} k!$

1.54. a) $(x_1 - a_1)(x_2 - a_2)...(x_n - a_n)(1 + a_1/(x_1 - a_1) + a_2/(x_2 - a_2) + ... +$

$+ a_n/(x_n - a_n)$. *Ko'rsatma.* $x_i = (x_i a_i) + a_i$ deb olish kerak;

b) $(a_1-x_1)(a_2-x_2)..(a_n-x_n)-a_1a_2...a_n$. *Ko'rsatma.* Yuqori chap burchakda 0=1-1 deb qabul qilish va determinantni birinchi satrga nisbatan yiђindisi ko'rinishija ifodalash lozim.

s)$(x_1-a_1 b_1)(x_2-a_2 b_2)...(x_n-a_n b_n)(1+(a_1b_1)/x_1 + (a_2b_2)/x_2 + ... + (a_nb_n)/x_n)$.

1.55. a) $x_1x_2...x_n\prod_{i>j}(x_i - x_j)$; b) $2^{\frac{n(n-1)}{2}}\prod_{i<j}\cos\frac{\varphi_i+\varphi_j}{2}\sin\frac{\varphi_i-\varphi_j}{2}$;

c) $\prod_{i>j}(x_i - x_j)$. *Ko'rsatma.* Ikkinchi satrdan birinchi satr, uchinchisidan almashtirish bajarilgan ikkinchi satr ayriladi va h.k. ;

d) $\left[(x_1-1)x_2...x_n+\prod_{i=2}^{n}(x_i-1)\right]\cdot\prod_{2\le j<i\le n}(x_i-x_j)$. *Ko'rsatma.* Birinchi ustun elementlarini (x_1-1)+1,0+1,...,0+1 yiђindi ko'rinishida ifodalash va berilgan determinantni determinantlar yiђindisi ko'rinishida yoyish mumkin;

ye) $\prod_{k=1}^{n}k!$.

1.56. a) 216; b) 1; c) –106; d) 120; e) –11; f) –2; g) 1; h) 15;

i) $-(ayz+bxz+cxy)$; j)-$(aa^1+bb^1+cc^1)$;

l) $(x_2-x_1)\sin(\gamma-\beta)+(y_2-y_1)\sin(\alpha-\gamma)+(z_2-z_1)\sin(\beta-\alpha)$;

m) $(x_4-x_3)[(x_3-x_4)(x_4-x_2)-2(x_3-x_1)(x_4-x_1)]$; n) $(-1)^n\prod_{n\ge i>k\ge 1}(x_i-x_k)^2$.

1.57. a) –12; b)16; c) 1; d) –400: e) –36; f) 0. *Ko'rsatma.* Ikkinchi, uchinchi, to'rtinchi satrlardan birinchi satr ayriladi; g) -84; h) 81; i) 14;
j) $(-1)^n(nx+1)x^{n.}$

1.58. a) 18; b) $(a+b+c+d)(a+b-c-d)(a-b+c-d)(a-b-c+d)$.

1.59. a) 256; b) 78400.

1.60. $(a^2+b^2+c^2+d^2+e^2+f^2+g^2+h^2)^4$.

1.61. a) $n>2$ da 0; $n=2$ da $(x_2-x_1)(y_2-y_1)$.

K*o'rsatma.* $\begin{vmatrix}1 & x_1 & 0 ... 0\\ 1 & x_2 & 0 ... 0\\ 1 & x_n & 0 ... 0\end{vmatrix}$ va $\begin{vmatrix}1 & x_1 & 0 ... 0\\ 1 & x_2 & 0 ... 0\\ 1 & x_n & 0 ... 0\end{vmatrix}$ determinantlarni ko'paytma ko'rinishida ifodalash kerak;

b) agar $n>2$ bo'lsa, 0, agar n+2 bo'lsa, $\sin(\alpha_1-\alpha_2)\sin(\beta_1-\beta_2)$;

c) agar n>2 bo'lsa, 0, agar n=2 bo'lsa, $\sin^2(\alpha_1-\alpha_2)$;

d) agar $n>2$ bo'lsa, 0, agar n=2 bo'lsa, $-\sin^2(\alpha_1-\alpha_2)$;

e) $(-1)^{\frac{n(n-1)}{2}}[(n-1)!]^n$. K*o'rsatma.*. i-chi satr va k-chi ustunda joylashgan elementni quyidagi ko'rinishda yozing $[i+(k-1)]^{n-1}$; f) $\prod_{i=1}^{n}(x_i-x_k)^2$. K*o'rsatma.*. Satrlarni satrlarga ko'paytirish yo'li bilan quyidagi determinantlar ko'paytmasi ko'rinishida yozing:

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ x_1 & x_2 & \dots & x_n & x \\ x_1^2 & x_2^2 & \dots & x_n^2 & x^2 \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} & x_n^{n-1} \\ x_1^n & x_2^n & \dots & x_n^n & x_n^n \end{vmatrix} \text{ va } \begin{vmatrix} 1 & 1 & \dots & 1 & 0 \\ x_1 & x_2 & \dots & x_n & 0 \\ x_1^2 & x_2^2 & \dots & x_n^2 & 0^2 \\ \dots & \dots & \dots & \dots & \dots \\ x_1^{n-1} & x_2^{n-1} & \dots & x_n^{n-1} & 0 \\ 0 & 0 & \dots & 0 & 1 \end{vmatrix}.$$

II-bob.
Matritsalar algebrasi

1-§

2.1. a) 0; b) $\begin{pmatrix} 4 \\ -11 \\ -16 \end{pmatrix}$; c) $\begin{pmatrix} -3 & -8 & 21 & -29 \\ 3 & -8 & -19 & 19 \end{pmatrix}$; d) $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$; e) $\begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$;

f) $\begin{pmatrix} -1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & -1 & 1 \end{pmatrix}$.

2.2. a), b), d) lar o'rinlidir, agar matritsalar bir xil o'lchovli bo'lsa; c), e) lar doimo o'rinli.

2.3. a) (-1); b) $\begin{pmatrix} 8 & -12 & 0 \\ 6 & -9 & 0 \\ 2 & 3 & 0 \end{pmatrix}$; c) $\begin{pmatrix} 8 & 14 \\ 8 & 14 \end{pmatrix}$; d) (1 1); e) $(0 \;\; 3 \;\; 2)$;

f) $\begin{pmatrix} 3 \\ 6 \\ 4 \\ 3 \end{pmatrix}$; g) $\begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix}$; h)(6, 9, 12); i) $\begin{pmatrix} 4 \\ 3 \\ 1 \\ 2 \end{pmatrix}$; j) E; k) $\begin{pmatrix} 0 & & \lambda_1^2 \\ & \iddots & \\ \lambda_n^2 & & 0 \end{pmatrix}$;

l) $\begin{pmatrix} 0 & & \lambda_1\lambda_n \\ & \iddots & \\ \lambda_n\lambda_1 & & 0 \end{pmatrix}$; m) $\begin{pmatrix} 2 & -6 & -5 \\ 2 & -6 & -5 \\ -2 & 6 & 5 \end{pmatrix}$; n) $\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 0 & 1 & 2 & \dots & n-1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$;

o) $n\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 1 & \dots & 0 & 0 \end{pmatrix}$.

2.4. a) A ning bo'yi V ning balandligiga teng; b) A ning bo'yi V ning bo'yiga teng;
c) A ning bo'yi V ning bo'yiga teng, A ning balandligi V ning bo'yiga teng.
2.5. Ayniyatlar to'g'ridir, agar ularda foydalanilgan amallar bajarilsa.
2.6. a) mavjud emas; b) $\begin{pmatrix} 8 \\ 16 \end{pmatrix}$; c) $(8 \quad 16)$; d) $(-1200 \quad 1300)$.

2.7. a) $2^{n-1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$; b) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$; c) $\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$; d) 0 agar $n > 1$ bo'lsa;

e) $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$; f) $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$; g) $\begin{pmatrix} \lambda_1^n & & 0 \\ & \ddots & \\ 0 & & \lambda_n^n \end{pmatrix}$; h) E.

2.8. b), c), d) lar o'rinli, agar ularda foydalanilgan amallar bajarilsa; a) doimo o'rinlidir.
2.9. a) $\begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix}$; b) 0.
2.11. a) $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$; b) $\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$.
2.13. a) 0; b) 0; c) 0; d) $-E$; e) $\begin{pmatrix} 2 & -5 & -4 \\ -3 & 16 & 12 \\ 4 & -20 & -15 \end{pmatrix}$; f) $\begin{pmatrix} -4 & 0 & 0 \\ 0 & -74 & 0 \\ 0 & 0 & 2 \end{pmatrix}$; g) 0.
2.14. Yo'q.
2.15. a) $\left\{\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbf{C}\right\}$; b) $\left\{\begin{pmatrix} \alpha & \beta \\ -3\beta & \alpha+3\beta \end{pmatrix} \mid \alpha, \beta \in \mathbf{C}\right\}$;

c) $\left\{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} \mid \alpha, \beta \in \mathbf{C}\right\}$; d), e) $\left\{\begin{pmatrix} \alpha & \beta & \gamma & \delta \\ 0 & \alpha & \beta & \gamma \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & \alpha \end{pmatrix} \mid \alpha, \beta, \gamma, \delta \in \mathbf{C}\right\}$;

f) diagonal matritsalar.
2.18. $\begin{pmatrix} 3197 & -1266 \\ 7385 & -922 \end{pmatrix}$.
2.19. $\begin{pmatrix} 190 & 189 & -189 \\ 126 & 127 & -126 \\ 252 & 252 & -251 \end{pmatrix}$.
2.21. $\pm E$ yoki $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, bunda $a^2 + bc = 1$.

2.22. $\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, bunda *a, b, c* – lar ixtiyoriy sonlar va $a^2+bc=0$ tenglikni qanoatlantiradi.

2-§

2.24. *AB* matritsaning *k*-satri *A* ning *k*-ustuni bilan *B* matritsaning ko'paytmasiga teng.

2.26. *AB* matritsaning *k*-ustuni *A* ning *k*-satri koeffisiyentlari bilan *B* matritsa satrlarining chiziqli kombinasiyasiga teng.

2.27. *Ko'rsatma*: 24-masalaga qarang.

2.28. a) *B* matritsaning ikkita ustunlarining o'rinlari almashsa, *AB* ning ham mos ustunlarining o'rinlari almashadi. b) agar *B* matritsaning *k*-ustuni *λ* songa ko'paytirilsa, *AB* ning ham *k*-ustuni *λ* songa ko'paytiriladi c) agar *B* ning *i*-chi ustuniga *j*-chi ustunini qo'shsak, *AV* matritsa ustida ham xuddi shunday elementar almashtirish bajariladi.

2.29. *Ko'rsatma*: ikkita elementdan iborat ustun uchun almashtirishlar quyidagichadir:

$$\begin{pmatrix} a \\ b \end{pmatrix} \sim \begin{pmatrix} a+b \\ b \end{pmatrix} \sim \begin{pmatrix} a+b \\ b-(a+b) \end{pmatrix} \sim \begin{pmatrix} a+b \\ -a \end{pmatrix} \sim \begin{pmatrix} b \\ -a \end{pmatrix} \sim \begin{pmatrix} b \\ a \end{pmatrix}.$$

2.30. a_{ij}, agar $A=(a_{ij})$.

2.31. a) shunday matritsaki, uning hamma satrlari nollardir *i*-chi satrdan tashqari va uning o'rnida *j*-chi satr joydashgandir; b) shunday matritsaki, uning hamma ustunlari nollardir, *j*-chi ustundan tashqari va uning o'rnida *i*-chi ustun joylashgandir.

2.32. *Ko'rsatma. a* va *b* lar sifatida birlik matritsaning mumkin bo'lgan barcha ustunlarini olish lozim.

2.34. a) A matritsani o'ngdan $(1 \quad 0 \quad 0 \quad \ldots \quad 0)^T$ ustunga ko'paytirish kerak; b) *A* matritsani chapdan $(1 \quad 0 \quad \ldots \quad 0)$ satrga ko'paytirish kerak.

2.35, 2.36. *K* matritsa *E* matritsadan xudi shunday elementar almashtirish yordamida kelib chiqadi.

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2.38. *Ko'rsatma*. Agar $A=\begin{pmatrix} a & b \\ ta & tb \end{pmatrix}$ bo'lsa, $A^m=(a+tb)^{m-1}A$ bo'ladi.

2.39. $\begin{cases} a_{11}x_1+\ldots+a_{1n}x_n=0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{j1}x_1+\ldots+a_{jn}x_n=1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1+\ldots+a_{nn}x_n=0 \end{cases}$

yoki matritsaviy shaklda $AX = e_j$, bunda e_j – E matritsaning j – chi ustuni.

2.40. a) $\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$; b) $\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$; c) $\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$;

d) $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$; e) $\begin{pmatrix} -\frac{7}{3} & 2 & -\frac{1}{3} \\ \frac{5}{3} & -1 & -\frac{1}{3} \\ -2 & 1 & 1 \end{pmatrix}$; f) $\frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{pmatrix}$;

g) $\frac{1}{9}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{pmatrix}$; h) $\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$; i) $\begin{pmatrix} 22 & -6 & -26 & 17 \\ -17 & 5 & 20 & -13 \\ -1 & 0 & 2 & -1 \\ 4 & -1 & -5 & 3 \end{pmatrix}$;

j) $\begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$; k) $\begin{pmatrix} 1 & -1 & 1 & -1 & \dots & (-1)^{n-1} \\ 0 & 1 & -1 & 1 & \dots & (-1)^{n-2} \\ 0 & 0 & 1 & -1 & \dots & (-1)^{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$;

l) $\begin{pmatrix} 1 & -a & 0 & 0 & \dots & 0 \\ 0 & 1 & -a & 0 & \dots & 0 \\ 0 & 0 & 1 & -a & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$;

m) $\frac{1}{n+1}\begin{pmatrix} n & n-1 & n-2 & n-3 & \dots & 1 \\ n-1 & 2(n-1) & 2(n-2) & 2(n-3) & \dots & 2 \\ n-2 & 2(n-2) & 3(n-2) & 3(n-3) & \dots & 3 \\ n-3 & 2(n-3) & 3(n-3) & 4(n-3) & \dots & 4 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 2 & 3 & 4 & \dots & n \end{pmatrix}$;

n) $\begin{pmatrix} 2-n & 1 & 1 & \dots & 1 \\ 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & -1 \end{pmatrix}$; o) $\frac{1}{n-1}\begin{pmatrix} 2-n & 1 & 1 & \dots & 1 \\ 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & -1 \end{pmatrix}$;

p) $-\frac{1}{a(n+a)}\begin{pmatrix} 1-n-a & 1 & 1 & \dots & 1 \\ 1 & 1-n-a & 1 & \dots & 1 \\ 1 & 1 & 1-n-a & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 1-n-a \end{pmatrix}$;

2.42. f) tasdiq umumiy holda noto'g'ridir.

2.43. mumkin bo'lgan javoblarning bittasini keltiramiz.

a) $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$;

b) $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$; c) $\left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbf{C} \right\}$.

2.44. a) $-A+4E$; b) $A^2+5A-3E$; c) agar *detA*=0 bo'lsa, A^{-1} mavjud emas, agar $\det A \neq 0$ bo'lsa, $A = E = A^{-1}$.

2.45. a) $X = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}, Y = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}$;

b) $Y = 2X + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, bunda X – ikkinchi tartibli ixtiyoriy matritsa.

2.46. $A^{-1} = -(A+E)$.

2.47. a) $\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$; b) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$; c) $\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$;

d) $\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$; e) $\frac{1}{7}\begin{pmatrix} -5 & 2 & 2 & 2 \\ 2 & -5 & 2 & 2 \\ 2 & 2 & -5 & 2 \\ 2 & 2 & 2 & -5 \end{pmatrix}$; f) $\begin{pmatrix} 0 & -1 & 1 & -1 \\ 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 \end{pmatrix}$;

g) $\begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$; h) $\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ -1 & 1 & -1 & 1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & -1 & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & 1 & -1 & 1 \end{pmatrix}$;

i) $\begin{pmatrix} \frac{1}{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\lambda_n} \end{pmatrix}$; j) $\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$;

k) $\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}^T$; l) $\begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -2 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & -2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$.

2.48. $\begin{pmatrix} A \\ B \end{pmatrix}$ matritsaning ustunlari bilan B ni E ga o'tkazadigan elementar almashtirishlar bajaramiz. Natijada A matritsa o'rnida AB^{-1} matritsa hosil bo'ladi.

2.53. a) 0; b) $A^{-1}B - C$.

2.54. a) $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$; b) $\begin{pmatrix} 5 & -8 \\ 2 & -3 \end{pmatrix}$; c) $\begin{pmatrix} 21 & -14 & -10 \\ -10 & 7 & 5 \\ -4 & 3 & 2 \end{pmatrix}$; d) $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}$;

e) yechimlar mavjud emas; f) $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ bunda a va b ixtiyoriy sonlar;

g) $\begin{pmatrix} a & b & c \\ 1-a & 2-b & 4-c \end{pmatrix}$, a, b, c – lar ixtiyoriy sonlar;

h) $\begin{pmatrix} 0 & 0 \\ -1 & 0 \\ -2 & -1 \end{pmatrix}$; i) $\begin{pmatrix} -1 & 2 & 1 \\ 1 & -1 & 0 \end{pmatrix}$; j) yeichmlar mavjud emas;

k) $\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & \dots & 1 \\ 0 & 0 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$.

2.55. a) i-chi va j- chi ustunlarning o'rinlari almashadi;

b) i-chi ustun $\frac{1}{c}$ songa ko'paytiriladi;

c) j-chi ustundan i- chi ustunning s soniga ko'paytmasi ayriladi.

4-§

2.61. *Ko'rsatma*. 60chi masaladan foydalaning.

2.62. 61-masalaga qarang.

2.63. a) $-\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$; b) $\begin{pmatrix} 5 & i \\ -i & 1 \end{pmatrix}$; c) $\begin{pmatrix} 1 & 2-i \\ i & 1+2i \end{pmatrix}$; d) $\begin{pmatrix} 1-i\sqrt{2} & 1 \\ 3 & 1+i\sqrt{2} \end{pmatrix}$.

2.65. a) kosoermit; b) simmetrik; s), d) ermit; h) ortogonal; e) diagonal; f) uchburchakli; g) kososimmetrik.

2.68. a) $\begin{pmatrix} a & b+ic \\ b-ic & d \end{pmatrix}$; b) $\begin{pmatrix} ia & b+ic \\ -b+ic & id \end{pmatrix}$, bunda *a, b, c, d* – ixtiyoriy haqiqiy sonlar; c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

2.75. Teskari matritsa berilgan matritsaga ermitli qo'shma matritsadir.

2.76. Teskari matritsa berilgan matritsaga ermitli qo'shma matritsadir:

a) $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$; b) $\frac{1}{\sqrt{2}}\begin{pmatrix} -i & 0 & 0 & 1 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \\ 0 & i & 1 & 0 \end{pmatrix}$.

2.77. $A=(a_{ij}), B=(b_{ij}), C=AB=(c_{ij})$ bo'lsin. U holda S ning bosh dioganalida $c_{ii}=a_{ii}b_{ii}$, qo'shimcha dioganalida: $c_{i,i+1}=a_{ii}b_{i,i+1}+a_{i,i+1}b_{i+1,i+1}$, *m*-chi qo'shimcha dioganalida:

$$c_{i,i+m}=a_{ii}b_{i,i+m}+a_{i,i+1}b_{i+1,i+m}+\ldots+a_{i,i+m}b_{i+m,i+m}.$$

Bosh dioganalining pastida nollar.

2.80. *d) AV = - VA.*

2.82. Yoyilma yagonadir: $A=\frac{1}{2}\left(A+A^T\right)+\frac{1}{2}\left(A-A^T\right)$.

2.83. a) $\begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}+\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; b) $E+\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$;

c) $\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -2 \end{pmatrix}+\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$.

2.88. Yoyilma yagonadir: $A=\frac{1}{2}\left(A+\overline{A}^T\right)+\frac{1}{2}\left(A-\overline{A}^T\right)$.

2.92. Bu xossalar matritsaning ortogonalligini ta'minlaydi.

2.95. *Ko'rsatma*. 92-masalada ifodalangan ortogonal matritsalarning xossalarini tekshiring.

2.96. *Ko'rsatma*. Chap tomondan o'rinalmashtirish matritsasiga ko'paytirish ko'paytuvchi matritsada satrlarning o'rinlarini almashtirishga teng kuchlidir.

2.97. Diagonal elementlari 1 ga yoki -1 ga teng.

2.98. a), c), k) stoxastikdir; d), f), g), j), e) nilpotentlik darajasi mos ravishda 3, 2, 2, 3, *n* ga teng nilpotent matritsadir; s), h), i) davrlari mos ravishda 2, 4, 4 ga teng davriy matritsalardir; b) $\alpha=\frac{2\pi p}{q}$ bo'lganda davriydir, $p\neq 0$ bo'lganda (*r* – butun son, *q*– natural son kasr qisqarmasdir) va $\alpha=0$ bo'lganda davri 1 ga teng.

2.100. *Ko'rsatma*. 99 va 38 masalalardan foydalanilsin.

2.102. *Ko'rsatma*. Agar $A^k=0, B^l=0$ bo'lsa, $(AB)^{kl}=0$ va $(A+B)^{k+l}=0$ bo'ladi.

2.103. *AB* ning davri *k= lm* ga teng, bunda *l, m – A, B* larning davrlari.

2.104. *Ko'rsatma*. Tenglikning ikala tomonini $E-A$ ga ko'paytiring.

2.109. *Ko'rsatma*. 107 va 108-masalalarning natijalarini qo'llang.

2.110. Har doim emas. Misol: $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ teskarilanuvchi emas, $\left(\frac{1}{4}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}\right)^{-1}$ stox-astik emas, lekin o'rinalmashtirish matritsalari o'zlarining teskari matritsalari Bilan birga stoxastikdir.

2.111. Agar matritsa o'rinalmashtirish matritsasi bo'lsa.

2.113. $\sum_{1}^{n} a_{ii}^{m}$, agar $A=(a_{ij})(i, j=1,...,n)$ bo'lsa. *Ko'rsatma*. 77-masalaga qa-rang.

2.114. a) $\sum_{i,k} a_{ik}^2$; b) $\sum_{i,k} |a_{ik}|^2$, agar $A=(a_{ik})$ bo'lsa.

2.117. Agar $A=(A_{ij})$, $B=(B_{ij})$ $(i=1,2)$ bo'lsa, u holda AB ning mavjud bo'lishi uchun katakli matritsaning ta'rifidan kelib chiqadigan shartlardan tashqari, A_{11} ning bo'yi B_{11} ning balandligiga, A_{12} ning bo'yi B_{21} ning balandligiga teng bo'lishi za-rurdir.

2.118. $\begin{pmatrix} M & N \\ O & P \end{pmatrix}\begin{pmatrix} D & F \\ O & G \end{pmatrix} = \begin{pmatrix} MD & MF+NG \\ 0 & PG \end{pmatrix}$.

2.119. a) Agar $A=\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $B=\begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ bo'lsa, katakli matritsaning ta'rifidan kelib chiqadigan shartlardan tashqari A_{11} matrsaning bo'yi B_1 ning balandligiga, A_{12} ning bo'yi B_2 ning balandligiga teng bo'lishi zarurdir.

c) A□B□=$\begin{pmatrix} A_{11}B_1 + A_{12}B_2 \\ A_{21}B_1 + A_{22}B_2 \end{pmatrix}$□.

2.120. a), b), c) A, B matritsalarning dioganallarida kataklar soni bir xildir va bir xil nomerli dioganal kataklarning tartiblari ham o'zaro tengdir;
d) $AB=BA$ bo'lishi uchun a) shart va bir xil nomerli dioganal kataklarning o'rinal-mashinuvchi bo'lishi zarurdir.

2.122. a) $\begin{pmatrix} 0 & -1 & -2 & -3 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$; b) E; c) $\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 3 \\ 2 & 4 & 6 & 8 \end{pmatrix}$;

d) $\begin{pmatrix} 0 & 0 & -1 & -1 \\ 2 & 1 & -1 & -2 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$; e) $\begin{pmatrix} 7 & 10 & 15 & 25 & 20 \\ 15 & 22 & 18 & 30 & 36 \\ 0 & 0 & 3 & 5 & 6 \\ 0 & 0 & 5 & 9 & 11 \\ 0 & 0 & 6 & 11 & 14 \end{pmatrix}$.

2.123. $\begin{pmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{pmatrix}$.

2.124. a) □ h; b) $\begin{pmatrix} -D \\ E \end{pmatrix}$ □ $h+\begin{pmatrix} b \\ 0 \end{pmatrix}$ □ (E - birlik matritsa va tartibi s ga teng, 0 - noldan iborat ustun va tartibi s ga teng, h - tartibi s ga teng ixtiyoriy ustun).

2.125. a) $\begin{pmatrix} 12 & 20 \\ -6 & -10 \\ 20 & 36 \\ -10 & -18 \end{pmatrix}$; b) $\begin{pmatrix} 12 & 20 \\ 20 & 36 \\ -6 & -10 \\ -10 & -18 \end{pmatrix}$; c) $\begin{pmatrix} -1 & -2 \\ 1 & 2 \\ -3 & -4 \\ 3 & 4 \end{pmatrix}$; d) $\begin{pmatrix} -1 & -2 \\ -3 & -4 \\ 1 & 2 \\ 3 & 4 \end{pmatrix}$

e) $\begin{pmatrix} 3 & 5 & 6 & 10 \\ 5 & 9 & 10 & 18 \\ 9 & 15 & 12 & 20 \\ 15 & 27 & 20 & 36 \end{pmatrix}$; f) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}$.

2.126. a$\otimes$b=b$\otimes$a=ba.

III-bob

Kompleks sonlar nazariyasi

1-§.

3.1. a) $3+7i$; $-22+7i$; b) $-8-10i$; $21-24i$; c) 10; 28.

3.2. a) $4-2i$; $1-2i$; b) $\left(1-\sqrt{2}\right)+\left(\sqrt{6}-\sqrt{3}\right)i$; $\sqrt{2}$;

c) $2\sqrt{b}i$, $\dfrac{a^2-b}{a^2+b}+\dfrac{2a\sqrt{b}}{a^2+b}i$.

3.3. a) $7+17i$; b) $10-11i$; c) $14-5i$; d) $5+i$;

e) $\dfrac{13}{2}-\dfrac{1}{2}i$; f) $\dfrac{11}{5}-\dfrac{27}{5}i$; d) 4; h) 52 i; i) 2; j) 1.

3.4. a) –2; b) 0.

3.5. a) 0; b) $-\dfrac{11}{17}$.

3.7. a) $z_1=2$, $z_2=1-i$; b) $\varnothing$; c) $z_1=\dfrac{(2+i)z_2-i}{2}$.

3.8. a) 1; b) $n=4\kappa$ bo'lganda 0, $n=4\kappa+1$ bo'lganda i; $n=4\kappa+2$ bo'lganda i-1; $n=4\kappa+3$ bo'lganda -1; c) $-i$.

3.9. a) $a^2+b^2+c^2-(ab+bc+ac)$; b) a^3+b^3;

c) $2\left(a^3+b^3+c^3\right)-3\left(a^2b+a^2c+b^2a+b^2c+c^2a+c^2b\right)+12abc$.

3.10. a) $-1-i$; b) $0,-1$, $\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$; c) i;

d) $z_1 = \frac{\sqrt{15}}{2} + \frac{1}{2}i\,, z_2 = \frac{-\sqrt{15}}{2} + \frac{1}{2}i\,;$

e) $\{x + yi \mid -7 \le x \le 1,\ y = \pm\sqrt{7 - 6x - x^2}\}$; f) $\varnothing$.

3.11. a) $\pm(1+i)$; b) $\pm(2-2i)$; c) $\pm(2-i)$; d) $\pm(1+4i)$; e) $\pm(5+6i)$;

f) $\pm(1-3i)$; g) $\pm\left(\sqrt{\frac{\sqrt{13}+2}{2}} - i\sqrt{\frac{\sqrt{13}-2}{2}}\right)$; h) $\pm\left(\sqrt{\frac{3}{2}} - i\sqrt{\frac{1}{2}}\right)$;

i) $i^{\alpha}\left(\frac{1+\sqrt{3}}{2} + \frac{1-\sqrt{3}}{2}i\right),\ \alpha = 0,1,2,3$; j) $\frac{\sqrt{2}(\pm 1 \pm i)}{2}$.

3.12. a) $x_1 = 3-i$; $x_2 = -1+2i$; b) $x_1 = 2+i$, $x_2 = 1-3i$;

c) $x_1 = 1-i$; $x_2 = \frac{4-2i}{5}$; d) $x_1 = 2-i$, $x_2 = -2+i$, $x_3 = 2+i$, $x_4 = -2-i$;

e) $x_1 = x_2 = i\sqrt{17}$, $x_3 = x_4 = -i\sqrt{17}$; f) $x_1 = 3+2i$, $x_2 = 1+i$;

g) $x_1 = -\frac{5}{2} + \frac{\sqrt{11}}{2}i\,,\ x_2 = -\frac{5}{2} - \frac{\sqrt{11}}{2}i\,;$ h) $x_1 = -i$, $x_2 = -1+i$

2-§.

3.14. $z_4 = z_1 + z_3 - 2z_2$.

3.15. $t(7+i)$, t – ixtiyoriy musbat son.

3.16. a) $2 - \frac{3}{2}i$; b) 0, $3i$, $-3i$; c) $bi,\ b \in \boldsymbol{R}$.

3.17. $\frac{7}{6} + \frac{5}{6}i$.

3.18. $-\frac{3}{2} - \frac{17}{4}i\,;\ -\frac{3}{2} - 2i$.

3.19. a) $\pm\frac{1}{2} \pm \frac{1}{2}i$; b) -1, $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$;

c) $4 + i\sqrt{3},\ 3 + 2i\sqrt{3},\ 1 + 2i\sqrt{3},\ i\sqrt{3},\ 1,\ 3$.

3.20. a) radiusi 1 ga teng va markazi koordinatalar boshiga joylashgan aylana;

b) koordinatalar boshidan chiquvchi va musbat haqiqiy yarim o’q bilan $\frac{\pi}{3}$ ga teng burchak hosil qiluvchi nur;

s) radiusi 2 ga teng va markazi koordinatalr boshida bo’lgan yopiq doira;

d) radiusi 1 ga teng bo’lib, markazi 1 + i nuqtaga joylashgan doiraning ichki qismi;

ye) radiusi 5 ga teng bo’lib, markazi –3 - 4i nuqtaga joylashgan yopiq doira;

f) radiuslari 3 va 5 ga teng bo’lib, markazlari koordinatalar boshiga joylashgan aylanalar bilan chegaralangan halqaning ichki qismi;

g) radiuslari 1 va 2 ga teng, markazi 2i nuqtaga joylashgan aylanalar bilan chegaralangan halqa bo'lib, unga 1 radiusli aylana kiradi, 2 radiusli aylana kirmaydi;

h) koordinatalar boshidan o'tuvchi va musbat haqiqiy yarim o'q bilan $-\frac{\pi}{6}$ va $\frac{\pi}{6}$ burchaklar hosil qiluvchi nurlar hosil qilgan burchakning ichki qismi;

i) $x = \pm 1$ to'g'ri chiziqlar orasiga joylashgan polosa, bu to'g'ri chiziqlar ham kiradi;

j) $u = 1$ to'g'ri chiziqlar;

k) $y = \pm 1$ ikki to'g'ri chiziq;

l) $x + y = \pm 1$ to'g'ri chiziqlar orasiga joylashgan polosaichki qismi;

m) $\frac{4x^2}{9} + \frac{4y^2}{5} = 1$ ellips;

n) $\frac{4x^2}{9} - \frac{4y^2}{7} = 1$ giperbola;

o) $u^2 = 8x$ parabola;

p) mavhum o'qdan chapda yotuvchi ochiq yarimtekislik.

3.21. i .

3.22. $\frac{12}{5} + \frac{16}{5}i$.

3.23. a) $[A, B]$ kesmada, bu yerda A(-1,2), V(2,-1);

b) nuqtalar $y = -x^2 + 10$ parabolaga joylashgan, bunda $y \geq 4$.

3.24. $|z| = |\bar{z}|$, $arg z = -arg\,\bar{z}$.

3.25. $s = 7+i$, $C(7,1)$.

3.26. a) $arg z_1 = arg z_2$; b) $arg z_1 = -arg z_2$, $|z_1| \geq |z_2|$.

3.27. *Ko'rsatma*. t ni z orqali ifodalang va $\bar{t} = t$ bo'lishini isbotlang.

3.28. a) $7(cos 0 + isin 0)$; b) $\left(cos\frac{\pi}{2} + isin\frac{\pi}{2}\right)$; c) $3(cos\pi + isin\pi)$;

d) $5\left(cos\frac{3\pi}{2} + isin\frac{3\pi}{2}\right)$; e) $2\left(cos\frac{\pi}{3} + isin\frac{\pi}{3}\right)$; f) $2\left(cos\frac{2\pi}{3} + isin\frac{2\pi}{3}\right)$;

g) $2\left(cos\left(-\frac{\pi}{3}\right) + isin\left(-\frac{\pi}{3}\right)\right)$; h) $2\left(cos\frac{\pi}{6} + isin\frac{\pi}{6}\right)$;

i) $2\left(cos\frac{5}{6}\pi + isin\frac{5}{6}\pi\right)$; j) $2\left(cos\left(-\frac{5}{6}\pi\right) + isin\left(-\frac{5}{6}\pi\right)\right)$;

k) $2\left(cos\left(-\frac{\pi}{6}\right) + isin\left(-\frac{\pi}{6}\right)\right)$; l) $\frac{2}{\sqrt{3}}\left(cos\frac{\pi}{6} + isin\frac{\pi}{6}\right)$;

m) $2\sqrt{2+\sqrt{3}}\left(cos\frac{\pi}{12} + isin\frac{\pi}{12}\right)$ yoki $\sqrt{6} + \sqrt{2}\left(cos\frac{\pi}{12} + isin\frac{\pi}{12}\right)$;

modul uchun ikkinchi ifodani hosil qilish uchun $\sqrt{a\pm\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2-b}}{2}}$ formulani qo'llash lozim;

n) $2\left(\sqrt{2+\sqrt{3}}\right)\left(cos\left(-\frac{5}{12}\pi\right)+isin\left(-\frac{5}{12}\pi\right)\right)$; o) $cos(-\alpha)+isin(-\alpha)$;

p) $cos\left(\frac{\pi}{2}-\alpha\right)+isin\left(\frac{\pi}{2}-\alpha\right)$; q) $cos2\alpha+isin2\alpha$;

r) $0\le\alpha<\pi,\ \ 2cos\frac{\alpha}{2}\left(cos\frac{\alpha}{2}+isin\frac{\alpha}{2}\right)$;

$\pi\le\alpha<2\pi,$ $-2cos\frac{\alpha}{2}\left(cos\frac{\alpha+2\pi}{2}+isin\frac{\alpha+2\pi}{2}\right)$;

s) $0\le\alpha\le\pi,\ \ 2cos\frac{\alpha}{2}\left(cos\frac{3\pi-\alpha}{2}+isin\frac{3\pi-\alpha}{2}\right)$;

$\pi\le\alpha<2\pi,$ $-2cos\frac{\alpha}{2}\left(cos\frac{\pi-\alpha}{2}+isin\frac{\pi-\alpha}{2}\right)$

3.29. a) $1=cos0+isin0$; b) $-\frac{1}{2}-\frac{\sqrt{3}}{2}i=cos\frac{4}{3}\pi+isin\frac{4}{3}\pi$;

c) $\frac{1}{2}=\frac{1}{2}(cos0+sin0)$; d) $\frac{1}{2}-\frac{\sqrt{3}}{2}i=cos\frac{5}{3}\pi+isin\frac{5}{3}\pi$;

e) $-i=cos\frac{3}{2}\pi+isin\frac{3}{2}\pi$.

3.30. a) $\frac{5}{3}\left(cos230^0+isin230^0\right)$; b) $\sqrt{2}\,sin\frac{\pi}{5}\left(cos\frac{29}{20}\pi+isin\frac{29}{20}\pi\right)$.

3.31. Ayniyat geometriyadagi quyidagi teormeani ifodalaydi: parallelogram dioganallari kvadratlarining yig'indisi tomonlari kvadratlarining yig'indisiga teng.

3-§.

3.32. a) $2^9\left(1-i\sqrt{3}\right)$; b) $\left(2-\sqrt{3}\right)^{12}$; c) – 64;

d) 2, agar n – juft bo'lsa, –2, agar n – toq bo'lsa;

e) $\frac{1}{cos^4 1}(cos4+isin4)$; f) $\frac{1}{cos^4 2}(cos2+isin2)$; g) $-32icos^5\frac{3\pi}{5}$.

3.36. a) $cos\frac{(4k+1)\pi}{12}+isin\frac{(4k+1)\pi}{12}\quad(0\le k\le 5)$;

b) $(cos\frac{(6k-1)\pi}{30}+isin\frac{(6k-1)\pi}{30})\quad(0\le k\le 9)$;

s) $\sqrt{2}\left(cos\frac{(8k-1)\pi}{32}+isin\frac{(8k-1)\pi}{32}\right)(0\le k\le 7)$.

3.37. a) $\left\{1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right\}$; b) $\{\pm 1, \pm i\}$; c) $\left\{\pm 1, \pm\frac{1+i\sqrt{3}}{2}; \pm\frac{1-i\sqrt{3}}{2}\right\}$;

d) $\left\{\frac{\sqrt{3}}{2}+\frac{1}{2}i, -\frac{\sqrt{3}}{2}+\frac{1}{2}i, -i\right\}$; e) $\{1\pm i; -1\pm i\}$; f) $2\sqrt[6]{1}$;

g) $\{\pm\sqrt{2}, \pm\sqrt{2}i, \pm\sqrt{2}(1+i), \pm\sqrt{2}(1-i)\}$;

h) $\left\{\pm i\sqrt{3}, \pm\frac{\sqrt{3}}{2}\left(\sqrt{3}+i\right), \pm\frac{\sqrt{3}}{2}\left(\sqrt{3}-i\right)\right\}$;

i) $\{\sqrt{3}+i, -1+i\sqrt{3}, -\sqrt{3}-i, 1-i\sqrt{3}\}$; j) $\{3-i\sqrt{3}, \sqrt{3}+3i, -3+i\sqrt{3}, -\sqrt{3}-3i\}$;

k) $\left\{\frac{1}{2}\sqrt[6]{2}\left(\sqrt{2\sqrt{3}}+i\sqrt{2\sqrt{3}}\right), \frac{1}{2}\sqrt[3]{4}(i-1), \frac{1}{6}\sqrt[6]{2}\left(\sqrt{2-\sqrt{3}}+i\sqrt{2+\sqrt{3}}\right)\right\}$;

l) $\left\{\frac{1}{2}\sqrt{2}\left(\sqrt{2+\sqrt{3}}-i\sqrt{2-\sqrt{3}}\right), -\frac{1}{2}\sqrt{2}\left(\sqrt{2-\sqrt{3}}-i\sqrt{2+\sqrt{3}}\right), 1-i\right\}$;

m) $\{\pm\sqrt{3}+i, -2i\}$; n) $\left\{\frac{3}{2}\left(\pm\sqrt{3}+i\right), -3i\right\}$;

o) $\left\{\pm\left(\frac{3}{2}+i\frac{\sqrt{3}}{2}\right), \pm\left(\frac{\sqrt{3}}{2}-i\frac{3}{2}\right)\right\}$; p) $\left\{\pm\left(1-i\frac{\sqrt{3}}{3}\right), \pm\left(\frac{\sqrt{3}}{3}+i\right)\right\}$.

3.38.

a) $\sqrt[5]{2}\left(cos\frac{\pi}{15}+isin\frac{\pi}{15}\right)$; $\sqrt[5]{2}\left(cos\frac{7\pi}{15}+isin\frac{7\pi}{15}\right)$; $\sqrt[5]{2}\left(cos\frac{13\pi}{15}+isin\frac{13\pi}{15}\right)$;

$\sqrt[5]{2}\left(cos\frac{19\pi}{5}+isin\frac{19\pi}{5}\right)$; $\sqrt[5]{2}\left(cos\frac{5\pi}{3}+isin\frac{5\pi}{3}\right)$

b) $\sqrt{3}+i,\ 2i,\ -\sqrt{3}+i,\ -\sqrt{3}-i, -2i,\ \sqrt{3}-i$.

3.39. $-\frac{3}{2}i$;

3.40

40. $2\left(cos144^0+isin144\right), 2\left(cos216^0+isin216^0\right)$,

$\sqrt[5]{31}\left(cos108^0+isin108^0\right), -\sqrt[5]{31}\left(cos252^0+isin252^0\right)$.

3.41.

$a)\ cos^6x+6cos^5xsinx-15cos^4xsin^2x-20cos^3xsin^3x+$

$+15cos^2xsin^4x+6cosxsin^5x-sin^6x;$

c) $sin8x=8cos^7xsinx-56cos^5xsin^3x+56cos^3xsin^5x-8cosxsin^7x$;

b) $cos8x=cos^8x-28cos^6xsin^2x+70cos^4xsin^4x-28cos^2xsin^6x+sin^8x$.

3.42. $ctg7x=\frac{7tgx-35tg^3x+21tg^5x-tg^7x}{1-21tg^2x+35tg^4x-7tg^6x}$.

3.43. $tg\ nx=\dfrac{\sum\limits_{\kappa=0}^{m}(-1)^{\kappa}C_n^{2\kappa+1}tg^{2\kappa+1}x}{\sum\limits_{\kappa=0}^{\ell}(-1)^{\kappa}C_n^{2\kappa}tg^{2\kappa}x}$, bu yerda m,$\ell$ – shunday butun sonlarki, $\frac{n-1}{2}-1<m\le\frac{n-1}{2}$, $\frac{n}{2}-1<\ell\le\frac{n}{2}$.

3.44. $n=2\ell+1$ toq bo'lganda

$$sin^n x=(-1)^{\frac{n-1}{2}}\left(\frac{1}{2}\right)^{n-1}\sum_{k=0}^{\ell}(-1)^k C_n^k sin(n-2k)x.$$

$n=2\ell$ juft son bo'lganda

$$sin^n x=(-1)^{\frac{n}{2}}\left(\frac{1}{2}\right)^{n-1}\left[\sum_{k=0}^{\ell-1}(-1)^k C_n^k cos(n-2k)x+(-1)^{\ell}\frac{1}{2}C_n^{\ell}\right].$$

$n=2\ell+1$ toq son bo'lganda $cos^n x=\left(\frac{1}{2}\right)^{n-1}\sum\limits_{k=0}^{\ell}C_n^k cos(n-2k)x$.

$n=2\ell$ juft son bo'lganda $cos^n x=\left(\frac{1}{2}\right)^{n-1}\left[\sum\limits_{k=0}^{\ell}C_n^k cos(n-2k)x+\frac{1}{2}C_n^{\ell}\right]$.

Ko'rsatma. $z=cosx+isinx,\ \bar z=cosx-isinx$ larni qaraladi. Bu yerdan $cosx=\frac{z+\bar z}{2},\ sinx=\frac{z-\bar z}{2i},\ cos^n x=\frac{1}{2^n}(z+\bar z)^n=$ $=\frac{1}{2^n}\sum\limits_{k=0}^{n}C_n^k z^{n-k}\bar z^k,\ sin^n x=\frac{1}{(2i)^n}(z-\bar z)=\frac{1}{(2i)^n}\sum\limits_{k=0}^{n}(-1)^k C_n^k z^{n-k}\bar z^k$ kelib chiqadi. Keyin bir xil binomial umumiy ko'patuvchini qavsdan tashqariga chiqarish va Muavr formulasidan foydalanish lozim.

3.45. a) $\frac{3sinx-sin3x}{4}$; b) $\frac{cos4x-4cos2x+3}{8}$;

c) $\frac{cos5x+5cos3x+10cosx}{16}$; d) $\frac{cos6x+6cos4x+15cos2x+10}{32}$;

e) $\frac{1}{128}(sin8x+2sin6x-2sin4x-6sin2x)$;

f) $\frac{1}{64}[(cos7x-sin7x)+7(cos5x+sin5x)+21(cos3x-sin3x)+35(cosx+sinx)]$.

4-§.

3.46. $C_{2n}^{\kappa}=C_{2n}^{2n-\kappa}$ ekanligini hisobga olinsa, a) va b) tengliklar 1-misolga keltiriladi.

3.47. a) va b) tenglamalarni 2-misoldagidek hosil qilish mumkin, bunda: $2^n+\varepsilon(1+\varepsilon)^n+\varepsilon^2(1+\varepsilon^2)^n,\ 2^n+\varepsilon^2(1+\varepsilon)^n+\varepsilon(1+\varepsilon^2)^n$.

3.48. 4-misolga qarang.

3.49. 5-misoldagi (*) ayniyatdan $x=k$ bo'lganda kelib chiqadi.

3.50. *Yechish.* Chap tomondagi ifoda quyidagi ko'phaddagi x^n oldidagi koeffisiyentan iborat:

$$x^n(1+x)^n - x^{n-1}(1+x)^n + x^{n-2}(1+x)^n + \ldots + (-1)^k x^{n-k}(1+x)^n =$$
$$=(1+x)^n\left(x^n - x^{n-1} + \ldots + (-1)^k x^{n-k}\right) = (1+x)^{n-1}\left(x^{n+1} + (-1)^k x^{n-k}\right).$$

Oxirgi ifodada x^n oldidagi koeffisiyent $(-1)^k C_{n-1}^k$ ga tengligi ravshan.

3.51. 7-misoldan kelib chiqadi.

3.52. a) *Yechish.* $(1+x)^m(1-x)^m = \left(1-x^2\right)^m$ ko'paytmani qaraymiz. natijada, $\sum\limits_{s=0}(-1)^s C_m^s x^s \sum\limits_{t=0}^{m} C_m^t x = \sum\limits_{k=0}^{m}(-1)^k C_m^k x^{2k}$, shuning uchun

$$\sum_{s+t=2k}(-1)^s C_m^s C_m^t = (-1)^k C_m^k$$

Avvalo faraz qilaylik, m – juft, ya'ni $m=2n$, $k=n$ bo'lsin. U holda $\sum\limits_{s+t=2n}^{m}(-1)^s C_{2n}^s C_{2n}^{2-nt} = (-1)^n C_{2n}^n$. Bu yerdan $\sum\limits_{s=0}^{2n}(-1)^s\left(C_{2n}^s\right)^2 = (-1)^n C_{2n}^n$ ni hosil qilamiz;

b) agar m – toq bo'lsa, $m=2n+1$ deb olamiz.

$(1+x)^m(1-x)^m = \left(1-x^2\right)^m$ tenglikning chap tomonidagi x^{2n+1} oldidagi koeffisiyent $\sum\limits_{s+t=2n+1}(-1)^s C_{2n+1}^s C_{2n+1}^t = \sum\limits_{s=0}^{2n+1}(-1)^S\left(C_{2n+1}^s\right)^2$ ga teng. Lekin qaralayotgan tenglikning o'ng tomonidan ko'rinadiki, bu koeffisiyent nolga teng bo'lishi kerak (chunki yoyilmasida x ning toq darajali hadlari qatnashmaydi). Shuning uchun $\sum\limits_{s=0}^{2n+1}(-1)^s\left(C_{2n+1}^s\right)^2 = 0$ va tenglik isbot bo'ldi.

3.53. a) 9-misolda φ ni $\frac{\pi}{2}-\varphi$ ga almashtiring;

b) va s) lar ham 9-misol va a) ga o'xshash keltirib chiqariladi.

3.54. 10-misolga o'xshash.

3.55. a) $2^n cos^n \frac{x}{2} cos\frac{(n+2)x}{2}$; b) $2^n cos^n \frac{x}{2} sin\frac{(n+2)x}{2}$.

3.56. $\frac{n}{2}-\frac{sin4nx}{4sin2x}$. *Ko'rsatma*: $sin^2\alpha = \frac{1-cos2\alpha}{2}$ formuladan foydalaning.

3.58. a) $\dfrac{(n+1)cosnx - ncos(n+1)x - 1}{4sin^2\frac{x}{2}}$;

b) $\frac{(n+1)sinnx - nsin(n+1)x}{4sin^2\frac{x}{2}}$. *Ko'rsatma.* $1+2\alpha+3\alpha^2+\ldots+na^{n-1}$ ko'rinishdagi yig'indini hisoblash uchun uni $1-\alpha$ ga ko'paytirish foydali.

5-§.

3.59. a) ± 1; b) $1, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$; c) $\pm 1, \pm i$; d) $\pm 1, \pm i, \pm\frac{\sqrt{2}}{2}(1 \pm i)$;

e) $\pm 1, \pm i, \pm\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \ \pm\frac{\sqrt{3}}{2} \pm \frac{i}{2}$;

g) $\pm 1, \ \pm i, \ \pm\frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \ \pm\frac{\sqrt{3}}{2}(1 \pm i); \ \pm\frac{\sqrt{3}}{2} \pm \frac{i}{2}; \ \pm\frac{\sqrt{6}+\sqrt{2}}{4} \pm i\frac{\sqrt{6}-\sqrt{2}}{4}; \ \pm\frac{\sqrt{6}-\sqrt{2}}{4} \pm i\frac{\sqrt{6}+\sqrt{2}}{4}$

3.60. a) -1; b) $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$; c) $\pm i$; d) $\pm\frac{\sqrt{2}}{2}(1 \pm i)$; e) $\pm\frac{\sqrt{3}}{2} \pm \frac{i}{2}$;

f) $\pm\frac{\sqrt{6}+\sqrt{2}}{4} \pm i\frac{\sqrt{6}-\sqrt{2}}{4}; \ \pm\frac{\sqrt{6}-\sqrt{2}}{4} \pm i\frac{\sqrt{6}+\sqrt{2}}{4}$.

3.61. a) $\varepsilon_k = cos\frac{2\pi k}{16} + isin\frac{2\pi k}{16}$ belgilashni kiritib, quyidagilarni hosil qilamiz:

1 ko'rsatkichga ε_0 tegishli;

2 ko'rsatkichga ε_8 tegishli;

4 ko'rsatkichga $\varepsilon_4, \varepsilon_{12}$ tegishli;

8 ko'rsatkichga $\varepsilon_2, \varepsilon_6, \varepsilon_{10}, \varepsilon_{14}$ tegishli;

16-darajali boshlang'ich ildizlar $\varepsilon_1, \varepsilon_3, \varepsilon_5, \varepsilon_7, \varepsilon_9, \varepsilon_{11}, \varepsilon_{13}, \varepsilon_{15},$

b) $\varepsilon_{\kappa,} = cos\frac{2\pi\kappa}{20} + isin\frac{2\pi\kappa}{20}$ belgilashni kiritib, quyidagilarni hosil qilamiz:

1 ko'rsatkichga ε_0 tegishli;

2 ko'rsatkichga ε_{10} tegishli;

4 ko'rsatkichga $\varepsilon_5, \varepsilon_{15}$ tegishli;

5 ko'rsatkichga $\varepsilon_4, \varepsilon_8, \varepsilon_{12}, \varepsilon_{16}$ tegishli;

10 ko'rsatkichga $\varepsilon_2, \varepsilon_6, \varepsilon_{14}, \varepsilon_{18}$ tegishli;

20-darajali boshlang'iya ildizlar $\varepsilon_1, \varepsilon_3, \varepsilon_5, \varepsilon_7, \varepsilon_9, \varepsilon_{11}, \varepsilon_{13}, \varepsilon_{17}, \varepsilon_{19}$

s) $\varepsilon_{\kappa,} = cos\frac{2\pi\kappa}{24} + isin\frac{2\pi\kappa}{24}$ belgilashlarni kiritib, quyidagilarni hosil qilamiz:

1 ko'rsatkichga ε_0 tegishli;

2 ko'rsatkichga ε_{12} tegishli;

3 ko'rsatkichga $\varepsilon_8, \varepsilon_{16}$ tegishli;

4 ko’rsatkichga $\varepsilon_{6,}\varepsilon_{18}$ tegishli;

6 ko’rsatikichga $\varepsilon_{4,}\varepsilon_{20}$ tegishli;

8 ko’rsatkichga $\varepsilon_{3,}\varepsilon_{9,}\varepsilon_{15,}\varepsilon_{21}$ tegishli;

12 ko’rsatkichga $\varepsilon_{2,}\varepsilon_{10,}\varepsilon_{14,}\varepsilon_{22}$ tegishli

24-darajali boshlang’ich ildizlar $\varepsilon_{1,}\varepsilon_{5,}\varepsilon_{7,}\varepsilon_{11,}\varepsilon_{13,}\varepsilon_{17,}\varepsilon_{19,}\varepsilon_{23}$.

3.62. $\frac{2}{1-\varepsilon}$.

3.63. 0, agar $n>1$ bo’lsa.

3.64. $-\frac{n}{1-\varepsilon}$, agar $\varepsilon \neq 1$ bo’lsa; $\frac{n(n+1)}{2}$, agar $\varepsilon=1$ bo’lsa.

3.65. $-\frac{n^2(1-\varepsilon)+2n}{(1-\varepsilon)^2}$, agar $\varepsilon \neq 1$ bo’lsa;

$\frac{n(n+1)(2n+1)}{6}$, agar $\varepsilon=1$ bo’lsa.

3.66. a) $-\frac{n}{2}$; b) $-\frac{n}{2}ctg\frac{\pi}{n}$;

3.67. a) 1; b) 0; c) -1.

3.68. *Yechilishi:* Agar z berilgan tenglamani qanoatlantirsa, u holda $\left|\frac{z-a}{z-b}\right|=\sqrt[n]{\left|\frac{\mu}{\lambda}\right|}$ bo’ladi. Berilgan ikki nuqtalargacha bo’lgan masofalar nisbati o’zgarmas bo’lgan nuqtalar to’plami aylanadan iborat (xususiy holda, $|\lambda|=|\mu|$ bo’lsa, bu to’plam to’g’ri chiziq bo’ladi).

3.69. a) $-2ictg\frac{\pi k}{n}$ $(k=0,1,\ldots,n\text{-}1)$; b) $x=5ctg\frac{\pi k}{n}$ $(k=1,\ldots,n\text{-}1)$;

c) $x=3ctg\frac{4k+3}{4n}\pi$ $(k=0,1,\ldots,n\text{-}1)$. *Ko’rsatma.*

$$x+3i=(x-3i)\alpha_k,\ \alpha_k=\sqrt[n]{-i}=cos\frac{4k+3}{2n}\pi+isin\frac{4k+3}{2n}\pi\ (k=0,1,\ldots,n-1)$$

tenglamani qarang;

d) $actg\frac{2\pi k+\varphi}{2n}$ $(k=0,1,\ldots,n\text{-}1)$.

3.70. *Yechish.* $A=cos\varphi+isin\varphi$ bo’lsin. U holda $\frac{1+ix}{1-ix}=\eta_k^2$, bu yerda

$$\eta_k=cos\frac{\varphi+2\pi k}{2m}+isin\frac{\varphi+2\pi k}{2m}\ (k=0,1,\ldots,m\text{-}1).$$

Bundan $x=\frac{\eta_k^2-1}{i(\eta_k^2+1)}=\frac{\eta_k-\eta_k^{-1}}{i(\eta_k+\eta_k^{-1})}=tg\frac{\varphi+2\pi k}{2m}$.

3.71. *Yechish.* $\varepsilon -$ $x^a -1$ va $x^в -1$ larning umumiy ildizi; s - ε ildiz tegishli bo'lgan ko'rsatkich bo'lsin. U holda s - a va v ning umumiy bo'luvchisi bo'ladi shuning uchun faqat s=1 va ε=1 bo'lishi mumkin. Teskarisi ko'rinib turibdi.

3.72. α va β - 1 ning a va b-darajali boshlang'ich ildizlari bo'lsin. $(\alpha\beta)^s =1$ bo'lsin. U holda $\alpha^{bs} =1$; $\beta^{as} =1$. Demak bs a ga bo'linadi, as b ga bo'linadi. Natijada s ab ga bo'linadi. λ - 1 ning ab-darajali boshlang'ich ildizi bo'lsin. U holda $\lambda = \alpha^{\kappa}\beta^{s}$ (9-misolni qarang). α^{κ} ildiz $a_1 < a$ ko'rsatkichga tegishli bo'lsin. U holda $\lambda^{a_1 b} = \left(a^{\kappa}\right)^{a_1 b}\left(\beta^{s}\right)^{a_2 b} =1$, bu esa mumkin emas. Xuddi shunday, β^S - 1 ning b- darajali boshlang'ich ildizi bo'lishini ko'rsatish mumkin.

3.73. 72-masaladan kelib chiqadi.

3.74. *Yechish.* Avvalo r^{α} dan oshmaydigan barcha r ga karrali sonlarni yozib olamiz. Bular $1\cdot r$, $2\cdot r,\ldots,r^{\alpha\text{-}1}$ r. Bunday sonlar $p^{\alpha-1}$ ta. Natijada, Eyler funksiyasining ta'rifiga ko'ra, $\varphi\left(p^{\alpha}\right)= p\alpha - p^{\alpha-1} = p^{\alpha}\left(1-\frac{1}{p}\right)$. U holda 73-masalaga ko'ra

$$\varphi(n)=\varphi\left(p_1^{\alpha_1}\right)\varphi\left(p_1^{\alpha_2}\right)..\varphi\left(p_n^{\alpha\kappa}\right)= n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)...\left(1-\frac{1}{p_{\kappa}}\right).$$

3.75. *Yechish.* Agar ε - birning n-darajali boshlang'ich ildizi bo'lsa, u holda uning qo'shmasi $\bar{\varepsilon}$ ham 1 ning n-darajali boshlang'ich ildizi bo'ladi. Bunda $\varepsilon\neq\pm1$, chunki $n>2$.

3.76. a) $X_1(x)= x-1$; b) $X_2(x)= x+1$; c) $X_3(x)= x^2 + x+1$;
d) $X_4(x)= x^2 +1$; e) $X_5(x)= x^4 + x^3 + x^2 + x+1$; f) $X_6(x)= x^2 - x+1$; g) $X_7(x)= x^6 + x^5 + x^4 + x^3 + x^2 + x+1$;
h) $X_8(x)= x^4 +1$; i) $X_9(x)= x^6 + x^3 +1$;
j) $X_{10}(x)= x^4 - x^3 + x^2 - x+1$;
k) $X_{11}(x)= x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x+1$;
l) $X_{12}(x)= x^4 - x^2 +1$; m) $X_{15}(x)= x^8 - x^7 + x^5 - x^4 + x^3 - x+1$;
n) $X_{105}(x)= x^{48} + x^{47} - x^{46} + x^{43} - x^{42} - 2x^{41} - x^{40} - x^{39} + x^{36} + x^{35} + x^{34}$
$+ x^{33} + x^{32} + x^{31} - -x^{28} - x^{26} - x^{24} - x^{22} - x^{20} + x^{17} + x^{16} + x^{15} + x^{14} +$
$x^{13} + x^{12} - x^9 - x^8 - 2x^7 - x^6 - x^5 + x^2 + x+1$

3.77. $X_p(x)= x^{p-1} + x^{p-2} +...+ x+1.$

3.78. $X_{p^m}(x)= x^{(p-1)p^{m-1}} + x^{(p-2)p^{m-1}} +...+ x^{p^{m-1}} +1$. *Ko'rsatma.* $x^{p^{m-1}} -1$ ning barcha ildizlari va faqat ular $x^{p^m} -1$ ning boshlang'ich ildizlari bo'ladi.

3.79. *Yechish.* $\alpha_1, \alpha_2, .. \alpha_{\varphi(n)}$ - 1 ning *n*-darajali boshlang'ich ildizlari bo'lsin. U holda (72-masalaga qarang) $(-\alpha_1)$, $(-\alpha_2), ..., (-\alpha_{\varphi(n)})$ sonlar 1 ning $2n$ – darajali boshlang'ich ildizlari bo'ladi.

$X_{2n}(x) = (x+\alpha_1)(x+\alpha_2)...(x+\alpha_{\varphi(n)}) = (-1)^{\varphi(n)}(-x-\alpha_1)(-x-\alpha_2)...(-x-\alpha_{\varphi(m)})$,

yoki (75-masalaga qarang) $X_{2n}(x) = X_n(-x)$.

3.80. *Yechish.* $\varepsilon_\kappa = cos\frac{2\pi\kappa}{nd} + isin\frac{2\pi\kappa}{nd}$ - 1 ning *nd*-darajali boshlang'ich ildizi bo'lsin, ya'ni *k* va *n* o'zaro tub sonlar. *k* ni *n* ga bo'lib, *k = nq+r, 0<r<n* ni hosil qilamiz. Bu yerdan:

$$\varepsilon_\kappa = cos\frac{2q\pi + \frac{2r\pi}{n}}{d} + isin\frac{2q\pi + \frac{2r\pi}{n}}{d},$$

ya'ni ε_κ - *d* darajali ildizning qiymatlaridan biri bo'ladi va $\eta_r = cos\frac{2\pi r}{n} + isin\frac{2\pi r}{n}$; η_r - 1 ning *n* darajali boshlang'ich ildizi bo'ladi, chunki *r* va *n* ning har bir umumiy bo'luvchisi *k* va *n* ning umumiy bo'luvchisi bo'ladi.

$\eta_r = cos\frac{2\pi r}{n} + isin\frac{2\pi r}{n}$ - 1 ning *n*-darajali boshlang'ich ildizi bo'lsin, ya'ni *r* va *n* o'zaro tub sonlar. Quyidagi sonlarni qaraymiz

$$\varepsilon_q = cos\frac{2q\pi + \frac{2r\pi}{n}}{d} + isin\frac{2q\pi + \frac{2r\pi}{n}}{d} = cos\frac{2\pi(r+nq)}{nd} + isin\frac{2\pi(r+nq)}{nd},$$

bu yerda $q = 0,1,2...,d-1$; ε_q- 1 ning *nd* darajali boshlang'ich ildizi bo'ladi. Haqiqatan, agar $r+nq$ va *nd* sonlar bir vaqtda *r* tub songa bo'linsa, *n* va *r* sonlar ham *p* ga bo'linar edi. Bu esa mumkin emas.

3.81. *Yechish.* $\varepsilon_1, \varepsilon_2, ...\varepsilon_{\varphi(n')}$ - 1 ning n' darajali ildizlari bo'lsin. U holda

$X_{n^/}\left(x^{n^{//}}\right) = \prod_{\kappa=1}^{\varphi\left(n^/\right)}\left(x^{n^{//}} - \varepsilon_\kappa\right)$. $(x-\varepsilon_{\kappa,1})(x-\varepsilon_{\kappa,2})...(x-\varepsilon_{\kappa,n^{//}})$ - $\left(x^{n^{//}} - \varepsilon_\kappa\right)$ ning

chiziqli ko'paytuvchilarga yoyilmasi bo'lsin. U holda $X_{n^/}\left(x^{n^{//}}\right) = \prod_{\substack{\kappa=1 \\ i=1}}^{\substack{\kappa=\varphi\left(n^/\right) \\ i=n^{//}}}\left(x - \varepsilon_{\kappa,.i}\right)$.

80-masalaga ko'ra har bir $x - \varepsilon_{\kappa,.i}$ chiziqli ko'paytuvchi $X_n(x)$ yoyilmaga kiradi va aksincha. Bundan tashqari, $\varphi(n) = n^{//}\varphi(n^/)$ bo'lganligi uchun $X_n(x)$ va $X_{n^/}\left(x^{n^{//}}\right)$ larning darajalari teng.

3.82. *Ko'rsatma*. 77, 78, 72- masalalardan foydalaning va

1) r – tub bo'lsa, $\mu(p) = -1$,;

2) r – tub, α>1 bo'lsa, $\mu(p^{\alpha})=0$;

3) a va b o'zaro tub bo'lsa, $\mu(ab)=\mu(a)\mu(b)$.

3.83. *Yechish.* 1 ning barcha n-darajali ildizlari yig'indisi 0 ga teng. 1 ning har bir n-darajali ildizi n ning bo'luvchisi bo'lgan d ko'rsatkichga tegishli i obratno, to $\sum_{d/n}\mu(d)=0$.

3.84. *Yechish.* $\varepsilon_{\kappa}=cos\frac{2\pi\kappa}{n}+isin\frac{2\pi\kappa}{n}$ ildiz n_1 ko'rsatkichga tegishli bo'lsin. U holda x-ε_κ ko'paytuvchi faqat shunday x^d-1 ikki hollarda qatnashadiki, d son n_1 ga bo'linadi. Bunda d n ning n_1 karrali barcha bo'luvchilari to'plamida, $\frac{n}{d}$ esa $\frac{n}{n_1}$ ning barcha bo'luvchilari to'plamida o'zgaradi. Shunday qilib, x-ε_κ ko'paytuvchi o'ng tomonda $\sum_{d_1/\frac{n}{n_1}}\mu(d_1)$ ko'rsatkich bilan qatnashadi. Agar $\frac{n}{n_1}\neq 1$ bo'lsa, bu yig'indi 0 ga, $n = n_1$ bo'lganda esa 1 ga teng.

3.85. *Yechish.* Agar $n=p^{\alpha}$, r – tub son bo'lsa, $X_n(1)=p$ bo'ladi. Agar $n=p_1^{\alpha_1}p_2^{\alpha_2}...p_\kappa^{\alpha_\kappa}$ ($p_1,p_2,....,p_\kappa$ – har xil tub sonlar) bo'lsa, u holda (81-masalaga qarang) $X_n(1)=X_{n^э}(1)$; bunda $n^{/}=p_1p_2...p_\kappa$.

Endi n= $p_1p_2...p_\kappa$; $\kappa\geq 2$; $n_1=\frac{n}{p_\kappa}$ bo'lsin. n_1 ning barcha bo'luvchilarini hosil qilish uchun n ning barcha bo'luvchilariga ularning r_k ga ko'paytmalarini qo'shish yetarli. Shuning uchun

$$X_n(x)=\prod_{d/n}\left(x^d-1\right)^{\mu\left(\frac{n}{d}\right)}=\prod_{d/n^{/}}\left(x^d-1\right)^{\mu\left(\frac{n}{d}\right)}\cdot\prod_{d/n_1}\left(x^{dp_k}-1\right)^{\mu\left(\frac{n}{dp_k}\right)}=$$
$$=\left[X_{n^{/}}(x)\right]^{-1}X_{n^{/}}\left(x^{p_\kappa}\right).$$

Bu yerdan $X_n(1)=1$.

3.86. *Yechish.* 1) n – birdan katta toq son bo'lsin. U holda (79-masalaga qarang) $X_n(-1)=X_{2n}(1)=1$;

2) n = 2ᴷ bo'lsin, u holda $X_n=\frac{x^n-1}{x^{\frac{n}{2}}-1}=x^{\frac{n}{2}}+1$ va $X_n(-1)$ k=1 bo'lganda 0 ga, k>1 bo'lganda 2 ga teng.

3) $n=2n_1$, n_1 - birdan katta toq son bo'lsin. U holda (79-masalaga qarang) $X_n(-1)=X_{n_1}(1)$ va natijada $X_n(-1)$ $n_1=p^{\alpha}$ bo'lganda (r – tub son) p ga, $n_1\neq p$ bo'lganda 1 ga teng.

4) $n=2^{\kappa}n_1$, κ>1, $n_1=p_1^{\alpha_1}p_2^{\alpha_2}....p_s^{\alpha_s}$ ($p_1,p_2,...p_s$- har xil toq sonlar) bo'lsin. Bu holda (81-masalaga qarang) $X_n(x)=X_{2}p_1p_2....p_s\left(x^{\lambda}\right)$, bunda $\lambda=2^{\kappa-1}p_1^{\alpha_1-1}...p_s^{\alpha_s-1}$. Bu yerdan kelib chiqadiki, $X_n(-1)=X_n(1)=1$.

3.87. *Yechish.* $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{\varphi(n)}$- 1 ning boshlang'ich ildizlari bo'lsin:

$$S = \varepsilon_1\varepsilon_2 + \varepsilon_1\varepsilon_3 + \ldots + \varepsilon_{\varphi(n)-1}\varepsilon_{\varphi(n)} = \frac{[\mu(n)]^2 - \left(\varepsilon_1^2 + \varepsilon_2^2 + \ldots + \varepsilon_{\varphi(n)}^2\right)}{2}.$$

1) m – toq son bo'lsin. Bu holda ε_i^2 1 ning n–darajali boshlang'ich ildizi va faqat $i = j$ bo'lganda $\varepsilon_i^2 = \varepsilon_j^2$ bo'ladi. Shuning uchun $\varepsilon_1^2 + \varepsilon_2^2 + \ldots + \varepsilon_{\varphi(n)}^2 = \mu(n)$ va $S = \frac{[\mu(n)]^2 - \mu(n)}{2}$.

2) $n = 2n_1; n_1$- toq son bo'lsin. Bu holda ε_i (72-masalaga qarang) 1 ning n_1 darajali boshlang'ich ildizi bo'ladi va shuning uchun (1) ga qarang) $\varepsilon_1^2 + \varepsilon_2^2 + \ldots + \varepsilon_{\varphi(n)}^2 = \mu(n_1) = -\mu(n)$. Shunday qilib, bu holda $S = \frac{[\mu(n)]^2 + \mu(n)}{2}$.

3) $n = 2^k n_1,\ k > 1,\ n_1$ - toq son bo'lsin. Bu holda ε_i^2 ildiz $\frac{n}{2}$ ko'rsatkichga tegishli bo'ladi. 80-masalaga ko'ra $\varepsilon_1, \varepsilon_2, \ldots \varepsilon_{\varphi(n)}$ lar $\eta_1, \eta_2, \ldots, \eta_{\varphi\left(\frac{n}{2}\right)}$ larning kvadrat ildizlaridan iborat bo'ladi, bu yerda $\eta_1, \eta_2, \ldots, \eta_{\varphi\left(\frac{n}{2}\right)}$- 1 ning $\frac{n}{2}$ - darajali boshlang'ich ildizlari. Bu yerdan kelib chiqadiki,

$$\varepsilon_1^2 + \varepsilon_2^2 + \ldots + \varepsilon_{\varphi(n)}^2 = 2\left(\eta_1 + \eta_2 + \ldots + \eta_{\varphi\left(\frac{n}{2}\right)}\right) = 2\mu\left(\frac{n}{2}\right);\ \ S = -\mu\left(\frac{n}{2}\right).$$

3.88. *Yechish.* y ning istalgan qiymatlarida $S = \sum_{x=0}^{n-1} \varepsilon^{x^2} = \sum_{x=y}^{y+n-1} \varepsilon^{x^2} = \sum_{S=0}^{n-1} \varepsilon^{(y+S)^2}$;

n ning toq qiymatlarida

$$S' = \sum_{y=0}^{n-1} \varepsilon^{-y^2},\ S'S = \sum_{y=0}^{n-1} \varepsilon^{-y^2} S = \sum_{y=0}^{n-1}\left(e^{-y^2} \sum_{s=0}^{n-1} \varepsilon^{(y+s)^2}\right) =$$

$$= \sum_{y=0}^{n-1}\sum_{s=0}^{n-1} \varepsilon^{2ys+s^2} = \sum_{s=0}^{n-1}\left(\varepsilon^{s^2} \sum_{y=0}^{n-1} \varepsilon^{2ys}\right) = n + \sum_{s=1}^{n-1} \varepsilon^{s^2} \sum_{y=0}^{n-1} \left(\varepsilon^{2s}\right)^y = n.$$

n juft bo'lganda $SS' = n + n\varepsilon^{\left(\frac{n}{2}\right)^2} = n\left(1 + (-1)^{\frac{n}{2}}\right)$, chunki n ga bo'linmaydigan $2s$ uchun $\sum_{y=0}^{n-1} \varepsilon^{2sy} = 0$. Shunday qilib, $|S| = \sqrt{n}$, agar n – toq bo'lsa va $|S| = \sqrt{n\left[1 + (-1)^{\frac{n}{2}}\right]}$, agar n – juft bo'lsa.

6-§.

3.89. *Yechish.* $u_n = 1 + \frac{a+bi}{n}$ ni trigonometrik shaklga keltiramiz va u_n^n ning absolyut qiymati va argumentining limitlarini topamiz. Natijada quyidagini hosil qilamiz:

$$r_n^n = \left|u_n^n\right| = \left(1 + \frac{2a}{n} + \frac{a^2+b^2}{n^2}\right)^{n/2} \to e^a;\ arg\, u_n^n = n\varphi_n,$$

bunda $sin\,\varphi_\mathrm{n} = \frac{b}{nr_n} \to 0$. $\varphi_n \to 0$, deb hisoblab, $n\varphi_n = \frac{b}{r_n} \cdot \frac{\varphi_n}{\sin\varphi_n} \to b$ ni hosil qilamiz. Shunday qilib, $\lim\limits_{n\to\infty} u_n^n = e^a\left(cosb + isinb\right)$.

3.90. *Yechish.*

$$\left(cos\varphi_1 + isin\varphi_1\right)\left(cos\varphi_2 + isin\varphi_2\right) = cos\left(\varphi_1 + \varphi_2\right) + isin\left(\varphi_2 + \varphi_2\right)$$

formula $e^{i\varphi_1}e^{i\varphi_2} = e^{i(\varphi_1+\varphi_2)}$ formulaga, ya'ni bir xil asosli darajalarni ko'paytirish qoidasiga aylanadi. Xuddi shunday, Muavr formulasi $\left(e^{i\varphi}\right)^n = e^{i\varphi n}$ formulaga aylanadi.

3.91. a) $1 + (2k+1)\pi\, i$; b) $ln2 + (2k+1)\pi\, i$; c) $(4k+1)\frac{\pi\, i}{2}$;

d) $(8k+1)\frac{\pi\, i}{4}$; e) –1; f) $e^{-2k\pi + i\ell n2}$; g) $e^{2k\pi + \frac{\pi}{4}}$.

3.92. $i\ arccosx + 2k\pi\, i$.

3.93. *Yechish.* $tg\varphi = \frac{sin\varphi}{cos\varphi} = \frac{1}{i} \cdot \frac{e^{i\varphi} - e^{-i\varphi}}{e^{i\varphi} + e^{-i\varphi}}$ ni hosil qilamiz. $tg\varphi = x$ bo'lsin. U holda $e^{2i\varphi} = \frac{1+ix}{1-ix}$, $\varphi = \frac{1}{2i} \cdot ln\frac{1+ix}{1-ix} + k\pi$.

IV- bob
Ko'phadlar halqasi

§ 1

4.1. a) $q(x) = x^2 - 2x + 4$, $r(x) = 3x + 11$;

b) $q(x) = 5x^2 - 15x + 34$, $r(x) = -72x - 62$;

s) $q(x) = 0$, $r(x) = 2x^2 - 3x + 1$;

d) $q(x) = 2x^2 + 3x + 11$, $r(x) = 25x - 5$.

4.2. a) $a = 0$, $q = -1$; b) $a = -3$, $q = 8$; c) $p = a^3 - 2a$, $q = a^2 - 1$;

d) $a=\pm 3,\quad q=1;\quad a=0,\quad q=-8.$

4.3. a) $r(x)=0,\quad q(x)=x^2+2\quad \boldsymbol{Z}_3[x];\quad r(x)=3x-3, q(x)=x^2+2\quad \boldsymbol{Z}_5[x]$ da va $\boldsymbol{Q}[x]$ da;

b) $r(x)=2x+1,\quad q(x)=2x+2$ v $\boldsymbol{Z}_3[x]; r(x)=2x, q(x)=2x^2$ v $\boldsymbol{Z}_5[x];\quad r(x)=2x+1, q(x)=2x^2+5$ v $\boldsymbol{Q}[x].$

4.5. EKUB: a) $x+1$; b) x^2+1; c) x^3+1; d) 1; e) 1. EKUK: a) $\frac{f(x)g(x)}{x+1}$; b) $\frac{f(x)g(x)}{3(x^2+1)}$; s) $\frac{f(x)g(x)}{3(x^3+1)}$; d) $f(x)g(x)$; e) $f(x)g(x)$.

4.6. a) $x^2-2=(-x-1)f(x)+(x+2)g(x);$

b) $x^3+1=-f(x)+(x+1)g(x);$

c) $x-1=-\frac{x-1}{3}f(x)+\frac{2x^2-2x-3}{3}g(x);$

d) $1=xf(x)+(-3x^3-x+1)g(x);$

e) $1=(-x-1)f(x)+(x^3+x^2-3x-2)g(x);$

f) $1=\frac{-x^2+3}{2}f(x)+\frac{x^4-2x^2-2}{2}g(x).$

4.9. a) $\varphi(x)=4-3x,\quad \psi(x)=1+2x+3x^2;$

b) $\varphi(x)=\frac{1}{16}(x^2-6x+9),\quad \psi(x)=-\frac{1}{16}(x^3-3x^2-4);$

c) $\varphi(x)=\frac{1}{17}(6x-11), \psi(x)=-\frac{1}{17}(6x^2-5x+25);$

d) $\varphi(x)=\frac{-16x^2+37x+26}{3},\quad \psi(x)=\frac{16x^3-53x^2-37x-23}{3}.$

4.10. a) $x^3-3x+3;$ b) $4x^4-27x^3+66x^2-65x+24;$ c) $x^4-3x^2+1.$

4.11. a) $\frac{-3+7\sqrt[3]{2}-\sqrt[3]{4}}{23}$; b) $1+3\sqrt[4]{2}+2\sqrt{2}-\sqrt[4]{8};$

c) $\frac{1}{5}(1+\sqrt{2}-\sqrt[3]{2})(1+\sqrt[3]{4}).$

4.12. a) $\varphi(x)=9x^2-26x-21,\quad \psi(x)=-9x^3+44x^2-39x-7;$

b) $\varphi(x)=3x^3+3x^2-7x+2,\quad \psi(x)=-3x^3-6x^2+x+2.$

4.13. $\varphi(x)=1+\frac{n}{1}x+\frac{n(n+1)}{1\cdot 2}x^2+\cdots+\frac{n(n+1)\cdots(n+m-2)}{1\cdot 2\cdots(m-1)}x^{m-1},$

$$\psi(x)=1+\frac{m}{1}(1-x)+\frac{m(m+1)}{1\cdot 2}(1-x)^2+\cdots+\frac{m(m+1)\cdots(m+n-2)}{1\cdot 2\cdots(n-1)}(1-x)^{n-1}=$$

$$=\frac{(m+1)(m+2)\cdots(m+n-1)}{(n-1)!}-\frac{m}{1}\frac{(m+2)\cdots(m+n-1)}{(n-2)!}x+$$

$$+\frac{m(m+1)}{1\cdot 2}\quad\frac{(m+3)\cdots(m+n-1)}{(n-3)!}x^2-\cdots+(-1)^{n-1}\frac{m(m+1)\cdots(m+n-2)}{(n-1)!}x^{n-1}.$$

Ko'rsatma. $(1-x)^n$ ga bo'linsin va $m-1$ marta differensiallab, har bir differensiallashdan so'ng $x=0$ deb olinsin. $\psi(x)$ darajasi m dan kichikligi, $\varphi(x)$ ning darajasi n dan kichikligidan foydalanish lozim.

4.14. a) $x+2$ $\boldsymbol{Z}_3[x]$ da , bir $\boldsymbol{Z}_5[x]$ va $\boldsymbol{Q}[x]$ da.

b) bir $\boldsymbol{Z}_3[x]$ da, x^3+3x+2 $\boldsymbol{Z}_5[x]$ da, $x+1$ $\boldsymbol{Q}[x]$ da.

s) bir $\boldsymbol{Z}_3[x]$ da, $x-2$ $\boldsymbol{Z}_5[x]$ da, bir $\boldsymbol{Q}[x]$ da.

4.15. a) $(f(x),g(x))=x^2+x+1,\quad \varphi(x)=x+1,\quad \psi(x)=x^2;$

b) $(f(x),g(x))=x+1,\quad \varphi(x)=x,\quad \psi(x)=x^2+1;$

s) $(f(x),g(x))=1,\quad \varphi(x)=x+1,\quad \psi(x)=x^2;$

d) $(f(x),g(x))=1,\quad \varphi(x)=x^3+x,\ \psi(x)=x^4+x+1.$

§ 2

4.16. a) 0; b) 1.

4.17. a) $q(x)=x^3-x^2+3x-3,\quad r(x)=5;$

b) $q(x)=2x^4-6x^2+13x^2-39x+109,\quad r(x)=-327;$

c) $q(x)=4x^2-(3+4i)x+7i-1,\quad r(x)=8-6i;$

d) $q(x)=x^2-2ix-5-2i,\quad r(x)=8i-9.$

4.18. a) 136; b) 286; c) $\frac{151}{16}$; d) -1-44i; e) $5+22i$.

4.19. a) $(x+1)^4-2(x+1)^3-3(x+1)^2+4(x+1)+1;$

b) $(x-1)^5+5(x-1)^4+10(x-1)^3+10(x-1)^2+5(x-1)+1;$

c) $(x-2)^4-18(x-2)+38;$

d) $(x+i)^4-2i(x+i)^3-(1+i)(x+i)^2-5(x+i)+7+5i;$

e) $(x+1-2i)^4-(x+1-2i)^3+2(x+1-2i)+1.$

4.20. a) $x^4+11x^3+45x^2+81x+55$; b) $2x^4+13x^3+35x^2+54x+39$;

c) $x^4-4x^3+6x^2+2x+8$; d) $x^5+15x^4+88x^3+255x^2+376x+229$.

4.21. a) 3; b) 4; c) 2.

4.22. x.

4.23. $-x^2-\frac{9}{2}x+\frac{3}{2}.$

4.24. $-\frac{1}{6}x^3+x^2+\frac{1}{6}x.$

4.25. a) $6x+1$; b) 1; c) $3x^3+3x+1.$

4.28. $b=9a^2,\quad 1728a^5+c^2=0.$

4.29. $a=-5.$

4.30. $a=3,\quad b=-4.$

4.31. a) $p=-\frac{8}{3},\ q=2,\ r=-\frac{1}{3}$; b) $p=-\frac{1}{3},\ q=2,\ r=-\frac{8}{3}$.

4.32. a) $a=-3,\quad k=3$; b) $a=-4,\quad k=2$; c) $a=4,\quad k=2$.

4.34. *Yechilishi.* $f(x)$ ko'phad $(x-1)^{k+1}$ ga bo'linishi uchun $f(1)=a_0+a_1+\cdots+a_n=0$ tenglikning o'rinli bo'liishi zarur va yetarlidir, $f'(x)$ ning $(x-1)^k$ ga bo'linishi uchun $f(1)=0$ shart bajarilganda $f_1(x)=nf(x)-xf'(x)$ ning $(x-1)^k$ ga bo'linishi zarur va yeatrlidir. $f_1(x)$ ni formal ravishda *n*-darajali ko'phad deb qarab yuqoridagi mulohazalarni *k* marta takrorlaymiz.

4.35. *Yechilishi*. Berilgan ko'phadning hosialsi $x^{n-m-1}[nx^m+(n-m)a]$ noldan farqli karrali ildizlarga ega emas.

4.36. *Yechilishi*. $(m,n)=d,\quad m=dm_1,\quad n=dn_1$ deb olib, $(-1)^{n_1}(n_1-m_1)^{n_1-m_1}m_1^{m_1}a^{n_1}=b^{m_1}n_1^{n_1}$ shartni hosil qilamiz.

4.37. *Ko'rsatma.* Matematik induksiya metodi bilan isbot qilinadi .

4.38. *Ko'rsatma.* $f(x)$ ko'phadning noldan farqli $(k-1)-$karrali ildizi $xf'(x)$ ko'phadning $(k-2)-$karrali ildizidan iborat va hakoza. Agar $x_1\neq 0-$ $a_1x^{m_1}+\cdots+a_kx^{m_k}$, ko'phadning $(k-1)-$karrali ildizidan ibyuorat bo'lsa, u holda $a_1x^{m_1},\cdots,a_kx^{m_k}$ sonlar quyidagi bir jinsli sistemaning yechimidan iborat bo'ladi:

$$\left.\begin{array}{l} z_1+z_2+\cdots+z_k=0\\ m_1z_1+m_2z_2+\cdots+m_kz_k=0\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ m_1^{k-2}z_1+m_2^{k-2}z_2+\cdots+m_k^{k-2}z_k=0 \end{array}\right\}$$

va demak, ular $\frac{\Delta}{\varphi'(m_1)},...,\frac{\Delta}{\varphi'(m_k)}$ - sonlarga proporsional bo'ladi, bu Δ - Vandermond determinantidan iborat.

4.39. *Yechilishi*. Agar $f(x)$ ko'phad $f'(x)$ ga bo'linsa, u holda bo'linma ko'phad bosh koeffisiyenti $\frac{1}{n}$ ga teng bo'lgan chiziqli ko'phaddan iborat, bu yerda $n-$ $f(x)$ ning darajasidan iboat. Shuning uchun $n\,f(x)=(x-x_0)\,f'(x)$. Bu tenglikni differensiallab, $(n-1)\,f'(x)=(x-x_0)\,f''(x)$ ni topamiz, va hakoza, bu yerdan $f(x)=\frac{(x-x_0)^n}{n!}f^{(n)}(x)=a_0(x-x_0)$. Teskarisi ko'0rinib turibdi.

4.40. *Yechilishi.* $f(x)=1+\frac{x}{1}+\cdots+\frac{x^n}{n!}$ ko'phadning karrali ildizi

$$f'(x)=1+\frac{x}{1}+\cdots+\frac{x^{n-1}}{(n-1)!}=f(x)-\frac{x^n}{n!}$$

Ko'phadning ildizidan iborat. Demak, agar $f(x_0)=f'(x_0)=0$ bo'lsa, u holda $x_0=0$, ammo nol $f(x)$ ko'phadning ildizi emas.

4.41. *Yechilishi. Agar* $f(x)=(x-x_0)^k\ f_1(x),$ bu yerda $f_1(x)-$kasr–rasional funksiya, ko'phad $x=x_0$ da nolga aylanmasa, u holda bevosita differensiallash natijasida quyidagini hosil qilamiz:

$$f(x_0)=f'(x_0)=\cdots=f^{(k-1)}(x_0)=0,\quad f^{(k)}(x_0)\neq 0.$$

4.42. *Yechilishi.*

$$g(x)=\frac{\psi(x)}{\omega(x)}=f(x)-f(x_0)-\frac{f'(x_0)}{1!}(x-x_0)-\cdots-\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

funksiya $g(x_0)=g'(x_0)=\cdots=g^{(n)}(x_0)=0$ shartni qanoatlantiradi. Demak, $\psi(x)=(x-x_0)^{n+1}\ F(x),$ bu yerda $F(x)-$ko'phad, shuni isbotlash talab etilgan edi.

4.43. *Yechilishi. Agar* $f_1(x)\,f_2(x_0)-f_2(x)\,f_1(x_0)$ ko'phad aynan nolga teng bo'lmasa, u holda $f_1(x_0)\neq 0$ deb hisoblash mumkin. $\dfrac{f_2(x)}{f_1(x)}-\dfrac{f_2(x_0)}{f_1(x_0)}$. kasr-rasional funksiyani qaraymiz. U aynan nolga teng emas va x_0 - uning ildizidan iborat. Uning karraligi $\dfrac{f_1(x)\,f_2'(x\ )-f_2(x)\,f_1'(x\ )}{[f_1(x)]^2}$ hosila ildizi x_0 ning karraliligidan bir birlikka katta. Bu yerdan berilgan tasdiqning isboti kelib chiqadi.

4.44. *Yechilishi.* x_0 -- $[f'(x)]^2-f(x)f''(x)$ ko'phadning k karrali ildizi bo'lsin. U holda $f(x_0)\neq 0,$ chunki, aks holda x_0 $f(x)$ va $f'(x)$ larningn umumiy ildizidan iborat bo'lar edi. Oldingi masalaga asosan x_0 -- darajasi n dan oshmaydigan $f(x)f'(x_0)-f(x_0)f'(x)$ ko'phadning $k+1$ karrali ildizidan iborat. Demak, $k+1\le n,\quad k\le n-1.$

4.45. *Yechilishi.* $f(x)f'(x_0)-f(x_0)f'(x)$ ko'phad $n-$karrali x_0 ildizga ega bo'lishi kerak, ya'ni u $A(x-x_0)^n$ ga teng bo'lishi lozim, bu yerda $A-$o'zgarmas son. $x-x_0$ ning darajalari bo'yicha yoyib, $x-x_0=z$ almashtirishdan so'ng $(a_0+a_1z+a_2z^2+\cdots+a_nz^n)a_1-(a_1+2a_2z+3a_3z^2+\cdots+na_nz^{n-1})a_0=Az^n$ ni hosil qilamiz, bu yerda $a_0=f(x_0)\neq 0.$ Bu yerdan esa $a_2=\dfrac{a_1^2}{2a_0},\quad a_3=\dfrac{a_1^3}{a_0^2 3!},...,a_n=\dfrac{a_1^n}{a_0^{n-1}n!}$ kelib chiqadi. $\dfrac{a_1}{a_0}=\alpha$ almashtirish olib,

$$f(x)=a_0\left[1+\frac{\alpha(x-x_0)}{1!}+\frac{\alpha^2(x-x_0)^2}{2!}+\cdots+\frac{\alpha^n(x-x_0)^n}{n!}\right]$$

ni hosil qilamiz.

4.46. *Yechilishi.* $u(x,y)=\dfrac{1}{2}(f(x+yi)+\bar f(x-yi))$ va $\upsilon(x,y)=\dfrac{1}{2i}(f(x+yi)-\bar f(x-yi))$ bo'lganligi uchun $u=0,\quad \upsilon=0$ sistema $f(x+yi)=0,\ \bar f(x-yi)=0$ sistemaga teng kuchli, bu yerdan $x+yi=z_k,$ $x-yi=\bar z_m$ и $x=\dfrac{z_k+\bar z_m}{2},\quad y=\dfrac{z_k-\bar z_m}{2i}.$ Bu yerda $z_k,\quad k=\overline{1,n}-$ sonlar $f(z)$

ko'phadning ildizlari, k va m indekslar esa o'zaro bog'liqsiz ravishda 1 dan n gacha bo'lgan qiymatlarni qabul qiladi.

4.47. a) $(x-1)^3(x+1)$;

b) $(x^2-4)^3(x^2+4)^2$;

s) $(x^2+1)(x^4+1)^3\cdot(x^6+1)^5(x^8+1)^7$.

4.48. $\frac{1}{2}(-1\pm i\sqrt{3}),\ \ \frac{1}{3}(1\pm i\sqrt{2})-\ f(x)$ ninng ildizlari, $\frac{1}{2}(1\pm i\sqrt{3}),\ \ \frac{1}{3}(1\pm i\sqrt{2})$- $g(x)$ ning ildizlaridan iborat.

4.49. a) $\varphi(x)=(x+1)(x-3),\quad f(x)=(x+1)^4(x-3)^2$;

b) $\varphi(x)=x^2-1,\quad f(x)=(x-1)^4(x+1)^2$;

s) $\varphi(x)=(x-1)(x^2+1),\quad f(x)=(x-1)^4(x+i)(x-i)$;

d) $\varphi(x)=(x-1)(x-2),\quad f(x)=(x-1)^2(x-2)^3$.

4.50. a) $x^4+4x^3-7x^2-22x+24$;

b) $x^4+(3-i)x^3+(3-3i)x^2+(1-3i)x-i$;

s) $x^4-3x^3+2x^2+2x-4$;

d) $x^4-19x^2-6x+72$.

4.51. a) $\frac{2}{3}$ va -$\frac{2}{3}$; b) a^2 va $(-1)^n b$.

4.52. –1 va $(-1)^{n-1}$.

4.53. a) –216; b) –25 yoki –16.

4.54. $p^3+4pq+8r=0$.

4.55. $p^2s=r$.

4.57. $\alpha_1=1,\quad \alpha_2=3,\quad \alpha_3=5,\quad \alpha_4=7$.

4.58. $a_1^2-2a_2,\quad -a_1^3+3a_1a_2-3a_3$.

4.59. $9x^3+20x^2+116x-100$.

4.60. a) $-\alpha_1,\quad \alpha_2,\ldots,\quad -\alpha_n$; b) $\frac{1}{\alpha_1},\quad \frac{1}{\alpha_2},\ldots,\quad \frac{1}{\alpha_n}$;

c) $\alpha_1-a,\quad \alpha_2-a,\ldots,\quad \alpha_n-a$; d) $b\alpha_1,\quad b\alpha_2,\ldots,\quad b\alpha_n$.

4.61. $\lambda=-3$.

4.62. $q^3+pq+q=0$.

4.63. $x^4-ax^2+1=0$, bu yerda $a=\frac{\alpha^4+1}{\alpha^2}$. *Ko'rsatma.* Berilgan tenglama x ni $-x$ ga va x ni $\frac{1}{x}$ ga almashtirish natijasida o'zgarmasligidan foydalanish kerak.

4.64. $(x^2-x+1)^3-a(x^2-x)^2=0$, bu yerda $a=\frac{(\alpha^2-\alpha+1)^3}{(\alpha^2-\alpha)^2}$. *Ko'rsatma.* Berilgan tenglama x ni $\frac{1}{x}$ ga va x ni $1-x$ ga almashtirish natijasida o'zgarmasligidan foydalanish kerak.

4.66. a) $-\frac{1}{3}(x-2)(x-3)(x-4)+\frac{1}{2}(x-1)(x-3)(x-4)-2(x-1)(x-2)(x-4)+$

$+\frac{1}{2}(x-1)(x-2)(x-3)=-\frac{4}{3}x^3+10x^2-\frac{65}{3}x+15$;

b) $\frac{1}{2}(5-(1-i)x-x^2-(1+i)x^3)$; c) x^3-3x+1; d) x^4-x^2+1.

4.67. a) $x+1+\frac{1}{24}x(x-1)(x-2)(x-3)$;

b) $-x^4+4x^3-x^2-7x+5$;

c) $1+\frac{2}{5}(x-1)-\frac{1}{105}(x-1)(4x-9)+\frac{1}{945}(x-1)(4x-9)(x-4)$,

$f(2)=1\frac{389}{945}$;

d) $x^3-9x^2+21x-8$.

4.68. $f(x)=\frac{n+1}{2}-\frac{1}{2}\sum_{k=1}^{n-1}(1-i\,ctg\,\frac{k\pi}{n})x^k$. *Ko'rsatma.* Lagranj formulasidan foydalaniladi. Natijaning har bir qo'shiluvchisida bo'lishni bajarib, o'xshash hadlarni ixchamlash kerak.

4.69. $f(x)=\sum_{k=1}^{n}\frac{y_k(x^n-1)}{(x-\varepsilon_k)n\varepsilon_k^{n-1}}=\frac{1}{n}\sum_{k=1}^{n}\frac{y_k(1-x^n)}{1-x\varepsilon_k^{-1}}$, $f(0)=\frac{1}{n}\sum_{k=1}^{n}y_k$.

4.70. *Yechilishi*. x^s ko'phad o'zining qiymatlari orqali Lagranjning interpolyasion formulasi orqali ifodalanadi: $x^s=\sum_{i=1}^{n}\frac{x_i^s\varphi(x)}{(x-x_i)\varphi'(x_i)}$. x^{n-1} ning koeffisiyentlarini taqqoslab: $\sum_{i=1}^{n}\frac{x_i^s}{\varphi'(x_i)}=0$ ni hosil qilamiz.

4.71. *Yechilishi*. $x^{n-1}=\sum_{i=1}^{n}\frac{x_i^{n-1}\varphi(x)}{(x-x_i)\varphi'(x_i)}$. x^{n-1} ning koeffisiyentlarini taqqoslab: $\sum_{i=1}^{n}\frac{x_i^{n-1}}{\varphi'(x_i)}=1$.

4.72. a) $f(x)=1+\frac{x}{1!}+\frac{x(x-1)}{2!}+\cdots+\frac{x(x-1)\cdots(x-n+1)}{n!}$;

b) $f(x)=1+\frac{(a-1)x}{1!}+\frac{(a-1)^2x(x-1)}{2!}+\cdots+\frac{(a-1)^nx(x-1)\cdots(x-n+1)}{n!}$.

Ko'rsatma. Nyuton usuli bo'yicha interpolyasion ko'phad tuzish lozim.

4.73. $f(x)=1-\frac{2x}{1!}+\frac{2x(2x-2)}{2!}+\cdots+\frac{2x(2x-2)\cdots(2x-4n+2)}{(2n)!}$.

Ko'rsatma. Izlanayotgan ko'phadning $x=0,1,2,3,...,2n$ lardagi qiymatlarini topish kerak.

4.74. $f(x)=1-\frac{x-1}{2!}+\frac{(x-1)(x-2)}{3!}-\cdots+(-1)^n\frac{(x-1)(x-2)\cdots(x-n+1)}{n!}=$ $=\frac{n!-(1-x)(2-x)\ldots(n-x)}{n!\,x.}$ *Ko'rsatma.* Masalani Nyuton usulidan foydalanib ham yechish mumkin. $F(x)=x\,f(x)-1$ ko'phadni qarash lozim, bu yerda $f(x)-$ izlanayotgan ko'phad.

4.75. $f(x)=\frac{\varphi(a)-\varphi(x)}{\varphi(a)(x-a)},$, bu yerda $\varphi(x)=(x-x_1)(x-x_2)\cdots(x-x_n)..$ *Ko'rsatma.* $(x-a)f(x)-1$ ko'phadni qarash kerak.

4.76. *Yechilishi.* Nyuton usuli bo'yicha ko'phad tuziladi. $f(x)$ ni

$$f(x)=A_0+A_1\frac{x-m}{1!}+A_2\frac{(x-m)(x-m-1)}{2!}+\cdots+A_m\frac{(x-m)(x-m-1)\cdots(x-m-n+1)}{n!},$$

ko'irinshda izlaymiz, bu yerda $m, m+1,...,m+n-$ x ning $f(x)$ butun qiymat qabul qiladigan qiymatlaridan iborat.

Ketma-ket $x=m,\,m+1,...,m+n$ deb olib, $A_0, A_1,..., A_n : A_0 = f(m)$ larni aniqlash uchun quyidagi tengliklarni hosil qilamiz:

$$A_k = f(m+k)-A_0-\frac{k}{1!}A_1-\frac{k(k-1)}{2!}A_2-\cdots-kA_{k-1}, \quad k=\overline{1,n},$$

Bu yerdan hamma A_k- koeffisiyentlarning butunligi kelib chiqadi.

x ning butun qiymatlarida $f(x)$ ning barcha qo'shiluvchilari butun A_k ko'paytuchvilarga ega binomial koeffisiyentlarga aylanadi va shuning uchun ular butun sonlardan iborat bo'ladi. Demak, $f(x)$ ko'phad x ning butun qiymatlaridan butun qiymatlarni qabul qiladi.

4.77. *Yechilishi.* $F(x)=f(x^2),$ ko'phadni qaraymiz, bu yerda $f(x)-$ izlanayotgan ko'phad. Darajasi $2n$ bo'lgan $F(x)$ ko'phad $2n+1$ ta $x=-n,\ \ -(n-1),...,-1,\,0,\,1,...,n$ qitymatlarda butun qiymatlarni qabul qiladi va oldingi masalaga asosan bu ko'phad x ning qolgan qiymatlarida ham butun qiymatlarni qabul qiladi.

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4.78. a) x^3+x+1; b) $(x+1)^3(x^2+x+1)$; c) $(x+3)(x^2+4x+2)$;

d) $(x^2+x+1)(x^2+2x+4)$.

4.79. $f_1(x)=x^2+1$; $f_2(x)=x^2+x+2$, $f_3(x)=x^2+2x+2$.

4.80. $f_1(x)=x^3+2x-1$, $f_2(x)=x^3+2x+2$, $f_3(x)=x^3+x^2+2$, $f_4(x)=x^3+2x^2+1$, $f_5(x)=x^3+x^2+x+2$, $f_6(x)=x^3+x^2+2x+1$, $f_7(x)=x^3+2x^2+x+1$, $f_8(x)=x^3+2x^2+2x+2$.

4.81. a) $(x-2)(x+1+i\sqrt{3})(x+1-i\sqrt{3})$, $(x-2)(x^2+2x+4)$;

b) $(x+2)(x-1+i\sqrt{3})(x-1-i\sqrt{3})$, $(x+2)(x^2-2x+4)$;

c) $(x-2)(x+2)(x-2i)(x+2i)$, $(x-2)(x+2)(x^2+4)$;

d) $(x-\sqrt{2}-i\sqrt{2})(x-\sqrt{2}+i\sqrt{2})(x+\sqrt{2}-i\sqrt{2})(x+\sqrt{2}+i\sqrt{2}),\quad (x^2-2\sqrt{2}x+4)(x^2+2\sqrt{2}x+4);$

$e)\quad (x-i\sqrt{3})(x+i\sqrt{3})(x-\frac{3}{2}-i\frac{\sqrt{3}}{2})(x+\frac{3}{2}-i\frac{\sqrt{3}}{2})(x-\frac{3}{2}+i\frac{\sqrt{3}}{2})(x+\frac{3}{2}+i\frac{\sqrt{3}}{2}),$

$(x^2+3)(x^2-3x+3)(x^2+3x+3);$

f) $(x-\sqrt[4]{3})^2(x-i\sqrt{3})^2(x+\sqrt[4]{3})(x+i\sqrt[4]{3})^2,\quad (x-\sqrt[4]{3})^2(x+\sqrt[4]{3})^2(x^2+\sqrt{3})^2;$

g) $\prod_{k=0}^{n-1}(x-(\cos\frac{1+8k}{4n}\pi\pm i\sin\frac{1+8k}{4n}\pi)),\quad \prod_{k=0}^{n-1}(x^2-2\sqrt[2n]{2}x\cos\frac{8k+1}{4n}\pi+\sqrt[n]{2});$

h) $\prod_{k=0}^{n-1}(x-(cos\frac{2\pi(1+3k)}{3n}\pm isin\frac{2\pi(1+3k)}{3n})),\quad \prod_{k=0}^{n-1}(x^2-2xcos\frac{2\pi(3k+1)}{3n}+1);$

i) $\prod_{k=0}^{2n-1}(x-(\cos\frac{k\pi}{n}\pm i\sin\frac{k\pi}{n})),\quad (x^2-1)\prod_{k=1}^{n-1}(x^2-2x\cos\frac{k\pi}{n}+1);$

j) $\prod_{k=0}^{2n}(x-(\cos\frac{3k\pi}{2n+1}+i\sin\frac{2k\pi}{2n+1})),\quad (x-1)\prod_{k=1}^{n}(x^2+2x\cos\frac{2k\pi}{2n+1}+1).$

4.83. a) $x^4-(1+i)x^3-(1-i)x^2+(1+i)x-i,\quad x^5-x^4-x+1;$

b) $x^3-3(1+2i)x^2-3(3+4i)x+(11-2i),\quad x^6-6x^5+27x^4-68x^3+135x^2-150x+125$;

c) $x^3+(1-i)x^2+(1-2i)x+1-i,\quad x^6+2x^5+4x^4+4x^3+5x^2+2x+2;$

d) $x^4+6x^3+(15+2i)x^2+(18+6i)x+8+6i,\quad x^8+12x^7+66x^6+216x^5+461x^4+$ $+660x^3+624x^2+360x+100$ ye) $x^3-ix^2-x+i,\quad x^4-1.$

4.84. a) $(x-1)^2(x+2);$ b) $(x+1)^2(x^2+1);$ c) $(x-1)^3.$

4.85. x^d-1, bu yerda $d=(m,n).$ *Ko'rsatma.* Umumiy ildizlar topilsin.

4.86. x^d+a^d, agar $\frac{m}{d}$ va $\frac{n}{d}$ -- sonlar toq bo'lsa; 1, agar bu sonlardan kamida biri juft bo'lsa; $d=(m,n).$

4.87. a) $m=3n+1$ va $m=3n+2;$ b) $m=6n+1$ va $m=6n+5;$

s) $m=6n+2$ va $m=6n+4.$

4.88. a) $m=6k+1;$ b) $m=6k+4.$

4.89. Yo'q, chunki birinchi va ikkinchi hosilalar bir vaqtda nolga aylanmaydi.

4.91. a) -3; b) -2; c) -2 – ikki karrali ildiz; d) 2; e) butun ildizlar yo'q.

4.92. *Yechilishi.* $\frac{k}{l}$ ni $f(x)$ ga qo'yib, l^n ga ko'paytirilgandan so'ng: $a_0k^n+a_1k^{n-1}l+\cdots+a_{n-1}kl^{n-1}+a_nl^n=0$ ni hosil qilamiz, bu yerdan a_0k^n ning l ga, a_nl^n ning esa k ga bo'linishi kelib chiqadi. k va l sonlar o'zaro tub. Demak, a_o koeffisiyent l ga, a_n esa k ga bo'linadi.

$f(x)$ ni $(x-m)$ ning darajalari bo'yicha yoyamiz:

$f(x)=a_0(x-m)^n+c_1(x-m)^{n-1}+\cdots+c_{n-1}(x-m)+c_n.$

$c_1,...,c_n$ – koeffisiyentlar butun sonlar, chunki m – butun son: $c_n=f(m).$

$x=\frac{k}{l}$ ni qo'yib

$$a_0(k-ml)^n + c_1(k-ml)^{n-1}l + \cdots + c_{n-1}(k-ml)l^{n-1} + c_n l^n = 0,$$

ni hosil qilamiz, bu yerdan $c_n\, l^n$ ning $k-ml$ ga bo'linishi kelib chiqadi. Demak, $c_n = f(m)$ ozod had $k-ml$ bo'linadi, chunki l va $k-ml$ lar o'zaro tub.

4.93. a) 2; b) -3; c) -2; d) -3, $\frac{1}{2}$; e) $\frac{5}{2}, -\frac{3}{4}$; f) 1, -2, 3; g), h) rasional ildizlar yo'q; i) $-\frac{1}{2}-$ikki karrali ildiz.

4.94. a), d) keltirilmaydigan; b) $(3x+2)(x^2+x+1)$;

c) $(2x+1)(5x-1)(3x+1)$.

4.95. *Ko'rsatma.* c) berilgan ko'phadni $(x-1)$ ning darajalari bo'yicha yoyish kerak; e) berilgan ko'phadni $(x-1)$ ning darajalari bo'yicha yoyish kerak (yoki $x = y+1$ deb olish kerak).

4.96. *Yechilishi.* 92 masalaga asosan k va $k-l-$ bir vaqtda toq. Demak, $l-$ juft son va birga teng bo'la olmaydi.

4.97. *Yechilishi.* 92 masalaga asosan $k - x_1\, l = \pm 1,\quad k - x_2\, l = \pm 1,$ bu yerdan $(x_2 - x_1)l = \pm 2$ yoki 0. 0 qiymat olinmaydi, chunki $q > 0,\quad x_2 \neq x_1.$ Aniqlik uchun $x_2 > x_1$ deb olib, $(x_2 - x_1)l = 2$ ni hosil qilamiz. Bu tenglik $x_2 - x_1 > 2$ da o'rinli emas. Endi $x_2 - x_1 = 1$ yoki 2 deb olamiz. k va l ning $(x_2 - x_1)\, q = 2$ tenglik o'rinli bo'ladigan yagona mumkin bo'lgan qiymati $p = x_1\, q + 1, q = \dfrac{2}{x_2 - x_1},$ bu yerdan $\dfrac{p}{q} = x_1 + \dfrac{1}{q} = \dfrac{x_1 + x_2}{2}$ rasional ildizning mumkin bo'lgan yagona qiymatini hosil qilamiz.

4.99. *Yechilishi.* 2 va 3 modullar bo'yicha keltirilmaydigan ko'paytuvchilarga ajratamiz:

$$x^5 - 6x^3 + 2x^2 - 4x + 5 \equiv (x+1)(x^4 + x^3 + x^2 + x + 1)(\text{mod } 2),$$
$$x^5 - 6x^3 + 2x^2 - 4x + 5 \equiv (x^2+1)(x^3 - x - 1)(\text{mod } 3).$$

Ko'paytuvchilar mos modullar bo'yicha keltirilmaydigan ko'phadlar va ularning darajalari har xil.

4.100. *Yechilishi.* Agar $f(x+a) = f(x+b),$ u holda $f(x) = f(x+c) = f(x+2c) = \cdots = f(x+(p-1)c),$ bu yerda $c = b-a.$ Agar $b \not\equiv a(modp)$ bo'lsa, u holda $0, c, 2c, ..., (p-1)c$ ketma-ketlik chegirmalar maydonining barcha elementlarini tashkil qiladi, chunki $f(x) = f(x+1) = \cdots = f(x+p-1).$

4.101. *Yechilishi.* $\varphi(x)-$ko'phad $f(x)$ ning keltirilmaydigan ko'paytuvchisi bo'lsin. Uning darajasi 1 dan katta. $\varphi(x), \varphi(x+1), ..., \varphi(x+p-1)$ ko'phadlarning hammasi keltirilmaydigan ko'phadlardan iborat bo'lib, ular $f(x)$ ning bo'luvchilaridan iborat. Ular juft-jufti bilan har xil bo'lishi mumkin emas, chunki $f(x)$ ularning ko'paytmasi bo'lgan va darajasi $2p$ dan katta yoki teng bo'lgan ko'phadga bo'linmaydi. Demak, $\varphi(x) = \varphi(x+1) = ... = \varphi(x+p-1).$ Shuning uchun

$x=0,1,\ldots,p-1$ larda $\varphi(x)-\varphi(0)=0$ bo'ladi. Shunday qilib, $\varphi(x)$ ning darajasi p dan kichik emas va $f(x)=\varphi(x)$.

4.102. a) $(x^2+x-1)(x^2-x-1)$; b), c) keltirilmaydi;

d) $(x^2-x-1)(x^2-2)$.

4.103. *Yechilishi.* Rasional ildizlarga ega bo'lmaydigan $x^4+ax^3+bx^2+cx+d$ ko'phad keltiriladigan holda faqat ikkinchi darajali butun koeffisiyentli ko'phadlar ko'paytmasiga yoyilishi mumkin:

$$x^4+ax^3+bx^2+cx+d=(x^2+\lambda x+m)(x^2+\mu x+n).$$

Ravshanki, m soni d ning bo'luvchisi bo'ladi: $mn=d$. x^3 va x larning koeffisiyentlarini taqqoslash natijasida $\lambda+\mu=a,\quad n\lambda+m\mu=c$ larni hosil qilamiz.

Agar $m\neq n$ bo'lsa, u holda $\lambda=\dfrac{c-am}{n-m}=\dfrac{cm-am^2}{d-m^2}$, shuni isbotlash talab etilgan edi.

Agarda $m=n$ bo'lsa, u holda $d=m^2,\quad c=am.$ Bu holda λ va μ sonlar $\lambda+\mu=a,\quad \lambda\mu+2m=b$ sistemadan topiladi.

4.104. *Yechilishi.* Berilgan ko'phad keltiriladigan holda $x^5+ax^4+bx^3+cx^2+dx+e=(x^2+\lambda x+m)(x^3+\lambda' x^2+\lambda'' x+n)$ bo'lishi kerak. Ko'paytuvchilarning koeffisiyentlari butun sonlar bo'lishi kerak.

Koeffisiyentlarni solishtirish $nm=e$ ni beradi, bu yerdan m ning e ning bo'luvchisi ekanligi kelib chiqadi. Endi

$$\lambda+\lambda'=a,$$
$$n\lambda+m\lambda''=d,$$
$$m+\lambda\lambda'+\lambda''=b,$$
$$n+\lambda\lambda''+m\lambda'=c,$$

bu yerdan

$$m\lambda''-n\lambda'=d-an,$$
$$\lambda(m\lambda''-n\lambda')+m^2\lambda'-n\lambda''=cm-bn$$

va demak, $(d-an)\lambda+m^2\lambda'-n\lambda''=cm-bn.$ Bu tenglamani $\lambda+\lambda'=a,\quad n\lambda+m\lambda''=d$ tenglamalar bilan birgalikda yechib, $\lambda=\dfrac{am^3-cm^2-dn+be}{m^3-n^2+ae-dm}$ ni hosil qilamiz.

4.105. a) $(x^2-2x+3)(x^2-2x-3)$; b) keltirilmaydi;

s) $(x^2-x-4)(x^2+5x+3)$; $(x^2-2x+2)(x^2+3x+3)$.

4.106. a) $x^5+mx^3-mx+1=(x+1)(x^4-x^3+(m+1)x^2-(m+1)x+1)$;

b) $x^5+mx^3-(m+2)x+1=(x-1)(x^4+x^3+(m+1)x^2+(m+1)x-1)$;

c) $x^5+x+1=(x^2+x+1)(x^3-x^2-1)$;

d) $x^5-2x^3-x+1=(x^2+x-1)(x^3-x^2-1)$;

e) $x^5+2x^3+x+1=(x^2-x+1)(x^3+x^2+2x+1)$;

f) $x^5-4x^3+3x+1=(x^2-x-1)(x^3+x^2-2x-1)$.

4.107. x^4+px^2+q ko'phadningn keltiriladigan ko'phad bo'lishi uchun quyidagi ikki shartdan kamida birining bajarilishi zarur va yetarlidir:

a) p^2-4q rasional sonning kvadratidan iborat bo'lishi;

b) q soni μ - rasional sonning kvadrati bo'lishi, $2\mu-p$ soni esa λ rasional soning kvadratidan iborat bo'lishi.

4.108. *Yechilishi.* $f(x)=\varphi(x)\psi(x)$ va $\varphi(x),\psi(x)$ lar butun koeffisiyentlarga ega bo'lsin. $f(a_i)=-1$ bo'lganligi uchun $\varphi(a_i)=1,$ $\psi(a_i)=-1,$ yoki $\varphi(a_i)=-1,$ $\psi(a_i)=1$ bo'lishi kerak, demak,

$$\varphi(a_i)+\ \psi(a_i)=0,\quad i=\overline{1,n}.$$

Agar $\varphi(x)$ va $\psi(x)$ larning ikkalasi ham o'zgarmas bo'lmasa, u holda $\varphi(x)+\psi(x)$ ning darajasi n dan kichik, bu yerdan $\varphi(x)+\psi(x)$=0 aynan tenglik kelib chiqadi. Shunday qilib, $f(x)=-[\varphi(x)]^2$ bo'lishi kerak. Buning bo'lishi mumkin emas, chunki $f(x)$ ning bosh koeffisiyenti musbat.

4.109. *Yechilishi.* Agar n-darajali $f(x)$ ko'phad $n=2m$ yoki $n=2m+1$ da keltiriladigan bo'lsa, u holda uning biror ko'paytuvchisi $\varphi(x)$ ning darajasi m dan oshmaydi. Agar $f(x)$ ko'phad ±1 qiymatni o'zgaruvchining $2m$ tadan ortiq qiymatlarida qabul qilsa, u holda $\varphi(x)$ ham o'zgaruvchining shu qiymatlarida ±1 qiymatni qabul qiladi. $\varphi(x)$ ning bu qiymatlari orasida m tadan ko'p +1 yoki -1 tenglari uchraydi. Ammo bu holda $\varphi(x)$ = +1 yoki -1 ga aynan teng bo'ladi.

4.110. *Yechilishi.* $f(x)$ ko'phad haqiqiy ildizlarga ega emas. Demak, agar u keltiriladigan ko'phad bo'lsa, uning ko'paytuvchilari $\varphi(x)+\psi(x)$ lar ham haqiqiy ildizlarga ega bo'lmaydi va shuning uchun ular x ning haqiqiy qiymatlarida ishoralarini o'zgartirmaydi. x ning barcha haqiqiy qiymatlarida $\varphi(x)>0,\quad \psi(x)>0$ deb hisoblash mumkin. $f(a_k)=1$ bo'lganligi uchun $\varphi(a_k)=\psi(a_k)=1,\quad k=\overline{1,n}.$ Agar $\varphi(x)$ (yoki $\psi(x)$) ning darajasi n dan kichik bo'lsa, $\varphi(x)$=1 (yoki $\psi(x)$=1) aynan tenglik o'rinli bo'ladi. Demak, $\varphi(x)$ va $\psi(x)$ ning darajalari n ga teng. Bu holda $\varphi(x)=1+\alpha(x-a_1)...(x-a_n),$ $\psi(x)=1+\beta(x-a_1)...(x-a_n),$ bu yerdaye α va β -- qandaydir butun sonlar. Ammo bu holda

$f(x)=(x-a_1)^2...(x-a_n)^2+1=1+(\alpha+\beta)(x-a_1)...(x-a_n)+\alpha\beta(x-a_1)^2...(x-a_n)^2.$ x^{2n} va x^n ning darajalarini solishtirish natijasida butun ildizlarga ega bo'lmaydigan $\alpha\beta=1,\quad \alpha+\beta=0,$ tenglamalar sistemasini hosil qilamiz. Demak, $f(x)$ keltirilmaydigan ko'phaddan iborat.

4.111. *Yechilishi.* $a[\varphi(x)]^2+b\varphi(x)+1=\psi(x)\omega(x)$ bo'lsin. Ko'paytuvchilardan birining darajasi $\leq n;$ $\psi(x)$ esa $x=a_1, a_2, ..., a_n$ larda +1 qiymatni qabul qiladi. $n\geq 7$ bo'lganligi uchun $\psi(x)$ ning barcha qiymatlari bir xil ishorali bo'lishi kerak. Demak,

$$\psi(x)=\pm1+\alpha(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)=\pm1+\alpha\varphi(x).$$

Agar $\alpha \neq 0$ bo'lsa, u holda $\omega(x)$ ning ham darajasi n ga teng va $\omega(x) = \pm 1 + \beta\varphi(x)$. Ammo $a[\varphi(x)]^2 + b\varphi(x) + 1 = [\pm 1 + \alpha\varphi(x)][\pm 1 + \beta\varphi(x)]$ tenglikning bajarilishi mumkin emas, chunki $ax^2 + bx + c$ ko'phad keltirilmaydigan ko'phad.

§ 4

4.112. a) $\frac{1}{12(x-1)} - \frac{4}{3(x+2)} + \frac{9}{4(x+3)}$;

b) $-\frac{1}{6(x-1)} + \frac{1}{2(x-2)} - \frac{9}{2(x-3)} + \frac{1}{6(x-4)}$;

s) $\frac{2}{x-1} + \frac{-2+i}{2(x-i)} + \frac{-2-i}{2(x+i)}$;

d) $\frac{1}{4(x-1)} - \frac{1}{4(x+1)} - \frac{i}{4(x-i)} + \frac{i}{4(x+i)}$;

e) $\frac{1}{3}\left(\frac{1}{x-1} + \frac{\varepsilon}{x-\varepsilon} + \frac{\varepsilon^2}{x-\varepsilon^2}\right), \varepsilon = -\frac{1}{3} + \frac{i\sqrt{3}}{2}$;

f) $-\frac{1}{16}\left(\frac{1+i}{x-1-i} + \frac{1-i}{x-1+i} + \frac{-1+i}{x+1-i} + \frac{-1+i}{x+1+i}\right)$;

g) $\frac{1}{n}\sum_{k=0}^{n-1}\frac{\varepsilon_k}{x-\varepsilon_k}, \quad \varepsilon_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$;

h) $-\frac{1}{m}\sum_{k=1}^{n}\frac{\eta_k}{x-\eta_k}, \ \eta_k = cos\frac{(2k-1)\pi}{n} + i\,sin\frac{(2k-1)\pi}{n}$;

i) $\sum_{k=0}^{n}\frac{C_n^k(-1)^{n-k}}{x-k}$;

j) $\sum_{k=-n}^{n}\frac{(-1)^{n-k}C_{2n}^{n+k}}{x-k}$. *Ko'rsatma.* Lagranj formulasi yordamida osonroq chiqariladi.

4.114. a) $\frac{1}{4(x-1)^2} - \frac{1}{4(x+1)}$;

b) $\frac{1}{4(x+1)} - \frac{1}{4(x-1)} + \frac{1}{4(x-1)^2} + \frac{1}{4(x+1)^2}$;

c) $\frac{3}{(x-1)^3} - \frac{4}{(x-1)^2} + \frac{1}{x-1} - \frac{1}{(x+1)^2} - \frac{2}{x+1} + \frac{1}{x+2}$;

d) $\frac{1}{n^2}\left(\sum_{k=0}^{n-1}\frac{\varepsilon_k}{(x-\varepsilon_k)^2} - (n-1)\sum_{k=0}^{n-1}\frac{\varepsilon_k}{x-\varepsilon_k}\right), \quad \varepsilon_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}$;

e) $\frac{1}{x^m}+\frac{\frac{n}{1!}}{x^{m-1}}+\frac{\frac{n(n+1)}{2!}}{x^{m-2}}+\cdots+\frac{\frac{n(n+1)\cdots(n+m-2)}{(m-1)!}}{x}+$

$+\frac{1}{(1-x)^n}+\frac{\frac{m}{1!}}{(1-x)^{n-1}}+\frac{\frac{m(m+1)}{2!}}{(1-x)^{n-2}}+\cdots+\frac{\frac{m(m+1)\cdots(m+n-2)}{(n-1)!}}{1-x};$

f) $\frac{1}{(-4a^2)^n}\sum_{k=0}^{n-1}(2a)^{n-k}\frac{n(n+1)\cdots(n+k-1)}{k!}\left(\frac{1}{(a-x)^{n-k}}+\frac{1}{(a+x)^{n-k}}\right);$

g) $\frac{1}{(4a^2)^n}\sum_{k=0}^{n-1}(2a)^{n-k}\frac{n(n+1)\cdots(n+k-1)}{k!}\left(\frac{1}{(a-ix)^{n-k}}+\frac{1}{(a+ix)^{n-k}}\right).$

Ko'rsatma. f) $\frac{a+x}{2a}=y$ deb olish kerak; d), h) Yoyilmani apniqmas koeffisiyentlar metodi bilan izlash kerak. Uning bir qismini umumiy maxrajga ko'paytirilgandan so'ng $x=x_1,x_2,...,x_n$ qiymatlar bilan topish kerak. So'ngra differensiallab yana $x=x_1,x_2,...,x_n$ qiymatlarni berish kerak.

4.115. a) $\frac{1}{3(x-1)}-\frac{x+2}{3(x^2+x+1)};$

b) $\frac{1}{8(x-2)}-\frac{1}{8(x+2)}+\frac{1}{2(x^2+4)};$

c) $\frac{1}{8}\cdot\frac{x+2}{x^2+2x+2}-\frac{1}{8}\cdot\frac{x-2}{x^2-2x+2};$

d) $\frac{1}{18}\cdot\left(\frac{1}{x^2+3x+3}+\frac{1}{x^2-3x+3}-\frac{2}{x^2+3}\right);$

ye) $\frac{1}{2n+1}\left(\frac{1}{x-1}+2\sum_{k=1}^{n}\frac{x\cos\frac{2k(m+1)\pi}{2n+1}-\cos\frac{2km\pi}{2n+1}}{x^2-2x\cos\frac{2k\pi}{2n+1}+1}\right);$

f) $\frac{(-1)^m}{2n+1}\left(\frac{1}{x+1}+2\sum_{k=1}^{n}\frac{x\cos\frac{2k(m+1)\pi}{2n+1}+\cos\frac{2km\pi}{2n+1}}{x^2+2x\cos\frac{2k\pi}{2n+1}+1}\right);$

g) $\frac{1}{2n}\left(\frac{1}{x-1}-\frac{1}{x+1}+2\sum_{n=1}^{n-1}\frac{x\cos\frac{k\pi}{n}-1}{x^2-2x\cos\frac{k\pi}{n}+1}\right);$

h) $\frac{1}{n}\sum_{k=1}^{n}\frac{\cos\frac{(2k-1)m\pi}{n}-x\cos\frac{(2k-1)(2m+1)}{2n}\pi}{x^2-2x\cos\frac{(2k-1)\pi}{2n}+1};$

i) $\frac{1}{(n!)^2 x}+2\sum_{k=1}^{n}\frac{(-1)^k x}{(n+k)!\,(n-k)!\,(x^2+k^2)}$.

Ko'rsatma. Lagranj formulasi yordamida yoyib, so'ngra qo'shma kompleks qo'shiluvchilarni birlashtirish kerak.

4.116. a) $-\frac{1}{4(x+1)}+\frac{x-1}{4(x^2+1)}+\frac{x+1}{2(x^2+1)^2}$;

b) $-\frac{1}{x}+\frac{7}{x+1}+\frac{3}{(x+1)^2}-\frac{6x+2}{x^2+x+1}-\frac{3x+2}{(x^2+x+1)^2}$;

c) $\frac{1}{16(x-1)^2}-\frac{3}{16(x-1)}+\frac{1}{16(x+1)^2}+\frac{3}{16(x+1)}+\frac{1}{4(x^2+1)}+\frac{1}{4(x^2+1)^2}$;

d) $\frac{1}{4n^2}\left(\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}-\frac{2n-1}{x-1}+\frac{2n-1}{x+1}\right)+$

$$+\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{\sin^2\frac{k\pi}{n}\left(1-2x\cos\frac{k\pi}{n}\right)}{\left(x^2-2x\cos\frac{k\pi}{n}+1\right)^2}+\frac{1}{n^2}\sum_{k=1}^{n-1}\frac{n-\sin^2\frac{k\pi}{n}-\left(n-\frac{1}{2}\right)x\cos\frac{k\pi}{n}}{x^2-2x\cos\frac{k\pi}{n}+1}.$$

4.117. a) $\frac{\varphi'(x)}{\varphi(x)}$; b) $\frac{x\varphi'(x)-n\varphi(x)}{\varphi(x)}$; c) $\frac{(\varphi'(x))^2-\varphi(x)\varphi''(x)}{(\varphi(x))^2}$.

4.118. a) 9; b) $-\frac{\varphi'(2)}{\varphi(2)}+\frac{\varphi'(1)}{\varphi(1)}=-\frac{17}{5}$; c) 17. *Ko'rsatma. 117 masaladan foydalanilsin*. b) $\frac{1}{x^2-3x+2}$ ni sodda kasrlarga yoyish kerak.

§ 5

4.119. a) $\sigma_1^3-3\sigma_1\sigma_2$; b) $\sigma_1\sigma_2-3\sigma_3$; c) $\sigma_1^4-4\sigma_1^2\sigma_2+8\sigma_1\sigma_3$;

d) $\sigma_1^3\sigma_2^2-2\sigma_1^4\sigma_3-3\sigma_1\sigma_2^3+6\sigma_1^2\sigma_2\sigma_3+3\sigma_2^3\sigma_3-7\sigma_1\sigma_3^2$;

e) $\sigma_1\sigma_2-\sigma_3$; f) $\sigma_1^2\sigma_2^2-2\sigma_1^3\sigma_3-2\sigma_2^3+4\sigma_1\sigma_2\sigma_3-\sigma_3^2$;

g) $2\sigma_1^3-9\sigma_1\sigma_2+27\sigma_3$;

h) $\sigma_1^2\sigma_2^2-4\sigma_1^3\sigma_3-4\sigma_2^3+18\sigma_1\sigma_2\sigma_3-27\sigma_3^2$.

4.120. a) $\sigma_1\sigma_2\sigma_3-\sigma_1^2\sigma_4-\sigma_3^2$; b) $\sigma_1^2\sigma_4+\sigma_3^2-4\sigma_2\sigma_4$;

c) $\sigma_1^3-4\sigma_1\sigma_2+8\sigma_3$.

4.121. a) $\sigma_1^2-2\sigma_2$; b) $\sigma_1\sigma_3-4\sigma_4$; c) $\sigma_2^2-2\sigma_1\sigma_3+2\sigma_4$;

d) $\sigma_1^2\sigma_2-\sigma_1\sigma_3-2\sigma_2^2+4\sigma_4$; e) $\sigma_1^4-4\sigma_1^2\sigma_2+2\sigma_2^2+4\sigma_1\sigma_3-4\sigma_4$;

f) $\sigma_2\sigma_3-3\sigma_1\sigma_4+5\sigma_5$; g) $\sigma_1^2\sigma_3-2\sigma_2\sigma_3-\sigma_1\sigma_4+5\sigma_5$;

h) $\sigma_1\sigma_2^2-2\sigma_1^2\sigma_3-\sigma_2\sigma_3+5\sigma_1\sigma_4-5\sigma_5$;

i) $\sigma_1^3\sigma_2-3\sigma_1\sigma_2^2-\sigma_1^2\sigma_3+5\sigma_2\sigma_3+\sigma_1\sigma_4-5\sigma_5$;

j) $\sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2 + 5\sigma_1^2\sigma_3 - 5\sigma_2\sigma_3 - 5\sigma_1\sigma_4 + 5\sigma_5$;

k) $\sigma_2\sigma_4 - 4\sigma_1\sigma_5 + 9\sigma_6$; l) $\sigma_3^2 - 2\sigma_2\sigma_4 + 2\sigma_1\sigma_5 - 2\sigma_6$;

m) $\sigma_1^2\sigma_4 - 2\sigma_2\sigma_4 - \sigma_1\sigma_5 + 6\sigma_6$;

n) $\sigma_1\sigma_2\sigma_3 - 3\sigma_1^2\sigma_4 - 3\sigma_1^2\sigma_4 - 3\sigma_3^2 + 4\sigma_2\sigma_4 + 7\sigma_1\sigma_5 - 12\sigma_6$;

o) $\sigma_3^2 - 3\sigma_1\sigma_2\sigma_3 + 3\sigma_1^2\sigma_4 + 3\sigma_3^2 - 3\sigma_2\sigma_4 - 3\sigma_1\sigma_5 + 3\sigma_6$;

p) $\sigma_1^3\sigma_3 - 3\sigma_1\sigma_2\sigma_3 - \sigma_1^2\sigma_4 + 3\sigma_3^2 + 2\sigma_2\sigma_4 + \sigma_1\sigma_5 - 6\sigma_6$;

q) $\sigma_1^2\sigma_2^2 - 2\sigma_1^3\sigma_3 - 2\sigma_2^3 + 4\sigma_1\sigma_2\sigma_3 + 2\sigma_1^2\sigma_4 - 3\sigma_3^2 + 2\sigma_2\sigma_4 - 6\sigma_1\sigma_5 + 6\sigma_6$;

r) $\sigma_1^4\sigma_2 - 4\sigma_1^2\sigma_2^2 - \sigma_1^3\sigma_3 + 2\sigma_2^3 + 7\sigma_1\sigma_2\sigma_3 + \sigma_1^2\sigma_4 - 3\sigma_3^2 + 6\sigma_2\sigma_4 - \sigma_1\sigma_5 + 6\sigma_6$;

s) $\sigma_1^6 - 6\sigma_1^4\sigma_2 + 9\sigma_1^2\sigma_2^2 + 6\sigma_1^3\sigma_3 - 2\sigma_2^3 - 12\sigma_1\sigma_2\sigma_3 - 6\sigma_1^2\sigma_4 - 3\sigma_3^2 + 6\sigma_2\sigma_4 + 6\sigma_1\sigma_5 - 6\sigma_6$

4.122. $\sigma_k^2 - 2\sigma_{k-1}\sigma_{k+1} + 2\sigma_{k-2}\sigma_{k+2} - 2\sigma_{k-3}\sigma_{k+3} + \cdots$.

4.123. a) $\dfrac{\sigma_1\sigma_2 - 3\sigma_3}{\sigma_3}$; b) $\dfrac{2(\sigma_1^2\sigma_3 - 3\sigma_1\sigma_3 - 2\sigma_2^3)}{\sigma_1\sigma_2 - \sigma_3}$;

c) $\dfrac{\sigma_2^3 + \sigma_1^3\sigma_3 - 6\sigma_1\sigma_2\sigma_3 + 9\sigma_2^3}{\sigma_3^2}$.

4.124. a) $\dfrac{\sigma_{n-1}}{\sigma_n}$; b) $\dfrac{\sigma_{n-1}^2 - 2\sigma_{n-1}\sigma_n}{\sigma_n^2}$; c) $\dfrac{\sigma_1\sigma_{n-1} - n\sigma_n}{\sigma_n}$.

4.125. – 4.

4.126. –35.

4.127. 16.

4.128. a) $\dfrac{25}{27}$; b) $\dfrac{35}{27}$; c) $-\dfrac{1679}{625}$.

4.129. a) $a_1^2a_2^2 - 4a_1^3a_3 - 4a_2^3a_0 + 18a_0a_1a_2a_3 - 27a_0^2a_3^2$; b) $a_1^3a_3 - a_2^3a_0$;

c) $\dfrac{a_1a_2}{a_0a_3} - 9$; d) $a_1^2a_2^2 - a_1^3a_3 - a_2^3a_0$.

4.130. $S_4 = \sigma_1^4 - 4\sigma_1^2\sigma_2 + 2\sigma_2^2 + 4\sigma_1\sigma_3 - 4\sigma_4$;

$S_5 = \sigma_1^5 - 5\sigma_1^3\sigma_2 + 5\sigma_1\sigma_2^2 + 5\sigma_1^2\sigma_3 - 5\sigma_2\sigma_3 - 5\sigma_1\sigma_4 + 5\sigma_5$;

$S_6 = \sigma_1^6 - 6\sigma_1^4\sigma_2 + 9\sigma_1^2\sigma_2^2 + 6\sigma_1^3\sigma_3 - 2\sigma_2^3 - 12\sigma_1\sigma_2\sigma_3 - 6\sigma_1^2\sigma_4 + 3\sigma_3^2 +$

$+ 6\sigma_2\sigma_4 + 6\sigma_1\sigma_5 - 6\sigma_6$.

4.131. $2\sigma_2 = S_1^2 - S_2$; $6\sigma_3 = S_1^3 - 3S_1S_2 + 2S_3$;

$24\sigma_4 = S_1^4 - 6S_1^2S_2 + 8S_1S_3 + 3S_2^2 - 6S_4$;

$120\sigma_5 = S_1^5 - 10S_1^3S_2 + 20S_1^2S_3 + 15S_1S_2^2 - 20S_2S_3 - 30S_1S_4 + 24S_5$;

$720\sigma_6 = S_1^6 - 15S_1^4S_2 + 40S_1^3S_3 + 45S_1^2S_2^2 - 120S_1S_2S_3 - 15S_2^3 - 90S_1^2S_4 + 40S_3^2 +$

$+ 90S_2S_4 + 144S_1S_5 - 120S_6$;

4.132. a) 859; b) 13; c) 621; d) 16; e) 24.

4.133. $S_1 = -1, S_2 = S_3 = \cdots = S_n = 0$.

4.134. $9x^3 + 20x^2 + 116x - 100$.

4.135. $8x^3 - 12x^2 + 726x - 291.$

4.136. $x^n - a = 0.$

4.137. $x^n - \frac{a}{1}x^{n-1} + \frac{a^2}{1\cdot 2}x^{n-2} - \cdots + (-1)^n \frac{a^n}{n!} = 0.$

4.138. *Ko'rsatma.* Ikkinchi ustunni $-S_1$ ga, uchinchisini – S_2, ..., k – nchisini – $(-1)^{k-1}S_k$ ga ko'paytirib birinchi ustunga qo'shiladi, so'ngra Nyuton formulasidan foydalaniladi.

140. $n!(x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} + \cdots + (-1)^n \sigma_n).$

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